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Multiple slowly oscillating periodic solutions in coupled lossless transmission lines

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Abstract

We study the coexistence and global continuation of several slowly oscillating periodic solutions for some systems of neutral FDEs due to the interaction of temporal delay and spatial dihedral symmetries. By using the equivariant degree theory we establish general results on the existence of multiple branches of nonconstant periodic solutions, classify their symmetries, and describe their maximal continuations. As an application, we study in detail a ring of identical oscillators with identical coupling between adjacent cells and prove the existence of large amplitude phase-locked and synchronous oscillations in these ring-structured systems. We also give an example to illustrate the possibility of the coexistence of several slowly oscillatory periodic solutions when the bifurcation parameter is far away from the bifurcation point. The key in our argument is the spectral theory for circulant matrices and the construction of a Liapunov function to exclude periodic solutions of a certain integer multiple of the delay.

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1. Introduction

Many important mathematical models from physics, chemistry, biology, engineering, etc., involve both time delay and spatial symmetry. The interaction of time lag and symmetries may have significant impact on a dynamical process and can result in formation of various patterns exhibiting certain particular symmetry properties. Prediction and description of these changing patterns constitutes a complex problem related to the so-called *symmetric bifurcation* phenomena. The methods of the equivariant topology and the representation theory of Lie groups are powerful mathematical techniques, which have been effectively applied to the study of bifurcation problems with symmetry (see [1–11,13–22,26–29]). In particular, the *equivariant degree theory* provides a complete topological description of zeros of en equivariant map in terms of equivariant topological obstructions, which can be effectively used to study symmetric bifurcation problems with symmetries; the occurrence and global continuation.

The general definition of the equivariant degree $\deg_G(f,\Omega)$ on a bounded invariant open set $\Omega \subset V$ for an admissible equivariant map $f: V \to W$ between two representations of a compact Lie group G ($\dim V \geqslant \dim W$) was introduced by Ize et al. (cf. [13–16]). In their work, the equivariant degree of f is defined as an element of the equivariant homotopy groups of spheres. It was proved that this equivariant degree has all the standard properties expected from a 'degree theory'. From the applications point of view, the most interesting case is where $f: V \oplus \mathbb{R}^n \to V$ (we assume here that G acts trivially on $\mathbb{R}^n$). In this case, by applying regular normal approximations (cf. [4,8,17,22]), the map f can be deformed on Ω to $\tilde{f}$, for which the set of zeros in Ω is composed of isolated disjoint compact subsets Z_α containing elements of the same orbit type $\alpha=(H)$. As the equivariant degree expresses the topological obstructions for the existence of equivariant extensions of a map without zeros, it follows from the additivity property that these obstructions depend on the orbit types in Ω. These obstructions are called *primary* if $\dim W(H)=n$ (where $W(H)=N(H)/H$ denotes the Weyl's group of H), and *secondary* if $\dim W(H)>n$.

Another version of an equivariant degree denoted by G-Deg(f,Ω), which we will call here the *primary G-degree*, or simply *G-degree*, was introduced (independently of the work of Ize et al.) by Geba et al. in [8]. As it turned out (see [2]), the primary degree is a part of the equivariant degree corresponding to the primary obstructions. The advantage of using the primary degree lies in the fact that it is relatively easy to compute, even in the case of many classical non-abelian compact groups. Additional feature of the primary degree, for certain types of groups, is the multiplicativity property that further reduces the computations of the primary degree and permits to express it in a form of a product.

In the case of an abelian symmetry group G, the G-degree was successfully applied to many symmetric local and global Hopf bifurcation problems for functional differential equations with symmetry (cf. [6,20–22,27–29]). However, for non-abelian symmetry groups there have been little progress for the existence of bifurcations of functional differential equations using the equivariant degree method. Recent advances in this direction (see [5,4,18]) provide new opportunities for possible applications of the one-parameter G-degree to the study of global bifurcation problems with non-abelian symmetries. In particular, computational formulas were established for the groups of type $G = S^1 \times \Gamma$, where Γ is a compact subgroup of a finite extension of $SO(3)$ or a finite group (cf. [4]). In fact, in some cases it is also possible to evaluate the secondary components of the equivariant degree for the non-abelian actions.

In [19], a local theory was developed for bifurcations of delayed functional differential equations with dihedral symmetry. In particular, the joint impact of temporal delay and spatial dihedral symmetry on the occurrence and multiplicity of Hopf bifurcations of delayed functional differential equations was discussed and the orbit type classification of possible Hopf bifurcations was established. The obtained results were applied to a ring of identical oscillators to describe the occurrence of several small amplitude nonconstant symmetric periodic solutions near a bifurcation point. The results obtained in [19] clearly indicate that an equivariance with respect to a non-abelian action can have a significant impact on the number of different branches of periodic solutions via a spontaneous bifurcation in a dynamical system.

The present paper is motivated by the work of Krawcewicz et al. [18,19]. The purpose of this paper is two-fold. First, by using the equivariant bifurcation theory developed by Geba et al. (cf. [8]), we study the existence, multiplicity and global continuations of symmetric periodic solutions for the following one parameter family of neutral functional differential equations (NFDEs) with dihedral symmetry

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - b(x_t, \alpha)] = F(x_t, \alpha), \qquad \alpha \in \mathbb{R}, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\tau \geqslant 0$ is a given constant and C_τ is the Banach space of continuous bounded functions from $(-\infty, \tau]$ into $\mathbb{R}^n$ equipped with the usual supremum norm, and $b, F : C_\tau \times \mathbb{R} \to \mathbb{R}^n$ are two continuously differentiable mappings specified later. Secondly, we apply our symmetric Hopf bifurcation theorems to a ring of identical oscillators with identical coupling between adjacent cells, which arises naturally from coupled lossless transmission lines, and is governed by a neutral functional differential equation. We will show how the temporal delay (both in kinetics and coupling) and the dihedral symmetries of the system may cause various types of oscillations in the case where each cell is described by only one state variable. In particular, we will prove the existence of large amplitude phase-locked and synchronous periodic solutions in these ring-structured neutral systems. More significantly, we shall obtain the existence of large number of slowly oscillating periodic solutions. To the best of our knowledge, slowly oscillatory periodic solutions play very important role in the description of global dynamics in functional differential equations (see [23] for delay equations) but little is known about the coexistence of several such solutions.

The remainder of this paper is organized as follows. In Section 2, we extend the results in [19] to neutral functional differential equations with dihedral symmetry and discuss the global continuations of the obtained multiple branches of nonconstant periodic solutions. These results are then applied, in Section 3, to a ring of identical cells governed by neutral equations and coupled by delayed diffusion along the sides of a polygon, and several unbounded branches of synchronous oscillations and phase-locked oscillations are obtained. Finally, in Section 4, an example is given to illustrate the possibility of the coexistence of several slowly oscillatory periodic solutions when the bifurcation parameter is far away from the bifurcation point.

2. Symmetric Hopf bifurcation theorems

Let $\tau \geqslant 0$ be a given constant, n a positive integer and C_τ the Banach space of continuous bounded functions from $(-\infty, \tau]$ into $\mathbb{R}^n$ equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\infty < \theta \leqslant \tau} |\varphi(\theta)|, \qquad \varphi \in C_\tau.$$

If $x : (-\infty, \tau + A] \to \mathbb{R}^n$ is a continuous function with $A > 0$ and if $t \in [0, A]$, then $x_t \in C_\tau$ is defined by

$$x_t(\theta) = x(t + \theta), \qquad \theta \in (-\infty, \tau].$$

Also, for any $x \in \mathbb{R}^n$, we will use $\bar{x}$ to denote the constant mapping from $(-\infty, \tau]$ into $\mathbb{R}^n$ with the value $x \in \mathbb{R}^n$.

In what follows, for a compact Lie group $G = \Gamma \times S^1$ and a (closed) subgroup $H \subset G$, we will denote by (H) the conjugacy class of H, which we will call an *orbit type*. We will denote by $A_1(G)$ the free $\mathbb{Z}$-module generated by the orbit types (H) such that the Weyl's group $W(H)$ is a one-dimensional manifold admitting invariant orientation (with respect to left and right translations on $W(H)$), and by $A(\Gamma)$ we will denote the Burnside ring of Γ. Let us recall that $A(\Gamma)$ is generated by the set $\Phi(\Gamma) = \{(H) \colon \dim W(H) = 0\}$ (see [20] for more details).

Assume that V is an orthogonal representation of G, $\Omega \subset V \oplus \mathbb{R}$ an invariant open bounded set, and $f : V \oplus \mathbb{R} \to V$ a G-equivariant map such that $f(x) \neq 0$ for $x \in \partial\Omega$. Then the primary degree G-Deg(f, Ω) is an element of the $\mathbb{Z}$-module $A_1(G)$, where $A_1(G)$ is generated by the set $\Phi_1(G) = \{(H) \colon \dim W(H) = 1$ and $W(H)$ is bi-orientable$\}$ (see [4,2,8] for more details). As we have mentioned in the introduction, in some special cases, the primary G-degree possesses an additional important property, which we call the *multiplicativity property*.

We state this property only in the case of $G = D_N \times S^1$, where D_N is the dihedral group of order $2N$. In the case $G := D_N \times S^1$, the G-degree computational formulas (including the $A(D_N)$-module tables) were developed in [18]. In fact, this property is also valid for a larger class of, the so-called *regularly twisted*, compact Lie groups of type $\Gamma \times S^1$ (see [5]).

Proposition 2.1. *Assume that V is an orthogonal $G = D_N \times S^1$-representation and U is an orthogonal D_N-representation. Let $f : V \oplus \mathbb{R} \to V$ (resp. $g : U \to U$) be a G-equivariant (resp. D_N-equivariant) map such that $f(x) \neq 0$ for $x \in \partial\Omega$ (resp. $g(x) \neq 0$ for $x \in \partial\mho$), where $\Omega \subset V \oplus \mathbb{R}$ (resp. $\mho \subset U$) is an invariant open bounded subset. Then*

$$G\text{-Deg}(g \times f, \mho \times \Omega) = D_N\text{-Deg}(g, \mho) \cdot G\text{-Deg}(f, \Omega),$$

where D_N-Deg$(g, \mho) \in A(D_N)$ and G-Deg$(f, \Omega) \in A_1(G)$, and the dot '$\cdot$' denotes the multiplication in the $A(D_N)$-module $A_1(G)$.

Let us describe the orbit types for the group D_N and the orbit types for $G = D_N \times S^1$ generating $A_1(G)$. The ring $A(D_N)$ is generated by the orbit types (H) of D_N as follows: If N is an odd number, then $\Phi_0(D_N) = \{(D_k), (\mathbb{Z}_k)\colon\ k|N\}$; and if N is even then

$$\tilde{D}_k = \mathbb{Z}_k \cup \kappa\xi_N \mathbb{Z}_k, \quad \xi_N = e^{2i\pi/N}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Notice that all the generators of $A_1(D_N \times S^1)$ are the *ℓ-folded θ-twisted* subgroups of the type $K^{(\theta,\ell)} := \{(\gamma, z) \in K \times S^1; \theta(\gamma) = z^\ell\}$, where K is a subgroup of D_N and $\theta : K \to S^1$ a homomorphism. See [5,18] for more details. We have the following ℓ-folded θ-twisted subgroups of $D_N \times S^1$ with non-trivial homomorphism θ:

(i) the subgroups $D_k^{(c,\ell)}$ and $\tilde{D}_k^{(c,\ell)}$, where $c : D_k \to \mathbb{Z}_2$ is a homomorphism such that $\ker c = \mathbb{Z}_k$;
(ii) the subgroup $D_k^{(d,\ell)}$ and $\tilde{D}_k^{(d,\ell)}$ (when k is even), where $d : D_k \to \mathbb{Z}_2$ is a homomorphism such that $\ker d = D_{k/2}$;
(iii) if k is divisible by 4, then there exists one more conjugacy class of the subgroup $D_k^{(\hat{d},\ell)}$, where $\ker \hat{d} = \hat{D}_{k/2} := \mathbb{Z}_{k/2} \cup \kappa\xi_k \mathbb{Z}_{k/2}$ with $\xi_k = e^{2i\pi/k}$;
(iv) the subgroups $\mathbb{Z}_k^{(\varphi_v,\ell)}$, corresponding to the homomorphism φ_v given by $\varphi_v(z) = z^v$, where v is an integer and $z \in \mathbb{Z}_k \subset S^1 \subset \mathbb{C}$;
(v) in the case where k is an even number, we have the homomorphism $d : \mathbb{Z}_k \to \mathbb{Z}_2$ such that $\ker d = \mathbb{Z}_{k/2}$, for which we have the ℓ-folded d-twisted subgroup $\mathbb{Z}_k^{(d,\ell)}$.

Let us point out that in the case of $G = D_N \times S^1$, the multiplication in $A(D_N)$ allows us to establish the $A(D_N)$-multiplication tables for $A_1(D_N \times S^1)$. Indeed, for two generators (K), $(H) \in A(D_N)$, knowing that

$$(K) \cdot (H) = \sum_{(L)} n_L \cdot (L) \quad \text{in } A(D_N),$$

implies that for the ℓ-folded θ-twisted subgroup $H^{(\theta,\ell)}$ we have the following multiplication formula:

$$(K) \cdot (H^{(\theta,\ell)}) = \sum_{(L)} n_L \cdot (L) \quad \text{in } A(D_N),$$

where the coefficient n_L in the both formulas are the same. The relevant multiplication tables are presented in Tables 1–3. We refer to [18,20] for more details.

Suppose that $\rho : \Gamma \to O(n)$ is an orthogonal representation of the group $\Gamma := D_N$, $N \geqslant 3$, on $V := \mathbb{R}^n$. Then ρ induces naturally an isometric Banach representation of Γ on the space C_τ with the action $\cdot : \Gamma \times C_\tau \to C_\tau$ given by

$$(\gamma\varphi)(\theta) := \rho(\gamma)(\varphi(\theta)), \quad \gamma \in \Gamma, \theta \in (-\infty, \tau].$$

Let us consider the following isotypical decomposition of V with respect to the action of D_N

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \tag{2.1}$$

Table 1
Multiplication table for $A(D_N)$

	$(\mathbf{D}_m)$ $2m\nmid N$	$(\mathbf{D}_m)$ $2m\mid N$	$(\tilde{\mathbf{D}}_m)$ $2m\nmid N$	$(\tilde{\mathbf{D}}_m)$ $2m\mid N$	$(\mathbb{Z}_m)$
$(\mathbf{D}_k)$ $2k\nmid N$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\mathbf{D}_k)$ $2k\mid N$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$2(D_l)+$ $\frac{Nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k\nmid N$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k\mid N$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbb{Z}_l)$	$2(\tilde{D}_l)+$ $\frac{Nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\mathbb{Z}_k)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{2Nl}{km}(\mathbb{Z}_l)$

Note: $l = gcd(m,k)$, $m|N$ and $k|N$.

Table 2
Multiplication table for $A(D_n)$

	$(\mathbf{D}_m)$ $2m\nmid n$	$(\mathbf{D}_m)$ $2m\mid n$	$(\tilde{\mathbf{D}}_m)$ $2m\nmid n$	$(\tilde{\mathbf{D}}_m)$ $2m\mid n$	$(\mathbb{Z}_m)$
$(\mathbf{D}_k)$ $2k\nmid n$	$(D_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$(D_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\mathbf{D}_k)$ $2k\mid n$	$(D_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$2(D_l)+$ $\frac{nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k\nmid n$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k\mid n$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+$ $\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$2(\tilde{D}_l)+$ $\frac{nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\mathbb{Z}_k)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{2nl}{km}(\mathbb{Z}_l)$

Note: $l = gcd(m,k)$, $m|n$ and $k|n$.

Table 3
Table of multiplication

	$\left(\mathbf{D}_k^{(d,l)}\right)$, $2\vert k$ $2k\nmid N$	$\left(\mathbf{D}_k^{(d,l)}\right)$, $2\vert k$ $2k\vert N$
$(\mathbf{D}_r)$, $2\vert r$ $2r\nmid N$	Excluded	$2\left(D_m^{(d,l)}\right)+\frac{lm-2kr}{2kr}\left(\mathbb{Z}_m^{(d,l)}\right)$
$(\mathbf{D}_r)$, $2\vert r$ $2r\vert N$	$2\left(D_m^{(d,l)}\right)+\frac{ml-2kr}{2kr}\left(\mathbb{Z}_m^{(d,l)}\right)$	$4\left(D_m^{(d,l)}\right)+\frac{ml-4kr}{2kr}\left(\mathbb{Z}_m^{(d,l)}\right)$

Here we assume that $m=gcd(k,r)$ is such that $2m|N$.

where $k=(N+1)/2$ if N is odd, or $k=(N+4)/2$ if N is even, and

(i) $V_0:=V^{\Gamma}=\{v\in V:\ \gamma v=v,\forall\gamma\in\Gamma\}$;
(ii) each isotypical component V_j, $j=1,\dots,k$, is a direct sums of all subrepresentations of V equivalent to a fixed irreducible orthogonal representation of D_N, which can be described as follows:
  (a1) For every integer number $1\leqslant j<[N/2]$, there is an orthogonal representation ρ_j (of real type) of D_N on $\mathbb{C}$ given by
  $$\gamma z:=\gamma^j\cdot z,\quad \text{for } \gamma\in\mathbb{Z}_N \text{ and } z\in\mathbb{C},$$
  $$\kappa z:=\bar{z},$$
  where $\gamma^j\cdot z$ denotes the usual complex multiplication.
  (a2) There is a representation $c:D_N\to\mathbb{Z}_2\subset O(1)$, such that $\ker c=\mathbb{Z}_N$.
  (a3) For N even, there is an irreducible representation $d:D_N\to\mathbb{Z}_2\subset O(1)$ such that $\ker d=D_{N/2}$.
  (a4) For N divisible by 4, there is an irreducible representation $\hat{d}:D_N\to\mathbb{Z}_2\subset O(1)$ such that $\ker\hat{d}=\hat{D}_{N/2}$.

We will denote by $U:=\mathbb{C}^n$ the complexification of $V=\mathbb{R}^n$. It is not difficult to see that the isotypical decomposition (2.1) induces the following isotypical decomposition of the complex representation U:

$$U=U_0\oplus U_1\oplus\cdots\oplus U_k, \tag{2.2}$$

where $U_0:=U^{\Gamma}$ and each of the isotypical components U_j is characterized by complex representation of the following types:

(b1) For $1\leqslant j<[N/2]$, the representation η_j on $\mathbb{C}\oplus\mathbb{C}$ is given by
$$\gamma(z_1,z_2):=(\gamma^j\cdot z_1,\gamma^{-j}\cdot z_2),\quad \text{for } \gamma\in\mathbb{Z}_N, \text{ and } z_1,z_2\in\mathbb{C},$$
$$\kappa(z_1,z_2):=(z_2,z_1).$$
(b2) The representation $c:D_N\to\mathbb{Z}_2\subset U(1)$, such that $\ker c=\mathbb{Z}_N$.
(b3) In the case when N is even, the representation $d:D_N\to\mathbb{Z}_2\subset U(1)$, such that $\ker d=D_{N/2}$.

