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BIFURCATION FROM A HOMOCLINIC ORBIT IN PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider a family of partial functional differential equations which has a homoclinic orbit asymptotic to an isolated equilibrium point at a critical value of the parameter. Under some technical assumptions, we show that a unique stable periodic orbit bifurcates from the homoclinic orbit. Our approach follows the ideas of Šil'nikov for ordinary differential equations and of Chow and Deng for semilinear parabolic equations and retarded functional differential equations.

1. Introduction.

For an ordinary differential equation

$$\dot{x} = g(x, \epsilon), \tag{1.1}$$

where $x \in R^n, \epsilon \in R$ is a parameter and g is a smooth function, it is known that if $x = 0$ is a hyperbolic equilibrium for $\epsilon = 0$ and the Jacobian matrix $D_x f(0, 0) = A$ has a unique eigenvalue $\lambda > 0$ which is simple and the real parts of all other eigenvalues are strictly less than $-\lambda$, then under certain additional transversality conditions, a unique stable periodic orbit bifurcates from the homoclinic orbit as the parameter ϵ changes. See, for example, Andronov et al. [AL73], Chow and Hale [CH86] and Kuznetsov [Ku95]. One of the approaches to the above bifurcation problem, originated in the work of Neimark and Šil'nikov [NS65] and Šil'nikov [Si68] for ordinary differential equations in R^n with $n \geq 3$, is to reduce the bifurcation problem to a problem of the continuation of fixed points for a one-parameter map in a small neighborhood of the hyperbolic equilibrium. This map resembles the well-known Poincaré map but the points on the stable manifold do not return. In

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what follows we shall call this map the *Šil'nikov map* and we refer to Kuznetsov [Ku95] for a detailed description of Sil'nikov's results and techniques.

The above result has been generalized to other kinds of equations, while several other methods have also been developed. These include the work of Blázquez [Bl86] for semilinear parabolic equations, and of Walther [Wa90] for retarded functional differential equations. We should particularly mention the work of Chow and Deng [CD89] for some infinite dimensional dynamical systems including semilinear parabolic differential equations and retarded functional differential equations, where they obtained some subtle estimates related to linear variational equations along semiorbits of the nonlinear equations and established the smoothness and the existence of a fixed point of the Šil'nikov map.

In this paper, we consider the following one-parameter family of partial functional differential equations:

$$\dot{u}(t) = Au(t) + L(u_t) + g(u_t, \epsilon), \tag{1.2}$$

where A is the generator of an analytic semigroup, L is a linear operator and g is a smooth nonlinear functional. g depends on not only the current but also the historic status of u. More specific descriptions will be given in next section. This kind of equations is motivated by reaction-diffusion equations where the reaction terms may involve time delay and have been studied by many researchers, see, for example, Faria [Fa99, Fa01], Faria et al. [FHW02], Hale [Ha86], Hale and Ladeira [HL93], He [He90], Martin and Smith [MS90], Memory [Me91], Travis and Webb [TW74, TW78], etc. For an introduction of the fundamental theory of such equations and some related references, we refer to the monograph by Wu [Wu96].

The purpose of this paper is to generalize Šil'nikov's theorem and Chow and Deng's techniques to the above partial functional differential equations. In section 2, we introduce the notations and present the main results. The differentiability of solutions of equation (1.2) with respect to the initial values and parameters and the smoothness of the stable and unstable manifolds are proved in section 3. The local analysis of equation (1.2) near the equilibrium is given in section 4. In section 5, we construct the Šil'nilov map and discuss some of its properties. The proof of the main theorem is presented in section 6.

2. The Main Results. Let X denote a Banach space over $R = (-\infty, \infty)$ and $B(X, X)$ the Banach space of bounded linear operators from X to X equipped with the operator norm. Let $r > 0$ be a given constant and $\mathcal{C} = C([-r, 0]; X)$ the Banach space of continuous X-valued functions on $[-r, 0]$ with the supremum norm $|\cdot|$. For any real numbers $a \le b$, $t \in [a, b]$ and any continuous mapping $u : [a - r, b] \to X$, u_t denotes the element of $\mathcal{C}$ given by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$.

Consider the following family of partial functional differential equations

$$\dot{u}(t) = Au(t) + L(u_t) + g(u_t, \epsilon), \tag{2.1}$$

where $\epsilon \in (-\epsilon_0, \epsilon_0)$ is a parameter, ϵ_0 is a given positive constant, A, L and g satisfy the following assumptions:

(H1) A is the infinitesimal generator of an analytic compact semigroup $\{S(t)\}_{t \ge 0}$ on X.

(H2) $L : \mathcal{C} \to X$ is given by

$$L\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta), \quad \phi \in \mathcal{C}$$

for a function $\eta : [-r, 0] \to B(X, X)$ of bounded variation.

(H3) $g \in C^3(\mathcal{C} \times (-\epsilon_0, \epsilon_0); X)$ and $g(0, \epsilon) = 0$, $D_\phi g(0, \epsilon) = 0$ for $\epsilon \in (-\epsilon_0, \epsilon_0)$.

The associated linear equation is given by

$$\dot{u}(t) = Au(t) + L(u_t). \tag{2.2}$$

For each complex number λ, define the linear operator $\Delta(\lambda) : \mathrm{Dom}(A) \to X$ by

$$\Delta(\lambda)u = Au - \lambda u + L(e^{\lambda \cdot} u), \quad u \in \mathrm{Dom}(A),$$

where $e^{\lambda \cdot} u \in \mathcal{C}$ is defined by

$$(e^{\lambda \cdot} u)(\theta) = e^{\lambda\theta} u, \quad \theta \in [-r, 0].$$

λ is called a *characteristic value* of equation (2.2) if there exists $u \in \mathrm{Dom}(A) \setminus \{0\}$ solving the *characteristic equation*

$$\Delta(\lambda)u = 0.$$

A characteristic value λ is *simple* if $\dim(\mathrm{Ker}(\Delta(\lambda))^n) = 1$ for all positive integer n. We further assume that

(H4) Equation (2.2) has a unique positive characteristic value $\lambda > 0$ which is simple and the real parts of all other characteristic values of (2.2) are smaller than $-\lambda$.

