Global attractivity of a synchronized periodic orbit in a delayed network ☆

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Abstract

We consider an artificial neural network where the signal transmission is of a digital (McCulloch–Pitts) nature and is delayed due to the finite switching speed of neurons (amplifiers). For a particular connection topology, we show that all solutions starting from nonoscillatory initial states will be eventually synchronized and stabilized at a unique limit cycle, and hence such a network can be used as a synchronized oscillator.

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1. Introduction

In this paper, we consider the following system of delay differential equations

$$\begin{cases} \dot{x} = -\mu x + a_{11} f(x(t-\tau)) + a_{12} f(y(t-\tau)), \\ \dot{y} = -\mu y + a_{21} f(x(t-\tau)) + a_{22} f(y(t-\tau)), \end{cases} \tag{1.1}$$

where μ and τ are given constants, $f : \mathbb{R} \to \mathbb{R}$ is the McCulloch–Pitts nonlinear function given by

$$f(\xi) = \begin{cases} \alpha & \text{if } \xi > \sigma, \\ \beta & \text{if } \xi \leqslant \sigma, \end{cases}$$

where $\alpha \neq \beta$ and σ are constants. Such a system describes the dynamics of a network of two neurons [6] where each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and where each neuron is connected to another via nonlinear activation function f multiplied by the synaptic weights a_{ij} ($i \neq j$). We also allow that a neuron may have self-feedback and we assume that signal transmission is delayed due to the finite switching speed of neurons. The McCulloch–Pitts nonlinearity reflects the fact that the signal transmission is of digital nature: a neuron is either fully active or completely inactive.

Despite the low number of units, two-neuron networks with delay often display the same dynamical behaviors as large networks and can thus be used as prototypes to improve our understanding of the computational performance of large networks with delayed feedback. The case where the function f is smooth has been studied in [1–3,9,11,13], but little has been done when f is of McCulloch–Pitts type, since most results in the dynamical systems theory requires the continuity and smoothness of nonlinear functions involved. Recently, in [4,7,8,10], model (1.1) with the piecewise constant activation function was studied when the synaptical connection topology satisfies either $[a_{11} = a_{22} = 0, a_{21} = a_{12} = 1]$ or $[a_{11} = a_{22} = 0, a_{21} = -a_{12} = 1]$, or more generally, $[a_{11} = a_{12} > 0, a_{21} > 0, -a_{21} < a_{22} \leqslant a_{21}]$.

We focus here on the asymptotical behaviors of model (1.1), where the activation function is given by

$$f(\xi) = \begin{cases} -\delta & \text{if } \xi > 0, \\ \delta & \text{if } \xi \leqslant 0, \end{cases} \tag{1.2}$$

where $\delta \neq 0$ is a given constant. To simplify the presentation, we first rescale the variables by

$$t^* = \mu t, \quad \tau^* = \mu \tau, \quad x^*(t^*) = \frac{\mu}{\delta} x(t), \quad y^*(t^*) = \frac{\mu}{\delta} y(t), \quad f^*(\xi) = \frac{1}{\delta} f\left(\frac{\delta}{\mu} \xi\right),$$

and then drop the $*$ to get

$$\begin{cases} \dot{x} = -x + a_{11} f(x(t-\tau)) + a_{12} f(y(t-\tau)), \\ \dot{y} = -y + a_{21} f(x(t-\tau)) + a_{22} f(y(t-\tau)) \end{cases} \tag{1.3}$$

with

$$f(\xi) = \begin{cases} -1 & \text{if } \xi > 0, \\ 1 & \text{if } \xi \leqslant 0. \end{cases} \tag{1.4}$$

It is natural to have the phase space $X = C([\tau, 0]; \mathbb{R}^2)$ as the Banach space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^2$ equipped with the sup-norm, see [5]. Note that for each given initial value $\Phi = (\varphi, \psi)^T \in X$, one can solve system (1.3) on intervals $[0, \tau]$, $[\tau, 2\tau], \ldots$ successively to obtain a unique mapping $(x^\Phi, y^\Phi)^T : [-\tau, \infty] \to \mathbb{R}^2$ such that $x^\Phi|_{[-\tau,0]} = \varphi$, $y^\Phi|_{[-\tau,0]} = \psi$, (x^Φ, y^Φ) is continuous for all $t \geqslant 0$, almost differentiate and satisfies (1.3) for $t > 0$. This gives a unique solution of (1.3) defined for all $t \geqslant -\tau$.

In applications, a network usually starts from a constant (or nearly constant) state. Therefore, we shall concentrate on the case where each component of Φ has no sign change on $[-\tau, 0]$. More precisely, we consider $\Phi \in X^{+,+} \cup X^{+,-} \cup X^{-,+} \cup X^{-,-} = X_0$, where

$$C^{\pm} = \left\{ \pm\varphi \colon \varphi \colon [-\tau, 0] \to [0, \infty) \text{ is continuous and} \atop \text{has only finitely many zeros on } [-\tau, 0] \right\}$$

and

$$X^{\pm,\pm} = \left\{ \Phi \in X \colon \Phi = (\varphi, \psi)^T, \ \varphi \in C^{\pm}, \text{ and } \psi \in C^{\pm} \right\}.$$

Clearly, all constantly initial values (except for 0) are contained in X_0.

Clearly, the connection weights have a fundamental effect on the dynamics of the networks. In this paper, we consider the following special connection topology:

(H1) $|a_{12}| < a_{11}, |a_{22}| < a_{21}, a_{11}a_{22} - a_{12}a_{21} = 0.$

In other words, we assume that the self-feedback to neuron 1 and the interaction from neuron 1 to neuron 2 are inhibitory and respectively dominate the interaction from neuron 2 to 1 and the self-feedback to neuron 2. We also assume that the determinant of the connection matrix is zero. We shall show that every solution starting from an initial state in X_0 will be eventually synchronized and stabilized at a unique attractive limit cycle, and hence such a network can be used as a synchronized oscillator.