(b4) In the case when N is even, the representation $\hat{d}: D_N \to \mathbb{Z}_2 \subset U(1)$, such that $\ker \hat{d} = \hat{D}_{N/2}$.

We are going to apply the equivariant bifurcation theory developed by Geba et al. (cf. [8], see also [2,5,18]) to establish the existence, multiplicity and global continuations of symmetric periodic solutions for the following one parameter family of equivariant neutral functional differential equations (NFDEs):

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - b(x_t, \alpha)] = F(x_t, \alpha), \quad \alpha \in \mathbb{R}, \tag{2.3}$$

where $x \in \mathbb{R}^n, b, F: C_\tau \times \mathbb{R} \to \mathbb{R}^n$ are two continuously differentiable mappings satisfying the following assumptions:

(A1) $|b(\varphi, \alpha) - b(\psi, \alpha)| \leqslant \bar{k}\|\varphi - \psi\|$, where $\bar{k} \in [0, 1)$ is a constant, $\varphi,\ \psi \in C_\tau,\ \alpha \in \mathbb{R}$.
(A2) F and b are Γ-equivariant, i.e., $b(\gamma\varphi, \alpha) = \rho(\gamma)b(\varphi, \alpha)$, $F(\gamma\varphi, \alpha) = \rho(\gamma)F(\varphi, \alpha)$.
(A3) $F(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$.

An element $(x, \alpha) \in V \times \mathbb{R}$ is called a *stationary solution* of (2.3) if $F(\bar{x}, \alpha) = 0$. A complex number $\lambda \in \mathbb{C}$ is said to be a *characteristic value* of the stationary solution (x, α) if it is a root of the following *characteristic equation*

$$\det \Delta_{(x,\alpha)}(\lambda) = 0, \tag{2.4}$$

where

$$\Delta_{(x,\alpha)}(\lambda) = \lambda[\mathrm{Id} - D_\varphi b(\bar{x}, \alpha)(\mathrm{e}^{\lambda\cdot}\,\mathrm{Id})] - D_\varphi F(\bar{x}, \alpha)(\mathrm{e}^{\lambda\cdot}\,\mathrm{Id}).$$

A stationary solution (x, α) is called *nonsingular* if $\det D_x\bar{F}(x, \alpha) \neq 0$, i.e., $D_x\bar{F}(x, \alpha): V \to V$ is an isomorphism, where $\bar{F}: V \times \mathbb{R} \to V$, the restriction of F on $V \times \mathbb{R}$, is defined by

$$\bar{F}(x, \alpha) = F(\bar{x}, \alpha), \quad x \in V,\ \ \alpha \in \mathbb{R}$$

and $D_x\bar{F}(x, \alpha)$ denotes the derivative of $\bar{F}$ with respect to x at (x, α). A nonsingular stationary point (x, α) is called a *center* if it has a purely imaginary characteristic value. We will call (x, α) an *isolated center* if it is the only center in some neighborhood of (x, α) in $V \times \mathbb{R}$.

We also make the following assumption:

(A4) There exists an $\alpha_0 \in \mathbb{R}$ such that $(0, \alpha_0)$ is an isolated center with $\lambda = \mathrm{i}\beta_0$, $\beta_0 > 0$, being a characteristic value of $(0, \alpha_0)$.

Let $\Omega_1 := (0, b) \times (\beta_0 - c, \beta_0 + c) \subset \mathbb{C}$. Under assumption (A2), the constants $b > 0$, $c > 0$ and $\delta > 0$ can be chosen to be sufficiently small so that for every $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, there are no characteristic values of $(0, \alpha)$ in $\partial\Omega_1$ except $\mathrm{i}\beta_0$ for $\alpha = \alpha_0$. Note that $\Delta_{(0,\alpha)}(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, it follows that $\det_{\mathbb{C}} \Delta_{(0,\alpha_0 \pm \delta)}(\lambda) \neq 0$ for $\lambda \in \partial\Omega_1$.

Since the mappings b and F are Γ-equivariant, for every $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, the operator $\Delta_{(0,\alpha)}(\lambda): \mathbb{C}^n \to \mathbb{C}^n$ is Γ-equivariant and consequently $\Delta_{(0,\alpha)}(\lambda)U_j \subset U_j$ for every isotypical component U_j of $U = \mathbb{C}^n$, $j = 0, 1, \ldots, k$.

We put $\Delta_{\alpha,j}(\lambda) := \Delta_{(0,\alpha)}(\lambda)|_{U_j} : U_j \to U_j$. Then we have

$$\det_{\mathbb{C}} \Delta_{(0,\alpha)}(\lambda) = \prod_{j=0}^{k} \det_{\mathbb{C}} \Delta_{\alpha,j}(\lambda).$$

Solutions $\lambda \in \mathbb{C}$ of the equation $\det_{\mathbb{C}} \Delta(\lambda) = 0$, $0 \leqslant j \leqslant k$, will be called the jth *isotypical characteristic value* of $(0, \alpha)$. We also define

$$c_{1,j}(\alpha_0, \beta_0) := \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0-\delta,j}(\cdot), \Omega_1) - \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0+\delta,j}(\cdot), \Omega_1)$$

for $0 \leqslant j \leqslant k$. The number $c_{1,j}(\alpha_0, \beta_0)$ will be called the jth *isotypical crossing number*, for the isolated center $(0, \alpha_0)$ corresponding to the characteristic value $i\beta_0$.

Since an integer multiple of $i\beta_0$ can also be an jth isotypical

$$c_{\ell,j}(\alpha_0, \beta_0) := \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0-\delta,j}(\cdot), \Omega_\ell) - \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0+\delta,j}(\cdot), \Omega_\ell),$$

where $\Omega_\ell := (0, b) \times (\ell\beta_0 - c, \ell\beta_0 + c) \subset \mathbb{C}$ and the constants $b > 0, c > 0$ and $\delta > 0$ are chosen to be sufficiently small so that there are no characteristic values of $(0, \alpha)$ in $\partial\Omega_\ell$ except perhaps $i\ell\beta_0$ for $\alpha = \alpha_0$. In other words, $c_{\ell,j}(\alpha_0, \beta_0) = c_{1,j}(\alpha_0, \ell\beta_0)$. If $i\ell\beta_0$ is not a jth isotypical characteristic value of $(0, \alpha_0)$, then $c_{\ell,j}(\alpha_0, \beta_0) = 0$.

In order to establish the existence of Hopf bifurcations at the stationary point $(0, \alpha_0)$, we will reformulate the Hopf bifurcation problem for Eq. (2.3) as a $\Gamma \times S^1$-equivariant bifurcation problem (with two parameters) in an appropriate isometric Banach representation of $G = \Gamma \times S^1$. For this purpose, we make the following change of variable $x(t) = z((\beta/2\pi)t)$ for $t \in \mathbb{R}$. Then Eq. (2.3) is equivalent to the following equation:

$$\frac{d}{dt}[z(t) - b(z_{t,\beta}, \alpha)] = \frac{2\pi}{\beta} F(z_{t,\beta}, \alpha), \tag{2.5}$$

where $z_{t,\beta} \in C_\tau$ is defined by

$$z_{t,\beta}(\theta) = z\left(t + \frac{\beta}{2\pi}\theta\right), \quad \theta \in (-\infty, \tau].$$

Evidently, $z(t)$ is a one-periodic solution of (2.5), if and only if $x(t)$ is a $2\pi/\beta$-periodic solution of (2.3).

Let us identify (via the exponential isomorphism) $\mathbb{R}^1/\mathbb{Z}$ with the group S^1, and consider the Banach spaces $\mathscr{V} := L^2(S^1, \mathbb{R}^n)$, $\mathscr{W} := C(S^1, \mathbb{R}^n)$ and the Sobolev space $H^1(S^1, \mathbb{R}^n)$. It is easy to see that the space $\mathscr{V}$ (resp. $\mathscr{W}$) is an isometric Banach representation of the group $G = D_N \times S^1$ with the action being given by

$$(\gamma, \theta)z(t) = \rho(\gamma)z(t + \theta),$$

$$(\gamma, \theta) \in D_N \times S^1, \quad \text{where } t \in S^1 \text{ and } z \in \mathscr{V} \text{ (resp. } \mathscr{W}). \tag{2.6}$$

Define

$$L : H^1(S^1, \mathbb{R}^n) \to \mathscr{V}, \quad Lz(t) = \dot{z}(t),$$

$$K : H^1(S^1, \mathbb{R}^n) \to \mathscr{V}, \quad Kz(t) = \int_0^1 z(s)\, ds, \quad z \in H^1(S^1, \mathbb{R}^n), \ t \in S^1.$$

Clearly, L and K are G-equivariant with respect to action (2.6). It is easy to show that the inverse of $L+K$, which is denoted by $(L+K)^{-1}:\mathscr{V}\to\mathscr{W}$, exists and is compact. Furthermore, $(L+K)^{-1}$ can be explicitly given by

$$(L+K)^{-1}z(t)=\int_0^t z(s)\,\mathrm{d}s+\int_0^1\left(\frac{1}{2}-t+s\right)z(s)\,\mathrm{d}s,\qquad z\in\mathscr{V},\ t\in S^1. \tag{2.7}$$

By using (2.7), we can easily verify that for every $\ell\geqslant 1$,

$$(L+K)^{-1}\sin 2\ell\pi\cdot\mathrm{Id}=-\frac{1}{2\ell\pi}\cos 2\ell\pi\cdot\mathrm{Id}, \tag{2.8}$$

$$(L+K)^{-1}\cos 2\ell\pi\cdot\mathrm{Id}=\frac{1}{2\ell\pi}\sin 2\ell\pi\cdot\mathrm{Id}, \tag{2.9}$$

Define $B:\mathscr{W}\times\mathbb{R}\times(0,\infty)\to\mathscr{W}$ and $N:\mathscr{W}\times\mathbb{R}\times(0,\infty)\to\mathscr{V}$ by

$$B(z,\alpha,\beta)(t)=b(z_{t,\beta},\alpha)N:\mathscr{W}\times\mathbb{R}\times(0,\infty)\to\mathscr{V}, \tag{2.10}$$

$$N(z,\alpha,\beta)(t)=\frac{2\pi}{\beta}F(z_{t,\beta},\alpha), \tag{2.11}$$

for $z\in\mathscr{W}$, $(\alpha,\beta)\in\mathbb{R}\times(0,\infty)$.

Let $\pi:\mathscr{W}\times\mathbb{R}^2\to\mathscr{W}$ be the projection, then it can be shown that $z\in\mathscr{W}$ is a solution of (2.5), if and only if $z=f(z,\alpha,\beta)$, where $f:\mathscr{W}\times\mathbb{R}\times(0,\infty)\to\mathscr{W}$ is defined by

$$f(z,\alpha,\beta)=B(z,\alpha,\beta)+(L+K)^{-1}[N+K(\pi-B)](z,\alpha,\beta). \tag{2.12}$$

By Conditions (A1) and (A2), we see that B is a G-equivariant condensing map. Moreover, (2.12) and the compactness of $(L+K)^{-1}$ implies that $f:\mathscr{W}\times\mathbb{R}\times(0,\infty)\to\mathscr{W}$ is also a G-equivariant condensing map.

With respect to the restricted S^1-action on $\mathscr{W}$, we have the following isotypical decomposition of the space $\mathscr{W}$

$$\mathscr{W}=\overline{\bigoplus_{\ell=0}^{\infty}\mathscr{W}_\ell},$$

where $\mathscr{W}_0=\mathscr{W}^{S^1}$ is the S^1-fixed point space consisting of all constant mappings from S^1 into $\mathbb{R}^n$, and $\mathscr{W}_\ell$ with $\ell\geqslant 1$ is the vector space of all mappings of the form $x\sin 2\ell\pi\cdot+y\cos 2\ell\pi\cdot$, $x,y\in V$.

For $\ell\geqslant 1$, we can complexify $\mathscr{W}_\ell$ by defining a complex structure on $\mathscr{W}_\ell$ as follows:

$$i\cdot(x\sin 2\ell\pi\cdot+y\cos 2\ell\pi\cdot)=x\cos 2\ell\pi-y\sin 2\ell\pi\cdot,\qquad x,y\in V \tag{2.13}$$

and the isotypical Γ-decomposition (2.2) of $U=\mathbb{C}^n$ induces the following isotypical Γ-decomposition of $\mathscr{W}_\ell$

$$\mathscr{W}_\ell=\mathscr{W}_{0,\ell}\oplus\mathscr{W}_{1,\ell}\oplus\cdots\oplus\mathscr{W}_{k,\ell},\qquad \ell\geqslant 1,$$

where for any fixed $1\leqslant j\leqslant k$, the isotypical components $\mathscr{W}_{j,\ell}$, $\ell\geqslant 1$ can be described exactly by the same conditions (b1)–(b4). We refer to [19] for more details.

Since $\mathscr{W}_0=V$, we also have the following isotypical Γ-decomposition of $\mathscr{W}_0$

$$\mathscr{W}_0=\mathscr{W}_{0,0}\oplus\mathscr{W}_{1,0}\oplus\cdots\oplus\mathscr{W}_{k,0},$$

where $\mathscr{W}_{j,0}:=V_j$, $0\leqslant j\leqslant k$.

Furthermore, $\mathscr{W}_{j,\ell}$, $0 \leqslant j \leqslant k$, $\ell \geqslant 0$, are G-invariant and thus are the isotypical G-components of the representation $\mathscr{W}$.

We need a more detailed description of the G-isotypical components $\mathscr{W}_{j,\ell}$ (see [19] for details). For every isotypical component $\mathscr{W}_{j,\ell}$, we denote by $Y_{j,\ell}$ the corresponding irreducible representation of G, (i.e., $Y_{j,\ell}$ is equivalent to every irreducible subrepresentation of $\mathscr{W}_{j,\ell}$).

The first type of $\mathscr{W}_{j,\ell}$ corresponds to the irreducible four-dimensional representations $Y_{j,\ell}$ of $G = D_N \times S^1$, where the action of G on the space $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{C} \oplus \mathbb{C}$ is given by

$$(\gamma,\tau)(z_1,z_2) = (\gamma^j \tau^\ell z_1, \gamma^{-j}\tau^\ell z_2), \quad \text{for } (\gamma,\tau) \in \mathbb{Z}_N \times S^1,$$

$$(\kappa\gamma,\tau)(z_1,z_2) = (\gamma^{-j}\tau^\ell z_2, \gamma^j \tau^\ell z_1), \quad \text{for } (\kappa\gamma,\tau) \in \kappa\mathbb{Z}_N \times S^1,$$

where $(z_1,z_2) \in \mathbb{C} \times \mathbb{C}$, $1 \leqslant j < [N/2]$. We put $h = \gcd(j,N)$:

(i1) If N/h is odd, we define the following element of $A_1(D_N \times S^1)$ by

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_h \times \mathbb{Z}_\ell) + (D_h^{(c,\ell)}) - (\mathbb{Z}_h \times \mathbb{Z}_\ell).$$

(i2) If $N/h \equiv 2 \pmod 4$, we define

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (D_{2h}^{(\hat{d},\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)}).$$

(i3) If $N/h \equiv 0 \pmod 4$, we put

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (\tilde{D}_{2h}^{(d,\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)}).$$

(i4) For an isotypical component $W_{j,\ell}$ on $\mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ which is given by

$$(\gamma,\tau)z = \tau^\ell z, \quad (\gamma,\tau) \in \mathbb{Z}_N \times S^1,$$

$$(\kappa\gamma,\tau)z = -\tau^\ell z, \quad (\kappa\gamma,\tau) \in \kappa\mathbb{Z}_N \times S^1,$$

we define

$$\deg_{j,\ell} := (\tilde{D}_N^{(c,\ell)}).$$

(i5) If N is even then there is a two-dimensional irreducible representation on $Y_{j,\ell} = \mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ given by

$$(g,\tau)z = \tau^\ell z, \quad (g,\tau) \in D_{N/2} \times S^1,$$

$$(g,\tau)z = -\tau^\ell z, \quad (g,\tau) \in (D_N \setminus D_{N/2}) \times S^1.$$

We put

$$\deg_{j,\ell} := (\tilde{D}_N^{(d,\ell)}).$$

(i6) Finally, for N even and $j = N/2$, there may also be an isotypical component $\mathscr{W}_{N/2,\ell}$ corresponding to the two-dimensional representation on $Y_{N/2,\ell} := \mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ given by

$$(\gamma,\tau)z = \gamma^{N/2}\tau^\ell z, \quad (\gamma,\tau) \in \mathbb{Z}_N \times S^1,$$

$$(\kappa\gamma,\tau)z = -\gamma^{N/2}\tau^\ell z, \quad (\kappa\gamma,\tau) \in \kappa\mathbb{Z}_N \times S^1.$$

We put

$$\deg_{j,\ell} := (D_N^{(\hat{d},\ell)}).$$

(j1) For the isotypical component corresponding to the type (a1) of the irreducible representations of D_N, i.e., $\mathscr{W}_{j,0} = V_j$, where $1 \leqslant j < [N/2]$. Let $h = \gcd(j,N)$ and $m := N/h$. If m is odd, we put

$$\deg_j := (D_h) - (\mathbb{Z}_h)$$

and if m is even, we put

$$\deg_j := (D_h) + (\tilde{D}_h) - (\mathbb{Z}_h).$$

(j2) For the isotypical component $\mathscr{W}_{j,0} = V_j$ corresponding to the irreducible representation $Y_{j,0}$ of type (a2), we put

$$\deg_j := (\mathbb{Z}_N).$$

(j3) For $\mathscr{W}_{j,0}$ corresponding to the irreducible representation $Y_{j,0}$ of type (a3), we put

$$\deg_j := (D_{N/2}).$$

(j4) In the case $j = N/2$, $\mathscr{W}_{j,0} = V_j$ corresponds to a one-dimensional irreducible representation $Y_{j,0}$ of type (a4), we put

$$\deg_j := (\hat{D}_{N/2}).$$

Let us notice that the elements $\deg_{j,\ell}$, which correspond to the G-isotypical components $\mathscr{W}_{j,\ell}$ of the space $\mathscr{W}$, are $\mathbb{Z}$-linearly independent in $A_1(G)$, i.e. they generate freely a $\mathbb{Z}$-submodule of $A_1(G)$. In this way, for every element μ of this submodule, we can indicate its (j,ℓ) coefficient $\mu_{j,\ell}$, which is an integer such that $\mu = \sum_{j,\ell} \mu_{j,\ell} \deg_{j,\ell}$. Let us point out, that the elements $\deg_{j,\ell}$ are the primary G-degrees of special G-equivariant maps (called *elementary*), which are associated with the irreducible G-representations (see [5]). In the case of the elements $\deg_{j,\ell}$, we will call the orbit type $(Z_N^{(\theta_j,\ell)})$, if $\deg_{j,\ell}$ is of the type (i1)–(i3), $(\tilde{D}_N^{(c,\ell)})$, if it is of type (i4), $(\tilde{D}_N^{(d,\ell)})$, if it is of type (i5), and $(D_N^{(\hat{d},\ell)})$, if it is of type (i6), the *leading orbit type* for $\deg_{j,\ell}$. Notice that each of the elements $\deg_{j,\ell}$ is uniquely identified by its leading orbit type. Let us also recall that (D_N) is the neutral (the unit) element for the Burnside ring $A(D_N)$, and every element in $a \in A(D_N)$ can be represented in a unique way as a linear combination

$$a = \sum_{(K)} n_K \cdot (K).$$

We will say that an element $a \in A(D_N)$ is *normalized*, if $a = \sum_{(K)} n_K \cdot (K)$ and we have $n_{D_N} = 1$, $n_{\mathbb{Z}_N} = 0$, i.e. $a = (D_N) + b$, where b does not contain terms corresponding to (D_N) or $(\mathbb{Z}_N)$.

