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ASYMPTOTIC SPEED OF PROPAGATION OF WAVE FRONTS IN A 2D LATTICE DELAY DIFFERENTIAL EQUATION WITH GLOBAL INTERACTION

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ABSTRACT. In this paper, we derive a lattice model for a single species in a two dimensional patchy environment with infinite number of patches connected locally by diffusion. Under the assumption that the death and diffusion rates of the mature population are age independent, we show that the dynamics of the mature population is governed by a lattice delay differential equation with global interactions. We obtain the existence of monotone travelling waves for wave speeds $c > c_*$ by the standard monotone iteration method and the construction of upper-lower solutions. We show that the minimal wave speed c_* is also the asymptotic speed of propagation.

1 Introduction The existence and (asymptotic) speed of travelling wave fronts of biological systems are of paramount importance due to their connection to geographically spread of infections or propagation of epidemics. One of the simplest models for the dynamics of a single species that accounts for spatial interaction as well as the age structure is of the following form

$$(1.1)\quad \frac{\partial}{\partial t}u(t,a,x) + \frac{\partial}{\partial a}u(t,a,x) = D(a)\frac{\partial^2}{\partial x^2}u(t,a,x) - d(a)u(t,a,x)$$

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for $x \in \mathbb{R}$, $t > 0$, where $u(t,a,x)$ is the population density at time t, age a and spatial location x per unit age and per unit spatial length, $D(a)$ is the diffusion coefficient accounting for spatial dispersion and $d(a)$ is the death rate at age $a \geq 0$. The total mature population at time t and location x is defined by

$$w(t,x) = \int_r^\infty u(t,a,x)\,da,$$

where r is the length of maturation period. The equation for w can be derived using equation (1.1) as

$$\frac{\partial}{\partial t}w(t,x) = u(t,r,x) + D_m\frac{\partial^2}{\partial x^2}w(t,x) - d_m w(t,x), \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

if we assume that $u(t,\infty,x) = 0$ and

$$D(a) = D_m = \text{const.} \quad \text{and} \quad d(a) = d_m = \text{const.} \quad \text{for } a \geq r. \tag{1.3}$$

Using the Fourier transform, one can obtain explicitly the function $u(t,r,x)$ as (see [**16**]):

$$u(t,r,x) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\int_0^r d(z)dz}\int_{-\infty}^{\infty} b(w(t-r,y))e^{-\frac{(x-y)^2}{4\alpha}}\,dy, \tag{1.4}$$

$$\alpha = \int_0^r D(z)\,dz, \quad \mu = \exp\left\{-\int_0^r d(z)\,dz\right\}, \tag{1.5}$$

where $b : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}_+$ is the birth rate.

This model with a birth function

$$b(w) = pwe^{-aw}, \quad w \geq 0, \tag{1.6}$$

was studied in [**16**], where it was shown that when $1 < \mu p/d_m \leq e$, there exist monotone travelling waves connecting two spatially homogeneous equilibria

$$w^0 = 0 \quad \text{and} \quad w^+ = \frac{1}{a}\ln\left(\frac{\mu p}{d_m}\right) > 0. \tag{1.7}$$

A discrete analog of the model (1.1) was developed in [**22**] as follows

$$(1.8)\quad \frac{\partial}{\partial t}u_j(t,a)+\frac{\partial}{\partial a}u_j(t,a) \\ = D(a)[u_{j+1}(t,a)+u_{j-1}(t,a)-2u_j(t,a)]-d(a)u_j(t,a)$$

for $t>0,\ j\in\mathbb{Z}:=\{0,\pm1,\pm2,\cdots\}$, where $u_j(t,a)$ denote the density of the population of the species of the jth patch at time $t\geq 0$ and age $a\geq 0$, $D(a)$ and $d(a)$ denote the diffusion rate and death rate of the population at age a respectively. Assuming that $u_j(t,\infty)=0$ for $t\geq 0,\ j\in\mathbb{Z}$. Note that

$$w_j(t)=\int_r^\infty u_j(t,a)\,da$$

is the total mature population at the jth patch. From (1.8) and assuming (1.3), it was obtained that

$$(1.9)\quad \frac{dw_j(t)}{dt}=u_j(t,r)+D_m[w_{j+1}(t)+w_{j-1}(t) \\ -2w_j(t)]-d_mw_j(t)\quad \text{for } t>0,$$

where

$$(1.10)\qquad u_j(t,r)=\frac{\mu}{2\pi}\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)\,b(w_k(t-r)),$$

and

$$(1.11)\quad \beta_\alpha(l)=\int_{-\pi}^{\pi}e^{i(l\omega)-4\alpha\sin^2(\frac{\omega}{2})}\,d\omega=e^{-2\alpha}\int_{-\pi}^{\pi}\cos(l\omega)e^{2\alpha\cos\omega}\,d\omega,$$

for any $l\in\mathbb{Z}$. Here μ and α are defined in (1.5), and i is the imaginary unit.

Assume that the birth function $b:\mathbb{R}_+\to\mathbb{R}_+$ satisfies the following properties

(H_1) b is continuous and $b(0)=0$, $b'(0)>d_m/\mu$, $b(w)\leq b'(0)w$ for $w\in\mathbb{R}_+$;

(H_2) b is non-decreasing on $[0,K]$, and $\mu b(w)=d_mw$ has a unique solution $w^+\in(0,K]$.

Weng *et al.* [**21**] showed that there exists a positive number c_* such that for any $c > c_*$, (1.9) has a monotone travelling wave solution connecting two spatially homogeneous equilibria $w^0 = 0$ and $w^+ > 0$. Furthermore, they introduced the concept of asymptotic speed of propagation into the model (1.9), and showed that c_* is exactly the asymptotic speed of propagation.

The concept "asymptotic speed of propagation" concerns with the asymptotic behavior (as $t \to \infty$) of solutions of (1.9) as follows: $c_* > 0$ is called the asymptotic speed if for any c_1, c_2 with $0 < c_1 < c_* < c_2$, the solutions tend to zero uniformly in the region $|j| \geq c_2 t$, whereas it is bounded away from zero uniformly in the region $|j| \leq c_1 t$ for t sufficiently large. The discussion of asymptotic speed of propagation can be found in [**1, 2, 3, 5, 6, 11, 12, 14, 18, 20**].

Note that equations (1.2) and (1.9) are in general nonlocal, involving integration over the whole spatial domain in (1.4) and summation over all integer $j \in \mathbb{Z}$ in (1.10). The idea of nonlocal interaction in a model with time delay was considered by Britton [**4**], by Gourley and Britton [**7, 8**], by Smith and Thieme [**15**], among others (see [**9**] for a short survey).

In this paper, we shall extend the work in [**21**] to higher dimensional dynamical systems with interaction between patches. We will consider the case in two space dimensions (2D) even though the results in this paper are valid for the n dimensional case. The rest of the paper is organized as follows. In Section 2, we derived the 2D lattice model. The existence of travelling wave fronts is given in Section 3 and the initial value problem is discussed in Section 4. The discussion on asymptotic speed of wave propagation is given in Section 5.

2 2D model derivation Let $u_{j_1,j_2}(t,a)$ denote the density of the population of the species of the (j_1, j_2)th patch at time $t \geq 0$ and age $a \geq 0$. Using $D(a)$ and $d(a)$ to denote the diffusion rate and death rate of the population at age a respectively, and assuming the patches are located at the integer set nodes of a 2-dimensional lattice and assuming spatial diffusion occurs only at the nearest neighborhood and is proportional to the difference of the densities of the population at adjacent patches, we obtain the following model

$$\begin{aligned}(2.1)\qquad \frac{\partial}{\partial t}u_{j_1,j_2}(t,a)+\frac{\partial}{\partial a}u_{j_1,j_2}(t,a) &= D(a)\Delta^2 u_{j_1,j_2}(t,a)-d(a)u_{j_1,j_2}(t,a),\\ &\qquad t>0,\ j=(j_1,j_2)\in\mathbb{Z}^2:=\mathbb{Z}\times\mathbb{Z},\end{aligned}$$

where

$$\begin{aligned}\Delta^2 u_{j_1,j_2}(t,a) &= u_{j_1+1,j_2}(t,a)+u_{j_1-1,j_2}(t,a)+u_{j_1,j_2+1}(t,a)\\ &\quad + u_{j_1,j_2-1}(t,a)+u_{j_1+1,j_2+1}(t,a)+u_{j_1-1,j_2+1}(t,a)\\ &\quad + u_{j_1+1,j_2-1}(t,a)+u_{j_1-1,j_2-1}(t,a)-8u_{j_1,j_2}(t,a).\end{aligned}$$

Clearly,

$$w_{j_1,j_2}(t)=\int_r^\infty u_{j_1,j_2}(t,a)\,da$$

is the total mature population at the (j_1,j_2)th patch. From (2.1), we obtain

$$\begin{aligned}(2.2)\qquad \frac{dw_{j_1,j_2}(t)}{dt} &= \int_r^\infty \frac{\partial}{\partial t}u_{j_1,j_2}(t,a)\,da\\ &= \int_r^\infty\left\{-\frac{\partial}{\partial a}u_{j_1,j_2}(t,a)+D(a)\Delta^2 u_{j_1,j_2}(t,a)\right.\\ &\qquad\qquad \left. - d(a)u_{j_1,j_2}(t,a)\right\}da.\end{aligned}$$

Assume (1.3) holds and that

$$u_{j_1,j_2}(t,\infty)=0\quad \text{for } t\ge 0,\ (j_1,j_2)\in\mathbb{Z}^2.$$

We obtain from (2.2) and (2.1) that

$$(2.3)\quad \frac{dw_{j_1,j_2}(t)}{dt}=u_{j_1,j_2}(t,r)+D_m\Delta^2 w_{j_1,j_2}(t)-d_m w_{j_1,j_2}(t)\quad \text{for } t>0.$$

In order to obtain a closed system for w_j, we need to evaluate $u_{j_1,j_2}(t,r)$. For fixed $s\ge 0$, let

$$V^s_{j_1,j_2}(t)=u_{j_1,j_2}(t,t-s)\quad \text{for } s\le t\le s+r.$$

Since only the mature population can reproduce, we have

$$V^s_{j_1,j_2}(s) = u_{j_1,j_2}(s,0) = b(w_{j_1,j_2}(s)), \tag{2.4}$$

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is the birth function. From (2.1),

$$\begin{aligned} \frac{d}{dt}V^s_{j_1,j_2}(t) &= \frac{\partial}{\partial t}u_{j_1,j_2}(t,a)\Big|_{a=t-s} + \frac{\partial}{\partial a}u_{j_1,j_2}(t,a)\Big|_{a=t-s} \\ &= D(t-s)\Delta^2 V^s_{j_1,j_2}(t) - d(t-s)V^s_{j_1,j_2}(t). \end{aligned} \tag{2.5}$$

Consider the discrete Fourier transform (see [**10, 19**]):

$$v^s(t,\omega_1,\omega_2) = \frac{1}{2\pi}\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty} e^{-i(j_1\omega_1+j_2\omega_2)}V^s_{j_1,j_2}(t), \tag{2.6}$$

$$V^s_{j_1,j_2}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(j_1\omega_1+j_2\omega_2)}v^s(t,\omega_1,\omega_2)\,d\omega_1\,d\omega_2, \tag{2.7}$$

where i is the imaginary unit. Applying (2.6) to equation (2.5) yields

$$\begin{aligned} \frac{\partial}{\partial t}v^s(t,\omega_1,\omega_2) &= [D(t-s)(e^{i\omega_1} + e^{-i\omega_1} + e^{i\omega_2} + e^{-i\omega_2} \\ &\quad + e^{i(\omega_1+\omega_2)} + e^{-i(\omega_1+\omega_2)} + e^{i(\omega_1-\omega_2)} \\ &\quad + e^{i(\omega_2-\omega_1)} - 8) - d(t-s)]v^s(t,\omega_1,\omega_2) \\ &= \Big\{ - 4D(t-s)\Big[\sin^2\Big(\frac{\omega_1}{2}\Big) + \sin^2\Big(\frac{\omega_2}{2}\Big) \\ &\quad + \sin^2\Big(\frac{\omega_1+\omega_2}{2}\Big) + \sin^2\Big(\frac{\omega_1-\omega_2}{2}\Big)\Big] \\ &\quad - d(t-s)\Big\}v^s(t,\omega_1,\omega_2). \end{aligned} \tag{2.8}$$