It is known that for each $\phi \in \mathcal{C}$, the initial value problem

$$\begin{aligned} u(t) &= S(t)\phi(0) + \int_0^t S(t-\alpha)L(u_\alpha)d\alpha, \quad t \geq 0, \\ u_0 &= \phi \end{aligned}$$

has a unique solution defined for $t \geq -r$. Denote this solution by

$$T(t)\phi = u_t(\phi),$$

then $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{C}$ with the generator denoted by A_T. Also, for each $\phi \in \mathcal{C}$ and $\epsilon \in (-\epsilon_0, \epsilon_0)$, there exists $\tau(\phi, \epsilon) > 0$ and a unique continuous map $u = u(\phi, \epsilon) : [-r, \tau(\phi, \epsilon)) \to X$ such that

$$u(\phi, \epsilon)(t) = S(t)\phi(0) + \int_0^t S(t-\alpha)[L(u_\alpha(\phi, \epsilon)) + g(u_\alpha(\phi, \epsilon), \epsilon)]d\alpha$$

for $t \in [0, \tau(\phi, \epsilon))$. Using the mapping $X_0 : [-r, 0] \to B(X, X)$ defined by

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0, \\ I, & \theta = 0, \end{cases}$$

we have the following variation of constants formula (see He [He90], Memory [Me91] and Wu [Wu96])

$$\begin{aligned} u(t) &= T(t)\phi + \int_0^t T(t-\alpha)X_0 g(u_\alpha, \epsilon)d\alpha, \\ u_0 &= \phi \end{aligned} \tag{2.3}$$

for $u(\phi,\epsilon)$ on $[0,\tau(\phi,\epsilon))$. By assumption **(H4)**, $\mathcal{C}$ can be decomposed as $\mathcal{C}=\mathcal{C}^s\oplus\mathcal{C}^u$, where $\mathcal{C}^u$ is the one-dimensional eigenspace of A_T associated with $\{\lambda\}$ and $\mathcal{C}^s$ is the generalized eigenspace associated with the remaining spectrum. Let $\phi_\lambda=\phi_\lambda(0)e^{\lambda\cdot}$ be the eigenvector of A_0 associated with λ and ϕ_λ^* be the eigenvector corresponding to $\{\lambda\}$ of the formal adjoint operator associated with the bilinear pairing (see Travis and Webb [Tw74])

$$\langle\psi,\phi\rangle=\psi(0)\phi(0)-\int_{-r}^0\int_0^\theta\psi(\xi-\theta)[d\eta(\theta)]\phi(\xi)d\xi,$$

where $\psi\in\mathcal{C}^*=C([0,r];X^*)$, X^* is the dual space of X and $\phi\in\mathcal{C}$. Then

$$\begin{aligned}\mathcal{C}^u&=\{\phi;\phi\in\mathcal{C},\ \phi=a\phi_\lambda \text{ for some } a\in R\},\\ \mathcal{C}^s&=\{\phi;\phi\in\mathcal{C},\ \ \langle\phi_\lambda^*,\phi\rangle=0\}.\end{aligned}$$

For every $\phi^s+\phi^u\in\mathcal{C}^s\oplus\mathcal{C}^u$, we have

$$\phi^u=\langle\phi_\lambda^*,\phi\rangle\phi_\lambda,\quad \phi^s=\phi-\phi^u.$$

Note that for $\phi\in\mathcal{C}$,

$$T(t)\phi(\theta)=\begin{cases}\phi(t+\theta), & -r\le t+\theta\le 0,\\ T(t+\theta)\phi(0), & t+\theta\ge 0\end{cases}\tag{2.4}$$

and for $\phi^u\in\mathcal{C}^u$,

$$\begin{aligned}T(t)\phi^u&=\phi^u e^{\lambda t},\quad t\in R,\\ \phi^u(\theta)&=\phi^u(0)e^{\lambda\theta},\quad \theta\in[-r,0].\end{aligned}\tag{2.5}$$

Let P^s and P^u be the projections of $\mathcal{C}$ onto $\mathcal{C}^s$ and $\mathcal{C}^u$, respectively, i.e. $\mathcal{C}^s=P^s\mathcal{C},\mathcal{C}^u=P^u\mathcal{C}$. It is shown that P^s and P^u can be applied to the elements X_0w with $w\in X$. Define X_0^s and X_0^u by

$$X_0^uw=P^uX_0w,\quad X_0^sw=P^sX_0w,\ \ w\in X.\tag{2.6}$$

Note that if $w\in X$, then $T(t)X_0^uw\in\mathcal{C}^u$ for all $t\in R$ and $T(t)X_0^sw\in\mathcal{C}^s$ for $t\ge r$. Moreover, there exist constants K_1 and $\mu>\lambda>0$ such that

$$\begin{aligned}&|T(t)\phi^s|\le K_1e^{-\mu t}|\phi^s|,\ \ t\ge 0,\ \ \phi^s\in\mathcal{C}^s;\quad |T(t)X_0^s|\le K_1e^{-\mu t},\quad t\ge 0.\\ &|T(t)\phi^u|\le K_1e^{\mu|t|}|\phi^u|,\ \ t\le 0,\ \ \phi^u\in\mathcal{C}^u;\quad |T(t)X_0^u|\le K_1e^{\mu|t|},\ \ t\le 0.\end{aligned}\tag{2.7}$$

