Electronic Journal of Differential Equations, Vol. 2003(2003), No. 61, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

CONVERGENCE AND PERIODICITY IN A DELAYED NETWORK OF NEURONS WITH THRESHOLD NONLINEARITY

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Abstract. We consider an artificial neural network where the signal transmission is of a digital (McCulloch-Pitts) nature and is delayed due to the finite switching speed of neurons (amplifiers). The discontinuity of the signal transmission functions, however, makes it difficult to apply the existing dynamical systems theory which usually requires continuity and smoothness. Moreover, observe that the dynamics of the network completely depends on the connection weights, we distinguish several cases to discuss the behaviors of their solutions. We show that the dynamics of the model can be understood in terms of the iterations of a one-dimensional map. As, a result, we present a detailed analysis of the dynamics of the network starting from non-oscillatory states and show how the connection topology and synaptic weights determine the rich dynamics.

1. Introduction

In this paper, we consider the following model for an artificial neural network of two neurons,

$$\begin{aligned}\dot{x} &= -\mu x + a_{11} f(x(t-\tau)) + a_{12} f(y(t-\tau)),\\ \dot{y} &= -\mu y + a_{21} f(x(t-\tau)) + a_{22} f(y(t-\tau)),\end{aligned}\tag{1.1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $x(t)$ and $y(t)$ denote the activation of two neurons, $\mu > 0$ is the decay rate, $\tau > 0$ is the synaptic transmission delay, a_{ij} with $1 \le i, j \le 2$ are the synaptic weights, $f : \mathbb{R} \to \mathbb{R}$ is the activation function. Such a model describes the computational performance of a Hopfield net [8] where each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and each neuron is connected to another via the nonlinear activation function f multiplied by the synaptic weights a_{ij} $(i \ne j)$. We also allow that a neuron has self-feedback and signal transmission is delayed due to the finite switching speed of neurons.

Networks of two neurons have been used as prototypes for us to understand the dynamics of large networks with delayed activation functions, but much of the existing work has concentrated on the case of a smooth activation function (see, for example, [1, 2, 3, 6, 11, 12, 15]). In this paper, we consider the McCulloch-Pitts

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2000 *Mathematics Subject Classification.* 34K25, 34K13, 92B20.
Key words and phrases. Neural networks, feedback, McCulloch-Pitts nonlinearity, one-dimensional map, convergence, periodic solution.

Submitted June 30, 2002. Revised January 23, 2003. Published May 26, 2003.

activation function

$$f(\xi) = \begin{cases} -\delta, & \text{if } \xi > 0, \\ \delta, & \text{if } \xi \le 0, \end{cases} \tag{1.2}$$

where $\delta \neq 0$ is a given constant. This case arises when the signal transmission is of digital nature: a neuron is either fully active or completely inactive. Very little has been done in this case since results of the aforementioned references cannot be applied as the dynamical systems theory heavily used in these references usually requires the continuity and smoothness of the nonlinear terms. In [4, 5, 9, 10, 13], model equation (1.1) with a piecewise constant activation function is studied when the synaptic connection topology satisfies $[a_{11} = a_{22} = 0,\ a_{21} = a_{12} = 1]$ or $[a_{11} = a_{22} = 0,\ a_{21} = -a_{12} = 1]$, and more generally, $[a_{11} + a_{12} = 0,\ a_{11} > 0,\ a_{21} < 0,\ a_{21} < a_{22} \le -a_{21}]$.

To simplify the presentation, we first rescale the variables by

$$t^* = \mu t, \quad \tau^* = \mu\tau, \quad x^*(t^*) = \frac{\mu}{\delta}x(t), \quad y^*(t^*) = \frac{\mu}{\delta}y(t), \quad f^*(\xi) = \frac{1}{\delta}f(\frac{\delta}{\mu}\xi), \tag{1.3}$$

and then drop the $*$ to get

$$\begin{aligned} \dot{x} &= -x + a_{11}f(x(t-\tau)) + a_{12}f(y(t-\tau)), \\ \dot{y} &= -y + a_{21}f(x(t-\tau)) + a_{22}f(y(t-\tau)) \end{aligned} \tag{1.4}$$

with

$$f(\xi) = \begin{cases} -1, & \text{if } \xi > 0, \\ 1, & \text{if } \xi \le 0. \end{cases} \tag{1.5}$$

Let

$$a = a_{11} + a_{12}, \quad b = a_{21} + a_{22}, \quad c = a_{11} - a_{12}, \quad d = a_{21} - a_{22}. \tag{1.6}$$

We can rewrite (1.4) as

$$\begin{aligned} \dot{x} &= -x + \frac{a}{2}\left[f(x(t-\tau)) + f(y(t-\tau))\right] + \frac{c}{2}\left[f(x(t-\tau)) - f(y(t-\tau))\right], \\ \dot{y} &= -y + \frac{b}{2}\left[f(x(t-\tau)) + f(y(t-\tau))\right] + \frac{d}{2}\left[f(x(t-\tau)) - f(y(t-\tau))\right]. \end{aligned} \tag{1.7}$$

To state our main results, we set the phase space $X = C([-\tau, 0]; \mathbb{R}^2)$ as the Banach space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^2$ equipped with the sup-norm, see [7]. Note that for each given initial value $\Phi = (\varphi, \psi)^T \in X$, one can solve system (1.7) on intervals $[0, \tau]$, $[\tau, 2\tau]$, $\cdots$ successively to obtain a unique mapping $\left(x^\Phi, y^\Phi\right)^T : [-\tau, \infty) \to \mathbb{R}^2$ such that $x^\Phi \mid_{[-\tau,0]} = \varphi$, $y^\Phi \mid_{[-\tau,0]} = \psi$, $\left(x^\Phi, y^\Phi\right)^T$ is continuous for all $t \ge -\tau$, almost differentiable and satisfies (1.7) for $t > 0$. This gives a unique solution of (1.7) defined for all $t \ge -\tau$. In applications, a network usually starts from a constant (or nearly constant) state. Therefore, in this paper, we shall concentrate on the case where each component of Φ has no sign change on $[-\tau, 0]$. More precisely, we consider $\Phi \in X^{+,+} \bigcup X^{+,-} \bigcup X^{-,+} \bigcup X^{-,-} = X_0$, where

$$\begin{aligned} C^{\pm} = \big\{ \pm\varphi :\ &\varphi : [-\tau, 0] \to [0, \infty) \text{ is continuous and} \\ &\text{has only finitely many zeros on } [-\tau, 0] \big\} \end{aligned}$$

and

$$X^{\pm,\pm} = \left\{\Phi \in X; \Phi = (\varphi, \psi)^T, \varphi \in C^{\pm} \text{ and } \psi \in C^{\pm}\right\}.$$

Clearly, all constant initial values (except 0) are contained in X_0. Our analysis shows that the semiflow defined by system (1.7) on X_0 (in other words, the behavior of a solution $\big(x^\Phi(t), y^\Phi(t)\big)^T$ of system (1.7) with initial value $\Phi \in X_0$) is completely determined by the value $(\varphi(0), \psi(0))^T$ and the synaptic connection topology.