2. Preliminary results

In this section, we establish several lemmas which will be needed later.

First, we further rescale the variables of (1.3) by

$$u(t) = \frac{x(t)}{a_{11} + a_{12}}, \qquad v(t) = \frac{y(t)}{a_{21} + a_{22}}, \qquad m = \frac{a_{11} - a_{12}}{a_{11} + a_{12}} > 0.$$

Then the rescaled system is equivalent to the following system:

$$\begin{cases} \dot{u} = -u + \frac{1}{2}[f(u(t-\tau)) + f(v(t-\tau))] + \frac{m}{2}[f(u(t-\tau)) - f(v(t-\tau))], \\ \dot{v} = -v + \frac{1}{2}[f(u(t-\tau)) + f(v(t-\tau))] + \frac{m}{2}[f(u(t-\tau)) - f(v(t-\tau))]. \end{cases} \tag{2.1}$$

As such, the simple form of (1.4) will enable us to carry out a direct elementary analysis of the dynamics of the network due to its obvious connection with the following systems of linear nonhomogeneous ordinary differential equations:

$$\begin{cases} \dot{u} = -u - 1, \\ \dot{v} = -v - 1, \end{cases} \tag{2.2}$$

$$\begin{cases} \dot{u} = -u + m, \\ \dot{v} = -v + m, \end{cases} \tag{2.3}$$

$$\begin{cases} \dot{u} = -u + 1, \\ \dot{v} = -v + 1, \end{cases} \tag{2.4}$$

$$\begin{cases} \dot{u} = -u - m, \\ \dot{v} = -v - m. \end{cases} \tag{2.5}$$

For the sake of simplicity, in the remaining part of this paper, for a given $s \in [0, \infty)$ and a continuous function $z : [-\tau, \infty) \to \mathbb{R}$, we define a mapping $z_s : [-\tau, 0] \to R$ by $z_s(\theta) = z(s + \theta)$ for $\theta \in [-\tau, 0]$.

First, we have the following observation:

Lemma 2.1. *If $(u(t), v(t))^T$ is a solution of system (2.1) with initial value $\Phi = (\varphi, \psi)^T \in X_0$, then the solution of (2.1) with the initial value $\Phi = (-\varphi, -\psi)^T \in X_0$ is $(-u(t), -v(t))^T$.*

Let $(u(t), v(t))^T$ be a solution of (2.1) with initial value in X_0. Then, we have

Lemma 2.2. *If there exists some $t_0 \geqslant 0$ such that $(u_{t_0}, v_{t_0})^T \in X^{-,+}$, then there exists some $t_0^* \geqslant t_0$ such that $(u_{t_0^*+\tau}, v_{t_0^*+\tau})^T \in X^{+,+}$ and $u(t_0^* + \tau) < v(t_0^* + \tau)$.*

Proof. In view of (2.1) and $(u_{t_0}, v_{t_0})^T \in X^{-,+}$, $(u(t), v(t))^T$ satisfies (2.3) for $t \in (t_0, t_0 + \tau)$. By the continuity of solutions, for $t \in [t_0, t_0 + \tau]$, we have

$$u(t) = \big(u(t_0) - m\big)e^{t_0-t} + m, \qquad v(t) = \big(v(t_0) - m\big)e^{t_0-t} + m, \tag{2.6}$$

Let t_1 be the first zero of $u(t)v(t)$ in $[t_0, \infty)$. Then (2.6) holds for all $t \in [t_0, t_1 + \tau]$. On the other hand, $u(t)v(t) = 0$ implies

$$t = t_0 + \ln\big(m - u(t_0)\big) - \ln m \quad \text{or} \quad t = t_0 + \ln\big(m - v(t_0)\big) - \ln m.$$

In view of $u(t_0) \leqslant 0$ and $v(t_0) > 0$, we have

$$t_1 = t_0 + \ln\big(m - u(t_0)\big) - \ln m.$$

This, together with (2.6), implies that

$$u(t_1 + \tau) = m - me^{-\tau} > 0, \qquad v(t_1 + \tau) = \frac{v(t_0) - m}{m - u(t_0)}me^{-\tau} + m > 0.$$

Moreover, it is easy to see that $(u_{t_1+\tau}, v_{t_1+\tau})^T \in X^{+,+}$ and $u(t_1 + \tau) < v(t_1 + \tau)$. Thus, the conclusion holds with $t_0^* = t_1$. This completes the proof. $\quad\square$

Lemma 2.3. *If there exists some $t_0 \geqslant 0$ such that $(u_{t_0}, v_{t_0})^T \in X^{-,-}$ and $u(t_0) < v(t_0)$, then there exists some $t_0^* \geqslant t_0$ such that $(u_{t_0^*+\tau}, v_{t_0^*+\tau})^T \in X^{+,+}$ and $u(t_0^* + \tau) < v(t_0^* + \tau)$.*

Proof. We distinguish two cases.

Case 1: $(1 - u(t_0))/(1 - v(t_0)) \geqslant e^\tau$. From (2.1) and $(u_{t_0}, v_{t_0})^T \in X^{-,-}$, $(u(t), v(t))^T$ satisfies (2.4) for $t \in (t_0, t_0 + \tau)$. By the continuity of solutions, for $t \in [t_0, t_0 + r]$, we have

$$u(t) = \big(u(t_0) - 1\big)e^{t_0-t} + 1, \qquad v(t) = \big(v(t_0) - 1\big)e^{t_0-t} + 1. \tag{2.7}$$

Let t_1 be the first zero of $u(t)v(t)$ in $[t_0, \infty)$. Then (2.7) holds for all $t \in [t_0, t_1 + \tau]$. On the other hand, $u(t)v(t) = 0$ implies

$$t = t_0 + \ln\big(1 - u(t_0)\big) \quad \text{or} \quad t = t_0 + \ln\big(1 - v(t_0)\big).$$

In view of $u(t_0) < v(t_0)$, we have

$$t_1 = t_0 + \ln\big[1 - v(t_0)\big].$$

This, together with (2.7), implies that

$$u(t_1 + \tau) = \frac{u(t_0) - 1}{1 - v(t_0)}e^{-\tau} + 1 \leqslant 0, \qquad v(t_1 + \tau) = 1 - e^{-\tau} > 0.$$

Moreover, it is easy to see that $(u_{t_1+\tau}, v_{t_1+\tau})^T \in X^{-,+}$. Thus, from Lemma 2.2 it follows that there exists some $t_0^* \geqslant t_1 + \tau$ such that $(u_{t_0^*+\tau}, v_{t_0^*+\tau})^T \in X^{+,+}$ and $u(t_0^* + \tau) < v(t_0^* + \tau)$.