Decompose $u_t(\phi,\epsilon)$ as

$$u_t(\phi,\epsilon)=u_t^s(\phi,\epsilon)+u_t^u(\phi,\epsilon)$$

with $u_t^s(\phi,\epsilon)\in\mathcal{C}^s$ and $u_t^u(\phi,\epsilon)\in\mathcal{C}^u$. Then we have the following variation of constants formula (see He [He90], Memory [Me91] or Wu [Wu96]):

$$\begin{aligned}u_t^s(\phi,\epsilon)&=T(t)\phi^s+\int_0^tT(t-\alpha)X_0^sg(u_\alpha(\phi,\epsilon),\epsilon)d\alpha,\\ u_t^u(\phi,\epsilon)&=e^{\lambda t}\phi^u+\int_0^te^{\lambda(t-\alpha)}X_0^ug(u_\alpha(\phi,\epsilon),\epsilon)d\alpha\end{aligned}\tag{2.8}$$

for $t \in [0, \tau(\phi, \epsilon))$.

Since g is C^3-smooth, by the differentiability of the solution with respect to initial values and parameters (see Theorem 3.1 in section 3), $u_t(\phi, \epsilon)$ is C^3-smooth in (ϕ, ϵ) for all $t \geq 0$ in the maximal interval of existence. Set

$$v_t^s(\phi, \epsilon) = D_\phi u_t^s(\phi, \epsilon), \quad v_t^u(\phi, \epsilon) = D_\phi u_t^u(\phi, \epsilon). \tag{2.9}$$

We have

$$\begin{aligned} v_t^s(\phi, \epsilon) &= T(t)P^s + \int_0^t T(t-\alpha)X_0^s D_\phi g(u_\alpha(\phi, \epsilon), \epsilon)v_\alpha(\phi, \epsilon)d\alpha, \\ v_t^u(\phi, \epsilon) &= T(t)P^u + \int_0^t T(t-\alpha)X_0^u D_\phi g(u_\alpha(\phi, \epsilon), \epsilon)v_\alpha(\phi, \epsilon)d\alpha \end{aligned} \tag{2.10}$$

with $T(t)P^u\phi = e^{\lambda t}\phi^u, \ \ \phi \in \mathcal{C}, \ \ t \in R$. By assumptions **(H3)** and **(H4)** there exist $\delta_1 > 0$ and $\epsilon_1 \in (0, \epsilon_0)$ such that the local stable and unstable manifolds $W_{\text{loc}}^s(\epsilon)$ and $W_{\text{loc}}^u(\epsilon)$ exist and are subsets of $B(\delta_1)$ for $\epsilon \in [-\epsilon_1, \epsilon_1]$, where $B(\delta_1) = \{\phi \in \mathcal{C}; |\phi^s| < \delta_1, |\phi^u| < \delta_1\}$ and $W_{\text{loc}}^s(\epsilon)$ and $W_{\text{loc}}^u(\epsilon)$ are given by

$$\begin{aligned} W_{\text{loc}}^s(\epsilon) &= \{\phi = \phi^s + \phi^u; \phi^u = h_s(\phi^s, \epsilon), \ |\phi^s| < \delta_1\}, \\ W_{\text{loc}}^u(\epsilon) &= \{\phi = \phi^s + \phi^u; \phi^s = h_u(\phi^u, \epsilon), \ |\phi^u| < \delta_1\}, \end{aligned} \tag{2.11}$$

where h_s and h_u are C^3-smooth (see Theorem 3.2 in section 3) and h_u is defined by

$$h_u(\phi^u, \epsilon) = \int_{-\infty}^0 T(-\alpha)X_0^s g(u_\alpha^*(\phi^u, \epsilon), \epsilon)d\alpha, \quad |\phi^u| < \delta_1, \quad \epsilon \in [-\epsilon_1, \epsilon_1] \tag{2.12}$$

(see Memory [Me91]), and $u_t^*(\phi^u, \epsilon)$ is the unique bounded solution of (2.1) on $(-\infty, 0]$ with

$$|u_t^*(\phi^u, \epsilon)| \leq K_2 e^{\mu t}|\phi^u|, \quad t \leq 0 \tag{2.13}$$

for some positive constant K_2 independent of (ϕ^u, ϵ).

In order to state the main theorem, we need one additional assumption:

(H5) When $\epsilon = 0$, equation (2.1) has a homoclinic orbit Γ_0 asymptotic to the equilibrium 0.

For a fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, let $W_+^u(\epsilon)$ be the orbit of equation (2.1) through a given $\phi_0 \in W_{\text{loc}}^u(\epsilon)$ with $\langle \phi_\lambda^*, \phi_0^u - h_s(\phi_0^s, \epsilon)\rangle > 0$. Without loss of generality, we assume that the homoclinic orbit $\Gamma_0 = W_+^u(0)$. Here, a homoclinic orbit Γ_0 asymptotic to 0 is a continuous mapping $u : R \to X$ satisfying

$$u(t) = S(t-s)u(s) + \int_s^t S(t-\alpha)[L(u_\alpha) + g(u_\alpha, 0)]d\alpha$$

for $t, s \in R$ with $t \geq s$, and $\lim_{t\to\pm\infty} u(t) = 0$. Now we can state our main theorem on homoclinic bifurcation of (2.1), which is a generalization of the results of Šil'nikov [Si68] and Chow and Deng [CD89] to abstract semilinear functional differential equations.