This equation can be solved easily as

$$v^s(t,\omega_1,\omega_2) = v_s(s,\omega_1,\omega_2)e^{-4g(\omega)\int_s^t D(z-s)dz - \int_s^t d(z-s)dz},$$

where

$$\begin{aligned} g(\omega) = g(\omega_1,\omega_2) &= \sin^2\Big(\frac{\omega_1}{2}\Big) + \sin^2\Big(\frac{\omega_2}{2}\Big) + \sin^2\Big(\frac{\omega_1+\omega_2}{2}\Big) \\ &\quad + \sin^2\Big(\frac{\omega_1-\omega_2}{2}\Big), \quad \omega = (\omega_1,\omega_2). \end{aligned}$$

Using the inverse discrete Fourier transform (2.7) we obtain

$$V^s_{j_1,j_2}(t) = \frac{1}{2\pi} e^{-\int_s^t d(z-s)dz} \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(j_1\omega_1+j_2\omega_2)-4\alpha_s g(\omega)} v^s(s,\omega_1,\omega_2)d\omega_1 d\omega_2,$$

where $\alpha_s = \int_s^t D(z-s)dz$. By (2.4) and (2.6), we obtain

$$v^s(s,\omega_1,\omega_2) = \frac{1}{2\pi}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} e^{-i(k_1\omega_1+k_2\omega_2)} b(w_{k_1,k_2}(s)).$$

Hence,

$$V^s_{j_1,j_2}(t) = \frac{1}{(2\pi)^2} e^{-\int_s^t d(z-s)dz} \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} b(w_{k_1,k_2}(s)) \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\{i[(j_1-k_1)\omega_1 + (j_2-k_2)\omega_2] - 4\alpha_s g(\omega)\}\, d\omega_1\, d\omega_2. \tag{2.9}$$

Let $s = t - r$, $\mu = e^{-\int_0^r d(z)dz}$, $\alpha = \int_0^r D(z)dz$. Then (2.9) yields

$$u_{j_1,j_2}(t,r) = \frac{\mu}{(2\pi)^2} \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} b(w_{k_1,k_2}(t-r)) \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\{i[(j_1-k_1)\omega_1 + (j_2-k_2)\omega_2] - 4\alpha g(\omega)\}\, d\omega_1\, d\omega_2. \tag{2.10}$$

Denote

$$G_\alpha(l) = G_\alpha(l_1,l_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(l_1\omega_1+l_2\omega_2)-4\alpha g(\omega)}\, d\omega_1\, d\omega_2, \tag{2.11}$$

where $l = (l_1, l_2)$. The following lemma describes the properties of $G(\alpha)$.

Lemma 2.1. *Let* $G_\alpha(l)$ *be given in* (2.11). *Then*

(i) $G_\alpha(-l_1, l_2) = G_\alpha(l_1, l_2)$, $G_\alpha(l_1, -l_2) = G_\alpha(l_1, l_2)$, $G_\alpha(l) = G_\alpha(-l)$ *for* $l \in \mathbb{Z}^2$, *that is,* $G_\alpha(l)$ *is an isotropic function for any* $\alpha \geq 0$;

(ii) $\frac{1}{(2\pi)^2} \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} G_\alpha(l) = 1$;

(iii) $G_\alpha(l) \geq 0$ *if* $\alpha = 0$ *and* $l \in \mathbb{Z}^2$; $G_\alpha(l) > 0$ *if* $\alpha > 0$ *and* $l \in \mathbb{Z}^2$.

Proof. For

$$g(\omega) = \sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right) + \sin^2\left(\frac{\omega_1+\omega_2}{2}\right) + \sin^2\left(\frac{\omega_1-\omega_2}{2}\right) = 2 - \frac{1}{2}(\cos\omega_1 + \cos\omega_2) - \cos\omega_1\cos\omega_2, \tag{2.12}$$

let $l \cdot \omega = l_1\omega_1 + l_2\omega_2$, then we have

$$\begin{aligned} G_\alpha(l) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(l\cdot\omega)-4\alpha[2-\frac{1}{2}(\cos\omega_1+\cos\omega_2)-\cos\omega_1\cos\omega_2]}\, d\omega_1\, d\omega_2 \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{2\alpha(\cos\omega_1+\cos\omega_2)+4\alpha\cos\omega_1\cos\omega_2-8\alpha} \\ &\qquad \times [\cos(l\cdot w) + i\sin(l\cdot w)]\, d\omega_1\, d\omega_2. \end{aligned}$$

Note that $e^{2\alpha(\cos\omega_1+\cos\omega_2)+4\alpha\cos\omega_1\cos\omega_2-8\alpha}$ is an even function of ω_1 and ω_2 in $[-\pi,\pi]\times[-\pi,\pi]$ and $\sin(l\cdot w)$ is an odd function of ω_1 and ω_2 in $[-\pi,\pi]\times[-\pi,\pi]$. Therefore,

$$G_\alpha(l) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{2\alpha(\cos\omega_1+\cos\omega_2)+4\alpha\cos\omega_1\cos\omega_2-8\alpha}\cos(l\cdot w)\, d\omega_1\, d\omega_2.$$

We complete the proof of (i).

Now we show conclusion (ii). Firstly, we give the 2D Fourier series expansion for function $f(x,y)$ as follows (see [**13**]),

$$f(x,y) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} C_{l_1,l_2} e^{i(l_1x+l_2y)}, \tag{2.13}$$

where

$$C_{l_1,l_2} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-i(l_1x+l_2y)}\, dx\, dy, \quad l_1, l_2 \in \mathbb{Z}.$$

Let

$$f(\omega_1, \omega_2) = \exp\{2\alpha(\cos\omega_1 + \cos\omega_2) + 4\alpha\cos\omega_1\cos\omega_2 - 8\alpha\}.$$

We can rewrite the expression of $G_\alpha(l)$ as

$$G_\alpha(l) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\omega_1, \omega_2)e^{i(l_1\omega_1 + l_2\omega_2)}d\omega_1 d\omega_2.$$

Let $C_{l_1,l_2} = (1/(2\pi)^2)G_\alpha(l)$. Then we have from (2.13)

$$\begin{aligned}\frac{1}{(2\pi)^2}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} G_\alpha(l) &= \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} C_{l_1,l_2}\cdot e^{i(l_1\cdot 0 + l_2\cdot 0)}\\ &= f(\omega_1,\omega_2)\Big|_{\omega_1=0,\omega_2=0} = 1.\end{aligned}$$

The conclusion of (ii) follows.

On the other hand, we have

$$\begin{aligned}G_\alpha(l) &\geq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(l_1\omega_1 + l_2\omega_2) + 2\alpha(\cos\omega_1 + \cos\omega_2) - 12\alpha}\, d\omega_1\, d\omega_2\\ &= e^{-8\alpha}\beta_\alpha(l_1)\beta_\alpha(l_2),\end{aligned}$$

where $\beta_\alpha(l_i)$, $i = 1, 2$ are defined in (1.11). From the following conclusions showed by Weng *et al.* (see [**21**]):

$$\begin{aligned}\beta_\alpha(m) &\geq 0 \quad \text{for} \quad m \in \mathbb{Z}, \quad \alpha = 0,\\ \beta_\alpha(m) &> 0 \quad \text{for} \quad m \in \mathbb{Z}, \quad \alpha > 0,\end{aligned}$$

we obtain conclusion (iii). This completes the proof. □

3 Existence of travelling waves In this section, we assume that the birth function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (H_1) and (H_2) in the first section.

A travelling wave of (2.3) is a solution of (2.3) of the form

$$w_{j_1,j_2}(t) = \phi(s), \tag{3.1}$$

where $s = j \cdot \nu + ct = j_1\nu_1 + j_2\nu_2 + ct$, $j = (j_1, j_2)$, $\nu = (\nu_1, \nu_2)$ is a given unit vector, and $c > 0$ is the wave speed. Substituting (3.1) into (2.3) yields

$$
\begin{aligned}
(3.2)\qquad c\frac{d\phi(s)}{ds} = D_m\Bigg[&\sum_{n=1}^{2}(\phi(s+\nu_n) + \phi(s-\nu_n)) + \phi(s+\nu_1+\nu_2) \\
&+ \phi(s-\nu_1-\nu_2) + \phi(s+\nu_1-\nu_2) \\
&+ \phi(s+\nu_2-\nu_1) - 8\phi(s)\Bigg] - d_m\phi(s) \\
&+ \frac{\mu}{(2\pi)^2}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} G_\alpha(l)\, b(\phi(s + l\cdot\nu - cr)).
\end{aligned}
$$

By (H_2), then we know that (3.2) has two equilibria $w^0 = 0$ and $w^+ > 0$. Define a subset $\mathcal{A}$ of $\mathbb{Z}^2$ as follows:

$$\mathcal{A} = \{(1,0),\ (0,1),\ (-1,0),\ (0,-1),\ (1,1),\ (-1,-1),\ (1,-1),\ (-1,1)\}.$$

Denote $p = (p_1, p_2)$, $\sum_{l\in Z^2} = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}$ and the characteristic equation (3.2) at w^0 by $\Delta(\lambda, c, w^0) = 0$, we have

$$
\begin{aligned}
(3.3)\quad \Delta(\lambda, c, w^0) \equiv -c\lambda + D_m\left[\sum_{p\in\mathcal{A}} e^{\lambda(p\cdot\nu)} - 8\right] - d_m \\
+ \frac{b'(0)\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)e^{\lambda(l\cdot\nu)}e^{-\lambda cr},
\end{aligned}
$$

which can be simplified as follows. Let

$$
\begin{aligned}
(3.4)\quad S(\alpha) &= \frac{1}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)e^{\lambda(l\cdot\nu)} \\
&= \frac{1}{(2\pi)^2}\sum_{l\in Z^2} e^{\lambda(l\cdot\nu)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{2\alpha[\cos\omega_1 + \cos\omega_2 + 2\cos\omega_1\cos\omega_2 - 4]} \\
&\qquad \times \cos(l\cdot w)d\omega_1 d\omega_2,
\end{aligned}
$$

then

$$\frac{dS(\alpha)}{d\alpha} = \frac{1}{(2\pi)^2}\sum_{l\in Z^2} e^{\lambda(l\cdot\nu)}$$
$$\times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[2(\cos\omega_1+\cos\omega_2)+4\cos\omega_1\cos\omega_2-8]$$
$$\times \cos(l\cdot\omega)e^{2\alpha(\cos\omega_1+\cos\omega_2)+4\alpha\cos\omega_1\cos\omega_2-8\alpha}\,d\omega_1\,d\omega_2$$
$$= \frac{1}{(2\pi)^2}\sum_{l\in Z^2} e^{\lambda(l\cdot\nu)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{2\alpha(\cos\omega_1+\cos\omega_2)+4\alpha\cos\omega_1\cos\omega_2-8\alpha}$$
$$\times\left\{\sum_{p\in\mathcal{A}}\cos[(l+p)\cdot\omega]-8\cos(l_1\omega_1+l_2\omega_2)\right\}d\omega_1\,d\omega_2$$
$$= S(\alpha)[e^{\lambda\nu_1}+e^{-\lambda\nu_1}+e^{\lambda\nu_2}+e^{-\lambda\nu_2}+e^{\lambda(\nu_1+\nu_2)}+e^{-\lambda(\nu_1+\nu_2)}$$
$$+e^{\lambda(\nu_1-\nu_2)}+e^{\lambda(\nu_2-\nu_1)}-8].$$

