SIAM J. MATH. ANAL.
Vol. 34, No. 4, pp. 836–860

NONLINEAR WAVES IN NETWORKS OF NEURONS WITH DELAYED FEEDBACK: PATTERN FORMATION AND CONTINUATION*

LIHONG HUANG† AND JIANHONG WU‡

Abstract. An on-center off-surround network of three identical neurons with delayed feedback is considered, and the effect of synaptic delay of signal transmission on the pattern formation and global continuation of nonlinear waves is described. The spontaneous bifurcation of multiple branches of periodic solutions is discussed, and their spatio-temporal patterns and mode interactions are studied by using the symmetric bifurcation theory of delay differential equations coupled with representation theory of standard dihedral groups, Liapunov's direct method, LaSalle's invariance principle, a priori estimates, and differential inequalities.

Key words. wave, neural network, delay, bifurcation, global continuation

AMS subject classifications. 34K15, 34K20, 34C25, 92B20

PII. S0036141001386519

1. Introduction.

We consider the system of delay differential equations

(1.1)
$$\epsilon\dot{x}_j = -x_j(t) + h(x_j(t-1)) - [g(x_{j-1}(t-1)) + g(x_{j+1}(t-1)) - 2g(x_j(t-1))],$$

where $j = 1, 2, 3 (\text{mod } 3), \epsilon = \tau^{-1} > 0, \dot{x}_j(t) = \frac{d}{dt}x_j(t), h, g \in C^2(\mathbb{R};\mathbb{R})$ with $h(0) = g(0) = 0$, or, equivalently, we consider

$$\dot{x}_j(t) = -x_j(t) + f(x_j(t-\tau)) - [g(x_{j-1}(t-\tau)) + g(x_{j+1}(t-\tau))] \tag{1.2}$$

with $f = h + 2g$ and $\tau = \epsilon^{-1} > 0$.

Such a system models the evolution of a network of three identical neurons with delayed feedback. There are several reasons why we are particularly interested in such a system. First, if h and g are monotonically increasing, then the network modeled by (1.2) has the property that the self-feedback is excitatory (positive) and the feedback from other neurons is inhibitory (negative). This property is called the on-center off-surround characteristic of a network, and such networks have been found in a variety of neural structures such as neocortex [1], cerebellum [2], and hippocampus [3]. The network described by system (1.2) is of the minimal size among all possible networks with such an on-center off-surround characteristic, and examples of a network of three neurons include the basic rhythm generating circuits of central

*Received by the editors March 16, 2001; accepted for publication (in revised form) July 22, 2002; published electronically February 25, 2003.
http://www.siam.org/journals/sima/34-4/38651.html

†College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P.R. China (lhhuang@hnu.net.cn). The research of this author was partially supported by the National Natural Science Foundation of P.R. China (10071016), the Doctor Programm Foundation of the Ministry of Education of P.R. China (20010532002), and the Foundation for University Excellent Teacher by the Ministry of Education of P.R. China.

‡Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada (wujh@mathstat.yorku.ca). The research of this author was partially supported by the Natural Sciences and Engineering Research Council of Canada, by the Canada Research Chairs Program, and by the Network of Centers of Excellence: Mathematics for Information Technology and Complex Systems.

pattern generators [4, 5] and the canonical cortical circuit proposed in [1, 6]. See also [7] for the motivation of the study of small neural populations. Second, much progress has been made for the theory of dynamics (and, in particular, for the local bifurcation and global continuation of periodic solutions) of scalar delay differential equations (see, for example, [8, 9, 10]), and it is natural to see how the results and methods for scalar delay differential equations can be extended to systems of delay differential equations. Some progress has been made in this direction for a network of two neurons without self-feedback and with delayed interaction (see, for example, [11, 12, 13, 14]). An important factor to the progress in [11, 12, 13] is the fact that such a system can be changed to the so-called unidirectional cyclic system of delay differential equations to which the recently developed powerful theory of Mallet-Paret and Sell [15, 16] and the geometric method developed in [17] can be applied. System (1.2), however, is bidirectional in the sense that the growth rate for the ith cell (component) depends on the feedback from the $(i-1)$th and $(i+1)$th cells, and *both* with a delay. We hope this detailed case study can provide motivation for a more general geometric theory for the global dynamics of bidirectional cyclic systems of delay differential equations. Third, we would like to use this detailed case study to demonstrate how systems with time delay can be used for coupled oscillators. In particular, we note that, in the classical work (see [18] for references), for a ring of cells coupled by diffusion along the sides of a polygon, it was observed that if the coupling is instantaneous, then Hopf bifurcations occur only when the state of each cell is described by at least two variables, and our case study here provides an example in which a ring of cells coupled by delayed nonlinear diffusion exhibits multiple symmetric Hopf bifurcations even when the state of each cell is described by a single variable.

According to the Cohen–Grossberg–Hopfield convergence theorem [19, 20], under standard assumptions on the sigmoid signal functions h and g and if $\tau = 0$, then every solution of system (1.2) is convergent to the set of equilibria. Such a convergence has important applications to a number of areas such as content addressable memory and pattern identification. On the other hand, it was observed in [21] and later confirmed in a number of papers (see [14, 22, 23] and references therein) that large delay may cause nonlinear oscillations in the network. Most of these nonlinear oscillations appear in the form of periodic solutions with certain spatio-temporal structures and, if stable under small perturbation, may represent memory of the network to be stored and retrieved. Therefore, it is important to discuss the spatio-temporal patterns of these periodic solutions and to describe the mode interaction along multiple branches of such periodic solutions.

Needless to say, this is a very difficult task due to the infinite-dimensional nature of the problem caused by the synaptic delay and the possible spatial structure of the system (equivariant with respect to a D_3-action). Some general theorems are available about the existence and global continuation of periodic solutions in symmetric delay differential equations; see [23] for local bifurcation and [24] for global continuation. However, applications of these general results to concrete systems such as (1.1) involve several highly nontrivial tasks: (i) distribution of zeros in characteristic equations which are usually transcendental and depend on parameters; (ii) symmetry analysis on certain generalized eigenspaces of the generator of a linearized system and identification of these spaces with a direct sum of two identical absolute irreducible representations of D_3; (iii) calculation of the so-called crossing numbers which are related to the usual transversality condition in a standard Hopf bifurcation theory (see section 2 for details); (iv) a priori estimation of the period and of the norm of a periodic solution.

In this paper, we show the following:

(a) The model equation (1.1) is equivariant with respect to a D_3-action.

(b) There exists a sequence of critical values $\{\tau_k\}$ at which the linearization of (1.1) at the zero solution has a pair of purely imaginary eigenvalues.

(c) The generalized eigenspace of the above eigenvalues is four-dimensional and is the direct sum of two identical absolutely irreducible representations of D_3.

(d) Near each τ_k, there exist eight branches of periodic solutions, two of which are phase-locked, three are standing waves, and three are mirror-reflecting waves.

(e) These bifurcations of periodic solutions exist for all $\tau > \tau_k$ (global continuation); the branches of mirror-reflecting waves and the branches of phase-locked oscillations do not coincide, but coincidence of some branches of mirror-reflecting waves and some branches of standing waves may occur through periodic doubling.

The local bifurcation and the asymptotic forms of the aforementioned waves will be described in section 2, and their global continuation will be studied in section 3.

2. The local existence and asymptotic forms of waves. We start by stating a general result due to [23]. Let C denote the Banach space of continuous mappings from $[-1, 0]$ into $\mathbb{R}^n$ equipped with the supremum norm $||\phi|| = \sup_{-1\le\theta\le 0} |\phi(\theta)|$ for $\phi \in C$. In what follows, if $\sigma \in \mathbb{R},\ A \geq 0$, and $x : [\sigma - 1, \sigma + A] \to \mathbb{R}^n$ is a continuous mapping, then $x_t \in C,\ t \in [\sigma, \sigma + A]$, is defined by $x_t(\theta) = x(t + \theta)$ for $-1 \le \theta \le 0$.

Suppose that $F : C \to \mathbb{R}^n$ is C^2-smooth with $F(0) = 0$. Consider the delay differential equation

$$\dot{x}(t) = \tau F(x_t),$$

where $\tau > 0$. Let $L\phi = DF(0)\phi$ with $\phi \in C$. It is well known that the linear system

$$\dot{x}(t) = \tau L x_t$$

generates a strongly continuous semigroup of linear operators with an infinitesimal generator $A(\tau)$. Moreover, the spectrum $\sigma(A(\tau))$ of $A(\tau)$ consists of eigenvalues which are solutions of the characteristic equation

$$\det \Delta(\tau, \lambda) = 0, \quad \lambda \in \mathbb{C},$$

where $\mathbb{C}$ is the set of all complex numbers, and the characteristic matrix $\Delta(\tau, \lambda)$ is given by

$$\Delta(\tau, \lambda) = \lambda I_n - \tau L(e^{\lambda\cdot} I_n),$$

where I_n is the identity matrix on $\mathbb{C}^n$, $e^{\lambda\cdot}z$ is the mapping from $[-1, 0]$ into $\mathbb{C}^n$ given by $e^{\lambda\cdot}z(\theta) = e^{\lambda\theta}z$ for $z \in \mathbb{C}^n$ and $\theta \in [-1, 0]$, and $L(e^{\lambda\cdot} I_n) = (L(e^{\lambda\cdot}e_1), \ldots, L(e^{\lambda\cdot}e_n))$ with $(e_1, \ldots, e_n)$ being the standard basis of $\mathbb{R}^n$ and $L(e^{\lambda\cdot}e_j)$ the image of $e^{\lambda\cdot}e_j$ under the complexification of the linear mapping L for each $j = 1, \ldots, n$.

We assume the following.

(G1) The characteristic matrix is continuously differentiable in $\tau \in (0, \infty)$, and there exist $\tau_0 \in (0, \infty)$ and $\beta_0 > 0$ such that (i) $A(\tau_0)$ has eigenvalues $\pm i\beta_0$; (ii) the generalized eigenspace, denoted by $U_{(i\beta_0,-i\beta)}(A(\tau_0))$, of these eigenvalues $\pm i\beta_0$

consists of only eigenvectors of $A(\tau_0)$ associated with $\pm i\beta_0$; (iii) all other eigenvalues of $A(\tau_0)$ are not integer multiples of $\pm i\beta_0$.

To state the next assumption that describes the possible (spatial) symmetry of the system considered, we need to introduce some group-theoretic preliminaries. We refer to [18, 25] for more details.

In what follows, by a (compact) Lie group Γ, we mean a closed subgroup of $GL(\mathbb{R}^n)$, the group of all invertible linear transformations of the vector space $\mathbb{R}^n$ into itself. Note that the space of $n\times n$ matrices may be identified with $\mathbb{R}^{n^2}$, which contains $GL(\mathbb{R}^n)$ as an open subset. We say that Γ is a closed subgroup of $GL(\mathbb{R}^n)$ if it is a closed subset of $GL(\mathbb{R}^n)$ as well as a subgroup of $GL(\mathbb{R}^n)$. A specific example of a Lie group is the special orthogonal group $SO(n)$ that consists of all $n \times n$ matrices A such that $AA^T = I_n$ and $\det A = 1$, where A^T is the transpose of A. In particular, $SO(2)$ consists precisely of the planar rotations

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In this way, $SO(2)$ may be identified with the circle group S^1, the identification being $R_\theta \to e^{i\theta}$. Two other Lie groups will be used in this paper. The first is Z_n, the cyclic group of order n. (The order of a finite group is the number of elements that it contains.) The second is the dihedral group D_n of order $2n$ that is generated by Z_n together with an element (called the flip) of order 2 that does not communicate with Z_n.

Let V be a topological vector space over the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$, and let $GL(V)$ be the group of isomorphisms of V onto itself. We say that a compact Lie group Γ acts on V if there is a continuous mapping $\Gamma \times V \ni (\gamma, v) \mapsto \gamma \cdot v \in V$ such that (a) for each $\gamma \in \Gamma$, the mapping $\rho_\gamma : V \to V$ given by $\rho_\gamma(v) = \gamma \cdot v$ is linear; (b) if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1\gamma_2) \cdot v$. The mapping that sends $\gamma \in \Gamma$ to $\rho_\gamma \in GL(V)$ is called a representation of Γ on V. In what follows, we shall write γv for $\gamma \cdot v$ for all $\gamma \in \Gamma$ and $v \in V$.

If Γ acts on both V and W and if there is a linear isomorphism $A : V \to W$ such that $A(\gamma v) = \gamma(Av)$ for all $v \in V$ and $\gamma \in \Gamma$, then we say the Γ actions on V and W are isomorphic, and such a linear isomorphism is called a Γ-isomorphism.