Theorem 2.1. *Suppose* **(H1)** $-$ **(H5)** *hold. Then there exist a neighborhood* $\mathcal{N}(\Gamma_0)$ *of* $\Gamma_0 \cup \{0\}$ *in* $\mathcal{C}$ *and* $\bar{\epsilon}_0 \in (0, \epsilon_0)$ *such that* $W_+^u(\epsilon) \cap W_{\text{loc}}^s(\epsilon) = \emptyset$ *if and only if there exists a periodic orbit in* $\mathcal{N}(\Gamma_0)$ *for given* $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$. *Furthermore, for the given* $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$ *this periodic orbit is unique and exponentially asymptotically stable.*

Corollary 2.2. *Under the same assumptions of Theorem 2.1, there exists a neighborhood $\mathcal{N}(\Gamma_0)$ of $\Gamma_0 \cup \{0\}$ in $\mathcal{C}$ and $0 < \bar{\epsilon}_0 < \epsilon_0$ such that if there is a homoclinic orbit in $\mathcal{N}(\Gamma_0)$ for equation (2.1) at ϵ with $|\epsilon| \leq \bar{\epsilon}_0$, then there exist no periodic orbits of equation (2.1) lying entirely in $\mathcal{N}(\Gamma_0)$ at ϵ.*

3. Preliminaries

3. Preliminaries. In this section, we prove the differentiability of solutions of equation (2.1) with respect to initial values and parameters and the smoothness of the stable and unstable manifolds.

3.1. Differentiability with Respect to Initial Values and Parameters. Let V be a neighborhood of 0 in $\mathcal{C}$, (a,b) be an open interval in R and $F \in C^k(V \times (a,b); X)$. Consider

$$\begin{aligned} u(t) &= T(t)\phi(0) + \int_0^t T(t-s)F(u_s, \alpha)ds, \\ u_0 &= \phi. \end{aligned} \tag{3.1}$$

Theorem 3.1. *The solution $u(\phi,\alpha)$ is C^k−smooth with respect to (ϕ,α) for t in any compact set of of the domain of definition of $u(\phi,\alpha)$. Moreover, for each $\psi \in \mathcal{C}$, $D_\phi u(\phi,\alpha)\psi(t)$ satisfies the linear variational equation*

$$\begin{aligned} v(t) &= T(t)\psi(0) + \int_0^t T(t-s)D_\phi F(u_s(\phi,\alpha),\alpha)v_s ds, \\ v_0 &= \psi. \end{aligned} \tag{3.2}$$

In the proof, we shall use Lemma 4.2 and the argument for Theorem 4.1 in Hale and Verduyn Lunel [HV93].

Proof. Fix $\xi \in V$ and $\alpha_0 \in (a,b)$. There exist constants $M > 0$, $\delta > 0$ and $N > 0$ such that

$$\begin{cases} |T(t) \leq M \text{ for } 0 \leq t \leq 1, \\ \overline{B_\delta(\xi)} \subseteq V \text{ with } B_\delta(\xi) = \{\psi \in \mathcal{C}; \|\psi - \xi\| < \delta\}, \\ [\alpha_0 - \delta, \alpha_0 + \delta] \subseteq (a,b), \\ |F(\psi,\alpha)| \leq N, \ |D_\phi F(\psi,\alpha)| \leq N \quad \text{for} \quad (\psi,\alpha) \in \overline{B_\delta(\xi)} \times [\alpha_0 - \delta, \alpha_0 + \delta]. \end{cases}$$

Now choose $\eta \in (0,1)$ and $\nu \in (0,1)$ so that

$$\begin{cases} \nu < \dfrac{\delta}{2}, \quad \eta < \dfrac{\nu}{MN}, \\ \sup\limits_{\substack{\theta,\theta' \in [-r,0] \\ |\theta'-\theta| \leq \eta}} |\xi(\theta') - \xi(\theta)| < \dfrac{\delta}{8}, \\ \sup\limits_{t \in [0,\eta]} \|T(t)\xi(0) - \xi(0)\| < \dfrac{\delta}{4}. \end{cases}$$

Let

$$K(\eta,\nu) = \{w \in C([-r,\eta]; X); w_0 = 0, \|w_t\| \leq \nu \text{ for } t \in [0,\eta)\}.$$

Clearly $K(\eta,\nu)$ is a closed subset of the Banach space

$$C_0([-r,\eta]) = \{\phi \in C([-r,\eta]; X); \ \phi(\theta) = 0 \text{ for } \theta \in [-r,0]\}$$

equipped with the super-norm.

For each $\phi \in \mathcal{C}$, define $\tilde{\phi} : [-r, \infty) \to X$ by $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}(t) = T(t)\phi(0)$ for $t \geq 0$. Now for fixed $\phi \in B_{\frac{\delta}{4(1+M)}}(\xi)$ and $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$, define $A(\phi, \alpha)$ on $K(\eta, \nu)$ by

$$A(\phi,\alpha)w(t) = \begin{cases} \int_0^t T(t-s)F(w_s + \tilde{\phi}_s, \alpha)ds, & w \in K(\eta,\nu), \quad t \in [0,\eta], \\ 0 & t \in [-r, 0]. \end{cases}$$