Case 2: $1 < (1 - u(t_0))/(1 - v(t_0)) < e^\tau$. Using a similar argument as above, we get

$$u(t_1 + \tau) = \frac{u(t_0) - 1}{1 - v(t_0)}e^{-\tau} + 1 > 0 \quad \text{and} \quad v(t_1 + \tau) = 1 - e^{-\tau} > 0.$$

Note that $u(t_0) < 0$ and (2.7) holds for $t \in (t_0, t_0 + \tau)$, we have some $t_2 \in [t_1, t_1 + \tau]$ such that $u(t_2) = 0$. In fact, by (2.7), we have

$$t_2 = t_0 + \ln\big(1 - u(t_0)\big),$$

and we can easily show that $u(t - \tau) < 0$ and $v(t - \tau) > 0$ for $t \in (t_1 + \tau, t_2 + \tau)$. Therefore, $(u(t), v(t))^T$ satisfies (2.3) for $t \in (t_1 + \tau, t_2 + \tau)$. Thus, for $t \in [t_1 + \tau, t_2 + \tau]$, we have

$$u(t) = \big[u(t_1 + \tau) - m\big]e^{t_1+\tau-t} + m = \left[\frac{u(t_0) - 1}{1 - v(t_0)}e^{-\tau} + 1 - m\right]e^{t_1+\tau-t} + m,$$

$$v(t) = \big[v(t_1 + \tau) - m\big]e^{t_1+\tau-t} + m = (1 - e^{-\tau} - m)e^{t_1+\tau-t} + m. \tag{2.8}$$

It follows that

$$u(t_2 + \tau) = m - e^{-\tau} + (1 - m)\frac{1 - v(t_0)}{1 - u(t_0)} > 0,$$

$$v(t_2 + \tau) = (1 - e^{-\tau} - m)\frac{1 - v(t_0)}{1 - u(t_0)} + m > 0,$$

and hence

$$v(t_2 + \tau) - u(t_2 + \tau) = \frac{v(t_0) - u(t_0)}{1 - u(t_0)}e^{-\tau} > 0.$$

Moreover, from (2.7) and (2.8), it is easy to see that $(u_{t_2+\tau}, v_{t_2+\tau})^T \in X^{+,+}$. This completes the proof. $\square$

Lemma 2.4. *If there exists some $t_0 \geqslant 0$ such that $(u_{t_0}, v_{t_0})^T \in X^{+,+}$ and $u(t_0) \leqslant v(t_0)$, then the first zero of $u(t)v(t)$ in $[t_0, \infty)$ is $t_1 = t_0 + \ln(1 + u(t_0))$. Moreover, we have $u(t_1) = 0$ and $v(t_1) = (v(t_0) - u(t_0))/(1 + u(t_0)) \geqslant 0$.*

The proof of Lemma 2.4 is similar to that of Lemma 2.3, and thus is omitted.

Construct a piecewise function $F : [0, \infty) \to [0, \infty)$ defined by

$$
F(x) = \begin{cases}
f_1(x) & \text{if } x \in [e^\tau - 1, \infty), \\
f_2(x) & \text{if } x \in \left[\frac{1-e^{-\tau}}{m}, e^\tau - 1\right), \\
f_3(x) & \text{if } x \in \left[\frac{1-e^{-\tau}}{m+e^{-\tau}}, \frac{1-e^{-\tau}}{m}\right), \\
f_4(x) & \text{if } x \in \left(0, \frac{1-e^{-\tau}}{m+e^{-\tau}}\right), \\
0 & \text{if } x = 0
\end{cases}
\tag{2.9}
$$

for $m \geqslant e^{-\tau}$, or

$$
F(x) = \begin{cases}
f_1(x) & \text{if } x \in [e^\tau - 1, \infty), \\
f_5(x) & \text{if } x \in \left[\frac{m+1-me^{-\tau}}{m+1-m^2}(e^\tau - 1), e^\tau - 1\right), \\
f_3(x) & \text{if } x \in \left[\frac{1-e^{-\tau}}{m+e^{-\tau}}, \frac{m+1-me^{-\tau}}{m+1-m^2}(e^\tau - 1)\right), \\
f_4(x) & \text{if } x \in \left(0, \frac{1-e^{-\tau}}{m+e^{-\tau}}\right), \\
0 & \text{if } x = 0
\end{cases}
\tag{2.10}
$$

for $0 < m < e^{-\tau}$, where

$$
f_1(x) = \frac{me^{-2\tau}x}{(m + 1 - e^{-\tau})(m + 1 - me^{-\tau})},
\tag{2.11}
$$

$$
f_2(x) = \frac{me^{-2\tau}x}{(m^2 - 1)e^{-\tau}x + (m + 1 - me^{-\tau})(1 - e^{-\tau}) + m + 1 - e^{-\tau}},
\tag{2.12}
$$

$$
f_3(x) = \frac{me^{-2\tau}x}{(m^2 - 1)(e^{-\tau} - m)x + (m + 1 - me^{-\tau})(m + 1 - e^{-\tau})},
\tag{2.13}
$$

$$
f_4(x) = \frac{e^{-2\tau}x}{2(1 - m)(1 - e^{-\tau})x + (1 - e^{-\tau})^2},
\tag{2.14}
$$

and

$$
f_5(x) = \frac{m^2 e^{-3\tau}x}{(m + 1 - me^{-\tau})[(1 - m^2)e^{-\tau}x + (me^{-\tau} - 1)(1 - e^{-\tau}) + m^2]}.
\tag{2.15}
$$