Lemma 2.1. *Let* $(D_N) + b_s$, *be normalized elements from* $A(D_N)$ *and* $d_s = \sum_{(j,\ell)} m_{j,\ell,s} \deg_{j,\ell}$, $m_{j,\ell,s} \in \mathbb{Z}$, *where* $s = 1, \ldots, k$. *Then*

$$((D_N) - (\mathbb{Z}_N)) \cdot \sum_{s=1}^{k} \left[((D_N) + b_s) \cdot \sum_{(j,\ell)} m_{j,\ell,s} \deg_{j,\ell} \right]$$

$$= 0 \;\; \Rightarrow \;\; \sum_{(j,\ell)} m_{j,\ell,s} \deg_{j,\ell} = 0,$$

where the dot '$\cdot$' *denotes the* $A(D_N)$*-multiplication in* $A_1(G)$.

Proof. Suppose that $b \in A(D_N)$ does not contain the (D_N)-component, i.e. $b = \sum_{(K)} n_K \cdot (K)$, and $n_{D_N} = 0$. Notice from the $A(D_N)$-multiplication tables for $A_1(G)$, that in the case $\deg_{j,\ell}$ is of the type (i4)–(i6), the element $b \cdot \deg_{j,\ell}$ does not contain any leading orbit type, so it cannot contribute to cancellation of the elements $\deg_{j',\ell'}$. On the other hand, if $\deg_{j,\ell}$ is of type (i1)–(i3), then leading orbit type in $b \cdot \deg_{j,\ell}$ can appear only if b contains component $m(\mathbb{Z}_N)$, $0 \neq m \in \mathbb{Z}$. On the other hand, it follows from the multiplication tables that $(\mathbb{Z}_N) \cdot \deg_{j,\ell} = 2(Z_N^{(\theta_j,\ell)})$, therefore in the product $[(D_N) + b] \cdot \deg_{j,\ell}$ the term $(Z_N^{(\theta_j,\ell)})$ appears with the coefficient $1 + 2m$. It is therefore clear that

$$((D_N) - (\mathbb{Z}_N)) \cdot \sum_{s=1}^{k} [((D_N) + b_s) \cdot m_{j,\ell,s} \deg_{j,\ell}] = 0 \;\Rightarrow\; \sum_{s=1}^{k} m_{j,\ell,s} \deg_{j,\ell} = 0.$$

This completes the proof. □

Let us point out, that it is possible to have $b \cdot \deg_{j,\ell} = 0$, for a not normalized element $0 \neq b \in A(D_N)$. For example, consider the case $N = 3$ and put $a = 2(D_3) - (\mathbb{Z}_3) + 2(D_1) - 2(\mathbb{Z}_1)$. Then, according to the above multiplication tables, we have

$$a \cdot \deg 1,1 = [2(D_3) - (\mathbb{Z}_3) + 2(D_1) - 2(\mathbb{Z}_1)] \cdot [(\mathbb{Z}_3^{(\theta_1,1)}) + (D_1 \times \mathbb{Z}_1)$$
$$+ (D_1^{(c,1)}) - (\mathbb{Z}_1 \times \mathbb{Z}_1)] = 0.$$

For every $0 \leqslant j \leqslant k$ and $\ell \geqslant 0$, we define

$$a_{j,\ell}(\alpha,\beta) := \mathrm{Id} - D_z B(0,\alpha,\beta) - (L+K)^{-1}[D_z N(0,\alpha,\beta)$$
$$+ K(\mathrm{Id} - D_z B(0,\alpha,\beta))]|_{W_{j,\ell}}$$
$$= (L+K)^{-1}[L(\mathrm{Id} - D_z B(0,\alpha,\beta)) - D_z N(0,\alpha,\beta)]|_{W_{j,\ell}}, \tag{2.14}$$

where $(\alpha,\beta) \in \mathbb{R} \times (0,\infty)$.

We have the following technical lemma (see [19]):

Lemma 2.1. *For any* $(\alpha,\beta) \in \mathbb{R} \times (0,\infty)$, *we have*

$$a_{j,0}(\alpha,\beta) = -\frac{2\pi}{\beta} D_x \bar{F}(0,\alpha)|_{V_j} \tag{2.15}$$

and

$$a_{j,\ell}(\alpha,\beta)=\frac{1}{i\ell\beta}\,\Delta_{\alpha,j}(i\ell\beta),\qquad \ell\geqslant 1. \tag{2.16}$$

Let $\lambda=\alpha+i\beta=(\alpha,\beta)\in\mathbb{R}^2=\mathbb{C}$ and $\lambda_0=\alpha_0+i\beta_0$, we define a *special neighborhood* $U(r,\rho)$ of the stationary solution $(0,\lambda_0)\in\mathscr{W}\times\mathbb{R}^2$ by

$$U(r,\rho):=\{(z,\lambda)\in\mathscr{W}\times\mathbb{C}:\|z\|<r,\ \ |\lambda-\lambda_0|<\rho\}.$$

It is clear that $U(r,\rho)$ is G-invariant with respect to action (2.6). By the implicit function theorem, we can choose sufficiently small $r>0$ and $\rho>0$ such that the equation

$$z-f(z,\lambda)=0,\qquad z\in W,\ \ \lambda\in\mathbb{C}=\mathbb{R}^2 \tag{2.17}$$

has no solution $(z,\lambda)\in\partial U(r,\rho)$ with $z\neq 0$ and $|\lambda-\lambda_0|=\rho$.

A G-invariant function $\xi:\overline{U(r,\rho)}\to\mathbb{R}$, defined by

$$\xi(z,\lambda):=|\lambda-\lambda_0|(\|z\|-r)+\|z\|$$

is called a *complementing function* with respect to $U(r,\rho)$. Define the mapping $F_\xi:\overline{U(r,\rho)}\to\mathscr{W}\times\mathbb{R}$ by $F_\xi(z,\lambda):=(z-f(z,\lambda),\xi(z,\lambda))$, where $(z,\lambda)\in\overline{U(r,\rho)}$. Then the mapping F_ξ is a G-equivariant condensing fields, and the G-equivariant degree of the map F_ξ with respect to the set $U(r,\rho)$, denoted by $G\text{-Deg}(F_\xi,U(r,\rho))$, is well defined and is an element of $A_1(D_N\times S^1)$ (see [22,29]).

By the excision property of the G-degree, it follows that $G\text{-Deg}(F_\xi,U(r,\rho))$ does not depend on the numbers $r>0$ and $\rho>0$ (for sufficiently small r and ρ), thus if $G\text{-Deg}(F_\xi,U(r,\rho))\neq 0$, then $(0,\lambda_0)$ is a bifurcation point of (2.17), i.e., there exists a continuum $C\subset U(r,\rho)$ of nonconstant periodic solution of (2.17) such that $(0,\lambda_0)\in\bar{C}$.

For each $1\leqslant j\leqslant k$, we define the numbers

$$v_j(\alpha_0,\beta_0)=\begin{cases}1 & \text{if sign}\det a_{j,0}(\alpha_0,\beta_0)=-1,\\ 0 & \text{if sign}\det a_{j,0}(\alpha_0,\beta_0)=1\end{cases} \tag{2.18}$$

and

$$v_0(\alpha_0,\beta_0)=\operatorname{sign}\det a_{0,0}(\alpha_0,\beta_0),$$

so

$$v_0(\alpha_0,\beta_0)=(-1)^{\dim\mathscr{V}_0}\operatorname{sign}\det D_x\bar{F}(0,\alpha_0)|_{\mathscr{V}_0}. \tag{2.19}$$

The exact value of $G\text{-Deg}(F_\xi,U(r,\rho))$ is given in the following lemma:

Lemma 2.2. *Assume that* (A1)–(A4) *are satisfied. Then*

$$G\text{-Deg}(F_\xi,U(r,\rho))=v_0\left(\prod_{j=1}^{k}((D_N)-v_j(\alpha_0,\beta_0)\deg_j)\right)\times\left(\sum_{\substack{j,\ell\\ \ell>0}}c_{\ell,j}(\alpha_0,\beta_0)\deg_{j,\ell}\right), \tag{2.20}$$

where $v_0 := v_0(\alpha_0, \beta_0)$ *and the products are given by the multiplication in the Burnside ring* $A(D_N)$ *and by the multiplication* $A(D_N) \times A_1(D_N \times S^1) \to A_1(D_N \times S^1)$, *respectively.*

Proof. The proof is similar to that of Theorem 3.1 in [19] and is omitted. □

Theorem 2.1. *Assume that* (A1)–(A4) *are satisfied. Then for every nonzero crossing number* $c_{\ell,j}(\alpha_0, \beta_0)$, *there exist, bifurcating from* $(0, \alpha_0, \beta_0)$, *branches of nonconstant periodic solutions of* (2.5). *More precisely, if* $h := \gcd(j, N)$, *then*

(i1) *if* $\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_h \times \mathbb{Z}_\ell) + (D_h^{(c,\ell)}) - (\mathbb{Z}_h \times \mathbb{Z}_\ell)$, *i.e.*, $N/h \equiv 1 \pmod 2$, *then there are* 2 *branches of periodic solutions with the orbit type* $(\mathbb{Z}_N^{(\theta_j,\ell)})$, $m = N/h$ *branches with the orbit type* $(D_h \times \mathbb{Z}_\ell)$, *and* $m = N/h$ *branches with the orbit type* $(D_h^{(c,\ell)})$;

(i2) *if* $\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (D_{2h}^{(\hat{d},\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)})$, *i.e.*, $N/h \equiv 2 \pmod 4$, *then there are* 2 *branches of periodic solutions with the orbit type* $(\mathbb{Z}_N^{(\theta_j,\ell)})$, $N/2h$ *branches with the orbit type* $(D_{2h}^{(d,\ell)})$, *and* $N/2h$ *branches with the orbit type* $(D_{2h}^{(\hat{d},\ell)})$;

(i3) *if* $\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (\tilde{D}_{2h}^{(d,\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)})$, *i.e.*, $N/h \equiv 0 \pmod 4$, *then there are* 2 *branches of periodic solutions with the orbit type* $(\mathbb{Z}_N^{(\theta_j,\ell)})$, $N/2h$ *branches with the orbit type* $(D_{2h}^{(d,\ell)})$, *and* $N/2h$ *branches with the orbit type* $(\tilde{D}_{2h}^{(d,\ell)})$;

(i4) *if* $\deg_{j,\ell} = (\tilde{D}_N^{(c,\ell)})$, *then there is one branch of periodic solutions of the orbit type* $(\tilde{D}_N^{(c,\ell)})$;

(i5) *if* $\deg_{j,\ell} = (\tilde{D}_N^{(d,\ell)})$, *then there is one branch of periodic solutions of the orbit type* $(\tilde{D}_N^{(d,\ell)})$;

(i6) *if* $\deg_{j,\ell} = (D_N^{(\hat{d},\ell)})$, *then there is one branch of periodic solutions of the orbit type* $(D_N^{(\hat{d},\ell)})$.

Proof. The proof is similar to that of Theorem 3.2 in [19] and thus is omitted. □

To describe the global continuation of the local bifurcation obtained in Theorem 2.1, we introduce the period p of a periodic solution as an additional parameter. In other words, we will put $p = 2\pi/\beta$ in system (2.5). With this in mind, we can rewrite (2.5) as

$$\frac{\mathrm{d}}{\mathrm{d}t}[z(t) - b(z_{t,2\pi/p}, \alpha)] = pF(z_{t,2\pi/p}, \alpha), \tag{2.21}$$

where $z_{t,2\pi/p} \in C_\tau$ is given by

$$z_{t,2\pi/p}(\theta) = z(t + \theta/p), \quad \theta \in (-\infty, \tau].$$

Using the same notations as in (2.12), we can define

$$\begin{aligned}\bar{f}(z, \alpha, p) &:= f(z, \alpha, 2\pi/p)\\ &= B(z, \alpha, 2\pi/p) + (L + K)^{-1}[N + K(\pi - B)](z, \alpha, 2\pi/p).\end{aligned}$$

Therefore, we can reduce (2.21) to the following fixed point problem

$$z=\bar{f}(z,\alpha,p),\quad z\in\mathscr{W}. \tag{2.22}$$

We also need the following assumptions:

(A5) $D_x\bar{F}(0,\alpha)\in GL(\mathbb{R}^n)$ for every $\alpha\in\mathbb{R}$.
(A6) The set $M^*:=\{\alpha\in\mathbb{R};\ (0,\alpha)$ has pure imaginary characteristic values$\}$ is complete and discrete in $\mathbb{R}$.

Let us define

$$M:=\{(0,\alpha,p);\alpha\in\mathbb{R},p>0\}\subset W\times\mathbb{R}^2.$$

We have the following global symmetric Hopf bifurcation theorem:

Theorem 2.2. *Assume that* (A1)–(A3), (A5) *and* (A6) *are satisfied. Let S denote the closure of the set of all nontrivial periodic solutions of* (2.21). *Then for each bounded connected component C of S, $C\cap M$ is a finite set and if*

$$C\cap M=\{(0,\alpha_1,p_1),\dots,(0,\alpha_q,p_q)\},$$

then

$$\sum_{s=1}^{q}v_0(\alpha_s,2\pi/p_s)c_{\ell,j}(\alpha_s,2\pi/p_s)=0,$$

for every $\ell\geqslant 1$ and $j=0,1,\dots,k$.

Proof. Note that every point of $C\cap M$ is a bifurcation point and by (A6), $C\cap M\subset\{(0,\alpha,p);\alpha\in M^*,\mathrm{i}2\pi/p$ is a pure imaginary characteristic value of $(0,\alpha)\}$ is complete and discrete in $\{0\}\times\mathbb{R}^2$. Since $C\cap M\subset C$ is also bounded, it follows that the set $C\cap M$ is finite. Suppose that $C\cap M=\{(0,\alpha_1,p_1),\dots,(0,\alpha_q,p_q)\}$ for some positive integer $q>0$. Choose $r>\rho>0$ to be sufficiently small. For each $s=1,2,\dots,q$, we define a special neighborhood U_s of $(0,\alpha_s,p_s)\in\mathscr{W}\times\mathbb{R}\times(0,\infty)$ by

$$\begin{aligned}U_s:=\{(z,\alpha_s,p_s)\in\mathscr{W}\times\mathbb{R}\times(0,\infty):\|z\|<r,(\alpha-\alpha_0)^2\\+4\pi^2(1/p-1/p_s)^2<\rho^2\}\end{aligned} \tag{2.23}$$

and a complementing function $\xi_s:\bar{U}_s\to\mathbb{R}$ with respect to U_s by

$$\xi_s(z,\alpha,p):=[(\alpha-\alpha_0)^2+4\pi^2(1/p-1/p_s)^2]^{1/2}(\|z\|-r)+\|z\|. \tag{2.24}$$

Without loss of generality, we can assume that $U_i\cap U_j=\emptyset$ for $i\neq j$. Put $U=U_1\cup U_2\cup\cdots\cup U_q$. Then the set U is open and G-invariant. We can find an open bounded G-invariant subset $\Omega_1\subset W\times\mathbb{R}^2$ such that $\bar{\Omega}_1\cap M=\emptyset$, $C\setminus U\subset\Omega_1$ and $(\partial\Omega_1\setminus U)\cap S=\emptyset$.

Put $\Omega=U\cup\Omega_1$. We define a complementing function $\xi:\bar{\Omega}\to\mathbb{R}$ with respect to Ω by

$$\xi(z,\alpha,p)=\begin{cases}\xi_s(z,\alpha,p), & \text{for } (z,\alpha,p)\in U_s,\\ r, & \text{for } (z,\alpha,p)\in\bar{\Omega}\setminus U.\end{cases} \tag{2.25}$$

Define $\bar{F}_\xi : \bar{\Omega} \to W \times \mathbb{R}$ by

$$\bar{F}_\xi(z,\alpha,p) := (z - \bar{f}(z,\alpha,p), \xi(z,\alpha,p)), \quad (z,\alpha,p) \in \bar{\Omega}. \tag{2.26}$$

Then $\bar{F}_\xi$ is a G-equivariant condensing field and $\bar{F}_\xi(z,\alpha,p) \neq 0$ for all $(z,\alpha,p) \in \partial\Omega$. Consequently, the G-degree $G\text{-Deg}(\bar{F}_\xi, \Omega)$ is well-defined.