Guo, Huang and Wu [4] showed that using form (1.2) and some simple changes of variables, we can see that the semiflow defined by the system

$$\begin{aligned}\dot{u} &= -u + \frac{1}{2} f(u(t-\tau)) - \frac{1}{2} f(v(t-\tau)),\\ \dot{v} &= -v - \frac{1+B}{2} f(u(t-\tau)) + \frac{1-B}{2} f(v(t-\tau))\end{aligned} \tag{1.8}$$

with $B \geq 0$ is topologically equivalent to that of (1.7)-(1.5) while one of the following four conditions is satisfied:

(A1) $a = 0$, $b \leq 0$, $c > 0$, $d < 0$
(A2) $a \leq 0$, $b = 0$, $c > 0$, $d < 0$
(A3) $a > 0$, $b > 0$, $c = 0$, $d \geq 0$
(A4) $a > 0$, $b > 0$, $c \leq 0$, $d = 0$.

Theorem 1.1 ([4]). *Let* $\omega = 2\ln(2e^\tau - 1)$, $M = (1 - e^{-\tau})(e^\tau - \frac{B}{B+1})$, $m = \frac{1-e^{-\tau}}{B+e^{-\tau}}$, $\eta = (\varphi(0) + \psi(0))/(1 - \varphi(0)) \geq 0$, *the* ω*-periodic function* $q : \mathbb{R} \to \mathbb{R}$ *be*

$$q(t) = \begin{cases} e^{-(t+\tau)} - 1, & \text{if } -\omega/2 \leq t \leq 0,\\ (e^{-\tau} - 2)e^{-t} + 1, & \text{if } 0 < t \leq \omega/2,\end{cases}$$

and polynomials

$$\begin{aligned} h(B) &= B^3(e^{-\tau} - 1 - e^\tau) + B^2(e^{2\tau} - 3e^\tau + e^{-\tau} + e^{-2\tau} - 3)\\ &\quad + B(2e^{2\tau} - e^\tau + e^{-2\tau} - 4) + e^{2\tau} + e^\tau - e^{-\tau} - 1,\\ g(x) &= (Be^\tau - B - 1)x^2 + [(1 + 3B)(e^\tau - 1) + Be^{-\tau} - B(B + e^{-\tau})(e^\tau + 1)]x\\ &\quad - B(B - 1)(e^\tau - 1).\end{aligned}$$

Then the behavior of the solution $(u(t), v(t))^T$ *of system (1.8) with initial value* $\Phi = (\varphi, \psi)^T \in X^{-,+}$ *and* $\varphi(0) + \psi(0) \geq 0$ *is as follows:*

(i) *Suppose that* $B = 2(1-e^{-\tau})$ *and* $\tau > \ln 2$. *If* $\eta \in [0, m]$, *then* $(u(t), v(t))^T$ *is eventually periodic with minimal period* ω*; If* $\eta \in (m, M)$, *then* $(u(t), v(t))^T$ *approaches the periodic solution corresponding to* $\eta = m$ *as* $t \to \infty$*; If* $\eta \in [M, \infty)$, *then* $(u(t), v(t))^T$ *tends to* $(0, B)^T$ *as* $t \to \infty$.

(ii) *Suppose that* $B = 2(1-e^{-\tau})$ *and* $\tau < \ln 2$. *If* $\eta \in [0, m]$, *then* $(u(t), v(t))^T$ *is eventually periodic with minimal period* ω*; If* $\eta \in (m, \infty)$, *then* $(u(t), v(t))^T$ *tends to* $(0, B)^T$ *as* $t \to \infty$.

(iii) *Suppose that* $0 \leq B < 2(1-e^{-\tau})$ *and* $\tau \geq \ln 2$ *or* $0 \leq B \leq B_1^*$ *and* $\tau < \ln 2$. *Then* $(u(t), v(t))^T$ *approaches the periodic solution* $(-q(t), q(t))^T$ *as* $t \to \infty$, *where* B_1^* *is the unique positive zero of* $h(B)$.

(iv) Suppose that $B > 2(1 - e^{-\tau})$ *and* $\tau \leq \ln 2$ *or* $B \geq B_2^*$ *and* $\tau > \ln 2$. *Then* $(u(t), v(t))^T$ *tends to* $(0, B)^T$ *as* $t \to \infty$, *where* B_2^* *is the unique positive zero of* $h(B)$.

(v) Suppose that $B_1^* < B < 2(1 - e^{-\tau})$ *and* $\tau < \ln 2$. *Then there must exist* $T_1 \geq 0$ *and* $\Phi_1 = (\varphi_1, \psi_1)^T \in X^{-,+}$ *with* $\varphi_1(0) + \psi_1(0) > 0$ *such that for* $t \geq T_1$, *the solution* $\big(u^1(t), v^1(t)\big)^T$ *of (1.8) with initial value* Φ_1 *is*

periodic. Moreover, as $t \to \infty$, every other solution $(u(t), v(t))^T$ of system (1.8) with initial value $\Phi = (\varphi, \psi)^T \in X^{-,+}$ and $\varphi(0) + \psi(0) > 0$ either tends to $(0, B)^T$ or approaches the periodic solution $(-q(t), q(t))^T$.