Elementary calculations lead to the following

Lemma 2.5. *The function F defined by (2.9) or (2.10) is continuous, monotonically increasing on the interval $[0, \infty)$, and satisfies that $F(x) < x$ for all $x \in (0, \infty)$. Moreover, the zero is a stable fixed point.*

3. Global attractivity of a periodic orbit

We start with the following

Theorem 3.1. *Assume that $(u^\Phi, v^\Phi)^T$ is a solution of system (2.1) with initial value $\Phi \in X_0$. If $\Phi = (\varphi, \psi)^T \in X^{+,+} \cup X^{-,-}$ and $\varphi(0) = \psi(0)$, then $u^\Phi(t) = v^\Phi(t)$ for all $t \geqslant 0$, and $u^\Phi(t) = v^\Phi(t) = q(t)$ for all $t \geqslant \tau + \ln[1 + \varphi(0)]$ (respectively, for $t \geqslant \tau + \ln[1 - \varphi(0)]$), where $q(t)$ is a periodic function with minimal period $\omega = 2\ln(2e^\tau - 1)$.*

Proof. We only consider the case where $\Phi = (\varphi, \psi)^T \in X^{+,+}$. The case where $\Phi = (\varphi, \psi)^T \in X^{-,-}$ can be dealt with analogously.

Using Eq. (2.1), we can easily obtain that $u(t) = v(t)$ for all $t \geqslant t_0$. Therefore, it suffices to show that the solution $u(t)$ of the equation

$$\dot{u} = -u + f\big(u(t - \tau)\big) \tag{3.1}$$

with the initial condition $\varphi \in C^+$ is eventually periodic with minimal period $2\ln(2e^\tau - 1)$. Let t_1 be the first nonnegative zero of $u(t)$ on $[0, \infty)$. Then for $t \in (0, t_1 + \tau)$ except at most finitely many t, we have

$$\dot{u} = -u - 1, \tag{3.2}$$

from which and the continuity of the solution it follows that

$$u(t) = e^{-t}\big[\varphi(0) + 1\big] - 1$$

for $t \in [0, t_1 + \tau]$, and in particular,

$$u(t_1) = e^{-t_1}\big[\varphi(0) + 1\big] - 1 = 0.$$

This implies

$$t_1 = \ln\big[1 + \varphi(0)\big]$$

and

$$u(t_1 + \tau) = e^{-(t_1+\tau)}\big[\varphi(0) + 1\big] - 1 = e^{-\tau} - 1 < 0.$$

Also

$$u_{t_1+\tau}(\theta) := u(t_1 + \tau + \theta) = e^{-(t_1+\tau+\theta)}\big[\varphi(0) + 1\big] - 1 = e^{-(\tau+\theta)} - 1 < 0$$

for $\theta \in (-\tau, 0]$. Therefore, $u_{t_1+\tau} \in C^-$.

To construct a solution of (3.1) beyond $[0, t_1 + \tau]$, we consider the solution of (3.1) with the new initial value $\varphi^* = u_{t_1+\tau}$. Let t_2 be the first zero after t_1 of u. Then $t_2 > t_1 + \tau$ and on $(t_1 + \tau, t_2 + \tau)$, we have

$$\dot{u} = -u + 1 \tag{3.3}$$

and hence

$$u(t) = e^{-(t-t_1-\tau)}\big[u(t_1 + \tau) - 1\big] + 1 = (e^{-\tau} - 2)e^{-(t-t_1-\tau)} + 1$$

for $t \in [t_1 + \tau, t_2 + \tau]$. In particular,

$$u(t_2) = (e^{-\tau} - 2)e^{t_1-t_2+\tau} + 1 = 0,$$

which implies

$$t_2 = t_1 + \tau + \ln(2 - e^{-\tau}).$$

Also,

$$u_{t_2+\tau}(\theta) = u(t_2 + \tau + \theta) = (e^{-\tau} - 2)e^{t_1-t_2-\theta} + 1 = 1 - e^{-(\tau+\theta)} > 0$$

for $\theta \in (-\tau, 0]$. Therefore, $u_{t_2+\tau} \in C^+$.

Repeating the above arguments and letting t_3 be the first zero after t_2 of u, $t_3 > t_2 + \tau$, and we know (3.2) holds on $(t_2 + \tau, t_3 + \tau$. Consequently

$$u(t) = e^{-(t-t_2-\tau)}\big[u(t_2 + \tau) + 1\big] - 1 = (2 - e^{-\tau})e^{-(t-t_2-\tau)} + 1$$

for $t \in [t_2 + \tau, t_3 + \tau]$. In particular,

$$u(t_3) = (2 - e^{-\tau})e^{t_2 - t_3 + \tau} - 1 = 0,$$

which implies

$$t_3 = t_2 + \tau + \ln(2 - e^{-\tau}).$$

Also,

$$u_{t_3+\tau}(\theta) = u(t_3 + \tau + \theta) = (2 - e^{-\tau})e^{t_2 - t_3 - \theta} - 1 = e^{-(\tau+\theta)} - 1 > 0$$

for $\theta \in [-\tau, 0]$. Therefore, $u_{t_3+\tau} \in C^-$.