We define a homotopy $H : \bar{\Omega} \times [0,1] \to \mathscr{W} \times \mathbb{R}$ by

$$H(z,\alpha,p,t) = (z - \bar{f}(z,\alpha,p), (1-t)\xi(z,\alpha,p) - t\rho), \quad (z,\alpha,p,t) \in \bar{\Omega} \times [0,1].$$

It is easy to see that $H(z,\alpha,p,t) \neq 0$ for all $(z,\alpha,p,t) \in \partial\Omega \times [0,1]$, and thus H is an Ω-admissible homotopy. Since $H(z,\alpha,p,0) = \bar{F}_\xi(z,\alpha,p)$ and $H(z,\alpha,p,1) = (z - \bar{f}(z,\alpha,p), -\rho) \neq 0$ for all $(z,\alpha,p) \in \bar{\Omega}$, it follows that $G\text{-Deg}(\bar{F}_\xi, \Omega) = 0$.

$$\bar{F}_\xi(z,\alpha,p) = (z - \bar{f}(z,\alpha,p), r) \neq 0, \quad \text{for all } (z,\alpha,p) \in \bar{\Omega} \setminus U.$$

Let $\bar{F}_{\xi_s} := \bar{F}_\xi|_{\bar{U}_s}$. By the excision and additivity properties of G-degree, we conclude that

$$0 = G\text{-Deg}(\bar{F}_\xi, \Omega) = \sum_{s=1}^{q} G\text{-Deg}(\bar{F}_{\xi_s}, U_s). \tag{2.27}$$

On the other hand, by Lemma 2.2, we have

$$G\text{-Deg}(\bar{F}_{\xi_s}, U_s) = v_0(\alpha_s, 2\pi/p_s) \left(\prod_{j=1}^{k} ((D_N) - v_j(\alpha_s, 2\pi/p_s) \deg_j) \right) \times \left(\sum_{\substack{j,\ell \\ \ell > 0}} c_{\ell,j}(\alpha_s, 2\pi/p_s) \deg_{j,\ell} \right). \tag{2.28}$$

Consider the coefficients $v_j(\alpha_s, 2\pi/p_s)$ corresponding to $\deg'_j$ of type (j2) (i.e. $\deg_{j'} = (\mathbb{Z}_N)$. If $v_j(\alpha_s, 2\pi/p_s) = -1$ for some s, then by (A5), $v_j(\alpha_s, 2\pi/p_s) = -1$ for all $s = 1, \ldots, q$. Consequently, $(D_N) - (\mathbb{Z}_N))$ can be factored out of (2.28). Since $\prod_{j \neq j'} ((D_N) - v_j(\alpha_s, 2\pi/p_s) \deg_j)$ is a normalized element of $A(D_N)$, it follows from (2.27), (2.28), and Lemma 2.1, that for every $\ell \geqslant 1$ and $j = 0, 1, \ldots, k$,

$$\sum_{s=1}^{q} v_0(\alpha_s, 2\pi/p_s) c_{\ell,j}(\alpha_s, 2\pi/p_s) = 0.$$

The proof is complete. □

Theorem 2.3. *Assume that* (A1)–(A3), (A5) *and* (A6) *are satisfied. Suppose that for some* $\ell > 0$ *and some* $0 \leqslant j \leqslant k$, $\deg_{j,\ell}$ *is an orbit type consisting of a single closed subgroup* $H_{j,\ell}$ *of* $D_N \times S^1$. *Let* $S^{j,\ell}$ *denote the closure of the set of all nontrivial periodic solutions of* (2.21) *with the orbit type* $\deg_{j,\ell}$. *Then for each bounded connected component* $C^{j,\ell}$, $C^{j,\ell} \cap M$ *is a finite set and if*

$$C^{j,\ell} \cap M = \{(0,\alpha_1,p_1), \ldots, (0,\alpha_q,p_q)\},$$

then

$$\sum_{s=1}^{q} v_0(\alpha_s, 2\pi/p_s)c_{\ell,j}(\alpha_s, 2\pi/p_s) = 0.$$

Proof. Since $C^{j,\ell} \cap M$ is complete and discrete in $\{0\} \times \mathbb{R}^2$, $C^{j,\ell} \cap M$ is a finite set, i.e., $C^{j,\ell} \cap M = \{(0, \alpha_1, p_1), \dots, (0, \alpha_q, p_q)\}$ for some positive integer $q > 0$. Suppose that for every $s = 1, 2, \dots, q$, we have defined a special neighborhood U_s and a complementing function ξ_s, which are given by (2.23) and (2.24), respectively. Then (2.28) holds for every $s = 1, 2, \dots, q$. In particular, the $\deg_{j,\ell}$-component of $G\text{-Deg}(\bar{F}_{\xi_s}, U_s)$ is equal to

$$v_0(\alpha_s, 2\pi/p_s)\left(\prod_{j=1}^{k} ((D_N) - v_j(\alpha_s, 2\pi/p_s)\deg_j)\right) c_{\ell,j}(\alpha_s, 2\pi/p_s)\deg_{j,\ell}.$$

Since $\deg_{j,\ell}$ is an orbit type consisting of a single closed subgroup $H_{j,\ell}$ of $D_N \times S^1$, it follows that $\mathscr{W}^{H_{j,\ell}}$ is G-invariant and $\bar{f} : \mathscr{W}^{H_{j,\ell}} \times \mathbb{R} \times (0, \infty) \to \mathscr{W}^{H_{j,\ell}}$. Let $U_s^{j,\ell} = U_s \cap \mathscr{W}^{H_{j,\ell}}$ and $\xi_s^{j,\ell} = \xi_s|_{U_s^{j,\ell}}$, then $U_s^{j,\ell}$ is a special neighborhood of $(0, \alpha_s, p_s)$ in $\mathscr{W}^{H_{j,\ell}}$ and $\xi_s^{j,\ell}$ is a complementing function with respect to $U_s^{j,\ell}$. By using a similar argument as in Theorem 2.2, we can show that

$$\sum_{s=1}^{q} G\text{-Deg}(\bar{F}_{\xi_s^{j,\ell}}, U_s^{j,\ell}) = 0, \tag{2.29}$$

where $\bar{F}_{\xi_s^{j,\ell}} = \bar{F}_{\xi_s}|_{U_s^{j,\ell}}$.

From the construction of the G-degree (cf. [8]), we see that the $\deg_{j,\ell}$-component of $G\text{-Deg}(\bar{F}_{\xi_s^{j,\ell}}, U_s^{j,\ell})$ is equal to the $\deg_{j,\ell}$-component of $G\text{-Deg}(F_{\xi_s}, U_s)$. Therefore, (2.29) implies that

$$\sum_{s=1}^{q} v_0(\alpha_s, 2\pi/p_s)\left(\prod_{j=1}^{k} ((D_N) - v_j(\alpha_s, 2\pi/p_s)\deg_j)\right) c_{\ell,j}(\alpha_s, 2\pi/p_s)\deg_{j,\ell} = 0.$$

Thus, we have

$$\sum_{s=1}^{q} v_0(\alpha_s, 2\pi/p_s)c_{\ell,j}(\alpha_s, 2\pi/p_s) = 0.$$

The proof is complete. □

Finally, we consider the restricted $\mathbb{Z}_N \times S^1$-action on $\mathscr{W}$ and the $\mathbb{Z}_N \times S^1$-equivariant bifurcation problem (2.21). We have the following global symmetric Hopf bifurcation theorem. We refer to [29] for the proof.

Theorem 2.4. *Assume that* (A1)–(A3), (A5) *and* (A6) *are satisfied. For* $0 \leqslant j \leqslant N-1$, *let* S^j *denote the closure of the set of all nontrivial periodic solutions of* (2.21) *in which each periodic solution* $z(t)$ *satisfies* $\rho(e^{i(2\pi/N)})z(t + j/N) = z(t)$. *Then for each* *bounded connected component* C^j, $C^j \cap M$ *is a finite set and if*

$$C^j \cap M = \{(0, \alpha_1, p_1), \dots, (0, \alpha_q, p_q)\},$$

then

$$\sum_{s=1}^{q} v_0(\alpha_s, 2\pi/p_s) c_{\ell,j}(\alpha_s, 2\pi/p_s) = 0.$$

3. Hopf bifurcations in a ring of identical oscillators

In this section, we consider a ring of identical oscillators with identical coupling between adjacent cells. Such a ring, which was studied by Turing (cf. [25]), provides models for various situations in biology, chemistry and electrical engineering. The local Hopf bifurcation of this Turing ring has been extensively analyzed in the literature, see [1,23,29] and references therein.

We will propose models of neutral functional differential equations as the kinetics and consider the delayed coupling and diffusion in the system. We will show how the temporal delay (both in kinetics and coupling) and the dihedral symmetries of the system may cause various types of oscillations in the case where each cell is described by only one state variable. In particular, we will prove the existence of large amplitude phase-locked and synchronous periodic solutions in these ring-structured neutral systems.

Let $N \geqslant 3$ be a positive integer. We consider now a ring of N identical cells that are coupled by diffusion along the sides of an N-gon (see Fig. 1 below).

We assume that the state of jth cell at the current time is completely specified by the value of one state variable at that instant. Each cell may be regarded as a chemical system and the state variable of the jth cell, denoted by x^j, may be regarded as the concentration of the chemical substance in the jth cell. We assume that coupling between cells occurs and the concentrations $x^j(t)$, $j = 1, 2, \dots, N$, of N cells satisfy

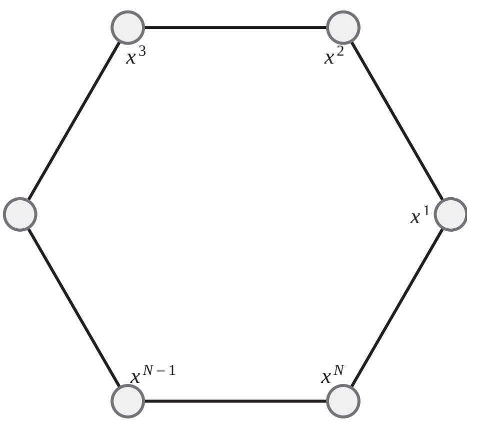

Fig. 1. Ring of N identical cells coupled by diffusion along the sides of an N-gon.

a system of neutral functional differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x^j(t) - b(x_t^j,\alpha)\right] = f\left(x_t^j,\alpha\right) + K(\alpha)\left(x_t^{j+1} + x_t^{j-1} - 2x_t^j\right), \tag{3.1}$$

where $j=1,2,\dots,N$ is expressed mod N, $t\in\mathbb{R}$ denotes the time, $\alpha\in\mathbb{R}$ is a parameter, and $b,f:C_\tau\times\mathbb{R}\to\mathbb{R}$, $C_\tau := C((-\infty,\tau],\mathbb{R})$, are continuously differentiable functionals which represent the kinetics within each cell, and $K(\alpha):C_\tau\to\mathbb{R}$ is a bounded linear functional such that the mapping $K:\mathbb{R}\to L(C_\tau,\mathbb{R})$ is continuously differentiable. The operator $K(\alpha)$ represents the coupling strength and the coupling term

$$K(\alpha)\left(x_t^{j+1} - x_t^j\right) + K(\alpha)\left(x_t^{j-1} - x_t^j\right)$$

in (3.1) is assumed to obey the ordinary *Fickian law of diffusion*.

We assume that

(H1) $f:C_\tau\times\mathbb{R}\to\mathbb{R}$ is completely continuous, $K(\alpha):C_\tau\to\mathbb{R}$ is compact for all $\alpha\in\mathbb{R}$ and there exists a constant $\bar{k}\in[0,1)$ such that

$$|b(\varphi,\alpha) - b(\psi,\alpha)| \leqslant \bar{k}\|\varphi-\psi\|, \quad \varphi,\psi\in C_\tau,\ \alpha\in\mathbb{R}.$$

(H2) $f(0,\alpha)=0$ for all $\alpha\in\mathbb{R}$.

By (H2), we see that $(0,\alpha)\in\mathbb{R}^N\times\mathbb{R}$ is a stationary solution of (3.1) and the linearization of (3.1) at $(0,\alpha)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x^j(t) - D_\varphi b(0,\alpha)x_t^j\right] = D_\varphi f(0,\alpha)x_t^j + K(\alpha)\left(x_t^{j+1} + x_t^{j-1} - 2x_t^j\right), \tag{3.2}$$

where $j=1,2,\dots,N \bmod N$. Therefore, the number $\lambda\in\mathbb{C}$ is a characteristic value if the following *characteristic equation* of (3.1) (see [12]):

$$\det_{\mathbb{C}} \Delta_\alpha(\lambda) = 0 \tag{3.3}$$

is satisfied, where for each $\alpha\in\mathbb{R},\lambda\in\mathbb{C}$, $\Delta_\alpha(\lambda):\mathbb{C}^N\to\mathbb{C}^N$ is given by

$$\Delta_\alpha(\lambda) := \operatorname{diag}(\lambda[1 - D_\varphi b(0,\alpha)\mathrm{e}^{\lambda\cdot}] - D_\varphi f(0,\alpha)\mathrm{e}^{\lambda\cdot}) - \delta(\lambda,\alpha), \tag{3.4}$$

where $\delta(\lambda,\alpha):\mathbb{C}^N\to\mathbb{C}^N$ is defined by

$$\{\delta(\lambda,\alpha)z\}_j = K(\alpha)[\mathrm{e}^{\lambda\cdot}(z^{j+1} + z^{j-1} - 2z^j)],$$

for $j=1,2,\dots,N \bmod N$ and $z=(z^1,z^2,\dots,z^N)\in\mathbb{C}^N$.

Let $\xi_N := \mathrm{e}^{\mathrm{i}\,2\pi/N}$. Then we have

$$\mathbb{C}^N = \mathbb{C}_0^N \oplus \mathbb{C}_1^N \oplus \cdots \oplus \mathbb{C}_{N-1}^N, \tag{3.5}$$

where for $j=0,1,\dots,N-1$, the subspace $\mathbb{C}_j^N$ is given by

$$\mathbb{C}_j^N = \left\{(1,\xi_N^j,\xi_N^{2j},\dots,\xi_N^{N-1})^{\mathrm{T}}z;\ z\in\mathbb{C}\right\}. \tag{3.6}$$

Put

$$\varpi_\alpha(\lambda) := \lambda[1 - D_\varphi b(0,\alpha)e^{\lambda\cdot}] - D_\varphi f(0,\alpha)e^{\lambda\cdot}. \tag{3.7}$$

Then for any $z \in \mathbb{C}$, $0 \leqslant j \leqslant N-1$ and $1 \leqslant k \leqslant N$, we have

$$\begin{aligned}
&\{\Delta_\alpha(\lambda)(1, \xi_N^j, \ldots, \xi_N^{(N-1)j})z\}_k \\
&= \Big\{\varpi_\alpha(\lambda)\xi_N^{(k-1)j} - K(\alpha)[e^{\lambda\cdot}(\xi_N^{kj} + \xi_N^{(k-2)j} - 2\xi_N^{(k-1)j}]\Big\}z \\
&= \Big\{\varpi_\alpha(\lambda) - K(\alpha)e^{\lambda\cdot}[\xi_N^j + \xi_N^j - 2]\Big\}\xi_N^{(k-1)j}z \\
&= [\varpi_\alpha(\lambda) + 4\sin^2(\pi j/N)K(\alpha)e^{\lambda\cdot}]\xi_N^{(k-1)j}z.
\end{aligned} \tag{3.8}$$

This implies that $\Delta_\alpha(\lambda)\mathbb{C}_j^N \subset \mathbb{C}_j^N$ and consequently we get the following lemma:

Lemma 3.1. *Let $\varpi_\alpha(\lambda)$ be given by (3.7). Then*

$$\det_{\mathbb{C}} \Delta_\alpha(\lambda) = \prod_{j=0}^{N-1}\left[\varpi_\alpha(\lambda) + 4\sin^2\frac{\pi j}{N}K(\alpha)e^{\lambda\cdot}\right]$$

and, consequently, $\lambda \in \mathbb{C}$ is a zero of (3.3) if and only if there exists a $j \in \{0, 1, \ldots, N-1\}$ such that

$$p_j(\lambda, \alpha) := \varpi_\alpha(\lambda) + 4\sin^2\frac{\pi j}{N}K(\alpha)e^{\lambda\cdot} = 0. \qquad \square \tag{3.9}$$

Remark 3.1. We call (3.9) the jth *characteristic equation* of (3.1). Note that $\sin^2(\pi j/N) = \sin^2(\pi(N-j)/N)$, $0 \leqslant j \leqslant N-1$. It follows that every zero of $p_j(\lambda, \alpha)$, $j \neq 0$, $N/2$, is of even multiplicity. This is due to the symmetry in the system, which forces characteristic values to be multiple.

We now make the following assumptions:

(H3) There exists an $\alpha_0 \in \mathbb{R}$ such that $\bar{f}'(0,\alpha_0) \neq 4\bar{K}(\alpha_0)\sin^2(\pi j/N)$ for every $j = 0, 1, \ldots, [N/2]$, here and in what follows, $\bar{f}'(0,\alpha)$ denote the derivative of $\bar{f}$ with respect to the first argument at $(0,\alpha)$ and $\bar{K}(\alpha) := K(\alpha)\bar{1}$.

(H4) There exist some $j \in \{0, 1, \ldots, [N/2]\}$ and positive constants $\beta_0 > 0$, $\varepsilon > 0$ and $\delta > 0$ such that

(i) $p_j(i\beta, \alpha) = 0$ for some $(\alpha, \beta) \in [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon] \times [\beta - \delta, \beta_0 + \delta]$ if and only if $\alpha = \alpha_0$ and $\beta = \beta_0$;

(ii) $p_j(u + iv, \alpha_0) = 0$ for some $(u, v) \in \partial\Omega$ with $\Omega := (0, \varepsilon) \times (\beta_0 - \delta, \beta_0 + \delta)$ if and only if $u = 0$ and $v = \beta_0$.

It is straightforward to obtain the following:

Lemma 3.2. *Assume* (H1)–(H4) *are satisfied, then the jth isotypical crossing number for the isolated center* $(0,\alpha_0)$ *corresponding to the value* $\mathrm{i}\beta_0$ *is equal to*

$$c_{1,j}(\alpha_0,\beta_0) = \begin{cases} 2[\deg_B(p_j(\cdot,\alpha_0-\delta),\Omega) \\ -\deg_B(p_j(\cdot,\alpha_0+\delta),\Omega)] & \text{if } 1 \leqslant j < [N/2], \\ \deg_B(p_j(\cdot,\alpha_0-\delta),\Omega) \\ -\deg_B(p_j(\cdot,\alpha_0+\delta),\Omega) & \text{if } j=0, \text{ or} \\ & N \text{ is even and } j=N/2, \end{cases} \tag{3.10}$$

where $p_j(\lambda,\alpha) = \varpi_\alpha(\lambda) + 4\sin^2(\pi j/N)K(\alpha)\mathrm{e}^{\lambda\cdot}$.

Now, we define an orthogonal representation $\rho: D_N \to O(N)$ of D_N on $\mathbb{R}^N$ by

$$(\xi_N x)_j := x_{j-1}, \quad \xi_N = \mathrm{e}^{\mathrm{i}2\pi/N}, \quad (\kappa x)_j := x_{N-j},$$
$$x = (x_1,x_2,\dots,x_N)^{\mathrm{T}} \in \mathbb{R}^N, \quad j = 1,2,\dots,N \,(\mathrm{mod}\, N). \tag{3.11}$$

Then it is easy to see that (3.1) is D_N-equivariant.