It follows that $A(\phi, \alpha)w \in C([-r, \eta]; X)$. Moreover, since for $s \in [0, \eta], \|w_s\| \leq \nu$ and

$$\begin{aligned} \|\tilde{\phi}_s - \xi\| \leq & \|\tilde{\phi}_s - \tilde{\xi}_s\| + \|\tilde{\xi}_s - \xi\| \\ \leq & \|\phi - \xi\| + \sup_{s\in[0,\eta]} \|T(s)\| |\phi(0) - \xi(0)| + \sup_{\substack{\theta\in[-r,0] \\ s\in[0,\eta] \\ s+\theta\in[-r,0]}} \|\xi(s+\theta) - \xi(\theta)\| \\ & + \sup_{\substack{\theta\in[-r,0] \\ s\in[0,\eta] \\ s+\theta\geq 0}} \|T(s+\theta)\xi(0) - \xi(0)\| + \sup_{\substack{\theta\in[-r,0] \\ s\in[0,\eta] \\ s+\theta\leq 0}} \|\xi(\theta) - \xi(0)\| \\ \leq & (1+M)\|\phi - \xi\| + \frac{\delta}{8} + \frac{\delta}{4} + \frac{\delta}{8} < \frac{\delta}{2}, \end{aligned}$$

we have $\|w_s + \tilde{\phi}_s - \xi\| < \nu + \frac{\delta}{2} < \delta$ and hence

$$|F(w_s + \tilde{\phi}_s, \alpha)| \leq N \text{ for } s \in [0, \eta], \alpha \in [\alpha_0 - \delta, \alpha_0 + \delta].$$

This shows that

$$|A(\phi,\alpha)w(t)| \leq MN\eta < \nu \quad \text{for } t \in [0,\eta].$$

Thus, $A(\phi, \alpha)w \in K(\eta, \nu)$ and $A(\phi, \alpha)K(\eta, \nu) \subseteq K(\eta, \nu)$. Moreover, using $|D_\phi F(\phi, \alpha)| \leq N$ for $(\phi, \alpha) \in \overline{B_\delta(\xi)} \times [\alpha_0 - \delta, \alpha_0 + \delta]$, we have for $w, \hat{w} \in K(\eta, \nu)$ that

$$\begin{aligned} |A(\phi,\alpha)w(t) - A(\phi,\alpha)\hat{w}(t)| \leq & \left| \int_0^t T(t-s)[F(w_s + \tilde{\phi}_s, \alpha) - F(\hat{w}_s + \tilde{\phi}_s, \alpha)]ds \right| \\ \leq & MN\eta \sup_{s\in[0,t]} \|w_s - \hat{w}_s\| \\ \leq & MN\eta \sup_{s\in[-r,\eta]} \|w(s) - \hat{w}(s)\| \\ < & \nu \sup_{s\in[-r,\eta]} \|w(s) - \hat{w}(s)\|. \end{aligned}$$

Since $\nu < 1$, we conclude that

$$A(\cdot) : \overline{B_{\frac{\delta}{4(1+M)}}(\xi)} \times [\alpha_0 - \delta, \alpha_0 + \delta] \to K(\eta, \nu)$$

is a uniform contraction. By Lemma 4.2 of Hale and Verduyn Lunel [HV93], for each fixed $(\phi, \alpha) \in \overline{B_{\frac{\delta}{4(1+M)}}(\xi)} \times [\alpha_0 - \delta, \alpha_0 + \delta]$, $A(\cdot)$ has a unique fixed point $w(\phi, \alpha) \in K(\eta, \nu)$ which is continuous in (ϕ, ν).

Note that $\overline{B_{\frac{\delta}{4(1+M)}}(\xi)} \times [\alpha_0 - \delta, \alpha_0 + \delta]$ is the closure of the open set $B_{\frac{\delta}{4(1+M)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta)$ and $A(\phi, \alpha)w$ has continuous $k-$th derivative with respect to $(\phi, \alpha, w) \in B_{\frac{\delta}{4(1+M)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta) \times K^0(\eta, \nu)$, where

$$K^0(\eta, \nu) = \{w \in K(\eta, \nu); \ \|w_t\| < \nu \text{ for } t \in [0, \alpha]\}$$

with $K^0(\eta,\nu)$ being open in $C_0([-r,\eta])$ and $K(\eta,\nu) = \overline{K^0(\eta,\nu)}$. Therefore, by Lemma 4.2, $w(\phi,\alpha)$ is C^k-smooth with respect to $(\phi,\alpha) \in B_{\frac{\delta}{4(1+M)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta)$. Hence, $u(\phi,\alpha) = \tilde{\phi} + w(\phi,\alpha)$ is C^k-smooth with respect to $(\phi,\alpha) \in B_{\frac{\delta}{4(1+M)}}(\xi) \times (\alpha_0 - \delta, \alpha_0 + \delta)$ for $t \in [0,\eta]$. Standard continuation argument then leads to the C^k-smoothness of $u(\phi,\alpha)$ with respect to (ϕ,α) for t in any compact subset of the domain of the definition of $u(\phi,\alpha)$. □

3.2. Smoothness of the Stable and Unstable Manifolds. In this subsection, we study the C^k-smoothness of the stable and unstable manifolds of equation (2.1) (Chow and Lu [CL88a, CL88b]). First, we modify assumption (**H3**) as follows:

(H3*) $g \in C^k(\mathcal{C} \times (-\epsilon_0,\epsilon_0); X)$ and $g(0,\epsilon) = 0$, $D_\phi g(0,\epsilon) = 0$ for $\epsilon \in (-\epsilon_0,\epsilon_0)$.