This also shows that $u_{t_3+\tau}(\theta) = u_{t_1+\tau}(\theta)$ for $\theta \in (-\tau, 0]$. Due to the uniqueness of the Cauchy initial value problem (see [5]), we have $u(t + t_3 + \tau) = u(t + t_1 + \tau)$ for $t \geqslant 0$. Namely, for $t \geqslant t_1 + \tau$, $u(t)$ is periodic with minimal period $\omega = t_3 - t_1 = 2\ln(2e^\tau - 1)$. This completes the proof. $\quad\square$

We remark that the aforementioned result was obtained also in [12]. We included a detailed proof for the sake of completeness. From Theorem 3.1, if $\Phi = (\varphi, \psi)^T \in X$ is synchronized (i.e., $\varphi = \psi$) then the solution $(u^\Phi, v^\Phi)^T : [-\tau, \infty) \to \mathbb{R}^2$ is synchronized, that is, $u^\Phi(t) = v^\Phi(t)$ for all $t \geqslant 0$, due to the uniqueness of the Cauchy initial value problem of (2.1) (see [5]). The above result shows that the solution $(u^\Phi, v^\Phi)^T$ of (2.1) is synchronized even if the initial value Φ is asynchronous but $\varphi(0) = \psi(0)$ and $(\varphi, \psi)^T \in X^{+,+} \cup X^{-,-}$. Moreover, a synchronized solution of (2.1) is characterized by the scalar equation (3.1) and Theorem 3.1 shows that a solution of (3.1) with initial value in $C^\pm$ is eventually periodic and is of the minimal period $\omega = 2\ln(2e^\tau - 1)$.

We now discuss the case where the initial value $\Phi = (\varphi, \psi)^T \in X^{+,+}$ and $\varphi(0) \leqslant \psi(0)$. In view of Lemma 2.4, the first zero of $u(t)v(t)$ in $[0, +\infty)$ is $t_1 = \ln(1 + \varphi(0))$. Moreover, $u(t_1) = 0$ and $v(t_1) = (\psi(0) - \varphi(0))/(1 + \varphi(0)) \geqslant 0$. It is easy to see that the values of $[t_1, u(t_1), v(t_1)]$ are completely determined by $\varphi(0)$ and $\psi(0)$. Without loss of generality, we let $u(0) = \varphi(0) = 0$ and $v(0) = \psi(0) = v \geqslant 0$. We will show that the behavior of $(u(t), v(t))^T$ as $t \to +\infty$ is completely determined by the value v. Recall that if $v = 0$, then by Theorem 3.1, $(u(t), v(t))^T$ is eventually periodic and coincides with the synchronized periodic solution $(q(t), q(t))^T$. Our analysis below shows that the behavior of $(u(t), v(t))^T$ as $t \to +\infty$ can be understood in terms of the iterations of a one-dimensional map in case $v > 0$.

We start with

Case 1: $v \geqslant e^\tau - 1$. In view of (2.1) and $\Phi = (\varphi, \psi)^T \in X^{+,+}$, $(u(t), v(t))^T$ satisfies system (2.2). By the continuity of the solution, for $t \in [0, \tau]$, we have

$$
\begin{aligned}
u(t) &= \big[u(0) + 1\big]e^{t_0 - t} - 1 = e^{-t} - 1, \\
v(t) &= \big[v(0) + 1\big]e^{t_0 - t} - 1 = (1 + v)e^{-t} - 1.
\end{aligned}
\tag{3.4}
$$

It follows that $u(\tau) = e^{-\tau} - 1 < 0$ and $v(\tau) = (1 + v)e^{-\tau} - 1 > 0$. From (3.4), we see that $(u_\tau, v_\tau)^T \in X^{-,+}$. This, together with the proof of Lemma 2.2, implies that the second zero of $u(t)v(t)$ on $[0, \infty)$ is

$$t_2 = \tau + \ln[m - u(\tau)] - \ln m = \tau + \ln[m + 1 - e^{-\tau}] - \ln m$$

and

$$u(t) = [u(\tau) - m]e^{\tau - t} + m = (e^{-\tau} - 1 - m)e^{\tau - t} + m,$$
$$v(t) = [v(\tau) - m]e^{\tau - t} + m = [(1 + v)e^{-\tau} - 1 - m]e^{\tau - t} + m \tag{3.5}$$

for $t \in [\tau, t_2 + \tau]$. This implies

$$u(t_2 + \tau) = m - me^{-\tau} > 0, \qquad v(t_2 + \tau) = \frac{me^{-2\tau}v}{m + 1 - e^{-\tau}} + m(1 - e^{-\tau}).$$

Moreover, we see that $(u_{t_2+\tau}, v_{t_2+\tau})^T \in X^{+,+}$ and $u(t_2 + \tau) < v(t_2 + \tau)$. This, together with Lemma 2.4, implies that the third zero of $u(t)v(t)$ on $[0, \infty)$ is

$$t_3 = t_2 + \tau + \ln[1 + u(t_2 + \tau)] = t_2 + \tau + \ln[m + 1 - me^{-\tau}]$$

and satisfies $u(t_3) = 0$ and

$$v(t_3) = \frac{v(t_2 + \tau) - u(t_2 + \tau)}{1 + u(t_2 + \tau)} = \frac{me^{-2\tau}v}{(m + 1 - e^{-\tau})(m + 1 - me^{-\tau})} = f_1(v) > 0,$$

where the function f_1 is defined as in (2.11).