Let Γ act on V, and let W be a subspace of V. We say that W is Γ-invariant if $\gamma w \in W$ for every $\gamma \in \Gamma$ and $w \in W$. We thus obtain a Γ-action on W called the restricted action of Γ on W.

Finally, if Γ acts on V, we say a linear mapping $F : V \to V$ is Γ-equivariant if $F(\gamma v) = \gamma F(v)$ for all $\gamma \in \Gamma$ and $v \in V$. A representation of Γ on V is absolutely irreducible if the only linear mappings on V that are Γ-equivariant are scalar multiples of the identity. A Γ-invariant subspace W of V is Γ-irreducible if the only invariant subspaces of W are $\{0\}$ and W. It is known that up to a Γ-isomorphism there are only a finite number of distinct Γ-irreducible subspaces, denoted by $U_1, \dots, U_s$. If we define W_k as the sum of all Γ-irreducible subspaces of V that are Γ-isomorphic to U_k, then $V = W_1 \oplus \cdots W_s$, and this is called an isotypical decomposition of V. In the case in which $\Gamma = Z_N$ and $V = \mathbb{C}^n$ for two fixed positive integers n and N, every irreducible subspace must be one-dimensional, and the restricted action of Γ to any such irreducible subspace is Γ-isomorphic to the Γ-action on $\mathbb{C}$ defined by $\rho \cdot z = \rho^j z$ for some nonnegative integer j and for all $z \in \mathbb{C}$, where ρ is the generator of $Z_N \leq S^1$.

With this short introduction to group-theoretic preliminaries, we can now state the next set of assumptions.

(G2) There exists a compact Lie group Γ acting on $\mathbb{R}^n$ such that F is Γ-equivariant; i.e., $F(\gamma\phi) = \gamma F(\phi)$ for $(\gamma, \phi) \in \Gamma \times C$, where $\gamma\phi \in C$ is given by $(\gamma\phi)(\theta) = \gamma\phi(\theta),\ \theta \in [-1, 0]$.

Note that the real Γ-action on $\mathbb{R}^n$ can be naturally extended to a Γ-action on $\mathbb{C}^n$ by

$$\gamma(u + iv) = \gamma u + i\gamma v, \qquad \gamma \in \Gamma, u, v \in \mathbb{R}^n.$$

This action is called the complexification of the Γ-action on $\mathbb{R}^n$. In what follows, we will simply call the complexification of Γ on $\mathbb{C}^n$ the Γ-action on $\mathbb{C}^n$. Due to the Γ-equivariance of F, we can easily show that $\mathrm{Ker}\Delta(\tau_0, i\beta_0)$ is an invariant subspace of $\mathbb{C}^n$ with respect to the complexification of the Γ-action on $\mathbb{R}^n$. We need the following assumption.

(G3) There exists a real m-dimensional absolutely irreducible representation of Γ on V such that the restricted action of Γ on $\mathrm{Ker}\Delta(\tau_0, i\beta_0)$ is isomorphic to the action of Γ on $V \oplus V$ defined by $\gamma(v_1, v_2) = (\gamma v_1, \gamma v_2)$ for $\gamma \in \Gamma, v_1, v_2 \in V$.

Let $\{b_{j1} + ib_{j2}\}_{j=1}^m$ be a basis for $\mathrm{Ker}\Delta(\tau_0, i\beta_0)$, and for any $\beta > 0$ define $\sin_\beta,\ \cos_\beta \in \mathrm{C}([-1, 0]; \mathbb{R})$ by

$$\sin_\beta(\theta) = \sin(\beta\theta),\ \cos_\beta(\theta) = \cos(\beta\theta),\ \theta \in [-1, 0].$$

Then the columns of $\Phi_{\tau_0} = (\varepsilon_1, \ldots, \varepsilon_{2m})$ form a basis for $U_{(i\beta_0, -i\beta_0)}(A(\tau_0))$, where

$$\begin{aligned}\varepsilon_j &= \sin_{\beta_0} b_{\mathrm{j}1} + \cos_{\beta_0} b_{\mathrm{j}2},\\ \varepsilon_{m+j} &= \cos_{\beta_0} b_{\mathrm{j}1} - \sin_{\beta_0} b_{\mathrm{j}2}, \quad 1 \le j \le m.\end{aligned}$$

It can be shown (see Lemma 2.1 of [23]) that there exist $\delta_0 > 0$ and a continuously differentiable function $\lambda : (\tau_0 - \delta_0, \tau_0 + \delta_0) \to \mathbb{C}$ such that $\lambda(\tau_0) = i\beta_0,\ \lambda(\tau)$ is an eigenvalue of $A(\tau)$, $U_{(\lambda(\tau), \overline{\lambda(\tau)})}(A(\tau))$ consists of eigenvectors of $A(\tau)$ associated with these eigenvalues, and $\dim U_{(\lambda(\tau), \overline{\lambda(\tau)})}(A(\tau)) = \dim U_{(i\beta_0, -i\beta_0)}(A(\tau_0))$.

We will require the following transversality condition.

(G4) $\frac{d}{d\tau}\mathrm{Re}\lambda(\tau) \mid_{\tau=\tau_0} \neq 0$.

Let $\omega = \frac{2\pi}{\beta_0}$. Denote by P_ω the Banach space of all continuous ω-periodic mappings $x : \mathbb{R} \to \mathbb{R}^n$. Then $\Gamma \times S^1$ acts on P_ω by

$$(\gamma, e^{i\theta})x(t) = \gamma x\left(t + \frac{\theta}{2\pi}\omega\right), \quad (\gamma, e^{i\theta}) \in \Gamma \times S^1,\ x \in P_\omega.$$

Denote by SP_ω the subspace of P_ω consisting of all ω-periodic solutions of $\dot{x}(t) = \tau_0 L x_t$. Then, for each subgroup $\Sigma \le \Gamma \times S^1$, the fixed point set

$$Fix(\Sigma, SP_\omega) = \{x \in SP_\omega; \quad (\gamma, \theta)x = x \text{ for all } (\gamma, \theta) \in \Sigma\}$$

is a subspace.

Under assumption (G1), the columns of $U(t) = \Phi_{\tau_0}(0)e^{B(\tau_0)t},\ t \in \mathbb{R}$, form a basis for SP_ω, where

$$B(\tau_0) = \begin{pmatrix} 0 & -\beta_0 I_m \\ \beta_0 I_m & 0 \end{pmatrix}.$$

Also, SP_ω is a $\Gamma \times S^1$-invariant subspace of P_ω (see Lemma 2.3 of [23]). We can now state the general symmetric local Hopf bifurcation theorem (Theorem 2.1 of [23]).

LEMMA 2.1. *Assume that* (G1)–(G4) *are satisfied and* $\dim Fix(\Sigma, SP_\omega) = 2$ *for some* $\Sigma \le \Gamma \times S^1$. *Then, for a chosen basis* $\{\delta_1, \delta_2\}$ *of* $Fix(\Sigma, SP_\omega)$, *there exist constants* $a_0 > 0$, $\tau_0^* > 0$, $\sigma_0 > 0$, C^1*-smooth functions* $\tau^* : \mathbb{R}^2_{a_0} \to \mathbb{R}$, $\omega^* : \mathbb{R}^2_{a_0} \to (0, \infty)$, *and a* C^1*-smooth mapping* $x^* : \mathbb{R}^2_{a_0} \to C(\mathbb{R}; \mathbb{R}^n)$, *where* $\mathbb{R}^2_{a_0} = \{a \in \mathbb{R}^2; |a| < a_0\}$ *and* $C(\mathbb{R}; \mathbb{R}^n)$ *is the Banach space of all continuous mappings from* $\mathbb{R}$ *into* $\mathbb{R}^n$ *equipped with the supremum norm such that, for each* $a \in \mathbb{R}^2_{a_0}$, $x^*(a)$ *is an* $\omega^*(a)$*-periodic solution of* $\dot{x}(t) = \tau F(x_t)$ *with* $\tau = \tau^*(a)$, *and*

$$\gamma x^*(a)(t) = x^*(a)\left(t - \frac{\omega^*(a)}{\omega}\theta\right), \quad (\gamma, \theta) \in \Sigma,$$
$$x^*(0) = 0,\ \omega^*(0) = \omega,\ \tau^*(0) = \tau_0^*,$$
$$x^*(a) = (\delta_1, \delta_2)a + o(|a|) \text{ as } |a| \to 0.$$

Furthermore, for $|\tau - \tau_0| < \tau_0^*$, $|\tilde{\omega} - \frac{2\pi}{\beta_0}| < \sigma_0$, *every* $\tilde{\omega}$*-periodic solution of* $\dot{x}(t) = \tau F(x_t)$ *with* $\|x_t\| < \sigma_0$, $\gamma x(t) = x(t - \frac{\tilde{\omega}}{\omega}\theta)$ *for* $(\gamma, \theta) \in \Sigma$, *and* $t \in \mathbb{R}$ *must be of the above type.*

We now consider the system (1.1). It arises from

$$\dot{y}_j(t) = -y_j(t) + h(y_j(t-\tau)) - [g(y_{j-1}(t-\tau)) + g(y_{j+1}(t-\tau)) - 2g(y_j(t-\tau))]$$

with $\epsilon = \tau^{-1}$ and by the change of variable $x_j(t) = y_j(\tau t)$. We will apply Lemma 2.1 to (1.1) with $F : C \to \mathbb{R}^3$ by

$$(F(\phi))_j = -\phi_j(0) + h(\phi_j(-1)) - [g(\phi_{j-1}(-1)) + g(\phi_{j+1}(-1)) - 2g(\phi_j(-1))]$$

for $\phi \in C := C([-\tau, 0]; \mathbb{R}^3)$ and $j (\text{mod } 3)$.

PROPOSITION 2.2. *Let* $\Gamma = D_3$ *be the dihedral group of order* 2×3. *Denote by* ρ *the generator of the cyclic subgroup* $Z_3 \le D_3$ *and by* κ *the flip. Define the action of* Γ *on* $\mathbb{R}^3$ *by*

$$\begin{cases} (\rho x)_j = x_{j+1}, \quad j (\text{mod } 3), \\ (\kappa x)_2 = x_3, (\kappa x)_3 = x_2, (\kappa x)_1 = x_1, x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

Then F *is* Γ*-equivariant.*

Proof. For $\phi \in C$ and $j (\text{mod } 3)$, we have

$$\begin{aligned}
&(F(\rho\phi))_j \\
&= -(\rho\phi)_j(0) + h((\rho\phi)_j(-1)) - [g((\rho\phi)_{j-1}(-1)) + g((\rho\phi)_{j+1}(-1)) - 2g((\rho\phi)_j(-1))] \\
&= -\phi_{j+1}(0) + h(\phi_{j+1}(-1)) - [g(\phi_j(-1)) + g(\phi_{j+2}(-1)) - 2g(\phi_{j+1}(-1))] \\
&= ((\rho F)(\phi))_j
\end{aligned}$$

and

$$\begin{aligned}
&(F(\kappa\phi))_1 \\
&= -(\kappa\phi)_1(0) + h((\kappa\phi)_1(-1)) - [g((\kappa\phi)_3(-1)) + g((\kappa\phi)_2(-1)) - 2g((\kappa\phi)_1(-1))] \\
&= -\phi_1(0) + h(\phi_1(-1)) - [g(\phi_2(-1)) + g(\phi_3(-1)) - 2g(\phi_1(-1))] \\
&= ((\kappa F)(\phi))_1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&(F(\kappa\phi))_2 \\
&= -(\kappa\phi)_2(0) + h((\kappa\phi)_2(-1)) - [g((\kappa\phi)_1(-1)) + g((\kappa\phi)_3(-1)) - 2g((\kappa\phi)_2(-1))] \\
&= -\phi_3(0) + h(\phi_3(-1)) - [g(\phi_1(-1)) + g(\phi_2(-1)) - 2g(\phi_3(-1))] \\
&= ((\kappa F)(\phi))_2.
\end{aligned}$$

Similarly, $(F(\kappa\phi))_3 = ((\kappa F)(\phi))_3$. This completes the proof. □

Let

$$\gamma = h'(0), \qquad \beta = g'(0). \tag{2.2}$$

Then the linearization of (1.1) at $x = 0 \in \mathbb{R}^3$ is

$$\frac{1}{\tau}\dot{X}_j(t) = -X_j(t) + \gamma X_j(t-1) - \beta[X_{j-1}(t-1) + X_{j+1}(t-1) - 2X_j(t-1)], \tag{2.3}$$

where $j = 1, 2, 3(\mathrm{mod}\, 3)$. The characteristic equation takes the form

$$\det \Delta(\tau, \lambda) = 0,$$

where

$$\Delta(\tau, \lambda) = (\lambda + \tau)I_3 - \tau M e^{-\lambda}, \quad \lambda \in \mathbb{C}, \tag{2.4}$$

and

$$M = \begin{pmatrix} \gamma + 2\beta & -\beta & -\beta \\ -\beta & \gamma + 2\beta & -\beta \\ -\beta & -\beta & \gamma + 2\beta \end{pmatrix}. \tag{2.5}$$

Proposition 2.3. $\det \Delta(\tau, \lambda) = (\lambda + \tau - \gamma\tau e^{-\lambda})[\lambda + \tau - (\gamma + 3\beta)\tau e^{-\lambda}]^2$.