Since $S(0)=1$,

$$S(\alpha) = \exp\{[e^{\lambda\nu_1}+e^{-\lambda\nu_1}+e^{\lambda\nu_2}+e^{-\lambda\nu_2}+e^{\lambda(\nu_1+\nu_2)}$$
$$+e^{-\lambda(\nu_1+\nu_2)}+e^{\lambda(\nu_1-\nu_2)}+e^{\lambda(\nu_2-\nu_1)}-8]\alpha\}$$
$$= e^{2\alpha[E(\lambda,\nu)-4]}, \tag{3.5}$$

where

$$E(\lambda,\nu) := \cosh(\lambda\nu_1)+\cosh(\lambda\nu_2)$$
$$+\cosh(\lambda(\nu_1+\nu_2))+\cosh(\lambda(\nu_1-\nu_2)).$$

Therefore, we have

$$\Delta(\lambda,c,w^0) = b'(0)\mu e^{2\alpha[E(\lambda,\nu)-4]-\lambda cr}-c\lambda+2D_m[E(\lambda,\nu)-4]-d_m.$$

Differentiating $\Delta(\lambda,c,w^0)$ with respect to λ, we obtain

$$\frac{\partial}{\partial\lambda}\Delta(\lambda,c,w^0)$$
$$= b'(0)\mu\{2\alpha[\nu_1\sinh(\lambda\nu_1)+\nu_2\sinh(\lambda\nu_2)+(\nu_1+\nu_2)\sinh(\lambda(\nu_1+\nu_2))$$
$$+(\nu_1-\nu_2)\sinh(\lambda(\nu_1-\nu_2))]-cr\}\exp\{2\alpha[E(\lambda,\nu)-4]-\lambda cr\}$$
$$-c+2D_m[\nu_1\sinh(\lambda\nu_1)+\nu_2\sinh(\lambda\nu_2)$$
$$+(\nu_1+\nu_2)\sinh(\lambda(\nu_1+\nu_2))+(\nu_1-\nu_2)\sinh(\lambda(\nu_1-\nu_2))],$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial\lambda^2}\Delta(\lambda, c, w^0)\\
&= b'(0)\mu\{2\alpha[\nu_1^2\cosh(\lambda\nu_1) + \nu_2^2\cosh(\lambda\nu_2)\\
&\quad + (\nu_1+\nu_2)^2\cosh(\lambda(\nu_1+\nu_2)) + (\nu_1-\nu_2)^2\cosh(\lambda(\nu_1-\nu_2))]\\
&\quad + [2\alpha[\nu_1\sinh(\lambda\nu_1) + \nu_2\sinh(\lambda\nu_2) + (\nu_1+\nu_2)\sinh(\lambda(\nu_1+\nu_2))\\
&\quad + (\nu_1-\nu_2)\sinh(\lambda(\nu_1-\nu_2))] - cr]^2\}\exp\{2\alpha[E(\lambda,\nu)-4] - \lambda cr\}\\
&\quad + 2D_m[\nu_1^2\cosh(\lambda\nu_1) + \nu_2^2\cosh(\lambda\nu_2) + (\nu_1+\nu_2)^2\cosh(\lambda(\nu_1+\nu_2))\\
&\quad + (\nu_1-\nu_2)^2\cosh(\lambda(\nu_1-\nu_2))].
\end{aligned}$$

Since $\frac{\partial^2}{\partial\lambda^2}\Delta(\lambda, c, w^0) > 0$ for $\lambda \in \mathbb{R}$, the graph of $\Delta(\lambda, c, w^0)$ as a function of $\lambda \in \mathbb{R}$ is convex. Furthermore, it can be easily verified that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\Delta(\lambda, c, w^0) = +\infty, \quad \Delta(0, c, w^0) = b'(0)\mu - d_m > 0,\\
&\frac{\partial}{\partial\lambda}\Delta(\lambda, c, w^0)\bigg|_{\lambda=0} = -\left(b'(0)\mu r + 1\right)c < 0
\end{aligned} \tag{3.6}$$

if $c > 0$ and (H_1)–(H_2) hold. In addition, we note that $E(\lambda,\nu) - 4 \geq 0$ for all $\lambda \in \mathbb{R}$, and then we can show that $\Delta(\lambda, 0, w^0) > 0$ and $\lim_{c\to\infty}\Delta(\lambda, c, w^0) < 0$ for any given λ, therefore, we have the following observations.

Lemma 3.1. *There exist a pair of c_* and λ_* such that*

(i) $\Delta(\lambda_*, c_*, w^0) = 0$; $\frac{\partial}{\partial\lambda}\Delta(\lambda_*, c_*, w^0) = 0$;

(ii) *for* $0 < c < c_*$ *and any* $\lambda > 0$, $\Delta(\lambda, c, w^0) > 0$;

(iii) *for any* $c > c_*$, *the equation* $\Delta(\lambda, c, w^0) = 0$ *has two positive real roots* $0 < \lambda_1 < \lambda_2$, *and there exists a* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *with*

$$0 < \lambda_1 < \lambda_1 + \epsilon < \lambda_2,$$

we have

$$\Delta(\lambda_1 + \epsilon, c, w^0) < 0. \tag{3.7}$$

We now define $C = C(\mathbb{R}, [0, K])$, and

$$S = \left\{ \phi \in C : \begin{array}{ll} \text{(i)} & \phi(s) \text{ is non-decreasing for } s \in \mathbb{R}, \\ \text{(ii)} & \lim_{s \to -\infty} \phi(s) = w^0, \quad \lim_{s \to \infty} \phi(s) = w^+, \end{array} \right\}$$

and an operator on C as

$$H(\phi)(s) = \frac{\mu}{(2\pi)^2} \sum_{l \in Z^2} G_\alpha(l)\, b(\phi(s + l \cdot \nu - cr)), \quad \phi \in C, \ s \in \mathbb{R}.$$

The following lemma summarizes some useful properties of H.

Lemma 3.2. *Assume that $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (H$_1$) *and* (H$_2$). *Then we have*

(i) *if $\phi \in S$, and $\phi(s) \geq 0$ for $s \in \mathbb{R}$, then $H(\phi)(s) \geq 0$ for $s \in \mathbb{R}$;*
(ii) *if $\phi \in S$, then $H(\phi)(s)$ is non-decreasing for $s \in \mathbb{R}$;*
(iii) *$H(\psi)(s) \leq H(\phi)(s)$ for $s \in \mathbb{R}$ provided that $\psi, \phi \in C$ and $\psi(s) \leq \phi(s) \leq K$ for $s \in \mathbb{R}$.*

Definition 3.1. A function $U \in C$ is called an *upper solution* of (3.2) if it is differentiable almost everywhere (a.e.) and satisfies the inequality

$$cU'(s) \geq D_m \left[\sum_{p \in \mathcal{A}} U(s + p \cdot \nu) - 8U(s) \right] - d_m U(s) + H(U)(s) \text{ a.e. in } \mathbb{R}.$$

Similarly, a function $L \in C$ is called a *lower solution* of (3.2) if it is differentiable almost everywhere and satisfies

$$cL'(s) \leq D_m \left[\sum_{p \in \mathcal{A}} L(s + p \cdot \nu) - 8L(s) \right] - d_m L(s) + H(L)(s) \text{ a.e. in } \mathbb{R}.$$

Suppose that

$$U(s) = \begin{cases} w^+, & s \geq 0, \\ e^{\lambda_1 s} w^+, & s \leq 0, \end{cases} \tag{3.8}$$

and

$$L(s) = \begin{cases} 0, & s \geq 0, \\ \zeta(1 - e^{\epsilon s}) e^{\lambda_1 s}, & s \leq 0, \end{cases} \tag{3.9}$$

where λ_1 and ϵ are given as in Lemma 3.1, $\zeta > 0$ is chosen small enough so that $L(s) \leq U(s)$ for $s \in \mathbb{R}$ and L is the lower solution of (3.2). Clearly, we have $0 \leq L(s) \leq U(s) \leq w^+ \leq K$ and $L(s) \not\equiv 0$ for $s \in \mathbb{R}$.

Lemma 3.3. *U given by (3.8) and L given by (3.9) are a pair of upper and lower solutions of (3.2).*

Proof. If $s \geq 0$, then we have from (iii) of Lemma 3.2 and the fact that $0 < (1/2\pi)\sum_{l\in Z^2} G_\alpha(l) \leq 1$, and $b(w) \leq b(w^+)$ for $0 \leq w \leq w^+$ the following holds:

$$-c\frac{dU(s)}{ds} + D_m\Big[\sum_{p\in\mathcal{A}} U(s+p\cdot\nu) - 8U(s)\Big] - d_m U(s) + H(U)(s)$$
$$\leq 0 + D_m(8w^+ - 8w^+) - d_m w^+ + b(w^+)\mu = 0.$$

Note that $U(s) \leq e^{\lambda_1 s} w^+$ for $s \in \mathbb{R}$ and $b(\phi) \leq b'(0)\phi$ for $\phi \geq 0$. Therefore, if $s \leq 0$, then

$$\begin{aligned}
&-c\frac{dU(s)}{ds} + D_m\Big[\sum_{p\in\mathcal{A}}(U(s+p\cdot\nu) - 8U(s)\Big] - d_m U(s) + H(U)(s)\\
&\leq -c\lambda_1 e^{\lambda_1 s} w^+ + D_m w^+\Big[\sum_{p\in\mathcal{A}} e^{\lambda_1(s+p\cdot\nu)} - 8e^{\lambda_1 s}\Big] - d_m w^+ e^{\lambda_1 s}\\
&\qquad + \frac{b'(0)\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)U(s+l\cdot\nu - cr)\\
&\leq e^{\lambda_1 s} w^+\Big\{ - c\lambda_1 + D_m\Big[\sum_{p\in\mathcal{A}} e^{\lambda_1(p\cdot\nu)} - 8\Big]\\
&\qquad - d_m + \frac{b'(0)\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)e^{\lambda_1(l\cdot\nu - cr)}\Big\}\\
&= 0.
\end{aligned}$$

Hence, U is an upper solution of (3.2).

Note that $L(s) \geq 0$ and thus $H(L)(s) \geq 0$ for $s \in \mathbb{R}$. Therefore, for $s \geq 0$, we have

$$-c\frac{dL(s)}{ds} + D_m\Big[\sum_{p\in\mathcal{A}} L(s+p\cdot\nu) - 8L(s)\Big] - d_m L(s) + H(L)(s) \geq 0.$$

Note also that $\zeta(1-e^{\epsilon s})e^{\lambda_1 s} \leq 0$ for $s \geq 0$ and

$$L(s) \geq \zeta(1-e^{\epsilon s})e^{\lambda_1 s} =: h(s) \quad \text{for } s \in \mathbb{R}.$$

Therefore,

$$H(L)(s) \geq H(h)(s) \quad \text{for } s \in \mathbb{R}.$$

Consequently, if $s \leq 0$, then

(3.10)
$$\begin{aligned}
&-c\frac{dL(s)}{ds} + D_m\Big[\sum_{p\in\mathcal{A}} L(s+p\cdot\nu) - 8L(s)\Big] - d_m L(s) + H(L)(s) \\
&\geq -c\lambda_1\zeta e^{\lambda_1 s} + c(\lambda_1+\epsilon)\zeta e^{(\epsilon+\lambda_1)s} + D_m \\
&\times \Big\{\sum_{p\in\mathcal{A}} \zeta[(1-e^{\epsilon(s+p\cdot\nu)})e^{\lambda_1(s+p\cdot\nu)} - 8\zeta(1-e^{\epsilon s})e^{\lambda_1 s}\Big\} \\
&- d_m\zeta(1-e^{\epsilon s})e^{\lambda_1 s} + H(h)(s).
\end{aligned}$$