   Case 2: $(1 - e^{-\tau})/m \leqslant v < e^\tau - 1$ and $m \geqslant e^{-\tau}$. Using a similar argument as above, we have that (3.4) holds for $t \in [0, \tau]$. Moreover, $u(\tau) = e^\tau - 1 < 0$ and $v(\tau) = (1 + v)e^{-\tau} - 1 < 0$. Recall that $v(0) = v > 0$, there exists $t_2 \in [0, \tau]$ such that $v(t_2) = 0$. From (3.4) we have

$$t_2 = \ln(1 + v).$$

Moreover, from (3.4), it follows that $u(t - \tau) < 0$ and $v(t - \tau) > 0$ for $t \in (\tau, t_2 + \tau)$. Thus, $(u(t), v(t))^T$ satisfies (2.3) for $t \in (\tau, t_2 + \tau)$. Namely, for $t \in [\tau, t_2 + \tau]$, (3.5) holds. It follows that

$$u(t_2 + \tau) = \frac{e^{-\tau} - m - 1}{1 + v} + m > 0, \qquad v(t_2 + \tau) = m + e^{-\tau} - \frac{1 + m}{1 + v} > 0.$$

Also since $u(\tau) < 0$ and $v(\tau) < 0$, there must exist the third and forth zeroes $t_3, t_4 \in (\tau, t_2 + \tau)$ of $u(t)v(t)$. From (3.5), we have

$$t_3 = \tau + \ln[m + 1 - (1 + v)e^{-\tau}] - \ln m, \qquad t_4 = \tau + \ln[m + 1 - e^{-\tau}] - \ln m,$$

such that $v(t_3) = 0$ and $u(t_4) = 0$. Moreover, from (3.4) and (3.5), we see that $u(t - \tau) < 0$ and $v(t - \tau) < 0$ for $t \in (t_2 + \tau, t_3 + \tau)$. Thus $(u(t), v(t))^T$ satisfies (2.4) for $t \in (t_2 + \tau, t_3 + \tau)$. Namely, for $t \in [t_2 + \tau, t_3 + \tau]$, we have

$$u(t) = [u(t_2 + \tau) - 1]e^{t_2 + \tau - t} + 1 = \left(m - 1 + \frac{e^{-\tau} - 1 - m}{1 + v}\right)e^{t_2 + \tau - t} + 1,$$

$$v(t) = [v(t_2 + \tau) - 1]e^{t_2 + \tau - t} + 1 = \left(m - 1 + e^{-\tau} - \frac{1 + m}{1 + v}\right)e^{t_2 + \tau - t} + 1. \tag{3.6}$$

This implies

$$u(t_3 + \tau) = \frac{(m-1)(1+v) + e^{-\tau} - 1 - m}{(m+1)e^\tau - (1+v)} m + 1 > 0,$$

$$v(t_3 + \tau) = \frac{(m - 1 + e^{-\tau})(1+v) - 1 - m}{(m+1)e^\tau - (1+v)} m + 1 > 0.$$

From (3.5) we see that $u(t - \tau) < 0$ and $v(t - \tau) > 0$ for $t \in (t_3 + \tau, t_4 + \tau)$. Thus $(u(t), v(t))^T$ satisfies system (2.3) for $t \in (t_3 + \tau, t_4 + \tau)$. It follows that for $t \in [t_3 + \tau, t_4 + \tau]$,

$$
\begin{aligned}
u(t) &= \big[u(t_3 + \tau) - m\big]e^{t_3 + \tau - t} + m \\
&= \left[\frac{(m-1)(1+v) + e^{-\tau} - 1 - m}{(m+1)e^\tau - (1+v)} m + 1 - m\right]e^{t_3 + \tau - t} + m, \\
v(t) &= \big[v(t_3 + \tau) - m\big]e^{t_3 + \tau - t} + m \\
&= \left[\frac{(m - 1 + e^{-\tau})(1+v) - 1 - m}{(m+1)e^\tau - (1+v)} m + 1 - m\right]e^{t_3 + \tau - t} + m.
\end{aligned}
\tag{3.7}
$$

This implies

$$u(t_4 + \tau) = \frac{(m^2 - 1)e^{-\tau}v + (1 + m - me^{-\tau})(1 - e^{-\tau})}{m + 1 - e^{-\tau}} > 0,$$

$$v(t_4 + \tau) = \frac{(m^2 - 1 + me^{-\tau})e^{-\tau}v + (1 + m - me^{-\tau})(1 - e^{-\tau})}{m + 1 - e^{-\tau}} > 0.$$

Moreover, from (3.5)–(3.7), we have

$$(u_{t_4+\tau}, v_{t_4+\tau})^T \in X^{+,+} \quad \text{and} \quad u(t_4 + \tau) < v(t_4 + \tau).$$

This, together with Lemma 2.4, implies that the fifth zero of $u(t)v(t)$ on $[0, \infty)$ is $t_5 = t_4 + \tau + \ln[1 + u(t_4 + \tau)]$ and $u(t_5) = 0$,

$$
\begin{aligned}
v(t_5) &= \frac{v(t_4 + \tau) - u(t_4 + \tau)}{1 + u(t_4 + \tau)} \\
&= \frac{me^{-2\tau}v}{(m^2 - 1)e^{-\tau}v + (m + 1 - me^{-\tau})(1 - e^{-\tau}) + m + 1 - e^{-\tau}} = f_2(v) > 0,
\end{aligned}
$$

where the function f_2 is defined as in (2.12).

Case 3: $(1 - e^{-\tau})/(m + e^{-\tau}) \leqslant v < (1 - e^{-\tau})/m$ and $m \geqslant e^{-\tau}$. Using a similar argument as above, we have $u(t_2 + \tau) = m + (e^{-\tau} - 1 - m)/(1 + v) < 0$ and $v(t_2 + \tau) = m + e^{-\tau} - (1 + m)/(1 + v) \geqslant 0$, which, together with the fact that $u(\tau) = e^{-\tau} - 1 < 0$ and $v(\tau) = (1 + v)e^{-\tau} - 1 < 0$, implies that there exists $t_3 \in (\tau, t_2 + \tau)$ such that $v(t_3) = 0$. It follows from (3.5) that