Proof. Let $\chi = e^{i\frac{2\pi}{3}}$ and

$$v_k = (1, \chi^k, \chi^{2k})^T, \quad k = 0, 1, 2. \tag{2.6}$$

Clearly, $v_0 = (1, 1, 1)^T$ and $v_2 = \overline{v}_1$. Let

$$\mathbb{C}_k = \{v_k z; z \in \mathbb{C}\}, \quad k = 0, 1, 2.$$

Then

$$\mathbb{C}^3 = \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$$

and

$$\begin{aligned}
&(\Delta(\tau, \lambda)v_k)_j \\
&= (\lambda + \tau - (\gamma + 2\beta)\tau e^{-\lambda})(v_k)_j + \tau\beta e^{-\lambda}(e^{i\frac{2\pi}{3}k} + e^{-i\frac{2\pi}{3}k})(v_k)_j \\
&= \left[\lambda + \tau - \tau(\gamma + 2\beta)e^{-\lambda} + 2\beta\tau\cos\left(\frac{2\pi}{3}k\right)e^{-\lambda}\right](v_k)_j \\
&= \left[\lambda + \tau - \left(\gamma + 4\beta\sin^2\left(\frac{\pi}{3}k\right)\right)\tau e^{-\lambda}\right](v_k)_j.
\end{aligned}$$

That is,

$$\begin{aligned}
&\Delta(\tau, \lambda)|_{\mathbb{C}_k} \\
&= \lambda + \tau - \left(\gamma + 4\beta\sin^2\left(\frac{\pi}{3}k\right)\right)\tau e^{-\lambda} \\
&= \begin{cases} \lambda + \tau - \gamma\tau e^{-\lambda} & \text{if } k = 0, \\ \lambda + \tau - (\gamma + 3\beta)\tau e^{-\lambda} & \text{if } k = 1, 2. \end{cases}
\end{aligned}$$

This completes the proof. □

We now make the following assumption.

(H1) $|\gamma| < 1, \gamma + 3\beta > 1$.

The critical values of τ where the characteristic equation has purely imaginary zeros are described in the following.

PROPOSITION 2.4. *Let $A(\tau)$ denote the infinitesimal generator of the semigroup generated by system* (2.3). *Assume that* (H1) *is satisfied. Define*

$$\begin{cases} \beta_k = 2k\pi - \arccos\dfrac{1}{\gamma+3\beta}, \\ \tau_k = -\beta_k \cot\,\beta_k, \quad k \geq 1. \end{cases}$$

Then the following hold.

(i) *For every fixed $\tau \geq 0$, all zeros of $\lambda + \tau - \gamma\tau e^{-\lambda}$ have negative real parts.*

(ii) *At (and only at) $\tau = \tau_k$, $A(\tau)$ has purely imaginary eigenvalues. These eigenvalues are given by $\pm i\beta_k$ with $\beta_k \in (2k\pi - \frac{\pi}{2}, 2k\pi)$.*

(iii) *All other eigenvalues of $A(\tau_k)$ are not integer multiples of $\pm i\beta_k$.*

(iv) *The generalized eigenspace $U_{(i\beta_k,-i\beta_k)}(A(\tau_k))$ consists of eigenvectors of $A(\tau_k)$ associated with $\pm i\beta_k$ only and*

$$U_{(i\beta_k,-i\beta_k)}(A(\tau_k)) = \left\{\sum_{i=1}^{4} x_i\epsilon_i; \quad x_i \in \mathbb{R}, i = 1,\dots,4\right\},$$

where, for $\theta \in [-1, 0]$,

$$\begin{aligned}
\epsilon_1(\theta) &= \mathrm{Re}(e^{i\beta_k\theta}v_1) = \cos(\beta_k\theta)\mathrm{Re}v_1 - \sin(\beta_k\theta)\mathrm{Im}v_1,\\
\epsilon_2(\theta) &= \mathrm{Im}(e^{i\beta_k\theta}v_1) = \sin(\beta_k\theta)\mathrm{Re}v_1 + \cos(\beta_k\theta)\mathrm{Im}v_1,\\
\epsilon_3(\theta) &= \mathrm{Re}(e^{i\beta_k\theta}v_2) = \cos(\beta_k\theta)\mathrm{Re}v_1 + \sin(\beta_k\theta)\mathrm{Im}v_1,\\
\epsilon_4(\theta) &= \mathrm{Re}(e^{i\beta_k\theta}v_2) = \sin(\beta_k\theta)\mathrm{Re}v_1 - \cos(\beta_k\theta)\mathrm{Im}v_1.
\end{aligned}$$

Proof. (i) Let $\lambda = u + iv$ be a zero of $\lambda + \tau - \gamma\tau e^{-\lambda}$. Then we get $v = -\gamma\tau e^{-u}\mathrm{sinv}$ and $u + \tau = \gamma\tau e^{-u}\mathrm{cosv}$, from which it follows that

$$\gamma^2\tau^2 e^{-2u} = v^2 + (u + \tau)^2.$$

Consequently, $u < 0$, for otherwise the left-hand side of the above equality is strictly less than τ^2, while the right-hand side is larger than or equal to τ^2.

To verify (ii)–(iv), let $\lambda = iv$ with $v > 0$ be a solution of $\lambda + \tau - (\gamma+3\beta)\tau e^{-\lambda} = 0$. Then

$$\begin{cases} \tau = (\gamma + 3\beta)\tau\cos v, \\ v = -(\gamma + 3\beta)\tau\sin v. \end{cases}$$

So

$$\tan v = -\frac{v}{\tau},$$

from which it follows that $\tan v < 0$, and hence $v \notin [0, \frac{\pi}{2}] + \mathbb{Z}\pi$; here $\mathbb{Z}$ is the set of all integers. Therefore, we must have

$$v = 2k\pi - \arccos\frac{1}{\gamma + 3\beta} = \beta_k, \quad k \geq 1,$$

and

$$\tau = -\beta_k \cot \beta_k = \tau_k.$$

Therefore, $\lambda + \tau - (\gamma + 3\beta)\tau e^{-\lambda} = 0$ has purely imaginary roots (given by $i\beta_k$) if and only if $\tau = \tau_k$ for some $k \geq 1$.

It is well known that $\phi \in C([-1, 0]; \mathbb{C}^3)$ is an eigenvector of $A(\tau_k)$ associated with the eigenvalue $i\beta_k$ if and only if $\phi(\theta) = e^{i\beta_k\theta}z, -1 \leq \theta \leq 0$, for some vector $z \in \mathbb{C}^3$ such that $\Delta(\tau_k, i\beta_k)z = 0$ (see, for example, pp. 198 in [9]). From the proof of Proposition 2.3, we then have $v \in \langle v_1, v_2\rangle$, the complex space spanned by v_1 and v_2. Similar arguments apply to $-i\beta_k$. Therefore, the eigenspace of $A(\tau_k)$ associated with $\pm i\beta_k$ is spanned by $e^{i\beta_k\theta}v_1, e^{i\beta_k\theta}v_2, e^{-i\beta_k\theta}v_1$, and $e^{-i\beta_k\theta}v_2$. Therefore, this space has the real basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. On the other hand, the eigenspace of $A(\tau_k)$ associated with $i\beta_k$ is of dimension 2 and the algebraic multiplicity of $\lambda = i\beta_k$ as a zero of $\det \Delta(\tau_k, \lambda) = 0$ is also 2. So the well-known folk theorem in functional differential equations (see [26] or Theorem 4.2 in [9]) implies that $U_{(i\beta_k, -i\beta_k)}(A(\tau_k))$ must coincide with the eigenspace of $A(\tau_k)$ associated with $\pm i\beta_k$. This completes the proof. ◻

PROPOSITION 2.5. *Let* $\Gamma = D_3$ *act on* $\mathbb{R}^2$ *by*

$$\rho\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$\kappa\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

Then $\mathbb{R}^2$ *is an absolutely irreducible representation of* Γ, *and the restricted action of* Γ *on* $\mathrm{Ker}\Delta(\tau_k, i\beta_k)$ *is isomorphic to the action of* Γ *on* $\mathbb{R}^2 \oplus \mathbb{R}^2$.

Proof. The proof for the absolute irreducibility of the representation of Γ on $\mathbb{R}^2$ is straightforward and can be found in, for example, [18]. Clearly,

$$\mathrm{Ker}\Delta(\tau_k, i\beta_k) = \{(x_1 + ix_2)v_1 + (x_3 + ix_4)v_2; \quad x_i \in \mathbb{R}, i = 1, \ldots, 4\}.$$

Define

$$J((x_1 + ix_2)v_1 + (x_3 + ix_4)v_2) = (x_1 + x_3, x_2 - x_4, x_2 + x_4, x_3 - x_1)^T.$$

Clearly, $J : \mathrm{Ker}\Delta(\tau_k, i\beta_k) \cong \mathbb{R}^4$ is a linear isomorphism. Note that

$$\begin{aligned}
&\rho[(x_1 + ix_2)v_1 + (x_3 + ix_4)v_2] \\
&= (x_1 + ix_2)e^{i\frac{2\pi}{3}}v_1 + (x_3 + ix_4)e^{-i\frac{2\pi}{3}}v_2 \\
&= \left[\left(-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2\right) + i\left(-\frac{1}{2}x_2 + \frac{\sqrt{3}}{2}x_1\right)\right]v_1 \\
&\quad + \left[\left(-\frac{1}{2}x_3 + \frac{\sqrt{3}}{2}x_4\right) + i\left(-\frac{1}{2}x_4 - \frac{\sqrt{3}}{2}x_3\right)\right]v_2
\end{aligned}$$

and

$$\kappa[(x_1 + ix_2)v_1 + (x_3 + ix_4)v_2] = (x_1 + ix_2)v_2 + (x_3 + ix_4)v_1.$$

Therefore,

$$\begin{aligned}
&J(\rho[(x_1+ix_2)v_1+(x_3+ix_4)v_2]) \\
&= \bigg(-\frac{1}{2}(x_1+x_3)-\frac{\sqrt{3}}{2}(x_2-x_4), -\frac{1}{2}(x_2-x_4)+\frac{\sqrt{3}}{2}(x_1+x_3), \\
&\quad -\frac{1}{2}(x_2+x_4)-\frac{\sqrt{3}}{2}(x_3-x_1), -\frac{1}{2}(x_3-x_1)+\frac{\sqrt{3}}{2}(x_2+x_4)\bigg)^T \\
&= \rho J((x_1+ix_2)v_1+(x_3+ix_4)v_2)
\end{aligned}$$

and

$$\begin{aligned}
&J(\kappa[(x_1+ix_2)v_1+(x_3+ix_4)v_2]) \\
&= (x_3+x_1, x_4-x_2, x_4+x_2, x_1-x_3)^T \\
&= \kappa J[(x_1+ix_2)v_1+(x_3+ix_4)v_2].
\end{aligned}$$

This completes the proof. ∎

PROPOSITION 2.6. *For each fixed $k \geq 1$, there exist $\delta_k > 0$ and a C^1-mapping $\lambda_k : (\tau_k-\delta_k, \tau_k+\delta_k) \to \mathbb{C}$ such that $\lambda_k(\tau_k) = i\beta_k$ and $\lambda_k(\tau)+\tau-(\gamma+3\beta)\tau e^{-\lambda_k(\tau)} = 0$ for all $\tau \in (\tau_k-\delta_k, \tau_k+\delta_k)$. Moreover, $\frac{d}{d\tau}\mathrm{Re}\lambda_k(\tau)|_{\tau=\tau_k} > 0$.*