(vi) *Suppose that $2(1 - e^{-\tau}) < B < B_2^*$ and $\tau > \ln 2$. Then there must exist $T_2 \geq 0$ and $\Phi_2 = (\varphi_2, \psi_2)^T \in X^{-,+}$ with $\varphi_2(0) + \psi_2(0) > 0$ such that for $t \geq T_2$, the solution $\big(u^2(t), v^2(t)\big)^T$ of (1.8) with initial value Φ_2 is periodic and the minimal period is $2\tau + \ln[(2 - 2e^{-\tau} - B)x_2^* + (1 - e^{-\tau})^2 + 3 - 2e^{-\tau}]$, where x_2^* is the positive zero of $g(x)$. Moreover, as $t \to \infty$, every solution $(u(t), v(t))^T$ of system (1.8) with initial value $\Phi = (\varphi, \psi)^T \in X^{-,+}$ and $\varphi(0) + \psi(0) > 0$ either tends to $(0, B)^T$ or approaches the periodic solution $\big(u^2(t), v^2(t)\big)^T$.*

(vii) *Suppose that $B = 1$ and $\tau = \ln 2$. If $\eta \in [0, M)$, then $(u(t), v(t))^T$ is eventually periodic; If $\eta \in [M, \infty)$, then $(u(t), v(t))^T \to (0, B)^T$ as $t \to \infty$.*

Guo, Huang and Wu [5] also showed that the semiflow defined by the system

$$\begin{aligned} \dot{u} &= -u + \frac{1+m}{2} f(u(t-\tau)) + \frac{1-m}{2} f(v(t-\tau)), \\ \dot{v} &= -v + \frac{1+m}{2} f(u(t-\tau)) + \frac{1-m}{2} f(v(t-\tau)). \end{aligned} \tag{1.9}$$

with $m > 0$ is topologically equivalent to that of (1.7)-(1.5) when one of the following four conditions are satisfied:

(B1) $a > 0$, $b > 0$, $c > 0$, $d > 0$, $ad = bc$
(B2) $a > 0$, $b < 0$, $c > 0$, $d < 0$, $ad = bc$
(B3) $a < 0$, $b > 0$, $c > 0$, $d < 0$, $ad = bc$
(B4) $a > 0$, $b > 0$, $c < 0$, $d < 0$, $ad = bc$.

Theorem 1.2 ([5]). *Every solution $(u(t), v(t))^T$ of system (1.9) with initial value $\Phi = (\varphi, \psi)^T \in X_0$ is either eventually periodic with minimal period ω or approaches the periodic solution $(q(t), q(t))^T$ as $t \to \infty$, where constant ω and ω-periodic function $q(t)$ are defined as in Theorem 1.1.*

In this paper, we consider the following cases:

(H1) $a \leq 0$, $b \leq 0$, $c \leq 0$, $d \geq 0$
(H2) $a > 0$, $b \leq 0$, $c \leq 0$, $d \geq 0$
(H3) $a \leq 0$, $b > 0$, $c \leq 0$, $d \geq 0$
(H4) $a \leq 0$, $b \leq 0$, $c \leq 0$, $d < 0$
(H5) $a \leq 0$, $b \leq 0$, $c > 0$, $d \geq 0$
(H6) $a < 0$, $b > 0$, $c > 0$, $d > 0$
(H7) $a > 0$, $b < 0$, $c < 0$, $d < 0$
(H8) $a < 0$, $b > 0$, $c < 0$, $d < 0$
(H9) $a > 0$, $b < 0$, $c > 0$, $d > 0$.

Let $(x(t), y(t))^T$ be a solution of (1.7) with initial value $\Phi \in X_0$. In this paper, we shall obtain the following results:

Theorem 1.3. *Suppose that (H1) holds. Then as $t \to \infty$, $(x(t), y(t))^T$ tends to $(-a, -b)^T$ provided $\Phi \in X^{+,+}$, to $(c, d)^T$ provided $\Phi \in X^{-,+}$, to $(a, b)^T$ provided $\Phi \in X^{-,-}$, and to $(-c, -d)^T$ provided $\Phi \in X^{+,-}$.*

Theorem 1.4. (i) *If (H2) holds, then as* $t \to \infty$, $(x(t), y(t))^T$ *tends to* $(c, d)^T$ *provided* $\Phi \in X^{+,+} \bigcup X^{-,+}$, *and to* $(-c, -d)^T$ *provided* $\Phi \in X^{+,-} \bigcup X^{-,-}$;

(ii) *If (H3) holds, then as* $t \to \infty$, $(x(t), y(t))^T$ *tends to* $(-c, -d)^T$ *provided* $\Phi \in X^{+,+} \bigcup X^{+,-}$, *and to* $(c, d)^T$ *provided* $\Phi \in X^{-,+} \bigcup X^{-,-}$;

(iii) *If (H4) holds, then as* $t \to \infty$, $(x(t), y(t))^T$ *tends to* $(-a, -b)^T$ *provided* $\Phi \in X^{+,+} \bigcup X^{+,-}$, *and to* $(-a, -b)^T$ *provided* $\Phi \in X^{-,+} \bigcup X^{-,-}$;

(iv) *If (H5) holds, then as* $t \to \infty$, $(x(t), y(t))^T$ *tends to* $(-a, -b)^T$ *provided* $\Phi \in X^{+,+} \bigcup X^{-,+}$, *and to* $(a, b)^T$ *provided* $\Phi \in X^{+,-} \bigcup X^{-,-}$.

Theorems 1.3 and 1.4 show that a simple network described by (1.7) can be used as an associative memory device because points representing the stored memories are locally stable in some sense, and from any initial state close to one of these attractors which represents partial knowledge of the memory stored at the attractor, the trajectory is driven by the system to the attractor, hence producing the full retrieval of the stored memory. By Theorems 1.3 and 1.4, system (1.7) has a point as the global attractor if we further restrict the parameters as follows:

Corollary 1.5. *Suppose that the parameters* a, b, c *and* d *satisfy one of the following conditions: (1)* $ab \leq 0$, $c = d = 0$; *(2)* $a = b = 0$, $cd \geq 0$. *Then trajectories of system (1.7) starting from non-oscillatory states converge to* $(0, 0)^T$.

We now consider the remaining cases.

Theorem 1.6. *If one of the two conditions (H6) and (H7) holds, then there exist* $\Phi_0 = (\varphi_0, \psi_0)^T \in X_0$ *and* $T_0 \geq 0$ *such that the solution* $(x^{\Phi_0}(t), y^{\Phi_0}(t))^T$ *of (1.7) with initial value* Φ_0 *is periodic for* $t \geq T_0$. *Moreover,* $\lim_{t\to\infty}[x^{\Phi}(t) - x^{\Phi_0}(t)] = 0$ *and* $\lim_{t\to\infty}[y^{\Phi}(t) - y^{\Phi_0}(t)] = 0$ *for every solution* $(x^{\Phi}(t), y^{\Phi}(t))^T$ *of (1.7) with* $\Phi = (\varphi, \psi)^T \in X_0$.