It is easily verified that with respect to restricted $\mathbb{Z}_N$-action, the $\mathbb{Z}_N$-isotypical decomposition of the complex representation $\mathbb{C}^N := \mathbb{R}^N + i\mathbb{R}^N$ is given by (3.5) and (3.6). Since κ sends $\mathbb{C}^N_j$ onto $\mathbb{C}^N_{-j}$, where $-j$ is taken $\mathrm{mod}\, N$, $\mathbb{C}^N$ has the following isotypical decomposition

$$C^N = U_0 \oplus U_1 \oplus U_2 \oplus \cdots \oplus U_{(N-1)/2} \qquad \text{if } N \text{ is odd},$$
$$C^N = U_0 \oplus U_1 \oplus U_2 \oplus \cdots \oplus U_{N/2-1} \oplus U_{N/2} \quad \text{if } N \text{ is even},$$

where

$$U_0 = \{(z,z,\dots,z) \in \mathbb{C}^N \colon z \in \mathbb{C}\},$$
$$U_j = \{(1,\xi_N^j,\dots,\xi_N^{(N-1)j})z \colon z \in \mathbb{C}\}$$
$$\cup\{(1,\xi_N^{-j},\dots,\xi_N^{-(N-1)j})z ; z \in \mathbb{C}\}, \quad 1 \leqslant j < N/2$$

and if N is even,

$$U_{N/2} = \{(1,\xi_N^{N/2},\dots,\xi_N^{(N-1)N/2})z \colon z \in \mathbb{C}\} = \{(1,-1,\dots,1,-1)z \colon z \in \mathbb{C}\}.$$

Every irreducible subrepresentation of U_j, $1 \leqslant j < N/2$, is equivalent to the irreducible (real) four-dimensional representation on $\mathbb{C}\times\mathbb{C}$:

$$\gamma(z_1,z_2) := (\gamma^j \cdot z_1, \gamma^{-j}\cdot z_2), \quad \kappa(z_1,z_2) := (z_2,z_1), \quad \gamma\in\mathbb{Z}_N, \ z_1,z_2\in\mathbb{C}.$$

Hence, for any fixed positive integer $\ell > 0$, the above irreducible representation corresponds to the irreducible (real) four-dimensional representation of $G = D_N \times S^1$ on $\mathbb{C}\times\mathbb{C}$:

$$(\gamma,\tau)(z_1,z_2) = (\gamma^j\tau^\ell z_1, \gamma^{-j}\tau^\ell z_2), \quad \text{for } (\gamma,\tau)\in\mathbb{Z}_N\times S^1,$$
$$(\kappa\gamma,\tau)(z_1,z_2) = (\gamma^{-j}\tau^\ell z_2, \gamma^j\tau^\ell z_1), \quad \text{for } (\kappa\gamma,\tau)\in\kappa\mathbb{Z}_N\times S^1, \tag{3.12}$$

where $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$, $1 \leqslant j < [N/2]$. Put $h = \gcd(j, N)$, then if N/h is odd, we have

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_h \times \mathbb{Z}_\ell) + (D_h^{(c,\ell)}) - (\mathbb{Z}_h \times \mathbb{Z}_\ell) \tag{3.13}$$

if $N/h \equiv 2 \pmod 4$, we have

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (D_{2h}^{(\hat{d},\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)}) \tag{3.14}$$

and if $N/h \equiv 0 \pmod 4$, we have

$$\deg_{j,\ell} = (\mathbb{Z}_N^{(\theta_j,\ell)}) + (D_{2h}^{(d,\ell)}) + (\tilde{D}_{2h}^{(d,\ell)}) - (\mathbb{Z}_{2h}^{(d,\ell)}). \tag{3.15}$$

When N is even, every irreducible subrepresentation of $U_{N/2}$ is equivalent to the irreducible two-dimensional representation

$$gz = \begin{cases} z, & g \in D_{N/2}, \\ -z, & g \in D_N \setminus D_{N/2}. \end{cases}$$

Hence, for any fixed positive integer $\ell > 0$, the above irreducible representation corresponds to the irreducible two-dimensional representation of $G = D_N \times S^1$ on $\mathbb{C} \times \mathbb{C}$:

$$(g, \tau)z = \tau^\ell z, \quad (g, \tau) \in \mathbb{Z}_{N/2} \times S^1,$$

$$(g, \tau)z = -\tau^\ell z, \quad (g, \tau) \in (\mathbb{Z}_N \setminus D_{N/2}) \times S^1. \tag{3.16}$$

Therefore, we have

$$\deg_{N/2,\ell} = (\tilde{D}_N^{(d,\ell)}). \tag{3.17}$$

By virtue of Theorem 2.1 and Lemma 3.2, we have the following

Theorem 3.1. *Assume that* (H1)–(H4) *are satisfied. If* $c_{1,j}(\alpha_0, \beta_0) \neq 0$, *then the stationary point* $(0, \alpha_0)$ *is a bifurcation point of* (3.1) *and several branches of nonconstant periodic solutions bifurcate from* $(0, \alpha_0, \beta_0)$. *More precisely, if* $h := \gcd(j, N)$, *then*

(i1) *if* $1 \leqslant j < N/2$ *and* $N/h \equiv 1 \pmod 2$, *then there are at least* 2 *branches of periodic solutions corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, N/h *branches of periodic solutions corresponding to the orbit type* $(D_h \times \mathbb{Z}_1)$, *and* N/h *branches of periodic solutions corresponding to the orbit type* $(D_h^{(c,1)})$;

(i2) *if* $1 \leqslant j < N/2$ *and* $N/h \equiv 2 \pmod 4$, *then there are at least* 2 *branches of periodic solutions corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(\hat{d},1)})$;

(i3) *if* $1 \leqslant j < N/2$ *and* $N/h \equiv 0 \pmod 4$, *then there are at least* 2 *branches of periodic solutions corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(\tilde{D}_{2h}^{(d,1)})$;

(i4) *if* N *is even and* $j = N/2$, *then there exists at least one branch of periodic solutions corresponding to the orbit type* $(\tilde{D}_N^{(d,1)})$;

(i5) *if* $j = 0$, *then there exists at least one branch of periodic solutions corresponding to the orbit type* $(D_N \times \mathbb{Z}_1)$.

Remark 3.2. We note that the obtained branches of periodic solutions are subsets of the so-called Fuller's space $BC(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}^2$. For an obtained branch of periodic solutions and a point (x, α, β) on it with $\beta > 0$, $p = 2\pi/\beta$ is a period of x, for the sake of convenience, we will call the periodic function $x \in BC(\mathbb{R}, \mathbb{R}^N)$ a *periodic solution on the branch* and the $p > 0$ a *corresponding period* of x. It can been shown that if for any positive integer $k > 1$, $ik\beta_0$ is not a characteristic value of the stationary point $(0, \alpha_0)$, then for every branch of periodic solutions bifurcating from $(0, \alpha_0, \beta_0)$ and every periodic solution $x(t)$ on it, the corresponding period $p > 0$ of x is also a minimal period of x. We refer to [6,20,21,29] for more detailed discussion in this aspect. Let $W_N := C(S^1, \mathbb{R}^N)$. Then the orthogonal representation $\rho : D_N \to O(N)$ of D_N on $\mathbb{R}^N$ induces an isometric Banach representation of $D_N \times S^1$ on W_N:

$$(\gamma, \theta)z(t) = \rho(\gamma)z(t + \theta) \quad (\gamma, \theta) \in G := D_N \times S^1, \ t \in S^1 \text{ and } z \in W_N.$$

We say that the obtained branch *corresponds* to the orbit type (H) in W_N, denoted by $C_{(H)}$, if for every periodic solution $x(t)$ belonging to the branch and with a corresponding period $p > 0$, $z \in W_N$ given by $z(t) = x(pt)$ satisfies $G_z \sim H$, i.e., the isotropy group G_z of z is conjugate to H in G.

The following Lemma 3.3 is obvious.

Lemma 3.3. *If* $\gamma \in \mathbb{Z}_N$, *then*

$$\kappa\gamma\kappa = \gamma^{-1}, \quad \kappa^{-1} = \kappa. \tag{3.18}$$

Lemma 3.4. *For every* $j \in \mathbb{Z}$, *the conjugacy class* $(\mathbb{Z}_N^{(\theta_j,1)})$ *contains exactly two closed subgroups of* $D_N \times S^1$:

$$\mathbb{Z}_N^{(\theta_j,1)} = \{(\gamma, \gamma^j) \in D_N \times S^1; \gamma \in \mathbb{Z}_N\}, \quad \mathbb{Z}_N^{(\theta_{-j},1)} = \{(\gamma, \gamma^{-j}) \in D_N \times S^1; \gamma \in \mathbb{Z}_N\}.$$

Proof. By definition, we have

$$\mathbb{Z}_N^{(\theta_j,1)} = \{(\gamma, \tau) \in \mathbb{Z}_N \times S^1; \theta_j(\gamma) = \tau = \gamma^j\} = \{(\gamma, \gamma^j) \in D_N \times S^1; \gamma \in \mathbb{Z}_N\}.$$

For every $(g, \theta) \in \mathbb{Z}_N \times S^1, \gamma \in \mathbb{Z}_N$, by Lemma 3.3, we have

$$(g, \theta)(\gamma, \gamma^j)(g, \theta)^{-1} = (g\gamma g^{-1}, \gamma^j) = (\gamma, \gamma^j)$$

and

$$(\kappa g, \theta)(\gamma, \gamma^j)(\kappa g, \theta)^{-1} = (\kappa g\gamma g^{-1}\kappa, \gamma^j) = (\kappa\gamma\kappa, \gamma^j) = (\gamma^{-1}, \gamma^j).$$

Therefore, $\mathbb{Z}_N^{(\theta_{-j},1)} = \{(\gamma, \gamma^{-j}) \in D_N \times S^1; \gamma \in \mathbb{Z}_N\}$ is conjugate to $\mathbb{Z}_N^{(\theta_j,1)}$, and $(\mathbb{Z}_N^{(\theta_j,1)})$ contains exactly two closed subgroups $\mathbb{Z}_N^{(\theta_j,1)}, \mathbb{Z}_N^{(\theta_{-j},1)}$ of $D_N \times S^1$. The proof is complete. □

Lemma 3.5. *If* $h|N$, *then the closed subgroup* $\mathbb{Z}_h \times \mathbb{Z}_1$ *of* $D_N \times S^1$ *is a subgroup of every element of the conjugacy classes* $(D_h \times \mathbb{Z}_1)$ *and* $(D_h^{(c,1)})$.

Proof. By definition, it is easy to show that

$$D_h \times \mathbb{Z}_1 = \{(\gamma, 1); \gamma \in \mathbb{Z}_h\} \cup \{(\kappa\gamma, 1); \gamma \in \mathbb{Z}_h\},$$
$$D_h^{(c,1)} = \{(\gamma, 1); \gamma \in \mathbb{Z}_h\} \cup \{(\kappa\gamma, -1); \Gamma \in \mathbb{Z}_h\}.$$

For every $(g, \theta) \in \mathbb{Z}_N \times S^1$ and $(\gamma, 1) \in \mathbb{Z}_h \times \mathbb{Z}_1$, by Lemma 3.3, we get

$$(g, \theta)(\gamma, 1)(g, \theta)^{-1} = (g\gamma g^{-1}, 1) = (\gamma, 1)$$

and

$$(\kappa g, \theta)(\gamma, 1)(\kappa g, \theta)^{-1} = (\kappa g\gamma g^{-1}\kappa, 1) = (\kappa\gamma\kappa, 1) = (\gamma^{-1}, 1).$$

Thus, $\mathbb{Z}_h \times \mathbb{Z}_1$ is a subgroup of each closed subgroup conjugate to $D_h \times \mathbb{Z}_1$ or $D_h^{(c,1)}$, and this completes the proof. □

Lemma 3.6. *If* $2h|N$, *then the closed subgroup* $\mathbb{Z}_{2h}^{(d,1)} = \{(\gamma, 1), (\xi_{2h}\gamma, -1) : \gamma \in \mathbb{Z}_h\}$ *of* $D_N \times S^1$ *is a subgroup of each closed subgroup conjugate to one of the following closed subgroups*:

$$D_{2h}^{(d,1)} = \{(\gamma, 1), (\xi_{2h}\gamma, -1), (\kappa\gamma, 1), (\kappa\xi_{2h}\gamma, -1); \gamma \in \mathbb{Z}_h\},$$

$$D_{2h}^{(\hat{d},1)} = \{(\gamma, 1), (\xi_{2h}\gamma, -1), (\kappa\gamma, -1), (\kappa\xi_{2h}\gamma, 1); \gamma \in \mathbb{Z}_h\},$$

$$\tilde{D}_{2h}^{(d,1)} = \{(\gamma, 1), (\xi_{2h}\gamma, -1), (\kappa\xi_N\gamma, 1), (\kappa\xi_N\xi_{2h}\gamma, -1); \gamma \in \mathbb{Z}_h\}.$$

Proof. The proof is similar to that of Lemma 3.5 and is omitted. □

Lemma 3.7. *If N is even, then the conjugacy class* $(\tilde{D}_N^{(d,1)})$ *contains exactly one closed subgroup given by*

$$\tilde{D}_N^{(d,1)} = \{(\gamma, 1), (\xi_N\gamma, -1), (\kappa\gamma, -1), (\kappa\xi_N\gamma, 1); \gamma \in \mathbb{Z}_{N/2}\}.$$

Proof. By definition, we can easily verify that

$$\tilde{D}_N^{(d,1)} = \{(\gamma, \tau) \in \tilde{D}_N \times S^1; d(\gamma) = \tau\}$$
$$= \{(\gamma, 1), (\xi_N\gamma, -1), (\kappa\gamma, -1), (\kappa\xi_N\gamma, 1); \gamma \in \mathbb{Z}_{N/2}\}.$$

For every $(g, \theta) \in \mathbb{Z}_N \times S^1$, by using the fact that $g^2 \in \mathbb{Z}_{N/2}$, we can show that

$$(g, \theta)\tilde{D}_N^{(d,1)}(g, \theta)^{-1} \subset \tilde{D}_N^{(d,1)}, \quad (\kappa g, \theta)\tilde{D}_N^{(d,1)}(\kappa g, \theta)^{-1} \subset \tilde{D}_N^{(d,1)}.$$

Therefore, the conjugacy class $(\tilde{D}_N^{(d,1)})$ contains only one closed subgroup $\tilde{D}_N^{(d,1)}$. The proof is complete. □

Theorem 3.2. *Assume that* (H1)–(H4) *are satisfied. If* $c_{1,j}(\alpha_0, \beta_0) \neq 0$, *then the stationary point* $(0, \alpha_0)$ *is a bifurcation point of* (3.1) *and several branches of nonconstant* *periodic solutions bifurcate from* $(0,\alpha_0,\beta_0)$. *More precisely, if* $h := \gcd(j,N)$, *then*

(j1) *if* $1 \leqslant j < N/2$, *then*

(j1a) *there are at least* 2 *branches of periodic solutions on which any periodic solution* $x(t)=(x_k(t))_{k=1}^N$ *satisfies*

$$x_{k+1}\left(t+\frac{jp}{N}\right)=x_k(t) \quad \text{and} \quad x_{k-1}\left(t+\frac{jp}{N}\right)=x_k(t), \tag{3.19}$$

respectively, where $k=1,2,\dots,N \pmod N$ *and* $p>0$ *is the corresponding period of* x;

(j1b) *if* $N/h \equiv 1 \pmod 2$, *then there are at least* $2N/h$ *branches of periodic solutions on which any periodic solution* $x(t)=(x_k(t))_{k=1}^N$ *satisfies*

$$x_{k-N/h}(t)=x_k(t), \quad k=1,2,\dots,N \pmod N; \tag{3.20}$$

(j1c) *if* $N/h \equiv 0 \pmod 2$, *there are at least* N/h *branches of periodic solutions on which any periodic solution* $x(t)=(x_k(t))_{k=1}^N$ *satisfies*

$$x_{k-N/(2h)}\left(t+\frac{p}{2}\right)=x_k(t), \quad k=1,2,\dots,N \pmod N, \tag{3.21}$$

where $p>0$ *is the corresponding period of* x.

(j2) *if* N *is even and* $j=N/2$, *then there exists at least one branch of periodic solutions on which any periodic solution* $x(t)=(x_k(t))_{k=1}^N$ *satisfies*

$$x_{k-1}\left(t+\frac{p}{2}\right)=x_k(t),$$
$$x_{N-k}\left(t+\frac{p}{2}\right)=x_k(t), \quad k=1,2,\dots,N \pmod N, \tag{3.22}$$

where $p>0$ *is the corresponding period of* x;

(j3) *if* $j=0$, *then there exists at least one branch of periodic solutions on which any periodic solution* $x(t)=(x_k(t))_{k=1}^N$ *satisfies*

$$x_1(t)=x_2(t)=\cdots=x_N(t). \tag{3.23}$$

Proof. By virtue of Theorem 3.1, we know that $(0,\alpha_0)$ is a bifurcation point and (i1)–(i5) in Theorem 3.1 are satisfied.