Proof. The existence of δ_k and the mapping λ_k follow from the implicit function theorem. We now substitute $\lambda = \lambda_k(\tau)$ into $\lambda+\tau-(\gamma+3\beta)\tau e^{-\lambda} = 0$, differentiating the equality with respect to τ, to get

$$\begin{aligned}
&\frac{d}{d\tau}\mathrm{Re}\lambda_k(\tau)|_{\tau=\tau_k} \\
&= \mathrm{Re}\frac{-1+(\gamma+3\beta)e^{-\lambda}}{1+\tau(\gamma+3\beta)e^{-\lambda}}\bigg|_{\lambda=i\beta_k,\tau=\tau_k} \\
&= \mathrm{Re}\frac{\lambda/\tau}{1+(\lambda+\tau)}\bigg|_{\lambda=i\beta_k,\tau=\tau_k} \\
&= \frac{\beta_k^2}{\tau_k[(1+\tau_k)^2+\beta_k^2]}.
\end{aligned}$$

This completes the proof. ∎

Fix $k \geq 1$. Let $\omega = \frac{2\pi}{\beta_k}$, and let P_ω be the Banach space of continuous ω-periodic mappings $x : \mathbb{R} \to \mathbb{R}^3$. $\Gamma \times S^1$ acts on P_ω by

$$(\gamma, e^{i\theta})x(t) = \gamma x(t+\theta), \quad e^{i\theta} \in S^1, x \in P_\omega, \gamma \in \Gamma.$$

We will write γx for $(\gamma, 1)x$ when $\gamma \in \Gamma$ and $x \in P_\omega$. Let SP_ω denote the subspace of P_ω consisting of all ω-periodic solutions of (2.3) with $\tau = \tau_k$. Then

$$SP_\omega = \{x_1\epsilon_1^* + x_2\epsilon_2^* + x_3\epsilon_3^* + x_4\epsilon_4^*; \quad x_i \in \mathbb{R}, i = 1, \ldots, 4\},$$

where

$$\begin{cases}
\epsilon_1^*(t) = \cos(\beta_k t)w_1 - \sin(\beta_k t)w_2, \\
\epsilon_2^*(t) = \sin(\beta_k t)w_1 + \cos(\beta_k t)w_2, \\
\epsilon_3^*(t) = \cos(\beta_k t)w_1 + \sin(\beta_k t)w_2, \\
\epsilon_4^*(t) = \sin(\beta_k t)w_1 - \cos(\beta_k t)w_2,
\end{cases}$$

and

$$w_1 = \left(1, -\frac{1}{2}, -\frac{1}{2}\right)^T, \quad w_2 = \left(0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)^T.$$

PROPOSITION 2.7. *With ϵ_i^* given above, we have*
(i) $\kappa\epsilon_1^* = \epsilon_3^*, \kappa\epsilon_2^* = \epsilon_4^*, \kappa\epsilon_3^* = \epsilon_1^*, \kappa\epsilon_4^* = \epsilon_2^*$;
(ii) $\rho\epsilon_1^* = -\frac{1}{2}\epsilon_1^* - \frac{\sqrt{3}}{2}\epsilon_2^*, \rho\epsilon_2^* = -\frac{1}{2}\epsilon_2^* + \frac{\sqrt{3}}{2}\epsilon_1^*, \rho\epsilon_3^* = -\frac{1}{2}\epsilon_3^* + \frac{\sqrt{3}}{2}\epsilon_4^*, \rho\epsilon_4^* = -\frac{1}{2}\epsilon_4^* - \frac{\sqrt{3}}{2}\epsilon_3^*$.

Proof. (i) is obvious from the definition of the action of κ in Proposition 2.2. To prove (ii), we note that

$$\rho\begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - \frac{\sqrt{3}}{2}\begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\rho\begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

So

$$\rho\epsilon_1^* = \cos(\beta_k t)\left[-\frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2\right] - \sin(\beta_k t)\left[\frac{\sqrt{3}}{2}w_1 - \frac{1}{2}w_2\right] = -\frac{1}{2}\epsilon_1^* - \frac{\sqrt{3}}{2}\epsilon_2^*,$$

$$\rho\epsilon_2^* = \sin(\beta_k t)\left[-\frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2\right] + \cos(\beta_k t)\left[\frac{\sqrt{3}}{2}w_1 - \frac{1}{2}w_2\right] = \frac{\sqrt{3}}{2}\epsilon_1^* - \frac{1}{2}\epsilon_2^*,$$

$$\rho\epsilon_3^* = \cos(\beta_k t)\left[-\frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2\right] + \sin(\beta_k t)\left[\frac{\sqrt{3}}{2}w_1 - \frac{1}{2}w_2\right] = -\frac{1}{2}\epsilon_3^* + \frac{\sqrt{3}}{2}\epsilon_4^*,$$

$$\rho\epsilon_4^* = \sin(\beta_k t)\left[-\frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2\right] - \cos(\beta_k t)\left[\frac{\sqrt{3}}{2}w_1 - \frac{1}{2}w_2\right] = -\frac{\sqrt{3}}{2}\epsilon_3^* - \frac{1}{2}\epsilon_4^*.$$

This completes the proof. ◻

Note that, if x is a periodic solution of (1.1), then so is $(\gamma, e^{i\theta})x$ for every $(\gamma, e^{i\theta}) \in \Gamma \times S^1$. If the symmetry of x is Σ_x for a subgroup of $\Gamma \times S^1$, that is, $\Sigma_x = \{(\gamma, e^{i\theta}) \in \Gamma \times S^1; (\gamma, e^{i\theta})x = x\}$, then the symmetry of $(\gamma, e^{i\theta})x$ is given by $(\gamma, e^{i\theta})\Sigma_x(\gamma, e^{i\theta})^{-1}$, which is conjugate to Σ_x. It is known that the subgroups of $D_3 \times S^1$, up to conjugacy, that describe the symmetry of periodic solutions of (1.1) which exhibit certain spatial-temporal patterns are given below (see, for example, p. 368 in [18]):

$$\Sigma^{\pm}_{(2,3)} = \langle(\kappa, \pm 1)\rangle,$$
$$\Sigma^{\pm}_{\rho} = \langle(\rho, e^{\pm i\frac{2\pi}{3}})\rangle.$$

More specifically, for example, $\Sigma^-_{(2,3)}$ is a group generated by $(\kappa, -1) \in D_3 \times S^1$.

PROPOSITION 2.8.

$$Fix(\Sigma^+_{(2,3)}, SP_\omega) = \{y\cos(\beta_k t)w_1 + z\sin(\beta_k t)w_1; \quad y, z \in \mathbb{R}\},$$
$$Fix(\Sigma^-_{(2,3)}, SP_\omega) = \{y\cos(\beta_k t)w_2 + z\sin(\beta_k t)w_2; \quad y, z \in \mathbb{R}\},$$
$$Fix(\Sigma^-_{\rho}, SP_\omega) = \{y\epsilon_1^* + z\epsilon_2^*; \quad y, z \in \mathbb{R}\},$$
$$Fix(\Sigma^=_{\rho}, SP_\omega) = \{y\epsilon_3^* + z\epsilon_4^*; \quad y, z \in \mathbb{R}\}.$$

Proof. First, $x \in Fix(\Sigma^+_{(2,3)}, SP_\omega)$ if and only if $\kappa x = x$. However, for $x = \sum_{i=1}^4 x_i\epsilon_i^*$, we have

$$\kappa x = x_1\epsilon_3^* + x_2\epsilon_4^* + x_3\epsilon_1^* + x_4\epsilon_2^*.$$

Therefore, $x \in Fix(\Sigma^+_{(2,3)}, SP_\omega)$ if and only if $x_1 = x_3$ and $x_2 = x_4$. This shows that $Fix(\Sigma^+_{(2,3)}, SP_\omega)$ is spanned by $\epsilon_1^* + \epsilon_3^*$ and $\epsilon_2^* + \epsilon_4^*$.

Second, $x \in Fix(\Sigma^-_{(2,3)}, SP_\omega)$ if and only if $\kappa x(t) = x(t + \frac{\omega}{2})$ for $t \in \mathbb{R}$. Let $x = \sum_{i=1}^4 x_i\epsilon_i^*$. Then, as $\cos(\beta_k t + \beta_k \frac{\omega}{2}) = -\cos(\beta_k t)$ and $\sin(\beta_k t + \beta_k \frac{\omega}{2}) = -\sin(\beta_k t)$, we get $\epsilon_i^*(t + \frac{\omega}{2}) = -\epsilon_i^*(t)$, and thus $x(t + \frac{\omega}{2}) = -\sum_{i=1}^4 x_i\epsilon_i^*$. This implies that $\kappa x(t) = x(t + \frac{\omega}{2})$ if and only if $x_1 = -x_3$ and $x_2 = -x_4$. Therefore, $Fix(\Sigma^-_{(2,3)}, SP_\omega)$ is spanned by $\epsilon_1^* - \epsilon_3^*$ and $\epsilon_2^* - \epsilon_4^*$.

Third, for $x = \sum_{i=1}^4 x_i\epsilon_i^*$, we have

$$\rho x = \left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2\right)\epsilon_1^* + \left(-\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)\epsilon_2^* + \left(-\frac{1}{2}x_3 - \frac{\sqrt{3}}{2}x_4\right)\epsilon_3^* + \left(\frac{\sqrt{3}}{2}x_3 - \frac{1}{2}x_4\right)\epsilon_4^*.$$

On the other hand, we have

$$\cos\left(\beta_k\left(t \pm \frac{\omega}{3}\right)\right) = \cos\left(\beta_k t \pm \frac{2\pi}{3}\right) = -\frac{1}{2}\cos(\beta_k t) \mp \frac{\sqrt{3}}{2}\sin(\beta_k t),$$
$$\sin\left(\beta_k\left(t \pm \frac{\omega}{3}\right)\right) = \sin\left(\beta_k t \pm \frac{2\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}\cos(\beta_k t) - \frac{1}{2}\sin(\beta_k t).$$

This, together with the expression of each ϵ_i^* and $x = \sum_{i=1}^4 x_i\epsilon_i^*$, leads to

$$\begin{aligned} &x\left(t \pm \frac{\omega}{3}\right) \\ &= \left(-\frac{1}{2}x_1 \pm \frac{\sqrt{3}}{2}x_2\right)\epsilon_1^* + \left(\mp\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)\epsilon_2^* \\ &\quad + \left(-\frac{1}{2}x_3 \pm \frac{\sqrt{3}}{2}x_4\right)\epsilon_3^* + \left(\mp\frac{\sqrt{3}}{2}x_3 - \frac{1}{2}x_4\right)\epsilon_4^*. \end{aligned}$$

Thus $x \in Fix(\Sigma^\pm_\rho, SP_\omega)$, i.e., $\rho x(t \pm \frac{\omega}{3}) = x(t)$, if and only if

$$\begin{cases} -\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 = -\frac{1}{2}x_1 \mp \frac{\sqrt{3}}{2}x_2, \\ -\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2 = \pm\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2, \\ -\frac{1}{2}x_3 - \frac{\sqrt{3}}{2}x_4 = -\frac{1}{2}x_3 \mp \frac{\sqrt{3}}{2}x_4, \\ \frac{\sqrt{3}}{2}x_3 - \frac{1}{2}x_4 = \pm\frac{\sqrt{3}}{2}x_3 - \frac{1}{2}x_4. \end{cases}$$

That is, $\rho x(t) = x(t + \frac{\omega}{3})$ if and only if $x_3 = x_4 = 0$, and $\rho x(t) = x(t - \frac{\omega}{3})$ if and only if $x_1 = x_2 = 0$. Therefore, $Fix(\Sigma_\rho^-, SP_\omega)$ is spanned by ϵ_1^* and ϵ_2^*, and $Fix(\Sigma_\rho^+, SP_\omega)$ is spanned by ϵ_3^* and ϵ_4^*. This completes the proof. ∎

We can now apply Lemma 2.1 to obtain the following main result of this section.