For a given $\epsilon \in (-\epsilon_0,\epsilon_0)$, a (mild) solution of equation (2.1) subject to the initial condition $u_0 = \phi \in \mathcal{C}$ on $[-r,\tau(\phi,\epsilon)), \tau(\phi,\epsilon) > 0$, is a continuous mapping $u = u(\phi,\epsilon) : [-r,\tau(\phi,\epsilon)) \to X$ such that

$$u(\phi,\epsilon)(t) = S(t)\phi(0) + \int_0^t S(t-\alpha)[L(u_\alpha(\phi,\epsilon)) + g(u_\alpha(\phi,\epsilon),\epsilon)]d\alpha$$

for $t \in [0,\tau(\phi,\epsilon))$. Note that $u(0,\epsilon)(t) = 0$ for all $t \geq 0$ is always a solution of (2.1) with $u_0 = 0$. Therefore, for a fixed $\tau_1 > r$, by the basic theory of (2.1) (see Wu [Wu96]) it follows that there exists an open neighborhood $\mathcal{N}_0$ of 0 in $\mathcal{C}$ and $\epsilon_1 \in (-\epsilon_0,\epsilon_0)$ such that for each $\epsilon \in (-\epsilon_1,\epsilon_1)$ and for every $\phi \in \mathcal{N}_0$, equation (2.1) has a solution $u(\phi,\epsilon)$ defined at least on $[-r,\tau_1]$. Let $\tilde{f} : \mathcal{N}_0 \times (-\epsilon_1,\epsilon_1) \to \mathcal{C}$ be given by

$$\tilde{f}(\phi,\epsilon) = u_{\tau_1}(\phi,\epsilon).$$

Then $\tilde{f}$ is completely continuous, C^k-smooth and $D_\phi\tilde{f}(0,0) = T(\tau_1)$ (defined in section 2). Let

$$\begin{aligned}
\Sigma_s &= \{\lambda \in \mathbb{C}; \lambda \text{ is a characteristic value of (2.2) with } \mathrm{Re}\lambda < 0\},\\
\Sigma_c &= \{\lambda \in \mathbb{C}; \lambda \text{ is a characteristic value of (2.2) with } \mathrm{Re}\lambda = 0\},\\
\Sigma_u &= \{\lambda \in \mathbb{C}; \lambda \text{ is a characteristic value of (2.2) with } \mathrm{Re}\lambda > 0\}
\end{aligned}$$

and assume that

$$\Sigma_u \neq \emptyset, \quad \Sigma_c = \emptyset.$$

For each $\lambda \in \Sigma_s \cup \Sigma_u$, let M_λ be the realized generalized eigenspace of A_T associated with λ and denote

$$\mathcal{C}^s = \bigoplus_{\lambda\in\Sigma_s} M_\lambda, \quad \mathcal{C}^u = \bigoplus_{\lambda\in\Sigma_u} M_\lambda.$$

Then we know that $\dim\mathcal{C}^u < \infty$, $\mathcal{C}^u$ and $\mathcal{C}^s$ are closed subspaces of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{C}^s \oplus \mathcal{C}^u$ and $T(\tau_1)\mathcal{C}^s \subseteq \mathcal{C}^s$, $T(\tau_1)\mathcal{C}^u \subseteq \mathcal{C}^u$.

Theorem 3.2. *We have the following results on the smoothness of the stable and unstable manifolds.*

(i) *There exist a constant $\epsilon_s \in (0,\epsilon_0)$, convex open bounded neighborhoods $\mathcal{N}_s$ of 0 in $\mathcal{C}^s$ and $\mathcal{N}_u$ of 0 in $\mathcal{C}^u$, and a C^k-smooth mapping $h_s : \mathcal{N}_s \times (-\epsilon_s,\epsilon_s) \to \mathcal{C}^u$ with $h_s(0,0) = 0$, $D_\phi h_s(0,0) = 0$, $h_s(\mathcal{N}_s \times (-\epsilon_s,\epsilon_s)) \subseteq \mathcal{N}_u$ such that for $\phi \in \mathcal{C}$ and $\epsilon \in (-\epsilon_s,\epsilon_s)$, if equation (2.1) has a solution $u(\phi,\epsilon)$ on*

$[-r,\infty)$ *satisfying* $u_t(\phi,\epsilon)\in\mathcal{N}_s\times\mathcal{N}_u$ *for* $t\geq 0$, *then* $\phi\in W^s_{\text{loc}}(\epsilon)$, *where* $W^s_{\text{loc}}(\epsilon)$ *is the stable manifold defined by*

$$W^s_{\text{loc}}(\epsilon)=\{\phi^s+h_s(\phi^s,\epsilon);\phi^s\in\mathcal{N}_s\}.$$

(ii) *There exist a constant* $\epsilon_u\in(0,\epsilon_0)$ *and a* C^k-*smooth mapping* $h_u:\mathcal{N}_u\times(-\epsilon_u,\epsilon_u)\to\mathcal{C}^u$ *with* $h_u(0,0)=0$, $D_\phi h_u(0,0)=0$, $h_u(\mathcal{N}_u\times(-\epsilon_u,\epsilon_u))\subseteq\mathcal{N}_s$ *such that for* $\phi\in\mathcal{C}$ *and* $\epsilon\in(-\epsilon_u,\epsilon_u)$, *if there exists* $u(\phi,\epsilon):(-\infty,0]\to X$ *satisfying* $u_0(\phi,\epsilon)=\phi$,

$$u(\phi,\epsilon)(t)=S(t-\theta)\phi(\theta)+\int_\theta^t S(t-\alpha)[L(u_\alpha(\phi,\epsilon))+g(u_\alpha(\phi,\epsilon),\epsilon)]d\alpha$$

for $t,\theta\leq 0$ *with* $t\geq\theta$, *and* $u_t(\phi,\epsilon)\in\mathcal{N}_s\times\mathcal{N}_u$ *for* $t\leq 0$, *then* $\phi\in W^u_{\text{loc}}(\epsilon)$, *where* $W^u_{\text{loc}}(\epsilon)$ *is the unstable manifold defined by*

$$W^u_{\text{loc}}(\epsilon)=\{\phi^u+h_u(\phi^u,\epsilon);\phi^u\in\mathcal{N}_u\}.$$