Let $K \subset G = D_N \times S^1$ be a closed subgroup. If there exists a branch of periodic solutions corresponding to the orbit type (K) and bifurcating from $(0,\alpha_0,\beta_0)$, then, corresponding to $|G/K|$ closed subgroups in the conjugacy class (K), there must exist $|G/K|$ different branches of periodic solutions corresponding to the orbit type (K) and bifurcating from $(0,\alpha_0,\beta_0)$ and each branch corresponds to a closed subgroup in (K). We say that a branch C_H *corresponds to* a closed subgroup H in the conjugacy class (K), if the branch C_H is corresponds to the orbit type (K), and any periodic solution $x(t)=(x_k(t))_{k=1}^N$ on the branch C_H, $z \in C(S^1,\mathbb{R}^N)$ given by

$$z(t)=(x_1(pt),\dots,x_N(pt)) \tag{3.24}$$

has the isotropy group H, i.e., $G_z=H$, where $p>0$ is the corresponding period of x. Therefore, if $x(t)=(x(_k(t))_{k=1}^N$ is a periodic solution on the branch C_H and has a corresponding period $p>0$, then $z \in C(S^1,\mathbb{R}^N)$ given by (3.24) satisfies

$$(g,\theta)z(t)=\rho(g)z(t+\theta)=z(t), \quad \forall (g,\theta)\in H.$$

Put $h = \gcd(j, N)$. If $1 \leqslant j < N/2$ and $N/h \equiv 1 \,(\mathrm{mod}\, N)$, then Theorem 3.1(i1) implies that there exist two branches of periodic solutions bifurcating from $(0, \alpha_0, \beta_0)$ and corresponding to the orbit type $(\mathbb{Z}_N^{(\theta_j,1)})$. By Lemma 3.4, we see that the conjugacy class $(\mathbb{Z}_N^{(\theta_j,1)})$ contains two closed subgroups $\mathbb{Z}_N^{(\theta_j,1)}$ and $\mathbb{Z}_N^{(\theta_{-j},1)}$, and hence those two branches corresponding to the orbit type $(\mathbb{Z}_N^{(\theta_j,1)})$ correspond to the closed subgroups $\mathbb{Z}_N^{(\theta_j,1)}$ and $\mathbb{Z}_N^{(\theta_{-j},1)}$, respectively. For every periodic solution $x(t) = (x_k(t))_{k=1}^N$ on the branch $C_{\mathbb{Z}_N^{(\theta_j,1)}}$, if $p > 0$ is the corresponding period of x, then $z \in C(S^1, \mathbb{R}^N)$ given by (3.24) satisfies

$$(\gamma, \gamma^j) z(t) = \rho(\gamma) z\left(t + \frac{j}{N}\right) = z(t), \quad \forall \gamma \in \mathbb{Z}_N.$$

In particular, we have

$$(\xi_N, \xi_N^j) z(t) = \rho(\xi) z\left(t + \frac{j}{N}\right) = z(t), \quad \xi_N = \mathrm{e}^{\mathrm{i}2\pi/N} \in \mathbb{Z}_N.$$

That is,

$$z_{k-1}\left(t + \frac{j}{N}\right) = z_k(t), \quad k = 1, 2, \ldots, N \,(\mathrm{mod}\, N).$$

Hence, we obtain

$$x_{k-1}\left(t + \frac{jp}{N}\right) = x_k(t), \quad k = 1, 2, \ldots, N \,(\mathrm{mod}\, N).$$

For every periodic solution $x(t) = (x_k(t))_{k=1}^N$ on the branch $C_{\mathbb{Z}_N^{(\theta_{-j},1)}}$, if $p > 0$ is the corresponding period of x, then a similar argument implies that

$$x_{k-1}\left(t - \frac{jp}{N}\right) = x_k(t), \quad k = 1, 2, \ldots, N \,(\mathrm{mod}\, N)$$

and hence

$$x_{k+1}\left(t + \frac{jp}{N}\right) = x_k(t), \quad k = 1, 2, \ldots, N \,(\mathrm{mod}\, N).$$

The rest of Theorem 3.2 can be proved in a similar way and thus is omitted. □

Remark 3.3. Following Alexander and Auchmuty [1], we call the periodic solution obtained in Theorem 3.2 *synchronous oscillations* if (3.23) holds and *phase-locked oscillations* if one equality of (3.19) holds for every $k = 1, \ldots, N \,(\mathrm{mod}\, N)$. Intuitively, synchronous oscillations occur when all the concentration oscillate in phase and phase-locked oscillations are those where each concentration oscillates just like the others except not necessarily in phase with each other. We refer to [1,7,9,21,26,29] for more details.

To detect the global continuation of the branches of periodic solutions obtained in Theorem 3.1, we further assume that

(H5) $\bar{f}'(0, \alpha) \neq 4\bar{K}(\alpha) \sin^2(\pi j/N)$ for every $\alpha \in \mathbb{R}$ and $j = 0, 1, \ldots, [N/2]$.

(H6) For every $j = 0, 1, \dots, [N/2]$, the set

$$M_j^* := \{\alpha \in \mathbb{R};\ \ p_j(i\sigma, \alpha) = 0 \text{ for some } \sigma > 0\}$$

is complete and discrete in $\mathbb{R}$.

From (H5) and Lemma 3.1, we can easily see that for any $\alpha \in \mathbb{R}$, 0 is not a characteristic value of $(0, \alpha)$. In other words, for every $\alpha \in \mathbb{R}$, we have

$$D_x \bar{F}(0, \alpha) \neq 0, \tag{3.25}$$

where $\bar{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is given by

$$\bar{F}_j(x, \alpha) = \bar{f}(x^j, \alpha) + \bar{K}(\alpha)(x^{j+1} + x^{j-1} - 2x^j),$$

$$x = (x^1, x^2, \dots, x^N) \in \mathbb{R}^N,\ \ j = 1, 2, \dots, N.$$

Therefore, (A5) is satisfied. By (H6), (A6) is also satisfied.

In particular, (3.25) implies that

$$\bar{f}'(0, \alpha) = D_x \bar{F}(0, \alpha)|_{V_0} \neq 0, \quad \forall \alpha \in \mathbb{R},$$

where $V_0 := (\mathbb{R}^N)^{D_N} = \{(c, c, \dots, c);\ c \in \mathbb{R}\}$. Therefore, by (2.19), we have

$$v_0(\alpha, 2\pi/p) = (-1)^{\dim V_0} \operatorname{sign} \det \bar{F}(0, \alpha)|_{V_0} = -\operatorname{sign} \bar{f}'(0, \alpha) \neq 0. \tag{3.26}$$

Since $\bar{f} : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with respect to the first argument, (3.26) implies that $v_0(\alpha, 2\pi/p)$ is a constant (1 or -1) for every $(\alpha, p) \in \mathbb{R} \times (0, \infty)$. Thus, by Theorems 2.3 and 2.4, we have the following global symmetric Hopf bifurcation theorem.

Theorem 3.3. *Assume that* (H1), (H2), (H5) *and* (H6) *are satisfied. For each* $j = 0, 1, \dots, [N/2]$, *put* $\tilde{M}_j = \{(\alpha, p) \in \mathbb{R} \times (0, \infty);\ \ p_j(\mathrm{i}2\pi/p, \alpha) = 0 \text{ for } p > 0\}$. *If there exists a* $j \in \{0, 1, \dots, [N/2]\}$ *such that for any finite subset* $\tilde{N}_j \subset \tilde{M}_j$,

$$\sum_{(\alpha, p) \in \tilde{N}} c_{1,j}(\alpha, 2\pi/p) \neq 0,$$

then for each $(\alpha, p) \in \tilde{M}_j$ *there exist, bifurcating from* $(0, \alpha, p)$, *unbounded branches of nonconstant periodic solutions of* (3.1). *More precisely, we have*

(j1) *if* $1 \leqslant j < N/2$, *then there are at least 2 unbounded branches of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *with a corresponding period* $p > 0$ *satisfies*

$$x_{k+1}\left(t + \frac{jp}{N}\right) = x_k(t) \quad \text{and} \quad x_{k-1}\left(t + \frac{jp}{N}\right) = x_k(t),$$

respectively, where $k = 1, 2, \dots, N \pmod N$;

(j2) *if* N *is even and* $j = N/2$, *then there exists at least one unbounded branch of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *with a corresponding period* $p > 0$ *satisfies*

$$x_{k-1}\left(t + \frac{p}{2}\right) = x_k(t), \quad x_{N-k}\left(t + \frac{p}{2}\right) = x_k(t), \quad k = 1, 2, \dots, N \pmod N;$$

(j3) *if* $j=0$, *then there exists at least one unbounded branch of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *satisfies*

$$x_1(t)=x_2(t)=\cdots=x_N(t).$$

4. The coexistence of multiple slowly oscillating periodic solutions

In this section, as a special example, we consider the following neutral functional differential equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[x_k(t)-qx_k(t-r)] = d[&(x_{k+1}(t)-qx_{k+1}(t-r)) \\ &+(x_{k-1}(t)-qx_{k-1}(t-r))-2(x_k(t)-qx_k(t-r))] \\ &-ax_k(t)-aqx_k(t-r)-g(x_k(t)-qx_k(t-r)), \qquad (4.1)\end{aligned}$$

where $k=1,2,\ldots,N \bmod N$, $N \geqslant 3$ is a positive integer, a, d, r are positive constants, $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable with $g(0)=0$, $q\in[0,1)$ is the bifurcation parameter.

We remark that the continuous version of (4.1)

$$\begin{aligned}\frac{\partial}{\partial t}[u(t,x)-qu(t-r,x)] = d\,&\frac{\partial^2}{\partial x^2}[u(t,x)-qu(t-r,x)] \\ &-au(t,x)-aqu(t-r,x)-g[u(t,x)-qu(t-r,x)],\end{aligned}$$

where $x\in S^1$, has been studied by Wu and Xia (cf. [28,26]), and (4.1) arises from coupled transmission lines (cf. [22]).

Let $Q:\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function such that

(i) $Q(\alpha)=0$ for all $\alpha\leqslant 0$.
(ii) $Q'(\alpha)>0$ for all $\alpha>0$ and $\lim_{\alpha\to\infty} Q(\alpha)=1$.

Then we can reparametrize system (4.1) to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[x_k(t)-Q(\alpha)x_k(t-r)] \\ &\quad= d[(x_{k+1}(t)-Q(\alpha)x_{k+1}(t-r))+(x_{k-1}(t)-Q(\alpha)x_{k-1}(t-r)) \\ &\qquad -2(x_k(t)-Q(\alpha)x_k(t-r)]-ax_k(t)-aQ(\alpha)x_k(t-r)-g(x_k(t) \\ &\qquad -Q(\alpha)x_k(t-r)), \qquad (4.2)\end{aligned}$$

for $k=1,2,\ldots,N \bmod N$. Then for any $q\in[0,1)$, $(x_k(t))_{k=1}^N$ is a periodic solution of (4.1) with q if it is a periodic solution of (4.2) with $\alpha=Q^{-1}(q)\geqslant 0$.

Let $C_0 := C((-\infty,0],\mathbb{R})$. Define $b,f: C_0\times\mathbb{R}\to\mathbb{R}$ by

$$b(\varphi,\alpha) := Q(\alpha)\varphi(-r), \qquad \varphi\in C_0$$

and

$$f(\varphi,\alpha) := -a\varphi(0) - aQ(\alpha)\varphi(-r) - g(\varphi(0) - Q(\alpha)\varphi(-r)), \quad \varphi \in C_0,$$

respectively, and for every $\alpha \in \mathbb{R}$, define $K(\alpha): C_0 \to \mathbb{R}$ by

$$K(\alpha)\varphi := d(\varphi(0) - Q(\alpha)\varphi(-r)).$$

Then (H1) and (H2) are satisfied and (4.2) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}[(x_j)(t) - b((x_j)_t,\alpha)] = f((x_j)_t,\alpha) + K(\alpha)((x_{j+1})_t + (x_{j-1})_t - 2(x_j)_t),$$

where $j = 1,2,\dots,N \bmod N$.

In what follows, for the sake of convenience, we put

$$g'(0) = v. \tag{4.3}$$

Then the characteristic equation at the zero solution of (4.1) takes the form

$$\det_{\mathbb{C}} \Delta(\lambda) = 0, \tag{4.4}$$

where for each $\lambda \in \mathbb{C}$, $\Delta(\lambda): \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$\Delta(\lambda) := \operatorname{diag}(\lambda(1 - q\mathrm{e}^{-\lambda r}) + a + aq\mathrm{e}^{-\lambda r} + v(1 - q\mathrm{e}^{-\lambda r})) - \delta(\lambda) \tag{4.5}$$

in which the discretized Laplacian $\delta(\lambda): \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$\{\delta(\lambda)z\}_j = d[(1 - q\mathrm{e}^{-\lambda r})z_{j+1} + (1 - q\mathrm{e}^{-\lambda r})z_{j-1} - 2(1 - q\mathrm{e}^{-\lambda r})z_j],$$

for $z = (z_1, z_2, \dots, z_N)^{\mathrm{T}} \in \mathbb{C}^N$, $j = 1,2,\dots,N \bmod N$.

Corresponding to Lemma 3.1, we have

Lemma 4.1. *Let* $\varpi_q(\lambda) := \lambda(1 - q\mathrm{e}^{-\lambda r}) + a + aq\mathrm{e}^{-\lambda r} + v(1 - q\mathrm{e}^{-\lambda r})$. *Then*

$$\det \Delta(\lambda) = \prod_{j=0}^{N-1} \left[\varpi_q(\lambda) + 4d(1 - q\mathrm{e}^{-\lambda r})\sin^2\frac{\pi j}{N}\right].$$

Therefore, $\lambda \in \mathbb{C}$ *is a zero of* (4.4) *if and only if there exists a* $j \in \{0,1,\dots,[N/2]\}$ *such that*

$$p_j(\lambda,q) := \varpi_q(\lambda) + 4d(1 - q\mathrm{e}^{-\lambda r})\sin^2\frac{\pi j}{N} = 0. \tag{4.6}$$

Let $\lambda = \mathrm{i}\beta$ with $\beta > 0$. Substituting it into (4.6), we get

$$\begin{cases} q\beta\sin\beta r - q(a - v + 4d\sin^2(\pi j/N))\cos\beta r = a + v - 4d\sin^2(\pi j/N), \\ q(a - 4d\sin^2(\pi j/N))\sin\beta r + q\beta\cos\beta r = \beta. \end{cases} \tag{4.7}$$

Since $D := q^2\beta^2 + q^2(a - v + 4d\sin^2(\pi j/N))^2 = q^2[\beta^2 + (a - v + 4d\sin^2(\pi j/N))^2] > 0$ for $q \in (0,1)$, it follows from (4.7) that

$$\begin{bmatrix} \sin\beta r \\ \cos\beta r \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 2aq\beta \\ q[\beta^2 - a^2 + (v - 4d\sin^2(\pi j/N))^2] \end{bmatrix}$$

or equivalently,

$$\cot\beta r = \frac{\beta^2 - a^2 + (v - 4d\sin^2(\pi j/N))^2}{2a\beta},$$

$$q^2 = \frac{\beta^2 + (a + v - 4d\sin^2(\pi j/N))^2}{\beta^2 + (a - v + 4d\sin^2(\pi j/N))^2}. \tag{4.8}$$

If there exists a $j \in \{0, 1, \ldots, [N/2]\}$ such that

$$v < 4d\sin^2\frac{\pi j}{N} < a + v, \tag{4.9}$$

then it is easy to show that for any real number $\beta > 0$,

$$\frac{\beta^2 + (a + v - 4d\sin^2(\pi j/N))^2}{\beta^2 + (a - v + 4d\sin^2(\pi j/N))^2} \in (0,1). \tag{4.10}$$

For each fixed $j \in \{0, 1, \ldots, [N/2]\}$ so that (4.9) holds, put

$$h_j(\beta) := \frac{\beta^2 - a^2 + (v - 4d\sin^2(\pi j/N))^2}{2a\beta}.$$

Then we have

$$h_j(0+) = -\infty, \quad h_j(+\infty) = +\infty$$

and

$$\begin{aligned} h_j'(\beta) &= \frac{4a\beta^2 - 2a(\beta^2 - a^2 + (v - 4d\sin^2(\pi j/N))^2)}{4a^2\beta^2} \\ &= \frac{\beta^2 + a^2 - (v - 4d\sin^2(\pi j/N))^2}{2a\beta^2} \\ &> \frac{a^2 - (v - 4d\sin^2(\pi j/N))^2}{2a\beta^2} > 0. \end{aligned}$$

Therefore, there exists a sequence of positive numbers $\{\beta_{j,k}\}_{k=0}^{\infty}$ such that

(1) $\beta_{j,k}$, $k = 0, 1, \ldots$, satisfy the first equation of (4.8);
(2) $\beta_{j,0} < \beta_{j,1} < \cdots < \beta_{j,k} < \cdots \to \infty$;
(3) $k\pi/r < \beta_{j,k} < (k+1)\pi/r$, $k = 0, 1, \ldots$, and hence

$$2r < 2\pi/\beta_{j,0} < \infty, \quad \frac{2r}{k+1} < 2\pi/\beta_{j,k} < \frac{2r}{k}, \quad k \geqslant 1.$$

Substituting this $\beta_{j,k}$ into the second equation of (4.8), and using (4.9) and (4.10), it follows that

$$q_{j,k} := \sqrt{\frac{\beta_{j,k}^2 + (a+v-4d\sin^2(\pi j/N))^2}{\beta_{j,k}^2 + (a-v+4d\sin^2(\pi j/N))^2}} \in (0,1), \tag{4.11}$$

where $0 < q_{j,0} < q_{j,1} < \cdots < q_{j,k} < \cdots \to \infty$.

If, in addition to (4.9), we assume that

$$a^2 > \frac{\pi^2}{4r^2} + \left(v - 4d\sin^2\frac{\pi j}{N}\right)^2, \tag{4.12}$$

then

$$h_j\left(\frac{\pi}{2r}\right) = \frac{r}{a\pi}\left[\frac{\pi^2}{4r^2} - a^2 + \left(v - 4d\sin^2\frac{\pi j}{N}\right)^2\right] < 0$$

and hence,

$$\frac{\pi}{2r} < \beta_{j,0} < \frac{\pi}{r},$$

which yields

$$2r < 2\pi/\beta_{j,0} < 4r.$$

Let $\lambda = \lambda(q)$ be a smooth curve of zeros of (4.6) so that $\lambda(q_{j,k}) = \mathrm{i}\beta_{j,k}$. Differentiating (4.6) with respect to q, we get

$$\lambda'(q) = \frac{\lambda - a + v - 4d\sin^2(\pi j/N)}{\mathrm{e}^{\lambda r} + r(\lambda + a + v - 4d\sin^2(\pi j/N))\mathrm{e}^{\lambda r} - q}.$$

It follows from (4.6) that

$$\mathrm{e}^{\lambda r} = \frac{q(\lambda - a + v - 4d\sin^2(\pi j/N))}{\lambda + a + v - 4d\sin^2(\pi j/N)},$$

therefore, we have

$$\lambda'(q) = \frac{\lambda - a + v - 4d\sin^2(\pi j/N)}{\dfrac{q(\lambda - a + v - 4d\sin^2(\pi j/N))}{\lambda + a + v - 4d\sin^2(\pi j/N)} + qr(\lambda - a + v - 4d\sin^2(\pi j/N)) - q}.$$

This leads to

$$\operatorname{sign}\operatorname{Re}\lambda'(q)|_{q=q_{j,k}} = \operatorname{sign}\operatorname{Re}\left.\frac{1}{\lambda'(q)}\right|_{q=q_{j,k}}$$

$$= \operatorname{sign}\operatorname{Re}\left.\left\{qr + \frac{q}{\lambda+a+v-4d\sin^2(\pi j/N)} - \frac{q}{\lambda - a + v - 4d\sin^2(\pi j/N)}\right\}\right|_{q=q_{j,k},\lambda=\mathrm{i}\beta_{j,k}}$$

$$= \operatorname{sign} \operatorname{Re} \left\{ r + \frac{1}{\mathrm{i}\beta_{j,k} + a + v - 4d\sin^2(\pi j/N)} - \frac{1}{\mathrm{i}\beta_{j,k} - a + v - 4d\sin^2(\pi j/N)} \right\}$$

$$= \operatorname{sign} \left\{ r + \frac{a + v - 4d\sin^2(\pi j/N)}{\beta_{j,k}^2 + (a + v - 4d\sin^2(\pi j/N))^2} + \frac{a - v + 4d\sin^2(\pi j/N)}{\beta_{j,k}^2 + (a - v + 4d\sin^2(\pi j/N))^2} \right\}$$

$$= \operatorname{sign} \left\{ r + \frac{2a\beta_{j,k}^2 + 2a(a^2 - (v - 4d\sin^2(\pi j/N))^2)}{[\beta_{j,k}^2 + (a + v - 4d\sin^2(\pi j/N))^2][\beta_{j,k}^2 + (a - v + 4d\sin^2(\pi j/N))^2]} \right\}$$

$$=1 > 0.$$

Let us summarize the above discussions for the sake of later reference.