THEOREM 2.9. *Assume that* (H1) *is satisfied. Then, near* $\tau = \tau_k$ *for each* $k \geq 1$, *system* (1.1) *has eight distinct branches of periodic solutions bifurcated from the trivial solution* $x = 0$. *More precisely, we have the following.*

(i) *There exist* $\epsilon_0^m > 0$ *and* $\delta_0^m > 0$ *such that, for each* $\theta \in [0, 2\pi], \alpha \in (0, \epsilon_0^m)$, *system* (1.1) *with* $\tau = \tau_k + \tau^m(\alpha, \theta)$ *has a periodic solution* $x^m = x^m(t; \alpha, \theta)$ *with period* $\omega^m(\alpha, \theta)$ *such that*

$$x_2^m(t; \alpha, \theta) = x_3^m(t; \alpha, \theta),$$

$$x^m(t; \alpha, \theta) = \alpha \cos(\beta_k t + \theta) \left(1, -\frac{1}{2}, -\frac{1}{2}\right)^{\mathrm{T}} + o(|\alpha|) \text{ as } \alpha \to 0.$$

The mapping $(x^m, \tau^m, \omega^m) : (0, \epsilon^m) \times [0, 2\pi] \to C(\mathbb{R}; \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ *is* C^1-*smooth, and*

$$\omega^m(0, \theta) = \frac{2\pi}{\beta_k}, \quad \tau^m(0, \theta) = 0.$$

Furthermore, if $|\tau - \tau_k| < \delta_0^m$ *and* $|\omega - \frac{2\pi}{\beta_k}| < \delta_0^m$, *then every* ω-*periodic solution of* (1.1) *satisfying* $x_2(t) = x_3(t)$ *and* $\sup_{t \in \mathbb{R}} |x(t)| < \delta_0^m$ *must be given by* $x^m(t; \alpha, \theta)$ *for some* $\alpha \in (0, \epsilon_0^m)$ *and* $\theta \in [0, 2\pi)$. *Similar results hold when we replace* $(2, 3)$ *by* $(1, 2)$ *or* $(1, 3)$.

(ii) *There exist* $\epsilon_0^s > 0$ *and* $\delta_0^s > 0$ *such that, for each* $\theta \in [0, 2\pi], \alpha \in (0, \epsilon_0^s)$, *system* (1.1) *with* $\tau = \tau_k + \tau^s(\alpha, \theta)$ *has a periodic solution* $x^s = x^s(t; \alpha, \theta)$ *with period* $\omega^s = \omega^s(\alpha, \theta)$ *such that*

$$x_1^s(t) = x_1^s\left(t - \frac{\omega^s}{2}\right), x_2^s(t) = x_3^s\left(t - \frac{\omega^s}{2}\right), x_3^s(t) = x_3^s(t + \omega^s),$$

$$x^s(t; \alpha, \theta) = \alpha \cos(\beta_k t + \theta) \left(0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)^{\mathrm{T}} + o(|\alpha|) \text{ as } \alpha \to 0.$$

The mapping $(x^s, \tau^s, \omega^s) : (0, \epsilon_0^s) \times [0, 2\pi] \to C(\mathbb{R}; \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ *is* C^1-*smooth, and*

$$\omega^s(0, \theta) = \frac{2\pi}{\beta_k}, \quad \tau^s(0, \theta) = 0.$$

Furthermore, if $|\tau - \tau_k| < \delta_0^s$ *and* $|\omega - \frac{2\pi}{\beta_k}| < \delta_0^s$, *then every* ω-*periodic solution of* (1.1) *satisfying* $x_1(t) = x_1(t - \frac{\omega}{2}), x_2(t) = x_3(t - \frac{\omega}{2})$, *and* $\sup_{t \in \mathbb{R}} |x(t)| < \delta_0^s$ *must be given by* $x^s(t; \alpha, \theta)$ *for some* $\alpha \in (0, \epsilon_0^s)$ *and* $\theta \in [0, 2\pi)$. *Similar results hold when we replace* $(1, 2, 3)$ *by* $(2, 1, 3)$ *or* $(3, 2, 1)$.

(iii) *There exist* $\epsilon_0^d > 0$ *and* $\delta_0^d > 0$ *such that, for each* $\theta \in [0, 2\pi], \alpha \in (0, \epsilon_0^d)$, *system* (1.1) *with* $\tau = \tau_k + \tau^d(\alpha, \theta)$ *has a periodic solution* $x^d = x^d(t; \alpha, \theta)$ *with period* $\omega^d = \omega^d(\alpha, \theta)$ *such that*

$$x_1^d(t) = x_2^d\left(t \pm \frac{\omega^d}{3}\right), x_2^d(t) = x_3^d\left(t \pm \frac{\omega^d}{3}\right),$$

$$x^d(t; \alpha, \theta) = \alpha \left(\cos(\beta_{\mathrm{k}} \mathrm{t} + \theta), \cos\left(\beta_{\mathrm{k}} \mathrm{t} + \theta \mp \frac{2\pi}{3}\right), \cos\left(\beta_{\mathrm{k}} \mathrm{t} + \theta \mp \frac{4\pi}{3}\right)\right)^T + o(|\alpha|)$$

as $\alpha \to 0$. *The mapping* $(x^d, \tau^d, \omega^d) : (0, \epsilon_0^d) \times [0, 2\pi] \to C(\mathbb{R}; \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ *is* C^1*-smooth, and*

$$\omega^d(0, \theta) = \frac{2\pi}{\beta_k}, \quad \tau^d(0, \theta) = 0.$$

Furthermore, if $|\tau - \tau_k| < \delta_0^d$ *and* $|\omega - \frac{2\pi}{\beta_k}| < \delta_0^d$*, then every* ω*-periodic solution of* (1.1) *satisfying* $x_1(t) = x_2(t \pm \frac{\omega}{3}), x_2(t) = x_3(t \pm \frac{\omega}{3})$*, and* $\sup_{t\in\mathbb{R}} |x(t)| < \delta_0^d$ *must be given by* $x^d(t; \alpha, \theta)$ *for some* $\alpha \in (0, \epsilon_0^d)$ *and* $\theta \in [0, 2\pi)$*. Similar results hold when we replace* $(1, 2, 3)$ *by* $(2, 1, 3)$ *or* $(3, 2, 1)$*.*

We call periodic solutions in (i)–(iii) *mirror-reflecting waves, standing waves*, and *discrete waves*, respectively. Note that Theorem 2.9 does not rule out the case in which $\tau^l(\alpha, \theta) \le 0$ $(l = m, s, d)$. In next section, we will use the global bifurcation theory to rule out this case. In fact, we will show that all eight branches of waves are *supercritical* and *global*; i.e., all eight branches of waves exist for $\tau > \tau_k$.

3. Global continuation of waves. We will need a general global symmetric Hopf bifurcation theorem developed in [24]. Namely, we consider the one-parameter family of retarded functional differential equations

$$\dot{x}(t) = \tau F(x_t), \tag{3.1}$$

where $x \in \mathbb{R}^n$, $\tau \in (0, \infty)$, and $F : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is continuously differentiable and completely continuous. Furthermore, we assume the following.

(A1) $\Gamma := Z_N$ for some integer N acts on $\mathbb{R}^n$ and $F : C \to \mathbb{R}^n$ is Γ-equivariant.

(A2) For every $x_0 \in M^\Gamma := \{x \in \mathbb{R}^n; \gamma x = x \text{ for } \gamma \in \Gamma, F(\bar{x}) = 0\}$, where $\bar{x} \in C$ is the constant mapping with the constant value $x \in \mathbb{R}^n$, $\det D\hat{F}(x_0) \neq 0$, where $\hat{F}$ is the C^1 mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$, induced by F according to $\hat{F}(x) = F(\bar{x})$ for $x \in \mathbb{R}^n$.

(A3) For every $\tau_0 > 0$ and $x_0 \in M^\Gamma$ such that the generator $A(\tau_0, x_0)$ of the linearized system of (3.1) with $\tau = \tau_0$ at $x = x_0$ has a pair of purely imaginary eigenvalues $\pm i\beta_0$, there exist positive constants b, c, and δ such that (i) the only possible eigenvalue $u + iv$ of $A(\tau_0, x_0)$ with $(u, v) \in \partial\Omega$ is $i\beta_0$, where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c)$; (ii) for $(\tau, \beta) \in [\tau_0 - \delta, \tau_0 + \delta] \times [\beta_0 - c, \beta_0 + c]$, $i\beta$ is an eigenvalue of $A(\tau, x_0)$ if and only if $\tau = \tau_0, \beta = \beta_0$.

(A4) $M^* := \{(\tau, x, \beta) \in (0, \infty) \times M^\Gamma \times (0, \infty); \pm i\beta \text{ are eigenvalues of } A(\tau, x)\}$ is a discrete set.

Note that the action of Γ on $\mathbb{R}^n$ induces an action on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, with respect to which we have the isotypical decomposition

$$\mathbb{C}^n = \mathbb{C}_0^n \oplus \mathbb{C}_1^n \oplus \cdots \oplus \mathbb{C}_j^n \oplus \cdots,$$

where $\mathbb{C}_j^n, j \ge 0$, is the direct sum of all one-dimensional Γ-irreducible subspaces V of $\mathbb{C}^n$ such that the restricted action Γ on V is isomorphic to the Γ-action on $\mathbb{C}$ defined by $\rho \cdot z = \rho^j z$ for the generator $\rho \in Z_N \le S^1$ and for $z \in \mathbb{C}$. Let

$$\Delta_{x_0}(\tau, \lambda) := \lambda I_n - \tau D_\phi F(\bar{x}_0)(e^{\lambda \cdot} I_n) \tag{3.2}$$

for $\tau > 0, x_0 \in M^\Gamma$, and $\lambda \in \mathbb{C}$. By assumption (A1), we have $\Delta_{x_0}(\tau, \lambda)\mathbb{C}_j^n \subset \mathbb{C}_j^n$ for $j \ge 0$ and for $\lambda \in \mathbb{C}$. Put

$$\Delta_{x_0,j}(\tau, \lambda) = \Delta_{x_0}(\tau, \lambda)|_{\mathbb{C}_j^n}, \quad j \ge 0. \tag{3.3}$$

Clearly, $\Delta_{x_0}(\tau,\lambda)$ is analytic in $\lambda\in\mathbb{C}$ and continuous in $\tau>0$. So, under assumption (A3), we may assume that $\det\Delta_{x_0}(\tau_0\pm\delta,u+iv)\neq 0$ for $(u,v)\in\partial\Omega$. Therefore, $\det\Delta_{x_0,j}(\tau_0\pm\delta,u+iv)\neq 0$ for $(u,v)\in\partial\Omega$ and for $j\geq 0$. Consequently, the following integers are well defined:

$$c_j(x_0,\tau_0,\beta_0)=\deg_B(\det\Delta_{x_0,j}(\tau_0-\delta,\cdot),\Omega)-\deg_B(\det\Delta_{x_0,j}(\tau_0+\delta,\cdot),\Omega), \tag{3.4}$$

where $\deg_B$ is the Brouwer degree. Let

$$\epsilon(x_0)=(-1)^n sign\det D\hat{F}(x_0). \tag{3.5}$$

We have the following global symmetric Hopf bifurcation theorem due to [24].

LEMMA 3.1. *Assume that* (A1)–(A4) *are satisfied and* $c_j(x_0,\tau_0,\beta_0)\neq 0$ *for some integer* $j\geq 0$ *and some* $(\tau_0,x_0,\beta_0)\in(0,\infty)\times M^\Gamma\times(0,\infty)$. *Let* S_j *denote the closure in* $[0,\infty)\times C(\mathbb{R};\mathbb{R}^n)\times[0,\infty)$ *of the set of all* $(\tau,z,\beta)\in[0,\infty)\times C(\mathbb{R};\mathbb{R}^n)\times\mathbb{R}\setminus M^*$ *such that* $x(t):=z(\frac{\beta}{2\pi}t)$ *is a* $\frac{2\pi}{\beta}$*-periodic solution of* (3.1) *with* $\rho x(t)=x(t-\frac{2\pi}{\beta}\frac{j}{N})$ *for* $t\in\mathbb{R}$. *Then* $S_j\neq\emptyset$, *and, for every bounded connected component* E_j *of* S_j, $(\Gamma\times S^1)E_j\cap M^*$ *is finite and*

$$\sum_{(\tau,x,\beta)\in(\Gamma\times S^1)E_j\cap M^*}\epsilon(x)c_j(x,\tau,\beta)=0; \tag{3.6}$$

here a set $E\subset(0,\infty)\times C(\mathbb{R};\mathbb{R}^n)\times(0,\infty)$ *is bounded if*

$$\sup\left\{\frac{1}{\tau}+\tau+\frac{1}{\beta}+\beta+\sup_{t\in\mathbb{R}}|x(t)|;\quad(\tau,x,\beta)\in E\right\}<\infty.$$

We now begin to apply the above result to discuss the global continuation of wave solutions of system (1.1). We need the following assumptions.

(H2) $\sup_{y\in\mathbb{R}}|h'(y)|<1$.

(H3) $g'(x)>0$ for all $x\in\mathbb{R}$.