This theorem shows that when we restrict the initial value Φ to X_0, then system (1.7) has a unique limit cycle which is the global attractor. Note that this represents significant improvement over a corresponding theorem in [10]. The proof, elementary but technical, will be presented in Section 3. The basic idea is to show that a typical trajectory of (1.7), when described in the 2-dimensional Euclidean space (not the phase space), is spiraling and rotates round the point $(0, 0)$ (Section 2).

Theorem 1.7. *If one of the two conditions (H8) and (H9) holds, then there exist* $\Phi_0 = (\varphi_0, \psi_0)^T \in X_0$ *and* $T_0 \geq 0$ *such that the solution* $(x^{\Phi_0}(t), y^{\Phi_0}(t))^T$ *of (1.7) with initial value* Φ_0 *is periodic for* $t \geq T_0$, *and* $(-x^{\Phi_0}(t), -y^{\Phi_0}(t))^T$ *is also a solution of (1.7). Moreover, either* $\lim_{t\to\infty}[x^{\Phi}(t) - x^{\Phi_0}(t)] = 0$ *and* $\lim_{t\to\infty}[y^{\Phi}(t) - y^{\Phi_0}(t)] = 0$ *or* $\lim_{t\to\infty}[x^{\Phi}(t) + x^{\Phi_0}(t)] = 0$ *and* $\lim_{t\to\infty}[y^{\Phi}(t) + y^{\Phi_0}(t)] = 0$ *for every solution* $(x^{\Phi}(t), y^{\Phi}(t))^T$ *of (1.7) with* $\Phi = (\varphi, \psi)^T \in X_0$.

Therefore, system (1.7) may have two stable limit cycles. Moreover, if we restrict initial value Φ to X_0, then every solution of system (1.7) approaches one of the limit cycles as $t \to \infty$. Theorems 1.6 and 1.7 show that a simple two neuron model network is capable of producing and sustaining periodic behaviors. It is worthy of noticing that periodic sequences of neural impulses are of fundamental significance for the control of dynamic functions of the human body. Therefore, it is of great interest to understand various mechanisms of neural networks which cause and sustain such periodic activities.

2. Preliminary results

In this section, we establish several technical lemmas, which play important roles in the proof of our main results. For the sake of simplicity, for the remaining part of this paper, for a given $s \in [0, \infty)$ and a continuous function $z : [-\tau, \infty) \to \mathbb{R}$, we define $z_s : [-\tau, 0] \to \mathbb{R}$ by $z_s(\theta) = z(s + \theta)$ for $\theta \in [-\tau, 0]$.

Lemma 2.1. *The semiflow defined by model (1.7) with parameters a, b, c and d satisfying (H2) is topologically equivalent to that defined by model (1.7) with parameters a, b, c and d satisfying any one of (H3), (H4) and (H5).*

Proof. We consider only the topological equivalence between the semiflow defined by (1.7) under the condition (H2) and that defined by model (1.7) under condition (H3). The remaining cases can be dealt with analogously.

If (H2) holds, we can further redefine variables in (1.7) by

$$x^*(t) = y(t), \quad y^*(t) = x(t), \quad a^* = b, \quad b^* = a, \quad c^* = -d, \quad d^* = -c$$

and then drop the $*$ to get (1.7) where the new parameters a, b, c and d satisfy (H3). The converse holds true as well. This justifies the claimed equivalence, according to the definition of topological equivalence in [7]. We complete the proof of Lemma 2.1. ☐

Using similar arguments, we can also establish the following lemmas.

Lemma 2.2. *The semiflow defined by the system*

$$\begin{aligned} \dot{x} &= -x + \frac{1-A}{2} f(x(t-\tau)) - \frac{1+A}{2} f(y(t-\tau)), \\ \dot{y} &= -y + \frac{1+B}{2} f(x(t-\tau)) + \frac{1-B}{2} f(y(t-\tau)), \end{aligned} \tag{2.1}$$

with $A > 0$ and $B > 0$ is topologically equivalent to that defined by model (1.7) with parameters a, b, c and d satisfying either (H6) or (H7).

Lemma 2.3. *The semiflow defined by the system*

$$\begin{aligned} \dot{x} &= -x - \frac{1+M}{2} f(x(t-\tau)) + \frac{1-M}{2} f(y(t-\tau)), \\ \dot{y} &= -y - \frac{1-N}{2} f(x(t-\tau)) + \frac{1+N}{2} f(y(t-\tau)), \end{aligned} \tag{2.2}$$

with $M > 0$ and $N > 0$ is topologically equivalent to that defined by model (1.7) with parameters a, b, c and d satisfying either (H8) or (H9).

Lemma 2.4. *If $(x(t), y(t))^T$ is the solution of system (1.7)(or (2.1), (2.2)) with initial value $\Phi = (\varphi, \psi)^T \in X_0$, then the solution of (1.7)(respectively, (2.1), (2.2)) with initial value $\Phi = (-\varphi, -\psi)^T \in X_0$ is $(-x(t), -y(t))^T$.*

We now describe the transition from one component of X_0 to another.

Lemma 2.5. *Suppose that $(x(t), y(t))^T$ is a solution of (2.1) with initial value in X_0. Then*

(i) *if there exists some $t_0 \geq 0$ such that $\left(x_{t_0}, y_{t_0}\right)^T \in X^{+,+}$, then there exists some $t_0^* \geq t_0$ such that $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{+,-}$*

(ii) *if there exists some $t_0 \geq 0$ such that $\left(x_{t_0}, y_{t_0}\right)^T \in X^{+,-}$, then there exists some $t_0^* \geq t_0$ such that $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{-,-}$*

(iii) *if there exists some* $t_0 \geq 0$ *such that* $(x_{t_0}, y_{t_0})^T \in X^{-,-}$, *then there exists some* $t_0^* \geq t_0$ *such that* $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{-,+}$

(iv) *if there exists some* $t_0 \geq 0$ *such that* $(x_{t_0}, y_{t_0})^T \in X^{-,+}$, *then there exists some* $t_0^* \geq t_0$ *such that* $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{+,+}$.