Proof. (i) Let $f:\mathcal{N}_0\times[-\epsilon_1,\epsilon_1]\to\mathcal{C}\times R$ be given by $f(\phi,\epsilon)=(\tilde{f}(\phi,\epsilon),\epsilon)$. Clearly, f is C^k-smooth, $f(0,0)=0$ and

$$L=Df(0,0)=(D_\phi\tilde{f}(0,0),\text{Id})=(T(\tau_1),\text{Id}).$$

Hence, $E=\mathcal{C}\times R$ has the following decomposition

$$E=E_s\oplus E_c\oplus E_u$$

with

$$E_s=\mathcal{C}^s\times\{0\},\ E_c=\{0\}\times R,\ E_u=\mathcal{C}^u\times\{0\}.$$

Clearly, $E_s\neq\{0\}$ is a closed subspace, $E_c\neq\{0\}$ and $E_u\neq\{0\}$ with $\dim E_c=1$ and $\dim E_u=\dim\mathcal{C}^u<\infty$. We have

$$\begin{cases} LE_s\subseteq E_s,\ \ LE_c\subseteq E_c,\ \ LE_u\subseteq E_u,\\ \sigma(L|_{E_s})\subseteq\{z\in\mathbb{C};|z|\leq a\}\text{ for some }a\in(0,1),\\ \sigma(L|_{E_c})=\{1\},\\ \sigma(L|_{E_u})\subseteq\{z\in\mathbb{C};|z|\geq 1\}.\end{cases}$$

By Theorem II.1 of Krisztin, Walther and Wu [KWW99], there exist open neighborhoods $\tilde{\mathcal{N}}_{sc}$ of 0 in $E_s\oplus E_c$, $\tilde{\mathcal{N}}_u$ of 0 in E_u and a C^k-smooth mapping $\tilde{h}:\tilde{\mathcal{N}}_{sc}\to E_u$ (Theorem II.1 ensures C^1-smoothness if f is C^1-smooth, the same argument there yields C^k-smoothness of $\tilde{h}$ if f is C^k-smooth) with

$$\tilde{h}(0,0)=0,\ \ D_{(\phi,\epsilon)}\tilde{h}(0,0)=0,\ \ \tilde{h}(\tilde{\mathcal{N}}_{sc})\subseteq\tilde{\mathcal{N}}_u$$

and

$$\bigcap_{n=0}^{\infty}f^{-n}(\tilde{\mathcal{N}}_{sc}\cup\tilde{\mathcal{N}}_u)\subseteq\tilde{W},$$

where

$$\tilde{W} = \{(\phi^s, \epsilon) + \tilde{h}(\phi^s, \epsilon); (\phi^s, \epsilon) \in \tilde{\mathcal{N}}_{sc}\}.$$

Let $\pi_u : E_u \to \mathcal{C}^u$ be the natural projection. Find open neighborhoods $\mathcal{N}_s$ of 0 in $\mathcal{C}^s$, $\mathcal{N}_u$ of of 0 in $\mathcal{C}^u$ and $\epsilon_s \in (0, \epsilon_0)$ so that

$$\mathcal{N}_s \times (-\epsilon_s, \epsilon_s) \subseteq \tilde{h}(\tilde{\mathcal{N}}_{sc}), \;\; \mathcal{N}_u \times \{0\} = \tilde{\mathcal{N}}_u.$$

Also let

$$h_s : \mathcal{N}_s \times (-\epsilon_s, \epsilon_s) \to \mathcal{C}^u$$

be given by

$$h_s(\phi^s, \epsilon) = \pi_u \tilde{h}(\phi^s, \epsilon).$$

Then h_s is C^k-smooth and satisfies

$$\begin{cases} h_s(0,0) =& \pi_u \tilde{h}(0,0) = 0, \\ D_\phi h_s(0,0) =& \pi_u D_\phi \tilde{h}(0,0) = 0, \\ h_s(\mathcal{N}_s \times (-\epsilon_s, \epsilon_s)) =& \pi_u \tilde{h}(\mathcal{N}_s \times (-\epsilon_s, \epsilon_s)) \\ & \subseteq \pi_u \tilde{h}(\tilde{\mathcal{N}}_{sc}) \subseteq \pi_u \tilde{\mathcal{N}}_u = \mathcal{N}_u. \end{cases}$$

Assume that $u(\phi, \epsilon)$ is a solution of equation (2.1) on $[-r, \infty)$ with $\epsilon \in (-\epsilon_s, \epsilon_s)$ and $u_t(\phi, \epsilon) \in \mathcal{N}_s \times \mathcal{N}_u$ for $t \geq 0$. Fix $t \geq 0$. Then for each integer $n \geq 0$,

$$u_{t+n\tau_1}(\phi, \epsilon) = \tilde{f}^n(u_t(\phi, \epsilon), \epsilon) \in \mathcal{N}_s \times \mathcal{N}_u.$$

Therefore,

$$f^n(u_t(\phi, \epsilon), \epsilon) = (\tilde{f}^n(u_t(\phi, \epsilon), \epsilon), \epsilon) \in (\mathcal{N}_s \times (-\epsilon_s, \epsilon_s)) \cup (\mathcal{N}_u \times \{0\}) \subseteq \tilde{\mathcal{N}}_{sc} \cup \tilde{\mathcal{N}}_u.$$

Consequently, $u_t(\phi, \epsilon) \in \tilde{W}$. In other words,

$$u_t(\phi, \epsilon) = (\tilde{\phi}^s, \epsilon) + \tilde{h}(\tilde{\phi}^s, \epsilon) \text{ for some } (\tilde{\phi}^s, \epsilon) \in \tilde{\mathcal{N}}_{sc}.$$

As $\tilde{\phi}^s = u_t^s(\phi, \epsilon) \in \mathcal{N}_s$ and $\epsilon \in (-\epsilon_s, \epsilon_s)$, we must have

$$u_t^u(\phi, \epsilon) = \pi_u \tilde{h}(\tilde{\phi}^s, \epsilon) = h_s(\tilde{\phi}^s, \epsilon) = h_s(u_t^s(\phi, \epsilon), \epsilon).$$

That is, $u_t(\phi, \epsilon) \in W$. This proves (i).