PROPOSITION 3.2. *Assume that* (H1)–(H3) *are satisfied. Then system* (1.1) *has no nonconstant* 1*-periodic solution.*

Proof. By way of contradiction, let x be a nonconstant periodic solution of system (1.1) with $x_i(t)=x_i(t-1)$ for all $t\in\mathbb{R}$ and $i=1,2,3$. Then we obtain a system of ordinary differential equations

$$\begin{cases}\frac{1}{\tau}\dot{x}_1(t)=-x_1(t)+h(x_1(t))+2g(x_1(t))-g(x_2(t))-g(x_3(t)),\\ \frac{1}{\tau}\dot{x}_2(t)=-x_2(t)+h(x_2(t))+2g(x_2(t))-g(x_1(t))-g(x_3(t)),\\ \frac{1}{\tau}\dot{x}_3(t)=-x_3(t)+h(x_3(t))+2g(x_3(t))-g(x_2(t))-g(x_1(t)).\end{cases} \tag{3.7}$$

Note that the above equation is exactly the model equation for the Hopfield net [20] of three identical neurons with self-feedback, and thus

$$\begin{aligned}&V(x_1,x_2,x_3)\\ &=-\frac{1}{2}\sum_{1\leq i<j\leq 3}[g(x_i)-g(x_j)]^2+\sum_{k=1}^{3}\int_0^{x_k}[s-h(s)]g\prime(s)ds\\ &=g(x_1)g(x_2)+g(x_2)g(x_3)+g(x_3)g(x_1)\\ &\quad-g^2(x_1)-g^2(x_2)-g^2(x_3)\\ &\quad+\int_0^{x_1}[s-h(s)]g\prime(s)ds+\int_0^{x_2}[s-h(s)]g\prime(s)ds+\int_0^{x_3}[s-h(s)]g\prime(s)ds\end{aligned}$$

is the so-called energy function. For such an energy function, we have

$$\begin{aligned}
&\dot{V}_{(15)}(x_1, x_2, x_3)\\
&= g\prime(x_1)\dot{x}_1[g(x_2) + g(x_3) - 2g(x_1) + x_1 - h(x_1)]\\
&\quad + g\prime(x_2)\dot{x}_2[g(x_1) + g(x_3) - 2g(x_2) + x_2 - h(x_2)]\\
&\quad + g\prime(x_3)\dot{x}_3[g(x_1) + g(x_2) - 2g(x_3) + x_3 - h(x_3)]\\
&= -\tau \sum_{i=1}^{3} g\prime(x_i)(\dot{x}_i)^2 \leq 0
\end{aligned}$$

and

$$\dot{V}_{(15)}(x_1, x_2, x_3) = 0 \text{ if and only if } \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0.$$

The LaSalle invariance principle [27] then implies that every solution of (3.6) converges to an equilibrium as $t \to \infty$. In particular, every 1-periodic solution of (1.1) must be constant. This completes the proof. ∎

PROPOSITION 3.3. *Under assumptions* (H1)–(H3)*, system* (1.1) *has no nonconstant* 2*-periodic solution.*

Proof. Assume that $x(t)$ is a 2-periodic solution. Let $x_4(t) = x_1(t-1), x_5(t) = x_2(t-1)$, and $x_6(t) = x_3(t-1)$. Then we obtain

$$\begin{cases}
\epsilon\dot{x}_1 = -x_1 + h(x_4) - g(x_5) - g(x_6) + 2g(x_4),\\
\epsilon\dot{x}_2 = -x_2 + h(x_5) - g(x_4) - g(x_6) + 2g(x_5),\\
\epsilon\dot{x}_3 = -x_3 + h(x_6) - g(x_4) - g(x_5) + 2g(x_6),\\
\epsilon\dot{x}_4 = -x_4 + h(x_1) - g(x_2) - g(x_3) + 2g(x_1),\\
\epsilon\dot{x}_5 = -x_5 + h(x_2) - g(x_1) - g(x_3) + 2g(x_2),\\
\epsilon\dot{x}_6 = -x_6 + h(x_3) - g(x_1) - g(x_2) + 2g(x_3).
\end{cases}$$

Then

$$\begin{cases}
\frac{1}{\tau}[x_1 - x_4]\prime = -[x_1 - x_4] + [h(x_4) - h(x_1)]\\
\qquad\qquad + [g(x_2) - g(x_5) + g(x_3) - g(x_6) - 2(g(x_1) - g(x_4))],\\
\frac{1}{\tau}[x_2 - x_5]\prime = -[x_2 - x_5] + [h(x_5) - h(x_2)]\\
\qquad\qquad + [g(x_1) - g(x_4) + g(x_3) - g(x_6) - 2(g(x_2) - g(x_5))],\\
\frac{1}{\tau}[x_3 - x_6]\prime = -[x_3 - x_6] + [h(x_6) - h(x_3)]\\
\qquad\qquad + [g(x_1) - g(x_4) + g(x_2) - g(x_5) - 2(g(x_3) - g(x_6))].
\end{cases}$$

Let D^+ denote the upper right Dini derivative; then

$$\begin{cases}
\frac{1}{\tau}D^+|x_1 - x_4| \leq -|x_1 - x_4| - 2|g(x_1) - g(x_4)| + |h(x_1) - h(x_4)|\\
\qquad\qquad + |g(x_2) - g(x_5)| + |g(x_3) - g(x_6)|,\\
\frac{1}{\tau}D^+|x_2 - x_5| \leq -|x_2 - x_5| - 2|g(x_2) - g(x_5)| + |h(x_2) - h(x_5)|\\
\qquad\qquad + |g(x_1) - g(x_4)| + |g(x_3) - g(x_6)|,\\
\frac{1}{\tau}D^+|x_3 - x_6| \leq -|x_3 - x_6| - 2|g(x_3) - g(x_6)| + |h(x_3) - h(x_6)|\\
\qquad\qquad + |g(x_1) - g(x_4)| + |g(x_2) - g(x_5)|.
\end{cases}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\tau}D^+[|x_1-x_4|+|x_2-x_5|+|x_3-x_6|] \\
&\le -[|x_1-x_4|+|x_2-x_5|+|x_3-x_6|] \\
&\quad + |h(x_1)-h(x_4)|+|h(x_2)-h(x_5)|+|h(x_3)-h(x_6)| \\
&\le -\left[1-\sup_{\theta\in\mathbb{R}}|h\prime(\theta)|\right][|x_1-x_4|+|x_2-x_5|+|x_3-x_6|].
\end{aligned}$$

This implies that

$$|x_1(t)-x_4(t)|+|x_2(t)-x_5(t)|+|x_3(t)-x_6(t)|\to 0 \quad \text{as } t\to\infty.$$

Therefore, for a 2-periodic solution x of (1), we must have $x_1(t) = x_1(t-1), x_2(t) = x_2(t-1)$, and $x_3(t) = x_3(t-1)$. So Proposition 3.2 can be applied to conclude that x must be constant. This completes the proof. □

It remains to obtain a priori bounds for the norm of periodic solutions of (1.1). We need the following assumption.

(H4) $\sup_{y\in\mathbb{R}}[|h(y)|+|g(y)|]<\infty$.

PROPOSITION 3.4. *Assume* (H1)–(H4) *are satisfied. Then there exists* $M = M(h,g) > 0$ *such that* $|x_1(t)|+|x_2(t)|+|x_3(t)| \le M$ *for all* $t\in\mathbb{R}$ *and for every periodic solution* x *of* (1.1).

Proof. Let $t^*\in\mathbb{R}$ and $j\in\{1,2,3\}$ be given so that $|x_j(t^*)| = \max_{t\in\mathbb{R}}\max_{1\le i\le 3}|x_i(t)|$. Then $\dot{x}_j(t^*)=0$. That is,

$$x_j(t^*) = h(x_j(t^*-1)) - [g(x_{j-1}(t^*-1)) + g(x_{j+1}(t^*-1)) - 2g(x_j(t^*-1))],$$

from which it follows that

$$|x_j(t^*)| \le \sup_{y\in\mathbb{R}}|h(y)| + 4\sup_{y\in\mathbb{R}}|g(y)| := \frac{M}{3} < \infty.$$

This completes the proof. □

We now apply Lemma 3.1 to investigate the global continuation of standing, mirror-reflecting, and discrete waves.

First, note that near $\tau=\tau_k$ system (1.1) has two bifurcations of discrete waves satisfying $x_{i-1}(t) = x_i(t\pm\frac{\omega}{3})$, where ω is a period. To look at the global continuation of such local bifurcations, we regard system (1.1) as a functional differential equation equivariant with respect to the action of $\Gamma = Z_3$, where the action is the cyclic permutation. We have

$$\begin{aligned}
M^\Gamma &= \{x\in\mathbb{R}^3; \gamma x = x \text{ for } \gamma\in\Gamma, F(\bar{x})=0\} \\
&= \{x\in\mathbb{R}^3; x_1=x_2=x_3 \text{ and } x_1 = h(x_1)\} = \{0\}
\end{aligned}$$

under assumption (H2). Clearly, (A1) and (A2) are satisfied.

Under assumption (H1), the discussions in the last section show that

$$M^* = \{(\tau_k, 0, \beta_k);\quad k\ge 1\}.$$

Therefore, M^* is discrete in $\mathbb{R}^3$.

Using Proposition 2.4 (ii), for a fixed integer k, we can choose positive constants b, c, and δ so that the only possible eigenvalue $u+iv$ of $A(\tau_k)$ with $(u,v)\in\partial\Omega$ is $i\beta_k$, where $\Omega = (0,b) \times (\beta_k - c, \beta_k + c)$, and if $(\tau, \beta) \in [\tau_k - \delta, \tau_k + \delta] \times [\beta_k - c, \beta_k + c]$, then $i\beta$ is an eigenvalue of $A(\tau)$ if and only if $\tau = \tau_k$ and $\beta = \beta_k$. Then, using Proposition 2.4 (i), we can conclude that the analytic function $p_\tau(\lambda) := \lambda + \tau - \gamma\tau e^{-\lambda}$ has no zero in $\bar{\Omega}$ for $\tau = \tau_k \pm \delta$. Also, by Propositions 2.4 and 2.6, the above b, c, and δ can be chosen so that, for the analytic function

$$q_\tau(\lambda) = \lambda + \tau - (\gamma + 3\beta)\tau e^{-\lambda},$$

we have that $q_{\tau_k - \delta}$ has no zero in $\bar{\Omega}$, while $q_{\tau_k+\delta}$ has exactly one zero in $\bar{\Omega}$, and this zero is simple and is in the interior of $\bar{\Omega}$. Therefore,

$$\deg_B(q_{\tau_k-\delta}, \Omega) = 0,$$

and

$$\deg_B(q_{\tau_k+\delta}, \Omega) = 1.$$

With respect to the complexification of the above $(\Gamma = Z_3)$-action in $\mathbb{R}^3$, we have the isotypical decomposition

$$\mathbb{C}^3 = \mathbb{C}^3_0 \oplus \mathbb{C}^3_1 \oplus \mathbb{C}^3_2,$$

where

$$\mathbb{C}^3_j = \{(1, e^{i\frac{2\pi}{3}j}, e^{i\frac{4\pi}{3}j})x; \quad x \in \mathbb{C}\}.$$

We have shown that

$$\begin{aligned} \Delta_{0,j} &:= \Delta_0(\tau,\lambda)|_{\mathbb{C}^3_j} = \Delta(\tau,\lambda)|_{\mathbb{C}^3_j} \\ &= \begin{cases} \lambda + \tau - \gamma\tau e^{-\lambda} & \text{if } j = 0, \\ \lambda + \tau - (\gamma + 3\beta)\tau e^{-\lambda} & \text{if } \quad j = 1, 2. \end{cases} \end{aligned}$$

Therefore, from the above discussions, we get

$$c_0(0, \tau_k, \beta_k) = \deg_B(p_{\tau_k-\delta}, \Omega) - \deg_B(p_{\tau_k+\delta}, \Omega) = 0,$$

and, for $j = 1, 2$,

$$c_j(0, \tau_k, \beta_k) = \deg_B(q_{\tau_k-\delta}, \Omega) - \deg_B(q_{\tau_k+\delta}, \Omega) = -1.$$

Let $S_j, j = 1, 2$, denote the closure in $[0, \infty) \times C(\mathbb{R}; \mathbb{R}^3) \times [0, \infty)$ of the set of all triples $(\tau, z, \beta) \notin M^*$ such that $x(t) := z(\frac{\beta}{2\pi}t)$ is a $\frac{2\pi}{\beta}$-periodic solution of (1.1) with $x_{k+1}(t) = x_k(t - \frac{2\pi}{\beta}\frac{j}{3})$ for $t \in \mathbb{R}$ and $k = 1, 2, 3 (\mathrm{mod}\, 3)$. Then Lemma 3.1 implies that S_j must have a nonempty connected component E_j passing through $(\tau_k, 0, \beta_k)$, and this component must be unbounded in the sense that

$$\sup_{(\tau,x,\beta)\in E_j} \left\{ \tau + \frac{1}{\tau} + \beta + \frac{1}{\beta} + \sup_{t\in\mathbb{R}} |z(t)| \right\} = \infty,$$

for otherwise, the summation (3.6) must hold, and this is clearly impossible as $c_j(0, \tau_k, \beta_k)$ has the same sign for all positive integers k.