Proof. We consider only the case where $(x_{t_0}, y_{t_0})^T \in X^{+,+}$ for some $t_0 \geq 0$, the remaining cases can be dealt with analogously. In view of (2.1), we have

$$\begin{aligned} \dot{x} &= -x + A, \\ \dot{y} &= -y - 1 \end{aligned} \tag{2.3}$$

for $t \in [t_0, t_0 + \tau]$ except at most finitely many t. Therefore, the variation-of-constants formula and the continuity of solutions yield

$$\begin{aligned} x(t) &= [x(t_0) - A]e^{-(t-t_0)} + A, \\ y(t) &= [y(t_0) + 1]e^{-(t-t_0)} - 1 \end{aligned} \tag{2.4}$$

for all $t \in [t_0, t_0 + \tau]$. Let t_1 be the first zero of $x(t) \cdot y(t)$ in (t_0, ∞). Then $(x(t), y(t))^T$ satisfies system (2.3) for $t \in (t_0, t_1 + \tau)$ except at most finitely many t, and so (2.4) holds for $t \in [t_0, t_1 + \tau]$. It follows from (2.4) that

$$t_1 = t_0 + \ln[y(t_0) + 1],$$

which implies that

$$\begin{aligned} x_{t_1+\tau}(\theta) &= \frac{x(t_0) - A}{y(t_0) + 1} e^{-(\tau+\theta)} + A > 0, \\ y_{t_1+\tau}(\theta) &= e^{-(\tau+\theta)} - 1 < 0 \end{aligned}$$

for $\theta \in (-\tau, 0]$, and so $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{+,-}$. Thus, claim (i) holds with $t_0^* = t_1$. ☐

Lemma 2.6. *Suppose that* $(x(t), y(t))^T$ *is a solution of (2.2) with initial value in* X_0. *Then*

(i) *if there exists some* $t_0 \geq 0$ *such that* $(x_{t_0}, y_{t_0})^T \in X^{+,+}$, *then there exists some* $t_0^* \geq t_0$ *such that* $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{+,-}$

(ii) *if there exists some* $t_0 \geq 0$ *such that* $(x_{t_0}, y_{t_0})^T \in X^{+,-}$, *then there exists some* $t_0^* \geq t_0$ *such that* $\left(x_{t_0^*+\tau}, y_{t_0^*+\tau}\right)^T \in X^{+,+}$.

Proof. (i) Using equations (2.2), (1.5) and the fact that $(x_{t_0}, y_{t_0})^T \in X^{+,+}$, we have

$$\begin{aligned} \dot{x} &= -x + M, \\ \dot{y} &= -y - N \end{aligned} \tag{2.5}$$

for $t \in [t_0, t_0 + \tau]$ except at most finitely many t. Therefore, the variation-of-constants formula and the continuity of solutions yield

$$\begin{aligned} x(t) &= [x(t_0) - M]e^{-(t-t_0)} + M, \\ y(t) &= [y(t_0) + N]e^{-(t-t_0)} - N \end{aligned} \tag{2.6}$$

for all $t \in [t_0, t_0 + \tau]$. Let t_1 be the first zero of $x(t) \cdot y(t)$ in (t_0, ∞). Then $(x(t), y(t))^T$ satisfies system (2.5) for $t \in (t_0, t_1 + \tau)$ except at most finitely many t, and so (2.6) holds for $t \in [t_0, t_1 + \tau]$. It follows from (2.6) that

$$t_1 = t_0 + \ln[y(t_0) + N] - \ln N,$$

which implies that

$$x_{t_1+\tau}(\theta) = \frac{x(t_0) - M}{y(t_0) + N} N e^{-(\tau+\theta)} + M > 0$$
$$y_{t_1+\tau}(\theta) = e^{-(\tau+\theta)} - 1 < 0$$

for $\theta \in (-\tau, 0]$, and so $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{+,-}$.
(ii) Using equations (2.1) and (1.5), as well as the fact that $(x_{t_0}, y_{t_0})^T \in X^{+,-}$, we have

$$\begin{aligned} \dot{x} &= -x + 1, \\ \dot{y} &= -y + 1 \end{aligned} \tag{2.7}$$

for $t \in [t_0, t_0 + \tau]$ except at most finitely many t. Therefore, the variation-of-constants formula and the continuity of solutions yield

$$\begin{aligned} x(t) &= [x(t_0) - 1]e^{-(t-t_0)} + 1, \\ y(t) &= [y(t_0) - 1]e^{-(t-t_0)} + 1 \end{aligned} \tag{2.8}$$

for all $t \in [t_0, t_0 + \tau]$. Let t_1 be the first zero of $x(t) \cdot y(t)$ in (t_0, ∞). Then $(x(t), y(t))^T$ satisfies system (2.7) for $t \in (t_0, t_1 + \tau)$ except at most finitely many t, and so (2.8) holds for $t \in [t_0, t_1 + \tau]$. It follows from (2.8) that

$$t_1 = t_0 + \ln[1 - y(t_0)],$$

which implies

$$x_{t_1+\tau}(\theta) = \frac{x(t_0) - 1}{1 - y(t_0)} e^{-(\tau+\theta)} + 1 > 0,$$
$$y_{t_1+\tau}(\theta) = 1 - e^{-(\tau+\theta)} > 0$$

for $\theta \in (-\tau, 0]$, and so $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{+,+}$. This completes the proof. □

In what follows, we will need the following continuous functions:

$$f_1(x) = \frac{(AB + 1 - e^{-\tau})e^{-\tau}x + (A+1)(AB+1)(1 - e^{-\tau})}{(B+1)e^{-\tau}x + (A+1)(B+1)(1 - e^{-\tau}) + 2e^{-\tau} - e^{-2\tau}}, \tag{2.9}$$

$$T(x) = 2\tau + \ln\left[(B+1)e^{-\tau}x + (A+1)(B+1)(1 - e^{-\tau}) + 2e^{-\tau} - e^{-2\tau}\right], \tag{2.10}$$

$$f_2(x) = A + \frac{B(x - A)e^{-2\tau} + B(1 - A)(1 + B)(1 - e^{-\tau})}{[1 + B(1 - e^{-\tau})](1 + B - e^{-\tau})} \tag{2.11}$$

for $x \in [0, \infty)$.