Applying Taylor's expansion of $b(u)$ about zero, we can write

$$H(h(s)) = \frac{\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)\{b'(0)h(s+l\cdot\nu - cr) + Q(h(s+l\cdot\nu - cr))\}, \tag{3.11}$$

where

$$\left|\frac{\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l) Q(h(s+l\cdot\nu_2 - cr))\right| \leq \zeta^2 Re^{2\lambda_1 s}, \tag{3.12}$$

and R is a constant. From (3.10)–(3.12) and Lemma 3.1 we obtain

$$\begin{aligned}
&-c\frac{dL(s)}{ds} + D_m\Big[\sum_{p\in\mathcal{A}} L(s+p\cdot\nu) - 8L(s)\Big] - d_m L(s) + H(L)(s) \\
&\geq \zeta e^{\lambda_1 s}\Delta(\lambda_1, c, w^0) - \zeta\Delta(\lambda_1+\epsilon, c, w^0)e^{(\lambda_1+\epsilon)s} - \zeta^2 Re^{2\lambda_1 s} > 0,
\end{aligned}$$

if ζ and ϵ are chosen small enough. Hence, L is a lower solution of (3.2). This completes the proof. □

Consider the following equivalent form of equation (3.2):

$$\frac{d\phi(s)}{ds} + \eta\phi(s) = F(\phi)(s), \tag{3.13}$$

where

$$F(\phi)(s) = \left(\eta - \frac{d_m}{c} - \frac{8D_m}{c}\right)\phi(s) + \frac{D_m}{c}\left[\sum_{p\in\mathcal{A}} \phi(s+p\cdot\nu)\right] + \frac{1}{c}H(\phi)(s),$$

and $\eta > 0$ is chosen so that $\eta - d_m/c - 8D_m/c > 0$. Then, $F(\phi)(s) \geq F(\psi)(s)$ for $s \in \mathbb{R}$ provided that $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$. Moreover,

$$F(w^0) = \eta w^0, \qquad F(w^+) = \eta w^+.$$

For any bounded solutions $\phi : \mathbb{R} \to \mathbb{R}$, (3.13) is equivalent to

$$\phi(s) = e^{-\eta s}\int_{-\infty}^{s} e^{\eta t}F(\phi)(t)\,dt. \tag{3.14}$$

It is natural to define an operator $T : S \to C$ by

$$(T\phi)(s) = e^{-\eta s}\int_{-\infty}^{s} e^{\eta t}F(\phi)(t)\,dt, \quad \phi \in S,\ t \in \mathbb{R}, \tag{3.15}$$

and it is straightforward to verify the following.

Lemma 3.4. *The operator T defined in* (3.15) *has the following properties:*

(i) *if $\phi \in S$, then $T\phi \in S$;*
(ii) *if ϕ is an upper (a lower) solution of* (3.2), *then $\phi(s) \geq (T\phi)(s)$ $\times$ $(\phi(s) \leq (T\phi)(s))$ for $s \in \mathbb{R}$;*
(iii) *if $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$, then $(T\phi)(s) \geq (T\psi)(s)$ for $s \in \mathbb{R}$;*
(iv) *if ϕ is an upper (a lower) solution of* (3.2), *then $T\phi$ is also an upper (a lower) solution of* (3.2).

We now construct a series of functions by the following iterative scheme: $U_n = T^n U_{n-1}$, $n \geq 1$ with $U_0 = U$. By Lemma 3.4, we have

$$w^0 \leq L(s) \leq \cdots \leq U_n(s) \leq U_{n-1}(s) \leq \cdots \leq U_0(s) \leq w^+.$$

Using Lebesgue's dominated convergence theorem, we know that the limit function $U_*(s) = \lim_{n\to\infty} U_n(s)$ exists and is a fixed point of T.

This gives a solution of (3.2). Furthermore, U_* lies in S and is non-decreasing with the limits

$$\lim_{s\to-\infty} U_*(s) = w^0, \quad \lim_{s\to\infty} U_*(s) = w^+. \tag{3.16}$$

Summarizing the above discussions, we obtain the following existing theorem of travelling waves.

Theorem 3.1. *Assume that $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (H_1) *and* (H_2). *Then there exists $c_* > 0$, such that for every $c > c_*$,* (2.3) *has a monotone travelling wave solution $\phi : \mathbb{R} \to \mathbb{R}$ satisfying the boundary condition*

$$\lim_{s\to-\infty} \phi(s) = w^0, \quad \lim_{s\to\infty} \phi(s) = w^+.$$

4 Solutions of initial value problem In this section, we shall investigate the existence and isotropic properties of solutions for the initial value problem of model (2.3) with $u_{j_1,j_2}(t,r)$ defined in (2.10). For the convenience of discussion, we first list some notations to be used.

$$|j|_\nu = |j_1\nu_1| + |j_2\nu_2|, \quad B_N = \{j \in \mathbb{Z}^2 \mid |j|_\nu \le N, N \in \mathbb{N}\},$$
$$C_K^+[-r,0] = C([-r,0],[0,K]), \quad C_K^+[-r,T) = C([-r,T),[0,K]),$$
$$w_j(t) = w(t,j) = w_{j_1,j_2}(t), \quad j = (j_1,j_2) \in \mathbb{Z}^2,$$
$$W(t) = W(t,\cdot) = \{w_j(t)\}_{j\in\mathbb{Z}^2},$$
$$\mathit{supp}\, W(t,\cdot) = \{j \mid w(t,j) \ne 0\} \text{ is the support of } W(t,\cdot),$$
$$W(t) \ge V(t) \text{ if } w_{j_1,j_2}(t) \ge v_{j_1,j_2}(t) \text{ for } j = (j_1,j_2) \in \mathbb{Z}^2,$$
$$W(t) \succ V(t) \text{ if } W(t) \ge V(t) \text{ and } w_{j_1,j_2}(t) > v_{j_1,j_2}(t)$$
$$\text{for } j \in \mathit{supp}\, V(t,\cdot).$$

Also we say W is isotropic on an interval I if $w_{-j_1,j_2}(t) = w_{j_1,j_2}(t)$ and $w_{j_1,-j_2}(t) = w_{j_1,j_2}(t)$ for $j \in \mathbb{Z}^2$ and $t \in I$.

In the remaining part of this paper, we assume that the birth function $b : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and satisfies

(H_2) b is non-decreasing on $[0,K]$, and $\mu b(w) = d_m w$ has a unique solution $w^+ \in (0,K]$;

(H$_3$) $b(0) = 0, \quad b'(0) > d_m/\mu, \quad |b(w) - b(v)| \le b'(0)|w - v|$ for $w, v \in \mathbb{R}_+$;

(H$_4$) $\mu b(w) > d_m w$ for $w \in (0, w^+)$, and $\mu b(w) < d_m w$ for $w \in (w^+, \infty)$.

Clearly, the birth function $b(w) = pwe^{-aw}$ in Nicholson's blowflies model satisfies the above assumptions, when the parameters are in appropriate ranges.

The initial value problem of (2.3) can be written as

$$\text{(4.1)} \quad \begin{cases} w_j(t) = e^{-\delta t} w_j(0) + \displaystyle\int_0^t e^{-\delta(t-s)} \\ \qquad \times \Big\{ D_m \Big[\displaystyle\sum_{p \in \mathcal{A}} w_{j+p}(s) \Big] + \dfrac{\mu}{(2\pi)^2} \\ \qquad \times \displaystyle\sum_{l \in Z^2} G_\alpha(l) b(w_{l+j}(s-r)) \Big\} ds, & j \in \mathbb{Z}^2, \ t \ge 0, \\ w_j(t) = w_j^o(t), & j \in \mathbb{Z}^2, \ t \in [-r, 0], \end{cases}$$

where $\delta = 8D_m + d_m$, and $w_j^o(t)$, $t \in [-r, 0]$, $j \in \mathbb{Z}^2$ are given initial data. A simple change of variable yields an equivalent form of (2.3) as

$$\text{(4.2)} \quad w_j(t) = e^{-\delta t} w_j(0) + \int_0^t e^{-\delta s} \Big\{ D_m \Big[\sum_{p \in \mathcal{A}} w_{j+p}(t-s) \Big] + \frac{\mu}{(2\pi)^2} \sum_{l \in Z^2} G_\alpha(l) b(w_{l+j}(t-s-r)) \Big\} ds, \quad j \in \mathbb{Z}^2, \ t \ge 0.$$

The existence and isotropic properties of the solution to the initial value problem is given by the following theorem.

Theorem 4.1. *For any given function*

$$W^o = \{w_j^o\}_{j \in \mathbb{Z}^2}, \quad w_j^o \in C_K^+[-r, 0], \ j \in \mathbb{Z}^2,$$

(4.1) has a unique solution $W(t) = \{w_j(t)\}_{j \in \mathbb{Z}^2}$ with $w_j \in C_K^+[-r, \infty)$. If W^o is isotropic on $[-r, 0]$, then W is isotropic on $\mathbb{R}_+$.

Proof. For $W^o = \{w_j^o\}_{j \in \mathbb{Z}^2}$ with $w_j^o \in C_K^+[-r, 0]$ and for every $T \in (0, \infty]$, define a set

$$S_T = \{W = \{w_j\}_{j \in \mathbb{Z}^2} | \ w_j \in C_K^+[-r, T), \ w_j(t) = w_j^o(t), \ t \in [-r, 0]\}$$

and an operator $F^T = \{F_j^T\}_{j\in\mathbb{Z}^2}$ on S_T, where for every $W \in S_T, j \in \mathbb{Z}^2$,

$$F_j^T[W](t) = \begin{cases} e^{-\delta t}w_j(0) + \int_0^t e^{-\delta(t-s)} \\ \quad \times\Big\{D_m\Big[\sum_{p\in\mathcal{A}} w_{j+p}(s)\Big] + \frac{\mu}{(2\pi)^2} \\ \quad \times \sum_{l\in Z^2} G_\alpha(l)b(w_{l+j}(s-r))\Big\}\,ds, & j\in\mathbb{Z}^2,\ t\geq 0, \\ w_j^o(t), & j\in\mathbb{Z}^2,\ t\in[-r,0]. \end{cases}$$

Clearly, for fixed $T > 0$, $F^T[W](t)$ is continuous in $t \in [-r, T)$. Note that if $W \in S^T$, then we have

$$\begin{aligned} 0 \leq F_j^T[W](t) &\leq e^{-\delta t}K + [8D_mK + \mu b(K)]\int_0^t e^{-\delta(t-s)}\,ds \\ &\leq e^{-\delta t}K + \frac{1}{\delta}[8D_mK + d_mK](1-e^{-\delta t}) \\ &= K, \end{aligned}$$

for $t \in [0,T)$ and $j \in \mathbb{Z}^2$. Therefore, $F^T(S_T) \subseteq S_T$.