Lemma 4.2. *Assume that there exists a* $j \in \{0, 1, \ldots, [N/2]\}$ *so that* (4.9) *is satisfied. The following statements hold true*:

(i) (4.6) *has a sequence of purely imaginary solutions* $\pm\mathrm{i}\beta_{j,k}$ *with* $0 < \beta_{j,0} < \beta_{j,1} < \beta_{j,2} < \cdots$ *for* $q = q_{j,k} \in (0,1)$ *with* $0 < q_{j,0} < q_{j,1} < \cdots < q_{j,k} < \cdots \to \infty$ *given by* (4.11);
(ii) *if* $\lambda(q)$ *is a smooth curve of zeros of* (4.6) *with* $\lambda(q_{j,k}) = \mathrm{i}\beta_{j,k}$, *then* $\operatorname{Re} \lambda'(q_{j,k}) > 0$;
(iii) $2r < 2\pi/\beta_{j,0} < \infty, 2r/(k+1) < 2\pi/\beta_{j,k} < 2r/k, \; k \geqslant 1$;
(iv) *if, in addition to* (4.9), *we assume that* (4.12) *holds, then*

$$2r < 2\pi/\beta_{j,0} < 4r.$$

It is straightforward to obtain the following technical result.

Corollary 4.1. *Assume that there exists a* $j \in \{0, 1, \ldots, [N/2]\}$ *such that* (4.9) *holds. Then the jth isotypical crossing number for the isolated center* $(0, q_{j,k})$ $(k \geqslant 0)$ *corresponding to the value* $\mathrm{i}\beta_{j,k}$ *is equal to*

$$c_{1,j}(q_{j,k}, \beta_{j,k}) = \begin{cases} 2[\deg_B(p_j(\cdot, q_{j,k} - \delta), \Omega) & \\ -\deg_B(p_j(\cdot, q_{j,k} + \delta), \Omega)] = -2 & \text{if } 1 \leqslant j < [N/2], \\ \deg_B(p_j(\cdot, q_{j,k} - \delta), \Omega) & \\ -\deg_B(p_j(\cdot, q_{j,k} + \delta), \Omega) = -1 & \text{if } j = 0, \text{ or} \\ & N \text{ is even and } j = N/2, \end{cases}$$

where $p_j(\lambda, q) = \varpi(\lambda) + 4d(1 - q\mathrm{e}^{-\lambda r})\sin^2(\pi j/N)$, $\Omega = (0, b) \times (\beta_{j,k} - c, \beta_{j,k} + c) \subset \mathbb{C}$ *and the constants* $b > 0, c > 0$ *and* $\delta > 0$ *are sufficiently small.*

By Theorem 3.2 and Corollary 4.1, we have

Theorem 4.1. *Assume there exist a* $j \in \{0, 1, \ldots, [N/2]\}$ *such that* $v < 4d\sin^2(\pi j/N) < a+v$. *Then the stationary point* $(0, q_{j,k})$, $k \geqslant 0$, *is a bifurcation point of* (4.1). *More precisely, if* $h := \gcd(j, N)$, *then*

(j1) *if* $1 \leqslant j < N/2$, *then*

(j1a) *there are at least* 2 *branches of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *with a minimal period* p *satisfies*

$$x_{k+1}\left(t + \frac{jp}{N}\right) = x_k(t) \quad \text{and} \quad x_{k-1}\left(t + \frac{jp}{N}\right) = x_k(t),$$

respectively, where $k = 1, 2, \ldots, N \bmod N$;

(j1b) *if* $N/h \equiv 1 \pmod 2$, *then there are at least* $2N/h$ *branches of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *satisfies*

$$x_{k-N/h}(t) = x_k(t), \quad k = 1, 2, \ldots, N \pmod N;$$

(j1c) *if* $N/h \equiv 0 \pmod 2$, *there are at least* N/h *branches of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *with the minimal period* p *satisfies*

$$x_{k-N/(2h)}\left(t + \frac{p}{2}\right) = x_k(t), \quad k = 1, 2, \ldots, N \pmod N;$$

(j2) *if* N *is even and* $j = N/2$, *then there exists at least one branch of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *with the minimal period* p *satisfies*

$$x_{k-1}\left(t + \frac{p}{2}\right) = x_k(t), \quad x_{N-k}\left(t + \frac{p}{2}\right) = x_k(t), \quad k = 1, 2, \ldots, N \pmod N;$$

(j3) *if* $j = 0$, *then there exists at least one branch of periodic solutions on which any periodic solution* $(x_k(t))_{k=1}^N$ *satisfies*

$$x_1(t) = x_2(t) = \cdots = x_N(t).$$

Finally, we will investigate the maximal continuum of branches of nonconstant periodic solutions obtained in Theorem 4.1. To do this, we firstly establish some a priori bounds for possible nonconstant periodic solutions of (4.1).

Lemma 4.3. *Assume that*

$$\lim_{z \to \pm\infty} \frac{g(z)}{z} = +\infty.$$

Then there exists a nondecreasing function $M : [0, 1) \to [0, \infty)$ *such that any periodic solution* $(x_k(t))_{k=1}^N$ *of* (4.1) *with* $q \in [0, 1)$ *satisfies* $|x_k(t)| \leqslant M(q)$ *for all* $t \in \mathbb{R}$ *and* $k = 1, 2, \ldots, N$.

Proof. Suppose that $(x_k(t))_{k=1}^N$ is a nontrivial periodic solution of (4.1). Then there exist some $k_0 \in \{1, 2, \ldots, N\}$ and $t_0 \in \mathbb{R}$ so that

$$|x_k(t) - qx_k(t-r)| \leqslant |x_{k_0}(t_0) - qx_{k_0}(t_0 - r)|, \quad \text{for } t \in \mathbb{R} \text{ and } k = 1, 2, \ldots, N. \tag{4.13}$$

Without loss of generality, we may assume $x_{k_0}(t_0) - qx_{k_0}(t_0 - r) \neq 0$. If this is not the case, then $x_k(t) \equiv qx_k(t-r)$, which together with (4.1) yields $x_k(t) \equiv 0$, $\forall t \in \mathbb{R}$ $k = 1, \ldots, N$. This leads to a contradiction.

There are two possible cases:

Case 1: $x_{k_0}(t_0) - x_{k_0}(t_0 - r) > 0$.

In this case, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \\ &= d[(x_{k_0+1}(t_0) - qx_{k_0+1}(t_0 - r)) + (x_{k_0-1}(t_0) - qx_{k_0-1}(t_0 - r)) \\ &\quad - 2(x_{k_0}(t_0) - qx_{k_0}(t_0 - r))] - ax_{k_0}(t_0) \\ &\quad - aqx_{k_0}(t_0 - r) - g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \\ &\leqslant -ax_{k_0}(t_0) - aqx_{k_0}(t_0 - r) - g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)]. \end{aligned}$$

Hence,

$$ax_{k_0}(t_0) + aqx_{k_0}(t_0 - r) + g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \leqslant 0,$$

which can be rewritten as

$$a[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] + 2aqx_{k_0}(t_0 - r) + g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \leqslant 0. \tag{4.14}$$

Note that $x_{k_0}(t_0) - x_{k_0}(t_0 - r) > 0$, from (4.13) we get

$$a + 2aq\,\frac{x_{k_0}(t_0 - r)}{x_{k_0}(t_0) - qx_{k_0}(t_0 - r)} + \frac{g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)]}{x_{k_0}(t_0) - qx_{k_0}(t_0 - r)} \leqslant 0. \tag{4.15}$$

It follows from (4.13) that

$$\begin{aligned} |x_k(t)| &\leqslant q|x_k(t-r)| + |x_{k_0}(t_0) - qx_{k_0}(t_0 - r)| \\ &\leqslant q^2|x_k(t-2r)| + (1+q)|x_{k_0}(t_0) - qx_{k_0}(t_0 - r)| \\ &\leqslant \cdots \leqslant q^m|x_k(t-mr)| + (1 + q + \cdots + q^{m-1})|x_{k_0}(t_0) - qx_{k_0}(t_0 - r)|. \end{aligned}$$

Let $m \to \infty$, then we have

$$|x_k(t)| \leqslant \frac{1}{1-q}|x_{k_0}(t_0) - qx_{k_0}(t_0 - r)|, \tag{4.16}$$

for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, N$. Therefore, (4.15) and (4.16) imply that

$$a - \frac{2aq}{1-q} + \frac{g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)]}{x_{k_0}(t_0) - qx_{k_0}(t_0 - r)} \leqslant 0. \tag{4.17}$$

Since $\lim_{z\to\infty} f(z)/z = \infty$, by (4.17), we can find a nondecreasing function $M_1 : [0,1) \to [0,\infty)$ such that

$$|x_{k_0}(t_0) - qx_{k_0}(t_0 - r)| \leqslant M_1(q), \quad q \in [0,1)$$

and hence, from (4.16), it follows that

$$|x_k(t)| \leqslant \frac{1}{1-q} M_1(q), \tag{4.18}$$

for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, N$.

Case 2: $x_{k_0}(t_0) - x_{k_0}(t_0 - r) < 0$.

In this case, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \\
&= d[(x_{k_0+1}(t_0) - qx_{k_0+1}(t_0 - r)) + (x_{k_0-1}(t_0) - qx_{k_0-1}(t_0 - r)) \\
&\quad - 2(x_{k_0}(t_0) - qx_{k_0}(t_0 - r))] - ax_{k_0}(t_0) \\
&\quad - aqx_{k_0}(t_0 - r) - g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \\
&\geqslant -ax_{k_0}(t_0) - aqx_{k_0}(t_0 - r) - g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)].
\end{aligned}$$

Hence,

$$a[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] + 2aqx_{k_0}(t_0 - r) + g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)] \geqslant 0. \tag{4.19}$$

Note that $x_{k_0}(t_0) - x_{k_0}(t_0 - r) < 0$, from (4.19) we also get

$$a + 2aq \frac{x_{k_0}(t_0 - r)}{x_{k_0}(t_0) - qx_{k_0}(t_0 - r)} + \frac{g[x_{k_0}(t_0) - qx_{k_0}(t_0 - r)]}{x_{k_0}(t_0) - qx_{k_0}(t_0 - r)} \leqslant 0.$$

In a similar way, we can find a nondecreasing function $M_2 : [0, 1) \to [0, \infty)$ such that

$$|x_k(t)| \leqslant \frac{1}{1-q} M_2(q), \quad \text{for all } t \in \mathbb{R} \text{ and } k = 1, 2, \ldots, N. \tag{4.20}$$

Put

$$M(q) = \frac{1}{1-q} M_1(q) + \frac{1}{1-q} M_2(q).$$

Then $M : [0, 1) \to [0, \infty)$ is nondecreasing and any periodic solution $(x_k(t))_{k=1}^N$ of (4.1) satisfies

$$|x_k(t)| \leqslant M(q),$$

for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, N$. The proof is complete. □

Next, we exclude nontrivial $4r$-periodic solutions of (4.1).

Lemma 4.4. *Assume* $\varpi := a + \inf_{z \neq 0} g(z)/z > 0$. *Let*

$$\varrho = \sup \left\{ q \in [0, 1); \frac{2aq}{1+q} < \varpi \right\}.$$

Then for any $q \in [0, \varrho)$, (4.1) *has no nontrivial* $4r$*-periodic solutions.*

Proof. Suppose for the contradiction that $(x_k(t))_{k=1}^N$ is a nontrivial $4r$-periodic solution of (4.1).

We rewrite (4.1) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&[x_k(t) - qx_k(t-r)]\\ &= d[(x_{k+1}(t) - qx_{k+1}(t-r))\\ &\quad + (x_{k-1}(t) - qx_{k-1}(t-r)) - 2(x_k(t) - qx_k(t-r))]\\ &\quad - a(x_k(t) - qx_k(t-r)) - 2aqx_k(t-r) - g(x_k(t) - qx_k(t-r)),\end{aligned} \tag{4.21}$$

for $k = 1, 2, \ldots, N \bmod N$.

Let

$$y_k(t) := \begin{bmatrix} y_{k,1}(t) \\ y_{k,2}(t) \\ y_{k,3}(t) \\ y_{k,4}(t) \end{bmatrix} = \begin{bmatrix} x_k(t) - qx_k(t-r) \\ x_k(t-r) - qx_k(t-2r) \\ x_k(t-2r) - qx_k(t-3r) \\ x_k(t-3r) - qx_k(t) \end{bmatrix}. \tag{4.22}$$

Then

$$\begin{bmatrix} x_k(t-r) \\ x_k(t-2r) \\ x_k(t-3r) \\ x_k(t) \end{bmatrix} = \frac{1}{1-q^4} B \begin{bmatrix} y_{k,1}(t) \\ y_{k,2}(t) \\ y_{k,3}(t) \\ y_{k,4}(t) \end{bmatrix} = \frac{1}{1-q^4} B y_k(t), \tag{4.23}$$

where

$$B = \begin{bmatrix} q^3 & 1 & q & q^2 \\ q^2 & q^3 & 1 & q \\ q & q^2 & q^3 & 1 \\ 1 & q & q^2 & q^3 \end{bmatrix}.$$

Substituting (4.22) and (4.23) into (4.21), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{k,s} = d[y_{k+1,s} + y_{k-1,s} - 2y_{k,s}] - a y_{k,s} - \frac{2aq}{1-q^4}(By_k)_s - g(y_{k,s}), \tag{4.24}$$

where $1 \leqslant s \leqslant 4$, $k = 1, 2, \ldots, N \bmod N$. Denote $Y = (y_1, y_2, \ldots, y_N)^{\mathrm{T}}$ and take a Liapunov function

$$V(Y) = \frac{1}{2} \sum_{k=1}^{N} \sum_{s=1}^{4} y_{k,s}^2. \tag{4.25}$$

Let $\langle\cdot,\cdot\rangle$ denote the standard inner product in $\mathbb{R}^N$. Put

$$A=(a_{ij})_{N\times N}=\begin{bmatrix} 2+\dfrac{\varpi}{d} & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2+\dfrac{\varpi}{d} & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2+\dfrac{\varpi}{d} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\dfrac{\varpi}{d} & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2+\dfrac{\varpi}{d} \end{bmatrix}. \quad (4.26)$$

Since $\varpi>0$, we have

$$a_{ii}=2+\frac{\varpi}{d}>2=\sum_{i\neq j}|a_{ij}|,$$

which implies that A is positive definite, and hence there exists an orthogonal matrix T, such that

$$A=T^{-1}DT, \quad D=\mathrm{diag}[\lambda_1,\lambda_2,\ldots,\lambda_N], \quad (4.27)$$

where $\lambda_k>0$, $1\leqslant k\leqslant N$, are the eigenvalues of A.

Differentiating V along solutions of (4.24), we get

$$\begin{aligned}
\dot V_{(4.24)} &= \sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}\dot y_{k,s} \\
&= d\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}[y_{k+1,s}+y_{k-1,s}-(2+a/d)y_{k,s}] \\
&\quad -\frac{2aq}{1-q^4}\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}(By_k)_s-\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}g(y_{k,s}) \\
&\leqslant d\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}[y_{k+1,s}+y_{k-1,s}-(2+a/d)y_{k,s}] \\
&\quad -\frac{2aq}{1-q^4}\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}(By_k)_s-\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}^2\inf_{z\neq 0} g(z)/z \\
&= d\sum_{s=1}^{4}\sum_{k=1}^{N} y_{k,s}\left[y_{k+1,s}+y_{k-1,s}-\left(2+\frac{\varpi}{d}\right)y_{k,s}\right]
\end{aligned}$$

$$- \frac{2aq}{1-q^4} \sum_{k=1}^{N} \sum_{s=1}^{4} y_{k,s}(By_k)_s$$

$$= -d \sum_{s=1}^{4} \langle A\bar{y}_s, \bar{y}_s \rangle - \frac{2aq}{1-q^4} \sum_{k=1}^{N} \sum_{s=1}^{4} y_{k,s}(By_k)_s, \tag{4.28}$$

where $\bar{y}_s := (y_{1,s}, y_{2,s}, \ldots, y_{N,s})^{\mathrm{T}} \in \mathbb{R}^N$.

By (4.27), for $1 \leqslant s \leqslant 4$, we have

$$\langle A\bar{y}_s, \bar{y}_s \rangle = \langle T^{-1}DT\bar{y}_s, \bar{y}_s \rangle = \langle DT\bar{y}_s, T\bar{y}_s \rangle = \sum_{k=1}^{N} \lambda_k (T\bar{y}_s)_k^2. \tag{4.29}$$

If we denote by $\lambda_{\min}$ the minimal eigenvalue of A, then $\lambda_{\min} > 0$ and (4.29) implies that

$$\langle A\bar{y}_s, \bar{y}_s \rangle \geqslant \lambda_{\min} \langle T\bar{y}_s, T\bar{y}_s \rangle = \lambda_{\min} \langle \bar{y}_s, \bar{y}_s \rangle, \quad 1 \leqslant s \leqslant 4. \tag{4.30}$$

We need the following result that was proved in [26] using the Nussbaum's spectral theorem for circulant matrices (cf. [24]).