$$t_3 = \tau + \ln\big[m + 1 - (1 + v)e^{-\tau}\big] - \ln m.$$

From (3.4) and (3.5) we see that $u(t - \tau) < 0$ and $v(t - \tau) < 0$ for $t \in (t_2 + \tau, t_3 + \tau)$. Therefore, $(u(t), v(t))^T$ satisfies (2.4) for $t \in (t_2 + \tau, t_3 + \tau)$. Namely, (3.6) holds for $t \in [t_2 + \tau, t_3 + \tau]$. It follows that $u(t_3 + \tau) > 0$ and $v(t_3 + \tau) > 0$. Also since $u(t_2 + \tau) < 0$

and $v(t_2 + \tau) \geqslant 0$, there exists $t_4 \in (t_2 + \tau, t_3 + \tau)$ such that $u(t_4) = 0$. It follows from (3.6) that

$$t_4 = \tau + \ln\left[(1 - m)v + 2 - e^{-\tau}\right].$$

From (3.5) and (3.6), we have $u(t - \tau) < 0$ and $v(t - \tau) > 0$ for $t \in (t_3 + \tau, t_4 + \tau)$. Therefore, $(u(t), v(t))^T$ satisfies (2.3) for $t \in (t_3 + \tau, t_4 + \tau)$. By the continuity of solution, (3.7) holds for $t \in [t_3 + \tau, t_4 + \tau]$, which implies that

$$u(t_4 + \tau) = m - e^{-\tau} + \frac{1-m}{m} \frac{m + 1 - (1 + v)e^{-\tau}}{(1-m)v + 2 - e^{-\tau}} > 0,$$

$$v(t_4 + \tau) = m - e^{-\tau} + \frac{ve^{(-2\tau)}}{(1-m)v + 2 - e^{-\tau}} + \frac{1-m}{m} \frac{m + 1 - (1 + v)e^{-\tau}}{(1-m)v + 2 - e^{-\tau}} > 0.$$

Moreover, from (3.6) and (3.7), we see that

$$(u_{t_4+\tau}, v_{t_4+\tau})^T \in X^{+,+} \quad \text{and} \quad u(t_4 + \tau) < v(t_4 + \tau).$$

This, together with Lemma 2.4, implies that the fifth zero of $u(t)v(t)$ on $[0, \infty)$ is $t_5 = t_4 + \tau + \ln[1 + u(t_4 + \tau)]$ and satisfies $u(t_5) = 0$ and

$$v(t_5) = \frac{v(t_4 + \tau) - u(t_4 + \tau)}{1 + u(t_4 + \tau)}$$

$$= \frac{me^{-2\tau}v}{(m^2 - 1)(e^{-\tau} - m)v + (m + 1 - me^{-\tau})(m + 1 - e^{-\tau})} = f_3(v) > 0,$$

where the function f_3 is defined as in (2.13).

Case 4: $0 < v < (1 - e^{-\tau})/(m + e^{-\tau})$. Using a similar argument as in Case 2, we have $u(t_2 + \tau) = m + (e^{\tau} - 1 - m)/(1 + v) < 0$ and $v(t_2 + \tau) = m + e^{-\tau} - (1 + m)/(1 + v) < 0$. It follows from (3.4) and (3.5) that $(u_{t_2+\tau}, v_{t_2+\tau})^T \in X^{-,-}$ and $u(t_2 + \tau) < v(t_2 + \tau)$. We claim that $[1 - u(t_2 + \tau)]/[1 - v(t_2 + \tau)] < e^{\tau}$. Suppose to the contrary. If $0 < v < (1 - e^{-\tau})/(m + e^{-\tau})$ and $[1 - u(t_2 + \tau)]/[1 - v(t_2 + \tau)] \geqslant e^{\tau}$, then

$$\left[(1 - m)(e^{\tau} - 1) - 1\right]v + (2 - e^{-\tau})(e^{\tau} - 1) \leqslant 0.$$

Thus, we have $(1 - m)(e^{\tau} - 1) - 1 \leqslant 0$ and $v > (2 - e^{-\tau})(e^{\tau} - 1)/[(m - 1)(e^{\tau} - 1) + 1] > (1 - e^{-\tau})/(m + e^{-\tau})$, which is a contradiction. Therefore, $[1 - u(t_2 + \tau)]/[1 - v(t_2 + \tau)] < e^{\tau}$. Using similar arguments as in the proof of Lemma 2.3, we have

$$t_3 = t_2 + \tau + \ln\left[1 - v(t_2 + \tau)\right] = \tau + \ln\left[(1 - m - e^{-\tau})v + 2 - e^{-\tau}\right],$$

$$t_4 = t_2 + \tau + \ln\left[1 - u(t_2 + \tau)\right] = \tau + \ln\left[(1 - m)v + 2 - e^{-\tau}\right],$$

$$u(t_3 + \tau) = \frac{(m - 1)v + e^{-\tau} - 2}{(1 - m - e^{-\tau})v + 2 - e^{-\tau}} e^{-\tau} + 1 > 0,$$

$$v(t_3 + \tau) = 1 - e^{-\tau} > 0,$$

$$u(t_4 + \tau) = 1 - e^{-\tau} + \frac{(m - 1)e^{-\tau}v}{(1 - m)v + 2 - e^{-\tau}} > 0,$$

$$v(t_4 + \tau) = 1 - e^{-\tau} + \frac{(m - 1 + e^{-\tau})e^{-\tau}v}{(1 - m)v + 2 - e^{-\tau}} > 0,$$

where t_3 and t_4 represent the third and forth zeroes $u(t)v(t)$ on $[0, \infty)$, respectively. Moreover, it is easy to show that $(u_{t_4+\tau}, v_{t_4+\tau})^T \in X^{+,+}$ and $u(t_4 + \tau) < v(t_4 + \tau)$. This, together with the proof of Lemma 2.4, implies that the fifth zero of $u(t)v(t)$ on $[0, \infty)$ is $t_5 = t_4 + \tau + \ln[1 + u(t_4 + \tau)]$ and satisfies $u(t_5) = 0$ and

$$v(t_5) = \frac{v(t_4 + \tau) - u(t_4 + \tau)}{1 + u(t_4 + \tau)} = \frac{e^{-2\tau}v}{2(1 - m)(1 - e^{-\tau})v + (1 - e^{-\tau})^2} = f_4(v) > 0,$$

where the function f_4 is defined as in (2.14).