Lemma 2.7. *The function $f_1 : [0, \infty) \to \mathbb{R}$ is continuous and has a unique 2-period point x_1^* which is stable (that is $\lim_{n\to\infty} f_1^n(x) = x_1^*$ for all $x \in (0, \infty)$). Moreover, f_1 is monotonically increasing provided that $\max\{A, B\} < 1 - e^{-\tau}$ or $\min\{A, B\} > 1 - e^{-\tau}$, and is monotonically decreasing provided that $\min\{A, B\} < 1 - e^{-\tau} < \max\{A, B\}$.*

Proof. It is easy to see that f_1 is continuous and has a unique fixed point $x_1^* \in [0, \infty)$. We first consider the case where $\max\{A, B\} < 1 - e^{-\tau}$ or $\min\{A, B\} > 1 - e^{-\tau}$. Then it is easy to verify that f_1 is monotonically increasing, $f_1(x) > x$ for $x \in [0, x_1^*)$ and $f_1(x) < x$ for $x \in (x_1^*, \infty)$. Thus, the fixed point x_1^* is stable, i.e., $\lim_{n\to\infty} f_1^n(x) = x_1^*$ for all $x \in [0, \infty)$, where $f_1^n(x) = f_1\left(f_1^{n-1}(x)\right)$. It is obvious that x_1^* is also a 2-period point of $f_1(x)$, i.e., the fixed point of $f_1^2(x)$. We can claim that x_1^* is the unique 2-period point of $f_1(x)$. Suppose to the contrary, let

$u^* \in [0,\infty)$ be another 2-period point of $f_1(x)$, namely, $f_1^2(u^*) = u^* \neq x_1^*$. Since $\lim_{n\to\infty} f_1^n(u^*) = x_1^*$, let $n = 2k$, then

$$u^* = \lim_{k\to\infty} (f_1^2)^k(u^*) = \lim_{n\to\infty} f_1^{2k}(u^*) = x_1^*,$$

which is a contradiction. Therefore, The function f_1 has the unique 2-period point x_1^*. Also since $f_1^2(x) > f_1(x) > x$ for $x \in [0, x_1^*)$ and $f_1^2(x) < f_1(x) < x$ for $x \in (x_1^*, \infty)$, it is easy to see that x_1^* is the stable 2-period point of $f_1(x)$. On the other hand, if $\min\{A, B\} < 1 - e^{-\tau} < \max\{A, B\}$, then f_1 is monotonically decreasing. However, f_1^2 is monotonically increasing and has one and only one unique x_1^*, which implies that x_1^* is the stable 2-period point of $f_1(x)$. This completes the proof. $\square$

Lemma 2.8. *The function $f_2 : [0,\infty) \to \mathbb{R}$ is continuous, monotonically increasing and has a unique fixed point x_2^* which is stable.*

The proof of Lemma 2.8 is similar to that of Lemma 2.7 and thus it is omitted.

3. Proof of main results

Proof of Theorem 1.3. We consider only the case where $\Phi \in X^{+,+}$. The remaining cases can be dealt analogously. Using the definition of f and the fact $\Phi \in X^{+,+}$, $x(t)$ and $y(t)$ satisfy

$$\begin{aligned} \dot{x} &= -x - a, \\ \dot{y} &= -y - b \end{aligned} \tag{3.1}$$

for $t \in [0, \tau]$. Therefore, for $t \in [t_0, t_0 + \tau]$ we have

$$\begin{aligned} x(t) &= (\varphi(0) + a)e^{-t} - a, \\ y(t) &= (\psi(0) + b)e^{-t} - b, \end{aligned} \tag{3.2}$$

which implies that $x_\tau(\theta) = x(\tau + \theta) > 0$ and $y_\tau(\theta) = y(\tau + \theta) > 0$ for $\theta \in (-\tau, 0)$, and so $x_\tau \in C^+$ and $y_\tau \in C^+$. Repeating this argument on $[\tau, 2\tau]$, $[2\tau, 3\tau]$, $\cdots$, successively, we obtain that $x_t \in C^+$ and $y_t \in C^+$ for all $t \geq 0$. Therefore, (3.1) holds for almost all $t > 0$. It follows that $(x(t), y(t))^T \to (-a, -b)^T$ as $t \to \infty$. $\square$

Proof of Theorem 1.4. In view of Lemma 2.1, it suffices to consider system (1.7) under the condition (H2). We distinguish four cases in our discussions of the behaviors of a solution for (1.7).

Case 1. $\Phi \in X^{-,+}$. Using a similar argument to that of Theorem 1.3, we can show that $(x(t), y(t))^T \to (c, d)^T$ as $t \to \infty$.

Case 2. $\Phi \in X^{+,+}$. In view of the definition of $f(\xi)$, $x(t)$ and $y(t)$ satisfy (3.1) for $t \in (0, \tau)$. Therefore, (3.2) holds for $t \in [t_0, t_0 + \tau]$. Let t_1 be the first nonnegative zero of $x(t) \cdot y(t)$ on $[0, \infty)$. Then for $t \in (0, t_1 + \tau)$, (3.1) holds. Namely, (3.2) holds for $t \in [0, t_1 + \tau]$. In particular,

$$t_1 = \ln[\varphi(0) + a] - \ln a.$$

It follows that

$$\begin{aligned} x_{t_1+\tau}(\theta) &= ae^{-(\tau+\theta)} - a < 0, \\ y_{t_1+\tau}(\theta) &= \frac{\psi(0) + b}{\varphi(0) + a} ae^{-(\tau+\theta)} - b > 0 \end{aligned}$$

for $\theta \in (-\tau, 0]$, and so $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{-,+}$. Then from the result for Case 1, $(x(t), y(t))^T \to (c, d)^T$ as $t \to \infty$.

Case 3 $\Phi \in X^{-,+}$. From the result for Case 1 and by Lemma 2.4, it is easy to see that $(x(t), y(t))^T \to (-c, -d)^T$ as $t \to \infty$.
Case 4 $\Phi \in X^{-,-}$. From the result for Case 2 and by Lemma 2.4, it is easy to see that $(x(t), y(t))^T \to (-c, -d)^T$ as $t \to \infty$. Thus the proof of Theorem 1.4 is complete. □

Proof of Theorem 1.6. In view of Lemmas 2.2 and 2.5, it suffices to discuss the behavior of a solution $(x(t), y(t))^T$ of (2.1) with initial value $\Phi \in X^{++}$. For the sake of convenience, we introduce the parameter u:

$$u = \frac{\varphi(0) + A\psi(0)}{1 + \psi(0)}.$$

We can show that the behavior of the solution $(x(t), y(t))^T$ as $t \to \infty$ are completely determined by the value u. Let t_1 be the first zero of $x(t) \cdot y(t)$ on $[0, \infty)$, then from the proof of case (i) in Lemma 2.5 it follows that

$$\begin{gathered} t_1 = \ln(1 + \psi(0)), \quad x(t_1) = \frac{\varphi(0) + A\psi(0)}{1 + \psi(0)} = u \geq 0, \quad y(t_1) = 0, \\ x(t_1 + \tau) = (u - A)e^{-\tau} + A > 0, \quad y(t_1 + \tau) = e^{-\tau} - 1 < 0. \end{gathered}$$