For any $W \in S_T$ and $\lambda > 0$, define a norm as follows:

$$\|W\|_\lambda := \sup_{t\in[0,T),j\in\mathbb{Z}^2} |w_j(t)|e^{-\lambda t}.$$

For any $W, \overline{W} \in S_T$, let $\phi_j(t) = w_j(t) - \overline{w}_j(t)$ and $\Phi(t) = \{\phi_j(t)\}_{j\in\mathbb{Z}^2}$, then for $t \geq 0$ we have

$$\begin{aligned} F_j^T[W](t) - F_j^T[\overline{W}](t) &= \int_0^t e^{-\delta(t-s)}\Big\{D_m\sum_{p\in\mathcal{A}}\phi_{j+p}(s)\,ds \\ &\quad + \frac{\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)[b(w_{l+j}(s-r)) - b(\bar{w}_{l+j}(s-r))]\Big\}\,ds, \\ &= \int_0^t e^{-\delta(t-s)}D_m\sum_{p\in\mathcal{A}}\phi_{j+p}(s)ds \\ &\quad + \begin{cases} \frac{\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)\int_0^{t-r} e^{-\delta(t-s-r)} \\ \qquad \times[b(w_{l+j}(s)) - b(\bar{w}_{l+j}(s))]\,ds, & t-r>0, \\ 0, & t-r\leq 0. \end{cases} \end{aligned}$$

When $t - r > 0$, using property in (H$_3$), we have

$$\left|F_j^T[W](t) - F_j^T[\overline{W}](t)\right| \le \int_0^t e^{-\delta(t-s)} D_m \sum_{p\in\mathcal{A}} |\phi_{j+p}(s)|\, ds$$
$$+\frac{\mu b'(0)}{(2\pi)^2} \sum_{l\in Z^2} G_\alpha(l) \int_0^{t-r} e^{-\delta(t-s-r)} |\phi_{l+j}(s)|\, ds,$$

which leads to

$$\left|F_j^T[W](t) - F_j^T[\overline{W}](t)\right| e^{-\lambda t} \le D_m \int_0^t e^{-\lambda s} e^{-\lambda(t-s)} \sum_{p\in\mathcal{A}} |\phi_{j+p}(s)|\, ds$$
$$+\frac{\mu b'(0)}{(2\pi)^2} \sum_{l\in Z^2} G_\alpha(l) \int_0^{t-r} e^{-\lambda s} e^{-\lambda(t-s)} |\phi_{l+j}(s)|\, ds.$$

Thus

$$\begin{aligned} &\|F^T[W](t) - F^T[\overline{W}](t)\|_\lambda \\ &\le 8D_m \|\Phi\|_\lambda \int_0^t e^{-\lambda(t-s)}\, ds + \mu b'(0) \|\Phi\|_\lambda \int_0^{t-r} e^{-\lambda(t-s)}\, ds \\ &= \frac{8D_m}{\lambda} \|\Phi\|_\lambda (1 - e^{-\lambda t}) + \frac{\mu b'(0)}{\lambda} \|\Phi\|_\lambda (e^{-\lambda r} - e^{-\lambda t}). \end{aligned} \tag{4.3}$$

Since

$$\lim_{\lambda\to\infty} \frac{8D_m}{\lambda}(1 - e^{-\lambda t}) + \frac{\mu b'(0)}{\lambda}(e^{-\lambda r} - e^{-\lambda t}) = 0 \tag{4.4}$$

and S_T is a Banach space with norm $\|\cdot\|_\lambda$, we have from (4.3) and (4.4) that F^T is a contracting map and hence has a unique fixed point W in S_T if $\lambda > 0$ is sufficiently large. This shows that a unique solution of (4.1) exists on $[0, T)$ for any $T > 0$, which guarantees the uniqueness and existence of solution W to (4.1) on $[0, \infty)$.

The isotropic property of the solution on $[-r, \infty)$ starting from an isotropic initial data W^o on $[-r, 0]$ can be verified by noting that the subspace S_T^I of S_T, consisting all elements which are isotropic on $[-r, \infty)$, is closed and $F^T(S_T^I) \subset S_T^I$. □

5 Asymptotic speed of wave propagation Let

$$L_c(\lambda) = \frac{1}{\delta + \lambda c}\{2D_m E(\lambda, \nu) + \mu b'(0) e^{2\alpha[E(\lambda,\nu)-4]-\lambda c r}\}.$$

Then, we can rewrite $\Delta(\lambda, c, 0) = 0$ as

$$L_c(\lambda) = 1, \tag{5.1}$$

and the minimum speed defined in Lemma 3.1 can also be written as

$$c_* := \inf\{c > 0 \mid L_c(\lambda) = 1 \quad \text{for some } \lambda \in \mathbb{R}_+\}. \tag{5.2}$$

In the following, we will show that c_* is the asymptotic speed of wave propagation in the sense that the solution of (4.1) satisfies

$$\lim_{t\to\infty} \sup\{w_j(t) \mid |j|_\nu \geq ct\} = 0 \quad \text{for } c \in (c_*, \infty), \tag{5.3}$$

$$\liminf_{t\to\infty} \min\{w_j(t) \mid |j|_\nu \leq ct\} \geq w^+ \quad \text{for } c \in (0, c_*), \tag{5.4}$$

if the initial function W^o satisfies some biologically realistic conditions to be specified in the following theorems.

Theorem 5.1. *Assume that $W^o = \{w_j^o\}_{j\in\mathbb{Z}^2}$, with $w_j^o \in C_K^+[-r, 0]$ for $j \in \mathbb{Z}^2$, is isotropic on $[-r, 0]$, and there exists an integer $N_1 \in \mathbb{N}$ such that $suppW^o(t, \cdot) \subseteq B_{N_1}$ for $t \in [-r, 0]$; Then for any $c > c_*$, we have*

$$\lim_{t\to\infty} \sup\{w_j(t) \mid |j|_\nu \geq ct\} = 0.$$

Proof. Define a sequence of maps by

$$W^{(n)}(t) = F^\infty[W^{(n-1)}](t) \quad \text{for } n \in \mathbb{N},\ t \geq -r,$$

$$W^{(o)}(t) = \{w_j^{(o)}(t)\}_{j\in\mathbb{Z}^2},$$

$$w_j^{(o)}(t) = \begin{cases} w_j^o(t), & t \in [-r, 0], \\ w_j^o(0), & t \in (0, \infty). \end{cases}$$

Then $W^{(o)}$ is isotropic and $supp\, W^{(o)}(t, \cdot) \subset B_{N_1}$ for $t \geq -r$. By an argument similar to that for Theorem 4.1, we obtain the convergence of $\{W^{(n)}\}$ on $[0, \infty)$. Let

$$W(t) = \lim_{n\to\infty} W^{(n)}(t), \quad t \in [0, \infty).$$

Then W is a solution of (4.1) with the isotropic property due to Lebesgue's theorem of dominated convergence.

Using the assumption on $W^{(o)}$, we can find $M > 0$ and $N \in \mathbb{N}$ such that

$$w_j^{(o)}(t)e^{\lambda(j\cdot\nu)} \leq Me^{\lambda N} \quad \text{for } t \geq -r,\ j \in \mathbb{Z}^2. \tag{5.5}$$

For any $c_1 > c_*$, let $c_2 \in (c_*, c_1)$, for $t \geq 0$, we have from (5.5) and (H_3) that

$$\begin{aligned} w_j^{(1)}(t)e^{\lambda(j\cdot\nu - c_2 t)} &= e^{-(\delta+\lambda c_2)t}\Big\{w_j^{(o)}(0)e^{\lambda(j\cdot\nu)} + \int_0^t e^{\delta s} D_m \\ &\quad \times \sum_{p\in\mathcal{A}} w_{j+p}^{(o)}(s)e^{\lambda((j+p)\cdot\nu)}e^{-\lambda(p\cdot\nu)}\,ds + \frac{\mu}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l) \\ &\quad \times \int_0^t e^{\delta s} b(w_{l+j}^{(o)}(s-r))e^{\lambda((j+l)\cdot\nu)}e^{-\lambda(l\cdot\nu)}\,ds\Big\} \\ &\leq e^{-(\delta+\lambda c_2)t}Me^{\lambda N}\Big\{1 + \big(2D_m E(\lambda,\nu) \\ &\qquad + \mu b'(0)e^{2\alpha[E(\lambda,\nu)-4]}\big)\int_0^t e^{(\delta+\lambda c_2)s}\,ds\Big\} \\ &\leq e^{-(\delta+\lambda c_2)t}Me^{\lambda(N+c_2 r)}\Big\{1 + \big(2D_m E(\lambda,\nu) \\ &\qquad + \mu b'(0)e^{2\alpha[E(\lambda,\nu)-4)-\lambda c_2 r]}\int_0^t e^{(\delta+\lambda c_2)s}\,ds\Big\} \\ &\leq Me^{\lambda(N+c_2 r)}[1 + L_{c_2}(\lambda)]. \end{aligned} \tag{5.6}$$

By induction we obtain

$$w_j^{(n)}(t)e^{\lambda(j\cdot\nu - c_2 t)} \leq Me^{\lambda(N+c_2 r)}[1 + L_{c_2}(\lambda) + \cdots + (L_{c_2}(\lambda))^n]. \tag{5.7}$$

Since $c_2 > c_*$, we can choose $\lambda > 0$ such that $L_{c_2}(\lambda) < 1$. For this choice of λ, the right hand side of (5.7) is bounded from above uniformly for n. By taking the limit from (5.7) we obtain that for $j \in \mathbb{Z}$,

$$w_j(t) \leq \frac{Me^{\lambda(N+c_2 r)}}{1 - L_{c_2}(\lambda)}\, e^{\lambda[c_2 t - (j\cdot\nu)]},$$

for $t \geq 0$. Thus by using the isotropic property of W, we obtain

$$w_j(t) \leq \frac{Me^{\lambda(N+c_2r)}}{1-L_{c_2}(\lambda)}\, e^{\lambda(c_2t-|j|_\nu)}.$$

Therefore, we have

$$\sup\{w_j(t) \mid |j|_\nu \geq c_1 t\} \leq \frac{Me^{\lambda(N+c_2r)}}{1-L_{c_2}(\lambda)}\, e^{\lambda(c_2-c_1)t} \to 0 \text{ as } t \to \infty$$

which leads to

$$\lim_{t\to\infty} \sup\{w_j(t) \mid |j|_\nu \geq c_1 t\} = 0, \quad c_1 > c_*.$$

Therefore we complete the proof. □

In order to obtain (5.4), we follow the approaches used by Aronson [**2**], Aronson and Weiberger [**1, 3**], Diekmann [**5, 6**], Lui [**11, 12**], Radcliffe [**14**], Thieme [**17**] and Weinberger [**20**], to develop a comparison principle and to construct a suitable sub-solution of (4.2).

For any $T > 0$, we define a map on

$$M_\infty = \{\Phi = \{\phi_j\}_{j\in\mathbb{Z}^2} \mid \phi_j = \phi_{j_1,j_2} \in C_K^+[-r,\infty)\}$$

by

$$E^T = \{E_j^T\}_{j\in\mathbb{Z}^2},$$

where for $\Phi \in M_\infty, t \geq T, j \in \mathbb{Z}^2$,

$$E_j^T[\Phi](t) = \int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} \phi_{j+p}(t-s) + \frac{\mu}{(2\pi)^2} \sum_{l\in Z^2} G_\alpha(l) b(\phi_{l+j}(t-s-r))\Big\}\, ds.$$

Lemma 5.1. *Suppose that*

(5.8) $$E^T[\Phi](t) \succ \Phi(t) \quad \textit{for } t \geq T,$$

where $\Phi \in M_\infty$ *satisfies*

(i) *for any* $t' > 0$, *there exists an* $N = N(t') \in \mathbb{N}$ *such that for any* $t \in [0,t']$, $\operatorname{supp}\Phi(t,\cdot) \subset B_N$;

(ii) *if* $\left\{(t_n, j_1^{(n)}, j_2^{(n)})\right\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{Z}^2$, $j^{(n)} = (j_1^{(n)}, j_2^{(n)}) \in supp\, \Phi(t_n, \cdot)$, *and* $\lim_{n\to\infty}(t_n, j_1^{(n)}, j_2^{(n)}) = (t_0, j_1^{(0)}, j_2^{(0)})$, *then* $j^{(0)} = (j_1^{(0)}, j_2^{(0)}) \in supp\, \Phi(t_0, \cdot)$.

If there exists a $\bar{t} \geq 0$ *such that the solution* W *of* (4.2) *satisfies*

$$W(\bar{t}+t) \succ \Phi(t) \quad for\ t \in [0, T],$$

then

$$W(\bar{t}+t) \succ \Phi(t) \quad for\ t \in [0, \infty).$$

Proof. Let

$$t_0 = \sup\{t \geq T \mid W(\bar{t}+t) \succ \Phi(t)\}.$$

If $t_0 < \infty$, since $W(t)$ is non-negative, there exists $\left\{(t_n, j_1^{(n)}, j_2^{(n)})\right\}_{n=1}^{\infty}$ such that

(a) $t_n \downarrow t_0,\ n \to \infty$,
(b) $j^{(n)} = (j_1^{(n)}, j_2^{(n)}) \in supp\, \Phi(t_n, \cdot)$,
(c) $w_{j^{(n)}}(\bar{t}+t_n) \leq \phi_{j^{(n)}}(t_n)$.

Under assumption (i), $\{j^{(n)}\}$ must be bounded. Thus, $\{j^{(n)}\}$ is composed of finite groups of integer set $(j_1^{(n)}, j_2^{(n)})$ and hence contains a convergent sub-sequence, which is a constant set sequence $\{j^{(0)}\}$. By (b) and (c), we know that $j^{(0)} \in supp\, \Phi(t_0, \cdot)$ and $w_{j^{(0)}}(\bar{t}+t_0) \leq \phi_{j^{(0)}}(t_0)$.