Case 5: $((m + 1 - me^{-\tau})/(m + 1 - m^2))(e^\tau - 1) \leqslant v < e^\tau - 1$ and $0 < m < e^{-\tau}$. Using a similar argument as in Case 2, we have $u(t_2 + \tau) = m + (e^{-\tau} - 1 - m)/(1 + v) < 0$ and $v(t_2 + \tau) = m + e^{-\tau} - (1 + m)/(1 + v) > 0$, which, together with the fact that $u(\tau) = e^{-\tau} - 1 < 0$ and that $v(\tau) = (1 + v)e^{-\tau} - 1 < 0$, implies that there exists $t_3 \in (\tau, t_2 + \tau)$ such that $v(t_3) = 0$. It follows from (3.5) that

$$t_3 = \tau + \ln[m + 1 - (1 + v)e^{-\tau}] - \ln m.$$

From (3.4) and (3.5), we have $u(t - \tau) < 0$ and $v(t - \tau) < 0$ for $t \in (t_2 + \tau, t_3 + \tau)$. Therefore, $(u(t), v(t))^T$ satisfies (2.4) for $t \in (t_2 + \tau, t_3 + \tau)$. Namely, for $t \in [t_2 + \tau, t_3 + \tau]$, (3.6) holds. Also since $v \geqslant (m + 1 - me^{-\tau})(e^\tau - 1)/(m + 1 - m^2)$ and $0 < m < e^{-\tau}$, we have $u(t_3 + \tau) \leqslant 0$ and $v(t_3 + \tau) > 0$. Furthermore, it follows from (3.5) and (3.6) that $(u_{t_3+\tau}, v_{t_3+\tau})^T \in X^{-,+}$. This, together with the proof of Lemma 2.2, implies that

$$t_4 = t_3 + \tau + \ln[m - u(t_3 + \tau)] - \ln m,$$

$$u(t_4 + \tau) = m(1 - e^{-\tau}) > 0, \qquad v(t_4 + \tau) = \frac{v(t_3 + \tau) - m}{m - u(t_3 + \tau)}me^{-\tau} + m > 0,$$

where t_4 represents the forth zero of $u(t)v(t)$. Moreover, we have $(u_{t_4+\tau}, v_{t_4+\tau})^T \in X^{+,+}$ and $u(t_4 + \tau) < v(t_4 + \tau)$. This, together with the proof of Lemma 2.4, implies that the fifth zero of $u(t)v(t)$ on $[0, \infty)$ is $t_5 = t_4 + \tau + \ln[1 + u(t_4 + \tau)]$ and satisfies $u(t_5) = 0$ and

$$v(t_5) = \frac{v(t_4 + \tau) - u(t_4 + \tau)}{1 + u(t_4 + \tau)}$$

$$= \frac{m^2 e^{-3\tau}v}{(m + 1 - me^{-\tau})[(1 - m^2)e^{-\tau}v + (me^{-\tau} - 1)(1 - e^{-\tau}) + m^2]}$$

$$= f_5(v) > 0,$$

where the function f_5 is defined as in (2.15).

Case 6: $(1 - e^{-\tau})/(m + e^{-\tau}) \leqslant v < ((m + 1 - me^{-\tau})/(m + 1 - m^2))(e^\tau - 1)$ and $0 < m < e^{-\tau}$. Using a similar argument as in Case 2, we have $u(t_3 + \tau) > 0$ and $v(t_3 + \tau) > 0$. Then, we have $(u_{t_4+\tau}, v_{t_4+\tau})^T \in X^{+,+}$, $u(t_4 + \tau) < v(t_4 + \tau)$, $u(t_5) = 0$, and $v(t_5) = f_3(v) > 0$, where t_4 and t_5 represent the forth and fifth zeroes of $u(t)v(t)$, respectively.

As discussed above for Cases 1–6, we obtain a new value $F(v)$. Thus we can repeat the same analysis and construction to get $F^2(v) = F(F(v))$ assuming that the initial condition is the value $F(v)$. By Lemma 2.5, we continue to iterate F to get a sequence

$$v, F(v), F^2(v), \ldots, F^n(v), \ldots,$$

where $F^n(v) = F(F^{n-1}(v))$. Therefore, by using F and its iterates, we can characterize the behavior of the solution $(u(t), v(t))^T$ of system (2.1) with initial value $\Phi = (\varphi, \psi)^T \in X^{+,+}$ and $\varphi(0) \leqslant \psi(0)$. Using the properties of $F(x)$ stated in Lemma 2.5 that the zero is the unique stable fixed point, we have $\lim_{n \to \infty} F^n(x) = 0$ for all $x \in [0, \infty)$. Therefore, every solution $(u(t), v(t))^T$ of system (2.1) with initial value $\Phi = (\varphi, \psi)^T \in X^{+,+}$ and $\varphi(0) \leqslant \psi(0)$ is either ultimately periodic with minimal period $\omega = 2\ln(2e^\tau - 1)$ (i.e., to coincide with the synchronized solution $(q(t), q(t))^T$) or approaches the periodic solution $(q(t), q(t))^T$ as $t \to \infty$. Thus, from Lemmas 2.1–2.4, we obtain the following main theorem.

Theorem 3.2. *Every solution $(u(t), v(t))^T$ of system (2.1) with initial value $\Phi = (\varphi, \psi)^T \in X_0$ is either eventually periodic with minimal period $\omega = 2\ln(2e^\tau - 1)$ or approaches the periodic solution $(q(t), q(t))^T$ as $t \to \infty$.*

Theorem 3.2 implies that the synchronized solution $(q(t), q(t))^T$ is the global attractor for all solutions of (2.1) with initial values in X_0.

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