Moreover, it is easy to see that $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{+,-}$. Therefore, for $t \in (t_1 + \tau, t_1 + 2\tau)$, we have $x(t - \tau) > 0$, $y(t - \tau) < 0$ and $(x(t), y(t))^T$ satisfies

$$\begin{aligned} \dot{x} &= -x - 1, \\ \dot{y} &= -y - B, \end{aligned} \tag{3.3}$$

from which and the continuity of the solution it follows that

$$\begin{aligned} x(t) &= [x(t_1 + \tau) + 1]e^{t_1+\tau-t} - 1 \\ &= [(u - A)e^{-\tau} + A + 1]e^{t_1+\tau-t} - 1, \\ y(t) &= [y(t_1 + \tau) + B]e^{t_1+\tau-t} - B \\ &= [B - 1 + e^{-\tau}]e^{t_1+\tau-t} - B \end{aligned} \tag{3.4}$$

for $t \in [t_1 + \tau, t_1 + 2\tau]$. Let t_2 be the second zero of $x(t) \cdot y(t)$ on $[0, \infty)$. Then for $t \in (t_1+\tau, t_2+\tau)$, $(x(t), y(t))^T$ satisfies (3.3). Thus, (3.4) holds for $t \in [t_1+\tau, t_2+\tau]$. It follows from (3.4) that

$$\begin{gathered} t_2 = t_1 + \tau + \ln[(u - A)e^{-\tau} + A + 1], \\ x(t_2 + \tau) = e^{-\tau} - 1 < 0, \\ y(t_2 + \tau) = \frac{B - 1 + e^{-\tau}}{(u - A)e^{-\tau} + A + 1}e^{-\tau} - B < 0. \end{gathered}$$

Moreover, it is easy to see that $(x_{t_2+\tau}, y_{t_2+\tau})^T \in X^{+,-}$. Therefore, for $t \in (t_2 + \tau, t_2 + 2\tau)$, we have $x(t - \tau) > 0$, $y(t - \tau) < 0$ and $(x(t), y(t))^T$ satisfies

$$\begin{aligned} \dot{x} &= -x - A, \\ \dot{y} &= -y + 1. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} x(t) &= [x(t_2+\tau)+A]e^{t_2+\tau-t}-A \\ &= [e^{-\tau}+A-1]e^{t_2+\tau-t}-A, \\ y(t) &= [y(t_2+\tau)-1]e^{t_2+\tau-t}+1 \\ &= \Big(\frac{B-1+e^{-\tau}}{(u-A)e^{-\tau}+A+1}e^{-\tau}-B-1\Big)e^{t_2+\tau-t}+1 \end{aligned} \tag{3.6}$$

for $t\in[t_2+\tau,t_2+2\tau]$. Let t_3 be the third zero of $x(t)\cdot y(t)$ on $[0,\infty)$. Then (3.5) holds for $t\in(t_2+\tau,t_3+\tau)$. Namely, (3.6) holds for $t\in[t_2+\tau,t_3+\tau]$. Thus, it follows from (3.6) that

$$\begin{aligned} t_3 &= t_2+\tau+\ln(1-y(t_2+\tau)) \\ &= t_1+2\tau+\ln\big[(B+1)e^{-\tau}x+(A+1)(B+1)(1-e^{-\tau})+2e^{-\tau}-e^{-2\tau}\big] \\ &= t_1+T(u) \end{aligned}$$

and

$$\begin{aligned} x(t_3) &= \frac{x(t_2+\tau)+Ay(t_2+\tau)}{1-y(t_2+\tau)} \\ &= \frac{(e^{-\tau}-1-AB)e^{-\tau}u+(A+1)(AB+1)(e^{-\tau}-1)}{(B+1)e^{-\tau}u+(A+1)(B+1)(1-e^{-\tau})+2e^{-\tau}-e^{-2\tau}} \\ &= -f_1(u)<0, \end{aligned}$$

$$y(t_3)=0,$$

where the function T and f_1 are defined as (2.10) and (2.9), respectively. Let t_4 and t_5 be the next zeroes of $x(t)\cdot y(t)$. Then from the above arguments and by Lemma 2.1, we have $(x_{t_4+\tau},y_{t_4+\tau})^T\in X^{+,+}$ and

$$\begin{gathered} t_5=t_3+T\left(f_1(u)\right)=t_1+T(u)+T\left(f_1(u)\right), \\ x(t_5)=f_1^2(u), \\ y(t_5)=0. \end{gathered}$$

Thus, we can repeat the same argument to get a sequence

$$u,\quad f_1(u),\quad f_1^2(u),\quad \cdots,\quad f_1^n(u),\quad \cdots.$$

In particular, the behavior of $(x(t),y(t))^T$ is determined by the iteration f_1. By Lemma 2.7, the function f_1 has a 2-period point x_1^*. Namely, $f_1^2(x_1^*)=x_1^*$. Let $(x^*(t),y^*(t))^T$ be a solution of (2.1) with the initial value $\Phi^*=(\varphi^*,\psi^*)^T\in X^{+,+}$ satisfying $[\varphi^*(0)+A\psi^*(0)]/[1+\psi^*(0)]=x_1^*$. Then for $t\ge\ln(1+\psi^*(0))$, $(x^*(t),y^*(t))^T$ is periodic with minimal period $\omega=T(x_1^*)+T\left(f_1(x_1^*)\right)=2T(x_1^*)$. Also since x_1^* is the stable 2-period point, it is obvious that the periodic solution $(x^*(t),y^*(t))^T$ is attractive, i.e., every solution of initial value problem (2.1) approaches $(x^*(t),y^*(t))^T$ as $t\to\infty$. Therefore, it is a stable limit cycle and its uniqueness is guaranteed by the uniqueness of the 2-period point of f_1. Thus, we complete the proof of Theorem 1.6. □

Proof of Theorem 1.7. In view of Lemmas 2.3 and 2.4, it suffices to discuss the behavior of a solution $(x(t),y(t))^T$ of (2.2) with initial value $\Phi\in X^{++}$. For the sake of convenience, we introduce the two parameters

$$u = M + \frac{\varphi(0) - M}{\psi(0) + N} N,$$
$$\omega = 2\tau + \ln[1 + N(1 - e^{-\tau})] + \ln(1 + N - e^{-\tau}) - \ln N.$$