Noting that $t_0 \geq T$ and $\bar{t} \geq 0$, we obtain from the definition of t_0 and (5.9) that

$$\begin{aligned}
w_{j^{(0)}}(\bar{t}+t_0) \geq& \int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} w_{j^{(0)}+p}(\bar{t}+t_0-s) \\
&+ \frac{\mu}{(2\pi)^2} \sum_{l\in Z^2} G_\alpha(l)\, b(w_{j^{(0)}+l}(\bar{t}+t_0-s-r))\Big\}\, ds \\
\geq& \int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} \phi_{j^{(0)}+p}(t_0-s) \\
&+ \frac{\mu}{(2\pi)^2} \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} G_\alpha(l)\, b(\phi_{j^{(0)}+l}(t_0-s-r))\Big\}\, ds \\
=& E_{j^{(0)}}^T[\Phi](t_0) > \phi_{j^{(0)}}(t_0),
\end{aligned}$$

which is a contradiction. Therefore, $t_0 = \infty$. This completes the proof. □

Define $K_c = K_c(h,T,N,\lambda)$ as

$$K_c(h,T,N,\lambda) = \int_0^T e^{-(\delta+\lambda c)s}\Big\{2D_m E(\lambda,\nu) + \frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu \le N} G_\alpha(l)e^{\lambda(l\cdot\nu - cr)}\Big\}\,ds$$
$$= \frac{1-e^{-(\delta+\lambda c)T}}{\delta+\lambda c}\Big\{2D_m E(\lambda,\nu) + \frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu \le N} G_\alpha(l)e^{\lambda(l\cdot\nu - cr)}\Big\}, \tag{5.9}$$

then we have the following.

Lemma 5.2. *For any $c \in (0, c_*)$, there exist $h \in (0, b'(0)), T > 0$ and $N \in \mathbb{N}$, such that*

$$K_c(h,T,N,\lambda) > 1 \quad \textit{for } \lambda \in \mathbb{R}. \tag{5.10}$$

Proof. By the definition of $K_c(h,T,N,\lambda)$, we have

$$K_c(h,T,N,-\lambda) \ge K_c(h,T,N,\lambda) \quad \text{for } \lambda \ge 0.$$

Therefore, we only need to show that

$$K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \ge 0.$$

We claim that there exist $N_0 > 0, \lambda_0 > 0, h_0 \in (0, b'(0))$ and $T_0 > 0$ such that

$$K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \ge \lambda_0,\ N \ge N_0,\ h \ge h_0 \text{ and } T \ge T_0.$$

In fact, since

$$S(\alpha) = \frac{1}{(2\pi)^2}\sum_{l\in Z^2} G_\alpha(l)e^{\lambda(l\cdot\nu)} = e^{2\alpha[E(\lambda,\nu)-4]} \ge 1,$$

which holds uniformly for $\lambda \in \mathbb{R}$, we can choose $N_0 > 0$ and $h_0 \in (0, b'(0))$ (h_0 can be chosen arbitrarily), such that for $N \ge N_0$ and $h \ge h_0$, we have

$$\frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu \le N} G_\alpha(l)e^{\lambda(l\cdot\nu)} > 0.$$

Since

$$\lim_{\lambda\to\infty}\frac{e^{\lambda(|\nu_1|+|\nu_2|)}}{\lambda c_*+\delta}=\infty,$$

we can choose $T_0>0$ and $\lambda_0>0$ such that for $T\geq T_0$ and $\lambda\geq\lambda_0$, we have

$$\frac{D_m}{\lambda c+\delta}(1-e^{-\delta T})e^{\lambda(|\nu_1|+|\nu_2|)}\geq\frac{D_m}{\lambda c_*+\delta}(1-e^{-\delta T_0})e^{\lambda(|\nu_1|+|\nu_2|)}\geq 1.$$

Then for $N\geq N_0, T\geq T_0, h\geq h_0$ and $\lambda\geq\lambda_0$, we have

$$K_c(h,T,N,\lambda)>\frac{2D_m}{\lambda c_*+\delta}(1-e^{-\delta T_0})e^{\lambda(|\nu_1|+|\nu_2|)}\geq 1.$$

If (5.11) is not true, then there exist $\{h_n\},\{T_n\},\{\lambda_n\},\{N_n\}$ satisfying $h_n\uparrow b'(0),\ T_n\uparrow\infty,\ N_n\uparrow\infty,\ \{\lambda_n\}\subset[0,\lambda_0]$ and

$$K_c(h_n,T_n,N_n,\lambda_n)\leq 1,\quad n=1,2,\ldots.$$

Since $\{\lambda_n\}$ is bounded, we can choose a sub-sequence $\{\lambda_{n_k}\}$ which has a finite limit, say $\bar\lambda$. By Fatou's Lemma, we have

$$1<L_c(\bar\lambda)\leq\liminf_{k\to\infty}K_c(h_{n_k},T_{n_k},N_{n_k},\lambda_{n_k})\leq 1,$$

which is impossible. This completes the proof. □

Define a function with two parameters ω,β as

$$q(y;\omega,\zeta)=\begin{cases}e^{-\omega y}\sin(\zeta y) & \text{for } y\in[0,\frac{\pi}{\zeta}],\\ 0 & \text{for } y\in\mathbb{R}\setminus[0,\frac{\pi}{\zeta}].\end{cases}\tag{5.11}$$

We have the following lemma:

Lemma 5.3. *Let $c\in(0,c_*)$. There exist a $\zeta_0>0$, a continuous function $\widetilde\omega=\widetilde\omega(\zeta)$ defined on $[0,\zeta_0]$, and a positive number $\delta_1\in(0,1)$ such that*

$$\int_0^T e^{-\delta s}\Big\{D_m\sum_{p\in\mathcal{A}}q(m+cs+p\cdot\nu)+\frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu\leq N}G_\alpha(l)q(m+l\cdot\nu+cs+cr)\Big\}\,ds\geq q(m-\delta_1)\tag{5.12}$$

for $m\in\mathbb{R}$, where $q(y)=q(y;\widetilde\omega(\zeta),\zeta)$.

Proof. Define

$$L(\lambda)=\int_0^T e^{-\delta s}\Big\{D_m\sum_{p\in\mathcal{A}}e^{-\lambda(cs+p\cdot\nu)} + \frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu\le N}G_\alpha(l)e^{-\lambda(l\cdot\nu+cs+cr)}\Big\}\,ds,$$

where T, h and N are defined in Lemma 5.23. We mention here that one can choose N sufficiently large so that

$$-N+c_*(T+r)<0. \tag{5.13}$$

Using Lemma 5.3 we have

$$L(\lambda)=K_c(h,T,N,\lambda)>1\quad\text{for all }\lambda\in\mathbb{R}. \tag{5.14}$$

Let $\lambda=\omega+i\zeta$, we have $L(\lambda)|_{\lambda=\omega+i\zeta}=\mathrm{Re}\,[L(\lambda)]+i\,\mathrm{Im}\,[L(\lambda)]$, where

$$\mathrm{Re}\,[L(\lambda)]=D_m\int_0^T e^{-\delta s}\Big\{\sum_{p\in\mathcal{A}}e^{-\omega(cs+p\cdot\nu)}\cos\zeta(cs+p\cdot\nu)\Big\}\,ds+\frac{\mu h}{(2\pi)^2}\times\sum_{|l|_\nu\le N}G_\alpha(l)\int_0^T e^{-\delta s}e^{-\omega(l\cdot\nu+cs+cr)}\cos\zeta(l\cdot\nu+cs+cr)\,ds,$$

$$\mathrm{Im}\,[L(\lambda)]=-D_m\int_0^T e^{-\delta s}\Big\{\sum_{p\in\mathcal{A}}e^{-\omega(cs+p\cdot\nu)}\sin\zeta(cs+p\cdot\nu)\Big\}\,ds-\frac{\mu h}{2\pi}\times\sum_{|l|_\nu\le N}G_\alpha(l)\int_0^T e^{-\delta s}e^{-\omega(l\cdot\nu+cs+cr)}\sin\zeta(l\cdot\nu+cs+cr)\,ds.$$

Since $L''(\lambda)>0$ and $\lim_{|\lambda|\to\infty}L(\lambda)=\infty$, we conclude that $L(\lambda)$ can achieve its minimum, say at $\lambda=\theta$. Then we obtain

$$L'(\theta)=-D_m\int_0^T e^{-\delta s}\Big[\sum_{p\in\mathcal{A}}(cs+p\cdot\nu)e^{-\theta(cs+p\cdot\nu)}\Big]\,ds-\frac{\mu h}{(2\pi)^2}\times\sum_{|l|_\nu\le N}G_\alpha(l)\int_0^T e^{-\delta s}(l\cdot\nu+cs+cr)e^{-\theta(l\cdot\nu+cs+cr)}\,ds=0.$$

We now define a function $H = H(\omega, \zeta)$ by

$$\begin{cases} H(\omega,\zeta) = \dfrac{1}{\zeta}\mathrm{Im}\,[L(\lambda)] & \text{for } \zeta \neq 0, \\ H(\omega,0) = \lim\limits_{\zeta\to 0} H(\omega,\zeta) = L'(\omega). \end{cases}$$

Then $H(\theta,0) = 0$ and $\frac{\partial H}{\partial \omega}(\theta,0) = L''(\theta) > 0$. The implicit function theorem implies that there exist $\zeta_1 > 0$ and a continuous function $\widetilde{\omega} = \widetilde{\omega}(\zeta)$ defined on $[0,\zeta_1]$ with $\tilde{\omega}(0) = \theta$ such that $H(\widetilde{\omega}(\zeta),\zeta) = 0$ for $\zeta \in [0,\zeta_1]$. Hence, we have

$$\mathrm{Im}\,[L(\lambda)]\Big|_{\lambda=\widetilde{\omega}(\zeta)+i\zeta} = 0 \quad \text{for } \zeta \in [0,\zeta_1]. \tag{5.15}$$

By (5.15), we have $\mathrm{Re}\,[L(\omega+i\zeta)]\big|_{\omega=\theta,\zeta=0} = L(\theta) > 1$. Thus there exists $\zeta_2 > 0$ such that

$$\mathrm{Re}\,[L(\tilde{\omega}(\zeta)+i\zeta)] > 1 \quad \text{for } \zeta \in [0,\zeta_2]. \tag{5.16}$$

Let $0 < \zeta \le \zeta_0 := \min\{\zeta_1, \zeta_2, \pi/(2N + c_*(T+r))\}$. For $m \in [0,\pi/\zeta]$ and $s \in [0,T]$, we have

$$-\frac{\pi}{\zeta} < -2N \le m + p\cdot\nu + cs \le 2N + c_*T \le \frac{2\pi}{\zeta} \quad \text{for } p \in \mathcal{A},$$

and

$$-\frac{\pi}{\zeta} < -2N \le m + l\cdot\nu + cs + cr \le 2N + c_*T \le \frac{2\pi}{\zeta} \quad \text{for } |l|_\nu \le N.$$

If $m \in [0,\pi/\zeta]$, let $U = (-\pi/\zeta, 0) \cup (\pi/\zeta, 2\pi/\zeta)$, then we have

$$\begin{cases} \sin\zeta(m + p\cdot\nu + cs) < 0 \\ \qquad\qquad \text{for } \ m + l\cdot\nu + cs \in U, \ p \in \mathcal{A}, \\ \sin\zeta(m + l\cdot\nu + cs + cr) < 0 \\ \qquad\qquad \text{for } \ m + l\cdot\nu + cs + cr \in U, \ |l|_\nu \le N. \end{cases} \tag{5.17}$$