The projection of E_j onto the space $C(\mathbb{R};\mathbb{R}^3)$ is bounded due to Proposition 3.4. Near τ_k, (ii) of Proposition 2.4 shows that, for $(\tau, z, \beta) \in E_j$, we have

$$\frac{2\pi}{\beta} \in \left(\frac{2\pi}{2k\pi}, \frac{2\pi}{2k\pi - \frac{\pi}{2}}\right) \subset \left(\frac{1}{k}, \frac{1}{k-\frac{1}{4}}\right) \subset \left(\frac{1}{k}, \frac{4}{3}\right) \subset \left(\frac{1}{k}, 2\right).$$

On the other hand, Propositions 3.2 and 3.3 imply that the projection of E_j onto the β-plane can never reach the lines $\frac{2\pi}{\beta} = \frac{1}{k}$ (note that (1.1) has no $\frac{1}{k}$-periodic solution as it does not have a 1-periodic solution) and $\frac{2\pi}{\beta} = 2$. Therefore, the projection of E_j onto the β-plane always satisfies $\pi < \beta < 2k\pi$.

On the other hand, the result of [28] shows that there exists $\alpha^* > 0$ such that any period p of a periodic solution of (1.2) must satisfy $p \geq \alpha^*$. Consequently, for $(\tau, z, \beta) \in E_j$, we must have $\tau\frac{2\pi}{\beta} \geq \alpha^*$. That is, $\tau \geq \frac{\beta\alpha^*}{2\pi} > \frac{\alpha^*}{2}$ for every $\tau \in I$, the projection of E_j onto the τ-axis which must be an interval. Therefore, I must be unbounded from above. Clearly, I contains τ_k. This proves the following.

THEOREM 3.5. *For each $\tau > \tau_k$, system* (1.1) *always has two discrete waves satisfying $x_{j+1}(t) = x_j(t \pm \frac{\omega}{3})$ for $t \in \mathbb{R}$ and $j(\mathrm{mod}\ 3)$, where ω is a period of $x(t)$ and $\frac{1}{k} < \omega < 2$.*

Let us now consider the global continuation of mirror-reflecting waves and standing waves. For this purpose, we consider (1.1) as a functional differential equation equivariant with respect to the action of $\Gamma = Z_2$ on $\mathbb{R}^3$ defined by

$$\rho\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_3\\x_2\end{pmatrix}, \quad x_i \in \mathbb{R}, i = 1,2,3, Z_2 = \langle\rho\rangle.$$

In this case,

$$M^\Gamma = \{x \in \mathbb{R}^3; \quad x_2 = x_3, x_i = h(x_i) - g(x_{i-1}) - g(x_{i+1}) + 2g(x_i), i(\mathrm{mod}\ 3)\}.$$

The structure of M^Γ is explicitly described in the following proposition under the following assumption.

(H5) $yh''(y) < 0$ and $yg''(y) < 0$ for $y \neq 0$.

PROPOSITION 3.6. *Under* (H1)–(H5)*, the system of equations*

$$x_i = h(x_i) - g(x_{i-1}) - g(x_{i+1}) + 2g(x_i), \quad i(\mathrm{mod}\ 3), \tag{3.8}$$

and

$$x_2 = x_3 \tag{3.9}$$

for $x = (x_1, x_2, x_3)^T$ has exactly three solutions. They are

$$(0,0,0)^T, \quad (z^-, y^+, y^+)^T, \quad (z^+, y^-, y^-)^T,$$

where $y^+ > 0, y^- < 0, z^+ > 0, z^- < 0$ are the unique solutions of

$$\begin{cases} y^\pm - h(y^\pm) = u^\pm, \\ z^\mp - h(z^\mp) = -2u^\pm \end{cases} \tag{3.10}$$

and $u^+ > 0$ and $u^- < 0$ are the unique positive and negative solutions of

$$u + g[G^{-1}(-2u)] - g[G^{-1}(u)] = 0 \tag{3.11}$$

with $G : \mathbb{R} \to \mathbb{R}$ being given by the equation

$$G(\theta) = \theta - h(\theta), \quad \theta \in \mathbb{R}. \tag{3.12}$$

In other words,

$$M^\Gamma = \{(0,0,0)^T, (z^-, y^+, y^+)^T, (z^+, y^-, y^-)^T\}.$$

Proof. Under assumption (H2), $G : \mathbb{R} \to \mathbb{R}$ defined by (3.12) is an increasing function. Define

$$u = G(y), \quad v = G(z). \tag{3.13}$$

Then $x = (x_1, x_2, x_3)^T$ with $x_1 = z$ and $x_2 = x_3 = y$ satisfies (3.8) if and only if

$$u = g[G^{-1}(u)] - g[G^{-1}(v)] \tag{3.14}$$

and

$$v = -2u. \tag{3.15}$$

In other words, (u, v) is given by $v = -2u$ and $u = g[G^{-1}(u)] - g[G^{-1}(-2u)]$. Let

$$H(u) = u + g[G^{-1}(-2u)] - g[G^{-1}(u)], \quad u \in \mathbb{R}.$$

Then

$$H(0) = 0, \quad H(\pm\infty) = \pm\infty.$$

Note that

$$\begin{aligned} H'(u) &= 1 + g'[G^{-1}(-2u)](G^{-1})'(-2u)(-2) - g'[G^{-1}(u)](G^{-1})'(u) \\ &= 1 - 2g'[G^{-1}(-2u)](G^{-1})'(-2u) - g'[G^{-1}(u)](G^{-1})'(u). \end{aligned}$$

Implicitly differentiating $F(\theta) = \theta - h(\theta)$, we get

$$(G^{-1})'(\theta) = \frac{1}{1 - h'[G^{-1}(\theta)]}.$$

Therefore,

$$H'(u) = 1 - \frac{2g'[G^{-1}(-2u)]}{1 - h'[G^{-1}(-2u)]} - \frac{g'[G^{-1}(u)]}{1 - h'[G^{-1}(u)]}.$$

In particular, with $h'(0) = \gamma$ and $g'(0) = \beta$ and under assumption (H1), we have

$$H'(0) = 1 - \frac{2\beta}{1-\gamma} - \frac{\beta}{1-\gamma} = \frac{1 - (\gamma + 3\beta)}{1 - \gamma} < 0.$$

Therefore, there must be $u^+ > 0$ and $u^- < 0$ such that $H(u^\pm) = 0$.

It remains to show that there exists no other nonzero zero of H. By way of contradiction, if there exists $u^* > 0$ (the case in which $u^* < 0$ can be dealt with similarly) such that $H(u^*) = 0$ and $u^* \neq u^+$, then there must be $\theta > 0$ so that $H''(\theta) = 0$. However, we have

$$\begin{aligned} H''(u) &= -2g''[G^{-1}(-2u)][(G^{-1})'(-2u)]^2(-2) \\ &\quad -2g'[G^{-1}(-2u)](G^{-1})''(-2u)(-2) \\ &\quad -g''[G^{-1}(u)][(G^{-1})'(u)]^2 - g'[G^{-1}(u)](G^{-1})''(u) \\ &= 4g''[G^{-1}(-2u)][(G^{-1})'(-2u)]^2 + 4g'[G^{-1}(-2u)](G^{-1})''(-2u) \\ &\quad -g''[G^{-1}(u)][(G^{-1})'(u)]^2 - g'[G^{-1}(u)](G^{-1})''(u). \end{aligned}$$

Under assumption (H5), for $u > 0$ we have

$$g''[G^{-1}(-2u)] > 0, \quad g''[G^{-1}(u)] < 0.$$

Therefore, $H''(u) > 0$ if we can show that

$$(G^{-1})''(-2u) > 0 \text{ and } (G^{-1})''(u) < 0 \text{ for } u > 0. \tag{3.16}$$

The above holds by using (H5) since

$$(G^{-1})''(u) = \frac{h''(G^{-1}(u))(G^{-1})'(u)}{[1 - h'(G^{-1}(u))]^2}$$

has the opposite sign from u. (Recall that $G^{-1}(u)$ has the same sign as u.)

This completes the proof. □

To verify (A2) and (A4) in the case in which $\Gamma = Z_2$, we need the following condition.

(H6) $h'(\alpha) > 0, h'(\alpha) + 3g'(\alpha) < 1$, where $\alpha = y^\pm, z^\pm$.

The linearization of (1.1) at (z^*, y^*, y^*) with $z^* = z^\mp, y^* = y^\pm$ takes the form

$$\begin{cases} \frac{1}{\tau}\dot{X}_1(t) = -X_1(t) + h_1'(z^*)X_1(t-1) \\ \qquad\qquad - [g'(y^*)X_2(t-1) + g'(y^*)X_3(t-1) - 2g'(z^*)X_1(t-1)], \\ \frac{1}{\tau}\dot{X}_2(t) = -X_2(t) + h_1'(y^*)X_2(t-1) \\ \qquad\qquad - [g'(y^*)X_3(t-1) + g'(z^*)X_1(t-1) - 2g'(y^*)X_2(t-1)], \\ \frac{1}{\tau}\dot{X}_3(t) = -X_3(t) + h_1'(y^*)X_3(t-1) \\ \qquad\qquad - [g'(z^*)X_1(t-1) + g'(y^*)X_2(t-1) - 2g'(y^*)X_3(t-1)], \end{cases}$$

and the characteristic matrix becomes

$$\begin{aligned} &\Delta_{(z^*,y^*,y^*)}(\tau, \lambda) \\ &= \begin{pmatrix} A & \tau g'(y^*)e^{-\lambda} & \tau g'(y^*)e^{-\lambda} \\ \tau g'(z^*)e^{-\lambda} & B & \tau g'(y^*)e^{-\lambda} \\ \tau g'(z^*)e^{-\lambda} & \tau g'(y^*)e^{-\lambda} & B \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} A &= \lambda + \tau - \tau[h'(z^*) + 2g'(z^*)]e^{-\lambda}, \\ B &= \lambda + \tau - \tau[h'(y^*) + 2g'(y^*)]e^{-\lambda}. \end{aligned}$$

The isotypical decomposition of $\mathbb{C}^3$ with respect to the above $\Gamma = Z_2$ action is

$$\mathbb{C}^3 = \mathbb{C}^3_0 \oplus \mathbb{C}^3_1,$$

where

$$\mathbb{C}^3_0 = \{(x, y, y)^T; x, y \in \mathbb{C}\},$$
$$\mathbb{C}^3_1 = \{(0, z, -z)^T; z \in \mathbb{C}\}.$$

Therefore,

$$\Delta_{(z^*,y^*,y^*)}(\tau,\lambda)|_{\mathbb{C}^3_0} = \begin{pmatrix} \lambda + \tau - \tau[h'(z^*) + 2g'(z^*)]e^{-\lambda} & \tau g'(z^*)e^{-\lambda} \\ 2\tau g'(y^*)e^{-\lambda} & \lambda + \tau - \tau[h'(y^*) + g'(y^*)]e^{-\lambda} \end{pmatrix}$$

and

$$\Delta_{(z^*,y^*,y^*)}(\tau,\lambda)|_{\mathbb{C}^3_1} = \lambda + \tau - \tau[h'(y^*) + 3g'(y^*)]e^{-\lambda\tau}.$$

It is already shown in the proof of Proposition 2.4 (i) that, under assumption (H6), every zero of $\Delta_{(z^*,y^*,y^*)}(\tau,\lambda)|_{\mathbb{C}^3_1}$ has negative real part. Note that $\Delta_{(z^*,y^*,y^*)}(\tau,\lambda)|_{\mathbb{C}^3_0}$ is the characteristic matrix for the following linear system of delay differential equations:

$$\text{(3.17)}\qquad \begin{cases} \frac{1}{\tau}\dot{u}_1(t) = -u_1(t) + [h'(z^*) + 2g'(z^*)]u_1(t-1) - g'(z^*)u_2(t-1), \\ \frac{1}{\tau}\dot{u}_2(t) = -u_2(t) + [h'(y^*) + g'(y^*)]u_2(t-1) - 2g'(y^*)u_1(t-1). \end{cases}$$