Lemma 4.5. *For any* $z = (z_1, z_2, z_3, z_4)^{\mathrm{T}} \in \mathbb{R}^4$, *one has*

$$\sum_{s=1}^{4} z_s (Bz)_s \geqslant -(1-q)(1+q^2) \sum_{s=1}^{4} z_s^2. \tag{4.31}$$

Thus, from (4.28), (4.30) and (4.31), we find

$$\dot{V}_{(4.24)} \leqslant -d\lambda_{\min} \sum_{s=1}^{4} \langle \bar{y}_s, \bar{y}_s \rangle + \frac{2aq}{1-q^4} (1-q)(1+q^2) \sum_{k=1}^{N} \sum_{s=1}^{4} y_{k,s}^2$$

$$= -\left(d\lambda_{\min} - \frac{2aq}{1+q} \right) \sum_{k=1}^{N} \sum_{s=1}^{4} y_{k,s}^2. \tag{4.32}$$

We claim that

$$\lambda_{\min} \geqslant \varpi / d. \tag{4.33}$$

Let $z \in \mathbb{R}^N$ $(z \neq 0)$ be such that $Az = \lambda_{\min} z$ and take a fixed $k \in \{1, \ldots, N\}$ so that $|z_k| = \max_{1 \leqslant i \leqslant N} |z_i|$, then we have

$$\sum_{j=1}^{N} a_{kj} z_j = \lambda_{\min} z_k$$

and hence

$$(\lambda_{\min} - a_{kk}) z_k = \sum_{j \neq k} a_{kj} z_j.$$

Therefore,

$$\lambda_{\min} = a_{kk} + \sum_{j\neq k} a_{kj}\frac{z_j}{z_k} \geqslant (2+\varpi/d) - 2 = \varpi/d.$$

Thus (4.33) holds.

Now, from (4.32) and (4.33) it follows that

$$\dot{V}_{(4.24)} \leqslant -\left(\varpi - \frac{2aq}{1+q}\right)\sum_{k=1}^{N}\sum_{s=1}^{4} y_{k,s}^2. \tag{4.34}$$

Thus, if $q\in[0,\varrho)$, then

$$\frac{2aq}{1+q} < \frac{2a\varrho}{1+\varrho} \leqslant \varpi,$$

which, together with (4.34), yields $\dot{V}_{(4.24)} < 0$, and hence $x_k(t)\to 0$ for every $1\leqslant k \leqslant N$ as $t\to\infty$. This leads to a contradiction. The proof of Lemma 4.4 is complete. □

Lemma 4.6. *If* $\varpi := a + \inf_{z\neq 0} g(z)/z > 0$ *and* $q=0$, *then* (4.1) *has no nontrivial periodic solutions.*

Proof. If $q=0$, then (4.1) reduces to the following ordinary equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k = d[x_{k+1}+x_{k-1}-2x_k] - ax_k - g(x_k), \tag{4.35}$$

for $k=1,2,\ldots,N \bmod N$.

Denote $X=(x_1,x_2,\ldots,x_N)^{\mathrm{T}}$ and take a Liapunov function $V(X)=\frac{1}{2}\sum_{k=1}^{N} x_k^2$. By using a similar argument as in the proof of Lemma 4.4, we obtain

$$\dot{V}_{(4.35)} \leqslant -\varpi\sum_{k=1}^{N} x_k^2.$$

Therefore, every solution of (4.1) tends to zero as $t\to\infty$. In particular, (4.1) has no nontrivial periodic solutions. The proof is complete. □

Now, we are in a position to present the following results on the global continuations of branches of slowly oscillatory periodic solutions of (4.1). Here, by a slowly oscillatory solution of (4.1) we mean a solution with a period larger than $2r$.

Theorem 4.2. *Assume that*

(1) $\varpi := a + \inf_{z\neq 0} g(z)/z > 0$, $\lim_{z\to\pm\infty} g(z)/z = +\infty$;
(2) $2a > \pi/r$ *and there exists a* $j\in\{0,1,\ldots,[N/2]\}$ *such that*

(3) $$v < 4d\sin^2\frac{\pi j}{N} < v + \sqrt{a^2 - \frac{\pi^2}{4r^2}}, \quad v := g'(0);$$

$$q_j := \sqrt{\frac{\beta_j^2 + (a+v-4d\sin^2(\pi j/N))^2}{\beta_j^2 + (a-v+4d\sin^2(\pi j/N))^2}} < \varrho,$$

where $\varrho := \sup\{q \in [0,1) : 2aq/(1+q) < \varpi\}$ *and* β_j *is the unique solution in* $(0, \pi/r)$ *of the equation*

$$\cot \beta r = \frac{\beta^2 - a^2 + (v - 4d \sin^2(\pi j/N))^2}{2a\beta}.$$

Then for each $q \in (q_j, \varrho)$, *the following statements hold true*:

(j1) *If* $1 \leqslant j < N/2$, *then there are at least two slowly oscillatory periodic solutions* $(x_k^{(1)}(t))_{k=1}^N$ *and* $(x_k^{(2)}(t))_{k=1}^N$ *of* (4.1) *with minimal periods* $p_1, p_2 \in (2r, 4r)$ *and satisfying*

$$x_{k+1}^{(1)}\left(t + \frac{jp_1}{N}\right) = x_k^{(1)}(t) \quad \textit{and} \quad x_{k-1}^{(2)}\left(t + \frac{jp_2}{N}\right) = x_k^{(2)}(t), \tag{4.36}$$

respectively, where $k = 1, 2, \ldots, N \bmod N$.

(j2) *If* N *is even and* $j = N/2$, *then there exists at least one slowly oscillatory periodic solution* $(x_k(t))_{k=1}^N$ *of* (4.1) *with a minimal period* $p \in (2r, 4r)$ *and satisfying*

$$x_{k-1}\left(t + \frac{p}{2}\right) = x_k(t), \quad x_{N-k}\left(t + \frac{p}{2}\right) = x_k(t), \quad k = 1, 2, \ldots, N \pmod N. \tag{4.37}$$

(j3) *If* $j=0$, *then there exists at least one slowly oscillatory periodic solution* $(x_k(t))_{k=1}^N$ *of* (4.1) *with a minimal period in* $(2r, 4r)$ *and satisfying*

$$x_1(t) = x_2(t) = \cdots = x_N(t). \tag{4.38}$$

Proof. We consider the reparametrized system (4.2). It has been shown that (H1) and (H2) are satisfied. By (2), we see that

$$a + v > v + \sqrt{a^2 - \frac{\pi^2}{4r^2}} > 4d \sin^2 \frac{\pi j}{N} \geqslant 0$$

and hence

$$\begin{aligned} \bar{f}'(0, \alpha) &= -a - aQ(\alpha) - v(1 - Q(\alpha)) \\ &= -(a+v) - (a-v)Q(\alpha) \\ &\leqslant \begin{cases} -(a+v), & \text{if } a \geqslant v, \\ -2a, & \text{if } a < v \end{cases} \\ &< 0. \end{aligned}$$

As $\bar{K}(\alpha) = K(\alpha)\bar{1} = d(1 - Q(\alpha)) \geqslant 0$, it follows that $\bar{f}'(0,\alpha) \neq 4\bar{K}(\alpha) \sin^2(\pi j/N)$ for every $\alpha \in \mathbb{R}$ and every $j = 0, 1, \ldots, [N/2]$, that is, (H5) holds.

Since $v < 4d \sin^2(\pi j/N) < a + v$, it follows from Lemma 4.2 that for $\alpha = Q^{-1}(q_{j,k})$ $(k \geqslant 0)$, the stationary solution $(0, \alpha)$ of (4.2) has purely imaginary solution $\pm i\beta_{j,k}$ with $0 < \beta_{j,0} < \beta_{j,1} < \cdots < \beta_{j,k} < \cdots \to \infty$, and $0 < Q^{-1}(q_{j,0}) < Q^{-1}(q_{j,1}) < \cdots < Q^{-1}(q_{j,k}) < \cdots \to \infty$. Hence, the set

$$M_j^* = \{\alpha \in \mathbb{R};\ p_j(i\sigma, Q(\alpha)) = 0 \text{ for some } \sigma > 0\}$$

is complete and discrete in $\mathbb{R}$. Therefore, (H6) is also satisfied.

Since $v < 4d\sin^2(\pi j/N) < a + v$, it follows from Lemma 4.2 and Corollary 4.1 that for any integer $k \geqslant 0$, the jth isotypical crossing number for the isolated center $(0, Q^{-1}(q_{j,k}))$ corresponding to $\mathrm{i}\beta_{j,k}$

$$c_{1,j}(Q^{-1}(q_{j,k}), \beta_{j,k}) = c_{1,j}(q_{j,k}, \beta_{j,k}) = \begin{cases} -2 & \text{if } 1 \leqslant j < [N/2], \\ -1 & \text{if } j = 0, \text{ or } N \text{ is even and } j = N/2. \end{cases}$$

Therefore, by virtue of Theorem 3.3, there are unbounded branches of nonconstant periodic solutions of (4.2), bifurcating from $(0, Q^{-1}(q_{j,k}), 2\pi/\beta_{j,k})$ for every integer $k \geqslant 0$. In particular, if $1 \leqslant j < N/2$, then there are at least 2 unbounded branches of periodic solutions of (4.2) satisfying (4.36), if N is even and $j = N/2$, then there exists at least one unbounded branch of periodic solutions of (4.2) satisfying (4.37) and if $j = 0$, then there exists at least one unbounded branch of periodic solutions of (4.2) satisfying (4.38).

As $\lim_{z\to\pm\infty} g(z)/z = +\infty$, it follows from Lemma 4.3 that there exists a nondecreasing function $\bar{M} : \mathbb{R} \to [0, \infty)$ so that every periodic solution $(x_k(t))_{k=1}^N$ of (4.2) satisfies $|x_k(t)| \leqslant \bar{M}(\alpha)$ for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, N$. As $\varpi = a + \inf_{z\neq 0} g(z)/z > 0$, it follows from Lemma 4.4 that for every $\alpha \leqslant Q^{-1}(\varrho)$, (4.2) has no nontrivial $4r$-periodic solutions which implies that (4.2) has also no nontrivial $4r/n$-periodic solutions for every integer $n \geqslant 1$.

Since $0 < q_{j,0} = q_j < \varrho$, we have $Q^{-1}(q_{j,0}) = Q^{-1}(q_j) < Q^{-1}(\varrho)$. Moreover, by Condition (2), we easily show that

$$a^2 > \frac{\pi^2}{4r^2} + \left(v - 4d\sin^2\frac{\pi j}{N}\right)^2.$$

That is, (4.12) holds, and hence Lemma 4.2 implies that $2r < 2\pi/\beta_{j,0} < 4r$. Consequently, for each $\eta \in (q_j, \varrho)$, any unbounded connected branch Σ_j of nonconstant periodic solution bifurcating from $(0, Q^{-1}(q_{j,0}), 2\pi/\beta_{j,0})$ must satisfy

$$\Sigma_j|_\eta \subset \Big\{(x, \alpha, p) \in BC(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}^2;\ \ p \in (2r, 4r), \sup_{t\in\mathbb{R}} |x_k(t)| \leqslant \bar{M}(\alpha), k = 1, \ldots, N\Big\},$$

where

$$\Sigma_j|_\eta := \{(x, \alpha, p) \in \Sigma_j;\ \ \alpha \leqslant Q^{-1}(\eta)\}.$$

By Lemma 4.6, we see that (4.2) has no nontrivial periodic solution, that is, Σ_j does not intersect with the hyperplane $\alpha=0$. Therefore, the projection of $\Sigma_j|_\eta$ onto the α-space is contained in $[0, Q^{-1}(\eta)]$ and $\Sigma_j|_\eta \cap \{(x, Q^{-1}(\eta), p);\ (x, p) \in BC(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}\} \neq \emptyset$. This shows that for every $\alpha \in (Q^{-1}(q_j), Q^{-1}(\eta)]$, if $1 \leqslant j < N/2$, then (4.2) has at least 2 slowly oscillatory periodic solutions satisfying (4.36), if N is even and $j = N/2$, then (4.2) has at least one slowly oscillatory periodic solution satisfying (4.37) and if $j = 0$, then (4.2) has at least one slowly oscillatory periodic solutions satisfying (4.38). Thus, noting that $\eta \in (q_j, \varpi)$ is an arbitrary number, we conclude that the statements (j1)–(j3) hold true and the proof is complete. □

If we assume $\inf_{z\neq 0} g(z)/z \geqslant 0$ in Theorem 4.1, then $\varpi = a + \inf_{z\neq 0} g(z)/z \geqslant a$, and hence $\varrho = \sup\{q \in [0,1); 2aq/(1+q) < \varpi\} = 1$. As $v < 4d\sin^2(\pi j/N) < a+v$, it follows from (4.10) that

$$q_j = \sqrt{\frac{\beta_j^2 + (a+v-4d\sin^2(\pi j/N))^2}{\beta_j^2 + (a-v+4d\sin^2(\pi j/N))^2}} \in (0,1).$$

That is, condition (3) of Theorem 4.1 is satisfied.

Consequently, we have the following:

Theorem 4.3. *Assume that*

(1) $\inf_{z\neq 0} g(z)/z \geqslant 0$, $\lim_{z\to\pm\infty} g(z)/z = +\infty$;
(2) $2a > \pi/r$ *and there exists a* $j \in \{0, 1, \ldots, [N/2]\}$ *such that*

$$v < 4d\sin^2\frac{\pi j}{N} < v + \sqrt{a^2 - \frac{\pi^2}{4r^2}}, \qquad v := g'(0).$$

Let β_j *be the unique solution in* $(0, \pi/r)$ *of the equation*

$$\cot\beta r = \frac{\beta^2 - a^2 + (v - 4d\sin^2(\pi j/N))^2}{2a\beta}.$$

Put

$$q_j := \sqrt{\frac{\beta_j^2 + (a+v-4d\sin^2(\pi j/N))^2}{\beta_j^2 + (a-v+4d\sin^2(\pi j/N))^2}}.$$

Then $q_j \in (0,1)$ *and for each* $q \in (q_j, 1)$, *the conclusions of Theorem* 4.1 *hold true.*

The following Theorem 4.4 gives a sufficient condition on the coexistence of several slowly oscillatory periodic solutions of (4.1) when the parameter is far away from the bifurcation point.

Theorem 4.4. *Assume that*

(1) $\inf_{z\neq 0} g(z)/z = g'(0) = 0$, $\lim_{z\to\pm\infty} g(z)/z = +\infty$,
(2) $2a > \pi/r$, $4d < \sqrt{a^2 - \pi^2/4r^2}$.

Let $\bar{\beta}$ *be the unique solution in* $(0, \pi/r)$ *of the equation*

$$\cot\beta r = \frac{\beta^2 - a^2 - 16d^2\sin^4(\pi/N)}{2a\beta}.$$

Put

$$\bar{q} := \sqrt{\frac{\bar{\beta}^2 + (a - 4d\sin^2\pi/N)^2}{\bar{\beta}^2 + (a + 4\sin^2\pi/N)^2}} < 1.$$

Then for each $q \in (\bar{q}, 1)$, *system* (4.1) *has at least* $2[N/2]$ (*if* N *is odd*) *or* $2[N/2]-1$ (*if* N *is even*) *slowly oscillatory periodic solutions with minimal periods in* $(2r, 4r)$ *which are described below*:

(j1) *There are at least* $[N/2]$ (*if* N *is odd*) *or* $[N/2]-1$ (*if* N *is even*) *slowly oscillatory periodic solutions* $(x_k^{(j)}(t))_{k=1}^N$ *with minimal periods* $p_j \in (2r, 4r)$, $1 \leqslant j < N/2$, *and satisfying*

$$x_{k+1}^{(j)}\left(t + \frac{jp_j}{N}\right) = x_k^{(j)}(t),$$

for $k = 1, 2, \ldots, N \bmod N$.

(j2) *There are at least* $[N/2]$ (*if* N *is odd*) *or* $[N/2]-1$ (*if* N *is even*) *slowly oscillatory periodic solutions* $(x_k^{(j)}(t))_{k=1}^N$ *with minimal periods* $p_j \in (2r, 4r)$, $1 \leqslant j < N/2$, *and satisfying*

$$x_{k-1}^{(j)}\left(t + \frac{jp_j}{N}\right) = x_k^{(j)}(t),$$

for $k = 1, 2, \ldots, N \bmod N$.

(j3) *If* N *is even, there exists at least one slowly oscillatory periodic solution* $(x_k(t))_{k=1}^N$ *with a minimal period* $p \in (2r, 4r)$ *and satisfying*

$$x_{k-1}\left(t + \frac{p}{2}\right) = x_k(t), \quad x_{N-k}\left(t + \frac{p}{2}\right) = x_k(t),$$

for $k = 1, 2, \ldots, N \bmod N$.

Proof. Clearly, condition (1) of Theorem 4.3 holds, and by (2), condition (2) of Theorem 4.3 also holds for every $j \in \{1, \ldots, [N/2]\}$. Therefore, for every $j \in \{1, \ldots, [N/2]\}$, the conclusions of Theorem 4.3 holds true.

Now, we define a function as follows:

$$h(x, y) = \frac{x + (a - 4dy)^2}{x + (a + 4dy)^2}, \quad x \in (0, \infty), \ y \in [0, 1].$$

Then by (2), we can easily verify that for $y \in [0, 1]$,

$$\frac{\partial}{\partial x} h(x, y) = \frac{16ady}{[x + (a + 4dy)^2]^2}$$

and for $x > 0$ and $y \in [0, 1]$,

$$\frac{\partial}{\partial y} h(x, y) = \frac{-16ad[x + a^2 - 16d^2y^2]}{[x + (a + 4dy)^2]^2} < \frac{-16ad[a^2 - 16d^2]}{[x + (a + 4dy)^2]^2} < 0.$$

Therefore, $h(x, y)$ is nondecreasing in $x \in (0, \infty)$, and decreasing in $y \in [0, 1]$ if $x > 0$.

Now, for each $j \in \{1, \ldots, [N/2]\}$, let $\beta_j \in (0, \pi/r)$ and q_j be specified by Theorem 4.3. As

$$h_j(\beta) := \frac{\beta^2 - a^2 + 16d^2 \sin^4(\pi j/N))^2}{2a\beta} \geqslant \bar{h}(\beta) := \frac{\beta^2 - a^2 + 16d^2 \sin^4(\pi/N)}{2a\beta}$$

for every $1 \leqslant j \leqslant [N/2]$, and $\cot \beta r$ is decreasing in $\beta \in (0, \pi/r)$, it follows that $\beta_j \leqslant \bar{\beta}$ for every $j \in \{1, \ldots, [N/2]\}$. Therefore, for $1 \leqslant j \leqslant [N/2]$, we have

$$q_j = \sqrt{h\left(\beta_j^2, \sin^2 \frac{\pi j}{N}\right)} \leqslant \sqrt{h\left(\bar{\beta}^2, \sin^2 \frac{\pi}{N}\right)} = \bar{q} < 1.$$

Therefore, by Theorem 4.3, for each $j \in \{1, \ldots, [N/2]\}$ and each $q \in (\bar{q}, 1)$, the statements of Theorem 4.4 hold true and the proof is complete. □

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