Therefore, by (5.17) and the definition of $q(y)$, we obtain

$$
\begin{aligned}
(5.18)\quad & \int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} q(m+cs+p\cdot\nu) \\
&\qquad + \frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu\le N} G_\alpha(l)q(m+l\cdot\nu+cs+cr)\Big\}\,ds \\
&\ge D_m \int_0^T e^{-\delta s} \\
&\quad \times \Big\{\sum_{p\in\mathcal{A}} e^{-\tilde{\omega}(\zeta)(m+cs+p\cdot\nu)}\sin(\zeta(m+cs+p\cdot\nu))\,ds\Big\} \\
&\quad + \frac{\mu h}{(2\pi)^2}\int_0^T e^{-\delta s}\sum_{|l|_\nu\le N} G_\alpha(l)e^{-\tilde{\omega}(\zeta)(m+l\cdot\nu+cs+cr)} \\
&\quad \times \sin(\zeta(m+l\cdot\nu+cs+cr))ds.
\end{aligned}
$$

Using a trigonometric identity and (5.16), (5.17) and (5.18) we obtain

$$
\begin{aligned}
(5.19)\quad & \int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} q(m+cs+p\cdot\nu) \\
&\qquad + \frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu\le N} G_\alpha(l)q(m+l\cdot\nu+cs+cr)\Big\}\,ds \\
&= e^{-\tilde{\omega}(\zeta)m}\big\{\sin(\zeta m)\mathrm{Re}\,[L(\lambda)] \\
&\qquad + \cos(\zeta m)\mathrm{Im}\,[L(\lambda)]\big\}_{\lambda=\tilde{\omega}(\zeta)+i\zeta} \\
&\ge e^{-\tilde{\omega}(\zeta)m}\sin(\zeta m) \\
&= q(m).
\end{aligned}
$$

We should emphasize that (5.19) is a strict inequality for $m\in(0,\pi/\zeta)$. On the other hand, if $m=0$ or $m=\pi/\zeta$, (5.19) is also a strict inequality by using (5.18) and (5.19). In fact, if $m=\pi/\zeta$ and $l_1\nu_1+l_2\nu_2>0$, we have

$$m+l_1\nu_1+l_2\nu_2+c(s+r)>\frac{\pi}{\zeta}.$$

Similarly, if $m=0$ and $l\cdot\nu=-N$, we have from (5.14) that

$$m+l\cdot\nu+c(s+r)<-N+c_*(T+r)<0.$$

However, for both cases, we have from (5.18) and the definition of q that

$$q(m + l \cdot \nu + cs + sr) = 0 \quad \text{and} \quad \sin(\zeta(m + l \cdot \nu + cs + sr)) < 0,$$

and thus (5.19) is a strict inequality. Therefore, for $m \in [0, \pi/\zeta]$, we have

$$\int_0^T e^{-\delta s}\Big\{D_m \sum_{p\in A} q(m + cs + p \cdot \nu) + \frac{\mu h}{(2\pi)^2} \sum_{|l|_\nu \le N} G_\alpha(l) q(m + l \cdot \nu + cs + cr)\Big\}\, ds > q(m). \tag{5.20}$$

Then (5.13) follows for $m \in [0, \pi/\zeta]$ from the continuity consideration.

If $m \notin [0, \pi/\zeta]$, then $q(m) = 0$. Noting that $q(y) \ge 0$ for $y \in \mathbb{R}$, we distinguish four cases:

1^o $m \ge \frac{\pi}{\zeta} + (|\nu_1| + |\nu_2|)$. Then we have $m + cs + l \cdot \nu \ge \pi/\zeta$ for $l \in A$, $s \in [0,\ T]$, which leads to $q(m + cs + l \cdot \nu) \equiv 0$. Therefore, we can choose $\delta_1 \in (0,\ 1)$ such that (5.13) holds.

2^o $\pi/\zeta < m < \pi/\zeta + (|\nu_1| + |\nu_2|)$. In this case, there exists a $s' \in (0,\ T)$ small enough such that $m + cs' - (|\nu_1| + |\nu_2|) \in (0, \pi/\zeta)$ because of $m - (|\nu_1| + |\nu_2|) < \pi/\zeta$. So $\int_0^T q(m + cs - (|\nu_1| + |\nu_2|))\, ds > 0$ for the continuity of $q(y)$, thus (5.21) holds which leads to (5.13).

3^o $-[cT + (|\nu_1| + |\nu_2|)] < m < 0$. Since $m + cT + (|\nu_1| + |\nu_2|) > 0$, there exists a $s'' \in (0,\ T)$ such that $m + cs'' + (|\nu_1| + |\nu_2|) \in (0, \pi/\zeta)$. Similar to 2^o, (5.13) still holds since one has $\int_0^T q(m + cs + (|\nu_1| + |\nu_2|))\, ds > 0$.

4^o $m \le -[cT + (|\nu_1| + |\nu_2|)]$. Similar to 1^o, we have $q(m + cs + l \cdot \nu) \equiv 0$ since $m + cs + l \cdot \nu \le 0$ for $l \in A$, $s \in [0,\ T]$, so we can choose $\delta_1 > 0$ such that (5.13) holds.

Summarizing the above arguments, we conclude that inequality (5.13) holds for $m \in \mathbb{R}$. This completes the proof. □

Now we consider the following family of functions

$$R(y; \omega, \zeta, \gamma) := \max_{\eta \ge -\gamma} q(y + \eta; \omega, \zeta)$$

$$= \begin{cases} M & \text{for } y \le \gamma + \rho, \\ q(y - \gamma; \omega, \zeta) & \text{for } \gamma + \rho \le y \le \gamma + \frac{\pi}{\zeta}, \\ 0 & \text{for } y \ge \gamma + \frac{\pi}{\zeta}, \end{cases} \tag{5.21}$$

where

$$M = M(\omega,\zeta) := \max\left\{ q(y;\omega,\zeta) \,\middle|\, 0 \le y \le \frac{\pi}{\zeta} \right\}, \tag{5.22}$$

and $\rho = \rho(\omega,\zeta)$ is the point where the above maximum M is achieved. The following lemma gives a sub-solution of (4.2).

Lemma 5.4. *Let $c \in (0, c_*)$. be given, then there exist $T > 0$, $\zeta > 0$, $\omega \in \mathbb{R}$, $D > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0,\sigma_0)$ and for any $t \ge T$*

$$E^T[\sigma\Phi](t) \succ \sigma\Phi(t) \quad \textit{for } t \ge T, \tag{5.23}$$

where $\Phi(t) = \{\phi_j(t)\}_{j\in\mathbb{Z}^2}$, $\phi_j(t) = R(|j|_\nu \mid \omega, \zeta, D + ct)$.

Proof. Let $h \in (0, b'(0))$, $T > 0$, $N > 0$ be chosen such that

$$K_c(h, T, N, \lambda) > 1 \quad \text{for all } \lambda \in \mathbb{R}.$$

According to Lemma 5.4, we can choose $\zeta > 0$, $\omega = \widetilde{\omega}(\zeta)$ and $\delta_1 \in (0,1)$ such that (5.13) holds.

Let σ_h be the smallest positive root of the equation $b(w) = hw$. Then $b(w) > hw$ for $w \in (0, \sigma_h)$. Choose $\sigma_0 \in (0, \sigma_h M^{-1})$, where M is defined in (5.23). Let $\sigma \in (0,\sigma_0)$ and $t \ge T$, then

$$\begin{aligned} E_j^T[\sigma\Phi](t) &= \int_0^T e^{-\sigma s}\Big\{\sigma D_m \sum_{p\in\mathcal{A}} \phi_{j+p}(t-s) \\ &\quad + \frac{\mu}{(2\pi)^2} \sum_{l\in Z^2} G_\alpha(l)\, b(\sigma\phi_{j+l}(t-s-r))\Big\}\, ds \\ &\ge \int_0^T e^{-\sigma s}\Big\{\sigma D_m \sum_{p\in\mathcal{A}} \phi_{j+p}(t-s) \\ &\quad + \frac{\mu}{(2\pi)^2} \sum_{|l|_\nu \le N} G_\alpha(l)\, b(\sigma\phi_{j+l}(t-s-r))\Big\}\, ds. \end{aligned} \tag{5.24}$$

We now distinguish two cases.

Case (i) $|j|_\nu \le D+\rho+c(t-T)-N$. If $|l|_\nu \le N, s\in[0,T]$, then

$$|l+j|_\nu \le D+\rho+c(t-T) \le D+\rho+c(t-s)$$

and consequently

$$E_j^T[\sigma\Phi](t) \ge \left\{8D_m\sigma M + \frac{\mu}{(2\pi)^2}\sum_{|l|_\nu\le N} G_\alpha(l)b(\sigma M)\right\} \times \int_0^T e^{-\delta s}\,ds > \sigma M K_c(h,T,N,0) > \sigma M. \tag{5.25}$$

Case (ii) $D+\rho+c(t-T)-N \le |j|_\nu \le \frac{\pi}{\zeta}+D+ct$. If $|l|_\nu \le N$ and $t\ge T$, then

$$\begin{aligned}|l\pm j|_\nu &= |l\cdot\nu \pm j\cdot\nu| \\ &= \left((l\cdot\nu)^2 \pm 2(l\cdot\nu)(j\cdot\nu)+(j\cdot\nu)^2\right)^{1/2} \\ &\le \left(|l|_\nu^2 \pm 2(l\cdot\nu)(j\cdot\nu)+|j|_\nu^2\right)^{1/2} \\ &\le |j|_\nu \pm \frac{(j\cdot\nu)(l\cdot\nu)}{|j|_\nu} + \frac{(|l|_\nu)^2}{2|j|_\nu} \\ &\le |j|_\nu \pm \frac{(j\cdot\nu)(l\cdot\nu)}{|j|_\nu} + \frac{N^2}{2(D+\rho-N)} \\ &\le |j|_\nu \pm \frac{(j\cdot\nu)(l\cdot\nu)}{|j|_\nu} + \delta_1,\end{aligned} \tag{5.26}$$

provided $D \ge N^2/2\delta_1 - \rho + N$. Since $\phi_j(t)$ is decreasing with respect to $|j|_\nu$, we have from (5.25), (5.27) and the isotropic property that

$$\begin{aligned}&E_j^T[\sigma\Phi](t) \\ &\ge \sigma\int_0^T e^{-\delta s}\Big\{D_m\Big[\sum_{p\in\mathcal{A}}\max_{\eta\ge -D-c(t-s)} q(|j|_\nu + \frac{(p\cdot\nu)(j\cdot\nu)}{|j|_\nu}+\delta_1+\eta)\Big] \\ &+\frac{\mu h}{(2\pi)^2}\sum_{|l|_\nu\le N} G_\alpha(l)\max_{\eta\ge -D-c(t-s-r)} q(|j|_\nu + \frac{(j\cdot\nu)(l\cdot\nu)}{|j|_\nu}+\delta_1+\eta)\Big\}\,ds\end{aligned}$$

$$= \sigma \int_0^T e^{-\delta s} \Big\{ D_m \Big[\sum_{p \in \mathcal{A}} \max_{\eta \geq -D-ct} q(|j|_\nu + p \cdot \nu + cs + \delta_1 + \eta) \Big]$$

$$+ \frac{\mu h}{(2\pi)^2} \sum_{|l|_\nu \leq N} G_\alpha(l) \max_{\eta \geq -D-ct} q(|j|_\nu + l \cdot \nu + cs + cr + \delta_1 + \eta) \Big\} \, ds$$

$$> \sigma \max_{\eta \geq -D-ct} q(|j|_\nu + \eta).$$

Combining (i) and (ii), we obtain (5.24) and complete the proof. □

Lemma 5.5. *Assume that* $W = \{w_j\}_{j \in \mathbb{Z}^2}$ *is a solution of* (4.1), *and assume that*

(i) $W^o = \{w_j^o\}_{j \in \mathbb{Z}^2}$, *with* $w_j^o \in C_K^+[-r, 0]$, *is isotropic on* $[-r, 0]$;
(ii) *there exists* $N_1 \in \mathbb{N}$ *such that*

$$\operatorname{supp} W^o(t, \cdot) \subset B_{N_1} \quad \text{for } t \in [-r, 0], \qquad \text{and} \quad w_j^o(0) > 0 \quad \text{for } |j|_\nu \leq N_1.$$

Then there exists $t_0 > r$ *such that*

$$w_j(t) > 0 \quad \text{for } t \in [t_0, \infty) \text{ and } j \in \mathbb{Z}^2.$$

The conclusion of Lemma 5.6 is obtained directly from an observation from (4.1).