Let $V(u_1, u_2) = \max\{|u_1|, |u_2|\}$. For a given solution of (3.17), if at some $t \geq 0$ we have $V(u_1(t-1), u_2(t-1)) \leq V(u_1(t), u_2(t)) = |u_1(t)|$, then

$$\begin{aligned} &\frac{1}{\tau}D^+V(u_1(t), u_2(t)) \\ &\leq -|u_1(t)| + [h'(z^*) + 2g'(z^*)]|u_1(t-1)| + g'(z^*)|u_2(t-1)| \\ &\leq -|u_1(t)| + [h'(z^*) + 3g'(z^*)]|u_1(t)| \\ &= -[1 - h'(z^*) - 3g'(z^*)]V(u_1(t), u_2(t)). \end{aligned}$$

Similarly, for a given solution of (3.17), if at some $t \geq 0$ we have $V(u_1(t-1), u_2(t-1)) \leq V(u_1(t), u_2(t)) = |u_2(t)|$, then

$$\frac{1}{\tau}D^+V(u_1(t), u_2(t)) \leq -[1 - h'(y^*) - 3g'(y^*)]V(u_1(t), u_2(t)).$$

Therefore, using assumption (H6) and the Razumikhin-type LaSalle invariance principle in [27, 29], we can conclude that all solutions of (3.17) converge to zero as $t \to \infty$. This shows that all zeros of $\det\Delta_{(z^*,y^*,y^*)}(\tau,\lambda)|_{\mathbb{C}^3_0}$ have negative real parts. In particular, $\det\Delta_{(z^*,y^*,y^*)}(\tau,0)|_{\mathbb{C}^3_0} \neq 0$, and this determinant is exactly the determinant of the derivative of the corresponding F at (z^*, y^*, y^*). This shows that (A2) is satisfied and that (A3) is trivial.

Therefore, even in the case in which $\Gamma = Z_2$, we have

$$M^* = \{(\tau_k, 0, \beta_k); \quad k \geq 1\}.$$

Thus M^* is discrete and (A4) holds. Using similar arguments as for Theorem 3.5, we can get the following theorems.

THEOREM 3.7. *For each* $\tau > \tau_k, k \geq 1$, *system* (1.1) *has one standing wave satisfying* $x_1(t) = x_1(t - \frac{\omega}{2})$ *and* $x_2(t) = x_3(t - \frac{\omega}{2})$ *for* $t \in \mathbb{R}$, *where* ω *is a period of* x *and* $\frac{1}{k} < \omega < 2$.

THEOREM 3.8. *For each* $\tau > \tau_k, k \geq 1$, *system* (1.1) *has one mirror-reflecting wave satisfying* $x_2(t) = x_3(t)$ *and* $x_i(t) = x_i(t + \omega)$ *for* $t \in \mathbb{R}, i = 1, 2, 3$, *where* $\frac{1}{k} < \omega < 2$.

Remark 1. Due to the D_3-symmetry, Theorems 3.5–3.8 in fact imply the existence of three standing waves, three mirror-reflecting waves, and two discrete waves for each $\tau > \tau_k$. Note also that

$$\tau_1 < \tau_2 < \tau_3 < \cdots .$$

The above results establish the existence of $3k$ standing waves, $3k$ mirror-reflecting waves, and $2k$ discrete waves. It should be mentioned that, in the above theorems, ω is not necessarily the minimal period, and several branches of waves may coincide at some values of τ. In terms of the following five remarks, we can claim that for $\tau > \tau_1$, system (1.1) has three orbits of waves—one orbit of discrete waves, one orbit of standing waves, and one orbit of mirror-reflecting waves—and only the last two orbits may coincide through the mechanism of periodic doubling. Discounting the above possible coincidence, system (1.1) has at least five wave solutions for each $\tau > \tau_1$.

Remark 2. A branch of nontrivial discrete waves and a branch of mirror-reflecting waves cannot coincide at any value of τ, for otherwise there exists a nontrivial ω-periodic solution x of (1.1) such that $x_i(t) = x_{i-1}(t \pm \frac{\omega}{3})$ for $i (\text{mod } 3)$ and $x_j(t) = x_k(t)$ for some $j \neq k$. For simplicity, let $x_2(t) = x_3(t)$. Then $x_2(t) = x_3(t \pm \frac{\omega}{3})$ implies that $\frac{\omega}{3}$ is also a period of $x_2 = x_3$, and thus $x_1(t) = x_2(t \pm \frac{\omega}{3}) = x_2(t)(= x_3(t))$. So x must be spatially homogeneous. As $\sup_{x \in \mathbb{R}} |h'(x)| < 1$ implies that $y = 0$ is the global attractor of the scalar equation $y'(t) = -y(t) + h(y(t - \tau))$ for any $\tau \geq 0$ (see, for example, [16]), we have $x = 0$, which is a contradiction.

Remark 3. A branch of nontrivial discrete waves and a branch of standing waves cannot coincide at any value of τ, for otherwise there exists a nontrivial ω-periodic solution x of (1.1) such that $x_i(t) = x_{i-1}(t \pm \frac{\omega}{3})$ for $i(\text{mod } 3)$ and, say, $x_1(t) = x_1(t + \frac{\omega}{2}), x_2(t) = x_3(t + \frac{\omega}{2})$. Then $x_2(t) = x_3(t + \frac{\omega}{3}) = x_3(t + \frac{\omega}{2})$. (The other case in which $x_2(t) = x_3(t - \frac{\omega}{3})$ can be dealt similarly.) Therefore, $\frac{\omega}{6}$ is also a period of x_3 (and thus x_2). Consequently, $x_2(t) = x_3(t + \frac{\omega}{3}) = x_3(t)$ and $x_1(t) = x_2(t + \frac{\omega}{3}) = x_2(t) = x_3(t)$. Again, x must be spatially homogeneous, and thus $x = 0$, which is a contradiction.

Remark 4. A branch of nontrivial discrete waves of the form $x_i(t) = x_{i-1}(t - \frac{\omega}{3})$ and a branch of discrete waves of the form $x_i(t) = x_{i-1}(t + \frac{\omega}{3})$ for $i (\text{mod } 3)$ and $t \in \mathbb{R}$ cannot coincide at any value of τ. Again, this can be verified by way of contradiction. Namely, if there is a discrete wave satisfying simultaneously $x_i(t) = x_{i-1}(t + \frac{\omega}{3}) = x_{i-1}(t - \frac{\omega}{3})$ for $i (\text{mod } 3)$, then $\frac{2\omega}{3}$ and ω are periods of x, and so is $\frac{\omega}{3}$. This, together with $x_i(t) = x_{i-1}(t - \frac{\omega}{3})$, implies that x is spatially homogeneous, and thus $x = 0$, which is a contradiction.

Remark 5. As no nontrivial spatially homogeneous periodic solution exists, it is clear that a branch of nontrivial mirror-reflecting waves satisfying $x_i(t) = x_j(t)$ for some $i \neq j$ and a branch of mirror-reflecting waves satisfying $x_l(t) = x_m(t)$ for some $l \neq m$ cannot coincide at any value of τ if $(i, j) \neq (l, m)$. Similarly, a branch of nontrivial standing waves with $x_i(t) = x_i(t + \frac{\omega}{2}), x_j(t) = x_k(t + \frac{\omega}{2})$ for $i \neq j \neq k$ and a branch of nontrivial standing waves with $x_{i^*}(t) = x_{j^*}(t + \frac{\omega}{2}), x_{j^*}(t) = x_{k^*}(t + \frac{\omega}{2})$ for $i^* \neq j^* \neq k^*$ cannot coincide at any value of τ if $(i, j, k) \neq (i^*, j^*, k^*)$.

Remark 6. Unfortunately, the above arguments cannot be extended to rule out the possibility of the coincidence of a branch of nontrivial ω-periodic mirror-reflecting waves with $x_i(t) = x_j(t)$ for some $i \neq j$ and a branch of ω-periodic standing waves with $x_i(t) = x_j(t + \frac{\omega}{2})$ for some $i \neq j$. In fact, such a coincidence may occur at some value of τ where periodic doubling happens: $x_i(t) = x_i(t + \frac{\omega}{2}), i(\bmod 3), t \in \mathbb{R}$.

Acknowledgment. We wish to thank one reviewer for her or his valuable comments that led to truly significant improvement of the manuscript.

REFERENCES

[1] J. Szenfagothai, *The "module-concept" in cerebral cortex architecture*, Brain Research, 95 (1967), pp. 475–496.
[2] J. C. Eccles, M. Ito, and J. Szenfagothai, *The Cerebellum as Neuronal Machine*, Springer-Verlag, New York, 1967.
[3] P. Anderson, O. Gross, T. Lomo, and O. Sveen, *Participation of inhibitory interneurons in the control of hippocampal cortical output*, in The Interneuron, M. Brazier, ed., University of California Press, Los Angeles, 1969.
[4] F. C. Hoppensteadt, *An Introduction to the Mathematics of Neurons*, Cambridge University Press, New York, 1986.
[5] G. M. Shepherd, *The Synaptic Organization of the Brain*, Oxford University Press, New York, 1990.
[6] R. J. Douglas, A. C. Martin, and D. Whitteridge, *A canonical microcircuit for neocortex*, Neural Comp., 1 (1989), pp. 480–488.
[7] J. Milton, *Dynamics of Small Neural Populations*, AMS, Providence, RI, 1996.
[8] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
[9] J. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.
[10] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations. Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.
[11] Y. Chen and J. Wu, *Existence and attraction of a phase-locked oscillation in a delayed network of two neurons*, Differential Integral Equations, 14 (2001), pp. 1181–1236.
[12] Y. Chen and J. Wu, *Minimal instability and unstable set of a phase-locked orbit in a delayed neural network*, Phys. D, 134 (1999), pp. 185–199.
[13] Y. Chen, J. Wu, and T. Krisztin, *Connecting orbits from synchronous periodic solutions to phase-locked periodic solutions in a delay differential system*, J. Differential Equations, 163 (2000), pp. 130–173.
[14] J. Wu, *Introduction to Neural Dynamics and Signal Transmission Delay*, Walter de Gruyter, Berlin, 2001.
[15] J. Mallet-Paret and G. Sell, *Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations, 125 (1996), pp. 380–440.
[16] J. Mallet-Paret and G. Sell, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay*, J. Differential Equations, 125 (1996), pp. 441–489.
[17] T. Krisztin. H.-O. Walther, and J. Wu, *Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback*, AMS, Providence, RI, 1999.
[18] M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Springer-Verlag, New York, 1988.
[19] M. A. Cohen and S. Grossberg, *Absolute stability of global pattern formation and parallel memory storage by competitive neural networks*, IEEE Trans. Systems Man Cybernet., 13 (1983), pp. 815–826.
[20] J. J. Hopfield, *Neurons with graded response have collective computational properties like those of two-stage neurons*, Proc. Nat. Acad. Sci. USA, 81 (1984), pp. 3088–3092.
[21] C. M. Marcus and R. M. Westervelt, *Stability of analog neural networks with delay*, Phys. Rev. A, 39 (1989), pp. 347–359.
[22] J. Bélair, *Stability in a model of a delayed neural network*, J. Dynam. Differential Equations, 5 (1993), pp. 607–623.
[23] J. Wu, *Symmetric functional differential equations and neural networks with memory*, Trans. Amer. Math. Soc., 350 (1998), pp. 4799–4838.
[24] W. Krawcewicz and J. Wu, *Theory and applications of Hopf bifurcations in symmetric functional differential equations*, Nonlinear Anal., 35 (1999), pp. 845–870.

[25] W. Krawcewicz and J. Wu, *Theory of Degrees with Applications to Bifurcations and Differential Equations*, John Wiley and Sons, Boston, 1996.

[26] B. W. Levinger, *A folk theorem in functional differential equations*, J. Differential Equations, 4 (1968), pp. 612–619.

[27] J. LaSalle, *The Stability of Dynamical Systems*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 25, SIAM, Philadelphia, 1976.

[28] A. Lasota and J. A. Yorke, *Bounds for periodic solutions of differential equations in Banach spaces*, J. Differential Equations, 10 (1971), pp. 83–91.

[29] J. R. Haddock and J. Terjéki, *Liapunov-Razumikhin functions and invariance principle for functional differential equations*, J. Differential Equations, 48 (1983), pp. 95–122.