We show that the behavior of the solution $(x(t), y(t))^T$ as $t \to \infty$ is completely determined by the value u. Let t_1 be the first zero of $x(t) \cdot y(t)$ on $[0, \infty)$, then from the proof of Lemma 2.6 (i) we have

$$\begin{gathered}
t_1 = \ln(\psi(0) + N) - \ln N, \\
x(t_1) = M + \frac{\varphi(0) - M}{\psi(0) + N} N = u \geq 0, \\
y(t_1) = 0, \\
x(t_1 + \tau) = (u - M)e^{-\tau} + M > 0, \\
y(t_1 + \tau) = Ne^{-\tau} - N < 0.
\end{gathered}$$

Moreover, it is easy to see that $(x_{t_1+\tau}, y_{t_1+\tau})^T \in X^{+,-}$. This, together with the proof of Lemma 2.6 (ii), implies that the second zero of $x(t) \cdot y(t)$ is

$$t_2 = t_1 + \tau + \ln[1 - y(t_1 + \tau)] = t_1 + \tau + \ln[1 + N(1 - e^{-\tau})].$$

Moreover,

$$\begin{aligned}
x(t) &= [x(t_1 + \tau) - 1]e^{t_1 + \tau - t} + 1 \\
&= [(u - M)e^{-\tau} + M - 1]e^{t_1 + \tau - t} + 1, \\
y(t) &= [y(t_1 + \tau) - 1]e^{t_1 + \tau - t} + 1 \\
&= [Ne^{-\tau} - N - 1]e^{t_1 + \tau - t} + 1
\end{aligned} \tag{3.7}$$

for all $t \in [t_1 + \tau, t_2 + \tau]$. It follows that

$$\begin{gathered}
x(t_2 + \tau) = \frac{(u - M)e^{-\tau} + M - 1}{1 + N(1 - e^{-\tau})} e^{-\tau} + 1 > 0, \\
y(t_2 + \tau) = 1 - e^{-\tau} > 0.
\end{gathered}$$

Moreover, it is easy to see that $(x_{t_2+\tau}, y_{t_2+\tau})^T \in X^{+,+}$. Again from the proof of Lemma 2.6 (i), the third zero of $x(t) \cdot y(t)$ is

$$\begin{aligned}
t_3 &= t_2 + \tau + \ln[y(t_2 + \tau) + N] - \ln N \\
&= t_1 + 2\tau + \ln[1 + N(1 - e^{-\tau})] + \ln(1 + N - e^{-\tau}) - \ln N \\
&= t_1 + \omega.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x(t_3) &= M + \frac{x(t_2 + \tau) - M)}{y(t_2 + \tau) + N} N \\
&= M + \frac{(u - M)e^{-2\tau} + (1 + N)(1 - M)(1 - e^{-\tau})}{[1 + N(1 - e^{-\tau})](1 + N - e^{-\tau})} N \\
&= f_2(u) > 0,
\end{aligned}$$
$$y(t_3) = 0,$$

where the function f_2 is defined as (2.11). Thus, we can repeat the same argument to get a sequence

$$u, \quad f_2(u), \quad f_2^2(u), \quad \cdots, \quad f_2^n(u), \quad \cdots.$$

Therefore, the behavior of $(x(t), y(t))^T$ as $t \to \infty$ is determined by the iteration of the function f_2. By Lemma 2.8, the function f_2 has a fixed point x_2^*. Namely, $f_2(x_2^*) = x_2^*$. Let $(x^*(t), y^*(t))^T$ be a solution of (2.2) with initial value $\Phi^* = (\varphi^*, \psi^*)^T \in X^{+,+}$ satisfying $M + N[\varphi^*(0) - M]/[\psi^*(0) + N] = x_2^*$. Then for $t \geq \ln(\psi^*(0) + N) - \ln N$, $(x^*(t), y^*(t))^T$ is periodic and is of the minimal period ω. Also since x_2^* is the stable fixed point, the periodic solution $(x^*(t), y^*(t))^T$ is attractive, i.e., every solution of (2.2) with initial value $\Phi = (\varphi, \psi)^T \in X^{+,+} \bigcup X^{+,-}$ approaches $(x^*(t), y^*(t))^T$ as $t \to \infty$. By Lemma 2.4, every solution of (2.2) with initial value $\Phi = (\varphi, \psi)^T \in X^{-,-} \bigcup X^{-,+}$ approaches the solution $(-x^*(t), -y^*(t))^T$ as $t \to \infty$. Thus, we complete the proof of Theorem 1.6. □

4. Conclusions

The model equation (1.1) with the McCulloch-Pitts nonlinearity (1.2) describes a combination of analog and digital signal processing in a network of two neurons with delayed feedback. For the sake of convenience, we can transform system (1.1)–(1.2) to the form (1.4)–(1.5) by the appropriate change of variables (1.3). Observe that the dynamics of the network completely depends on the connection weights, we distinguish several cases and discuss the behaviors of solutions of (1.4). We show that the dynamics of the model (1.4) can be understood in terms of the iterations of a one-dimensional map. As a result, we obtain the convergence of solutions as well as the existence, multiplicity and attractivity of periodic solutions. Throughout the paper, we only consider the case where the initial value $\Phi = (\varphi, \psi)^T \in X$ does not change sign at the initial time interval. Moreover, the digital nature of the sigmoid function allows us to relate equation (1.4) to four systems of simple linear nonhomogeneous ordinary differential equations. In future work, we shall describe the dynamics of solutions of (1.1)–(1.2) with initial data in $X \setminus X_0$ (i.e., solutions whose initial states oscillate around 0 with high frequencies).

Acknowledgments. S. Guo and L. Huang's research was partially supported by the Science Foundation of Hunan University, by the National Natural Science Foundation of P. R. China (10071016), by the Foundation for University Excellent Teacher by the Ministry of Education, and by the Key Project of Chinese Ministry of Education (No [2002]78). J. Wu's research was partially supported by the Natural Sciences and Engineering Research Council of Canada, by the Network of Centers of Excellence "Mathematics for Information Technology and Complex Systems", and by Canada Research Chairs Program. The authors express their sincere gratitude to the anonymous referee who carefully read the manuscript and made remarks leading to improvements on the presentation of this paper.

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