Lemma 5.6. *Let* $\{Q_n(t, N)\}$ *be defined by* $Q_1(t, N) \equiv a \in [0, w^+)$, *and*

$$Q_{n+1}(t, N) = \frac{1}{\delta} \Big[8 D_m Q_n(t, N) + \frac{\mu}{(2\pi)^2} \sum_{|l|_\nu \leq N} G_\alpha(l) b(Q_n(t, N)) \Big] (1 - e^{-\delta t})$$

for $n = 1, 2, \cdots$. *Then for any* $\epsilon > 0$, *there exist* $\bar{t}(\epsilon)$, $\bar{N}(\epsilon)$ *and* $\bar{n}(\epsilon)$ *such that for any* $t \geq \bar{t}(\epsilon)$, $N \geq \bar{N}(\epsilon)$ *and* $n \geq \bar{n}(\epsilon)$,

$$Q_n(t, N) \geq w^+ - \epsilon.$$

Proof. First, we note that

$$\frac{8 D_m w^+ + \mu b(w^+)}{\delta} = w^+, \tag{5.27}$$

and

$$0 < Q_1(t,N) < w^+, \quad 0 < \frac{1}{\delta}(1-e^{-\delta t}) < 1,$$

$$0 < \frac{1}{(2\pi)^2} \sum_{|l|_\nu \le N} G_\alpha(l) < 1.$$

Therefore, we have by induction that $0 < Q_n(t,N) \le K$ for all $n \in \mathbb{N}$, $t \ge 0$ and $N \in \mathbb{N}$.

Let $\epsilon > 0$ be sufficiently small such that $\epsilon < a < \omega^+ - \epsilon$. Since

$$8D_m w + \mu b(w) > (8D_m + d_m)w \quad \text{for } 0 < w < w^+,$$

we have

$$\inf\left\{ \frac{8D_m w + \mu b(w)}{(8D_m + d_m)w} \;\middle|\; \epsilon \le w \le w^+ - \epsilon \right\} > 1.$$

Choose $\alpha(\epsilon) < 1$ so that

$$\alpha(\epsilon)[8D_m w + \mu b(w)] > (8D_m + d_m)w \quad \text{for } \epsilon \le w \le w^+ - \epsilon.$$

Define a sequence as follows:

$$M_1 \equiv a, \; M_{n+1} = \frac{\alpha(\epsilon)}{\delta}\left(8D_m M_n + \mu b(M_n)\right), \quad n \ge 2.$$

Then we have the following observations:

(i) if $\epsilon \le M_n \le w^+ - \epsilon$, then $M_{n+1} \ge M_n$;
(ii) if $M_n > w^+ - \epsilon$, then

$$M_{n+1} \ge \frac{\alpha(\epsilon)}{\delta}[8D_m(w^+ - \epsilon) + \mu b(w^+ - \epsilon)] \ge w^+ - \epsilon.$$

We now claim that $M_n > w^+ - \epsilon$ for large n. If not, then using (ii) we can assume that $M_n \le w^+ - \epsilon$ for all n. Then by (i), $\lim_{n\to\infty} M_n = \overline{M} < \omega^+ - \epsilon$ exists and we have

$$\overline{M} = \frac{\alpha(\epsilon)}{\delta}[8D_m\overline{M} + \mu b(\overline{M})],$$

which is impossible by (5.28). Therefore, there is $\bar{n}(\epsilon) > 0$ such that $M_n > w^+ - \epsilon$ for all $n > \bar{n}(\epsilon)$.

Choose $\bar{t}(\epsilon)$ and $\overline{N}(\epsilon)$ such that

$$\frac{1}{(2\pi)^2}(1-e^{-\delta\bar{t}(\epsilon)})\sum_{|l|_\nu\leq\overline{N}(\epsilon)} G_\alpha(l)\geq\alpha(\epsilon).$$

Then, for any $t\geq\bar{t}(\epsilon)$ and $N\geq\overline{N}(\epsilon)$, we have $Q_1(t,N)=a\geq M_1$ and

$$\begin{aligned}Q_{n+1}(t,N)&\geq\frac{1}{\delta}(1-e^{-\delta\bar{t}(\epsilon)})\Big(8D_mQ_n(t,N)\\&\qquad+\frac{\mu}{(2\pi)^2}\sum_{|l|_\nu\leq\bar{N}(\epsilon)}G_\alpha(l)b(Q_n(t,N))\Big)\\&>\frac{1}{\delta}\alpha(\epsilon)\left(8D_mQ_n(t,N)+\mu b(Q_n(t,N))\right).\end{aligned}$$

Using the monotonicity of b on $[0,K]$ and by induction, we obtain $Q_n(t,N)\geq M_n\geq w^+-\epsilon$ for all $n>\bar{n}(\epsilon)$. This completes the proof. □

Theorem 5.2. *Assume that W^o satisfies all conditions in Lemma* 5.6. *Then for any $c\in(0,c_*)$, we have*

$$\liminf_{t\to\infty}\min\{w_j(t)\mid|j|_\nu\leq ct\}\geq w^+.$$

Proof. Fix $c_1\in(0,c_*)$ and choose $c_2\in(c_1,c_*)$. According to Lemma 5.5, there exist $T>0$, $\zeta>0$, $\omega\in\mathbb{R}$, $D>0$ and $\sigma_0>0$ such that for any $\sigma\in(0,\sigma_0)$ and any $t\geq T$,

$$E^T[\sigma\Phi](t)\succ\sigma\Phi(t),$$

where $\Phi(t)=\{\phi_j(t)\}_{j\in\mathbb{Z}^2},\phi_j(t):=R(|j|_\nu\mid\omega,\zeta,D+c_2t)$. By Lemma 5.6, we can find $t_0>r$ so that

$$w_j(t)>0\quad\text{for }t\in[t_0,t_0+T],\ j\in\mathbb{Z}^2.$$

Then we can choose $\sigma_1\in(0,\sigma_0)$ such that

$$\sigma_1M<w^+,\quad w_j(t_0+t)>\sigma_1\phi_j(t)\quad\text{for }t\in[0,T],\quad j\in\mathbb{Z}^2. \tag{5.28}$$

We refer from the comparison principle (Lemma 5.2) that (5.29) holds for $t\geq0$. Hence by (5.22) and the definition of $\phi_j(t)$, we have

$$w_j(t_0+t)\geq\sigma_1M\quad\text{for }t\geq0,\ |j|_\nu\leq\rho+D+c_2t. \tag{5.29}$$

By (4.2), we obtain

$$(5.30)\quad w_j(t_0+t) \geq \int_0^t e^{-\delta s}\Big\{D_m \sum_{p\in\mathcal{A}} w_{j+p}(t_0+t-s) + \frac{\mu}{(2\pi)^2}\sum_{|l|_\nu\leq N} G_\alpha(l)b(w_{l+j}(t_0+t-s-r))\Big\}\, ds.$$

Let $a = \sigma_1 M = Q_1(t,N)$ and let $Q_n(t,N)$ be defined in Lemma 5.7. Then by induction and using (5.30)–(5.31), we have

$$w_j(t_0+t) \geq Q_n(t,N) \quad \text{for } t \geq 0,$$
$$|j|_\nu \leq \rho + D + c_2 t - 2nN.$$

Therefore, for any $\epsilon > 0$ small enough , we can find $\bar{t}(\epsilon), \overline{N}(\epsilon)$ and $\bar{n}(\epsilon)$ such that

$$(5.31)\qquad \begin{aligned} &w_j(t) \geq w^+ - \epsilon \quad \text{for } t \geq t_0 + \bar{t}(\epsilon),\\ &|j|_\nu \leq \rho + D + c_2(t-t_0) - 2\bar{n}(\epsilon)\bar{N}(\epsilon). \end{aligned}$$

Define

$$t_1 = \max\left\{t_0 + \bar{t}(\epsilon), \frac{2\bar{n}(\epsilon)\overline{N}(\epsilon) + c_2 t_0 - \rho - D}{c_2 - c_1}\right\}.$$

Since $c_2 > c_1$, we have from (5.32) that

$$w_j(t) \geq w^+ - \epsilon \quad \text{for } t \geq t_1, \quad |j|_\nu \leq c_1 t.$$

This completes the proof. □

REFERENCES

1. D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics, (J. A. Goldstein, ed.), Lecture Notes in Math. **446**, Springer-Verlag, Heidelberg, Berlin, 1975, 5–49.
2. D. G. Aronson, *The asymptotic speed of a propagation of a simple epidemic*, in Nonlinear Diffusion, (W. E. Fitzgibbon and H. F. Walker, ed.), Research Notes in Math. **14**, Pitman, London, 1977, 1–23.
3. D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Advances in Math. **30** (1978), 33–76.
4. N. F. Britton, *Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model*, SIAM J. Appl. Math. **50** (1990), 1663–1688.
5. O. Diekmann, *Thresholds and travelling waves for the geographical spread of infection*, J. Math. Biol. **69** (1978), 109–130.
6. O. Diekmann, *Run for your life, a note on the asymptotic speed of propagation of an epidemic*, J. Diff. Eqns. **33** (1979), 58–73.
7. S. A. Gourley and N. F. Britton, *A predator-prey reaction-diffusion system with nonlocal effects*, J. Math. Biol. **34** (1996), 297–333.
8. S. A. Gourley, *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol. **41** (2000), 272–284.
9. S. A. Gourley, J. W. H. So and J. Wu, *Non-locality of reaction-diffusion equations induced by delay : biological modeling and nonlinear dynamics*, Contemporary Mathematics, Thematic Surveys, Kluwer Plenum (D. V. Anosov and A. Skubachevskii eds.), **P129** (2003), 84–120.
10. R. R. Goldberg, *Fourier Transform*, Cambridge University Press, New York, 1965.
11. R. Lui, *Biological growth and spread modelled by systems of recursions*, I. Biological Theory, Mathematical Biosciences **93** (1989), 269–296.
12. R. Lui, *Biological growth and spread modelled by systems of recursions*, II. Biological Theory, Mathematical Biosciences **93** (1989), 297–312.
13. Wengjie Pan, *Fourier Analysis and Application* (in Chinese), Publishing House of Beijing University, Beijing, 2000, 32–33.
14. J. Radcliffe and L. Rass, *The asymptotic speed of propagation of the deterministic non-reducible n-type epidemic*, J. Math. Biol. **23** (1986), 341–359.
15. H. Smith and H. Thieme, *Strongly order preserving semiflows generated by functional differential equations*, J. Diff. Eqns. **93** (1991), 332–363.
16. J. W. H. So, J. Wu and X. F. Zou, *A reaction-diffusion model for a single species with age structure I: Travelling wavefronts on unbounded domains*, Proc. Roy. Soc. London **A457** (2001), 1841–1853.
17. J. W. H. So, J. Wu and X. F. Zou, *Structured population on two patches: modelling dispersal and delay*, J. Math. Biol. **43** (2001), 37–51.
18. H. R. Thieme, *Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations*, J. reine angew. Math. **306** (1979), 94–121.
19. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1962.
20. H. F. Weinberger, *Asymptotic behaviours of a model in population genetics*, in Nonlinear Partial Differential Equations and Applications (J. M. Chadam, ed.), Lecture Notes in Mathematics **648**, Springer-Verlag, Heidelberg, Berlin, 1978.

21. P. X. Weng, H. X. Huang and J. H. Wu, *Asymptotic Speed of Propagation of Wave fronts in a lattice delay differential equation with global interaction,* IMA of Applied Mathematics **68** (2003), 409–439.
22. J. H. Wu and X. F. Zou, *Travelling wave fronts of reaction-diffusion systems with delay,* J. Dynam. Diff. Eqns. **13** (2001), 651–687.

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