(ii) Using Theorem III.1 of Krisztin, Walther and Wu [KWW99], the smoothness of the unstable manifold can be proved similarly. $\square$

4. Local Analysis. Under hypothesis **(H3)**, we may assume, without loss of generality, that the constant K_2 defined in (2.13) is positive and that $\delta_1 > 0$ is chosen so that for $|\phi^s| < \delta_1, |\phi^u| \leq \delta_1$ and $\epsilon \in [-\epsilon_1, \epsilon_1]$, we have

$$|h_s(\phi^s, \epsilon)| \leq K_2 |\phi^s|^2, \tag{4.1}$$

$$|D_{\phi^s} h_s(\phi^s, \epsilon) \cdot \psi^s| \leq K_2 |\phi^s| |\psi^s|, \quad \psi^s \in \mathcal{C}^s, \tag{4.2}$$

$$|D^2_{\phi^s} h_s(\phi^s, \epsilon) \cdot (\psi_1^s, \psi_2^s)| \leq K_2 |\psi_1^s| \, |\psi_2^s|, \quad \psi_i^s \in \mathcal{C}^s, \, i = 1, 2 \tag{4.3}$$

and

$$|h_u(\phi^u,\epsilon)| \leq K_2|\phi^u|^2, \tag{4.4}$$
$$|D_{\phi^u}h_u(\phi^u,\epsilon)\cdot\psi^u| \leq K_2|\phi^u||\psi^u|, \quad \psi^u \in \mathcal{C}^u, \tag{4.5}$$
$$|D^2_{\phi^u}h_u(\phi^u,\epsilon)\cdot(\psi^u_1,\psi^u_2)| \leq K_2|\psi^u_1|\cdot|\psi^u_2|, \quad \psi^u_i \in \mathcal{C}^u, i=1,2. \tag{4.6}$$

Denote $u^*_t = u^*_t(\phi^u,\epsilon)$ and $h_u = h_u(\phi^u,\epsilon)$ for $|\phi^u| < \delta_1$ and $|\epsilon| \leq \epsilon_1$. By (2.4), the definition of X_0 and the fact that $X^s_0 + X^u_0 = X_0$, we have

$$T(-\alpha)X^s_0(\theta) = \begin{cases} X^s_0(\theta-\alpha) = -X^u_0(\theta-\alpha), & \theta-\alpha \leq 0, \\ X^s_0(0) + \int_0^{\theta-\alpha} L(T(\beta)X^s_0)d\beta, & \theta-\alpha > 0. \end{cases} \tag{4.7}$$

Thus, (2.12) can be rewritten as follows

$$\begin{aligned} h_u(\theta) = &\int_{-\infty}^{\theta} [X^s_0(0) + \int_0^{\theta-\alpha} L(T(\beta)X^s_0)d\beta] g(u^*_\alpha,\epsilon)d\alpha \\ &- \int_\theta^0 X^u_0(\theta-\alpha)g(u^*_\alpha,\epsilon)d\alpha. \end{aligned}$$

By the smoothness of the local unstable manifold $W^u_{\text{loc}}(\epsilon)$, differentiation of $h_u(\theta)$ with respect to $\theta \in [-r,0]$ leads to

$$\begin{aligned} \frac{d}{d\theta}h_u(\theta) = X_0(0)g(u^*_\theta,\epsilon) + L\left(\int_{-\infty}^0 T(-\alpha)X^s_0 g(u^*_{\alpha+\theta},\epsilon)d\alpha\right) \\ - \int_\theta^0 \frac{d}{d\theta}X^u_0(\theta-\alpha)g(u^*_\alpha,\epsilon)d\alpha. \end{aligned} \tag{4.3}$$

Similar to the proof of Proposition 3.2 in Chow and Deng [CD89], we have the following lemma.

Lemma 4.1. *For $\phi^u \in \mathcal{C}^u$ with $|\phi^u| < \delta_1, \theta \in [-r,0]$ and $\epsilon \in [-\epsilon_1,\epsilon_1]$, we have*

$$\frac{d}{d\theta}\left(D_{\phi^u}h_u\cdot\phi^u_1\right)(\theta) = \left(D_{\phi^u}\frac{d}{d\theta}h_u(\theta)\right)\cdot\phi^u_1, \quad \phi^u_1 \in \mathcal{C}^u, \tag{4.8}$$
$$\frac{d}{d\theta}(D^2_{\phi^u}h_u\cdot(\phi^u_1,\phi^u_2))(\theta) = \left(D^2_{\phi^u}\frac{d}{d\theta}h_u(\theta)\right)\cdot(\phi^u_1,\phi^u_2), \quad \phi^u_1,\phi^u_2 \in \mathcal{C}^u, \tag{4.9}$$
$$\frac{d}{d\theta}\left(D_{\phi^u}\frac{d}{d\theta}h_u\cdot\phi^u_1\right)(\theta) = \left(D_{\phi^u}\frac{d^2}{d\theta^2}h_u(\theta)\right)\cdot\phi^u_1, \quad \phi^u_1 \in \mathcal{C}^u. \tag{4.10}$$

Moreover, there exists a constant $K_3 > 0$ depending on $\delta_1, \epsilon_1, K_1$ and K_2 such that

$$\left|\frac{d}{d\theta}[(D^2_{\phi^u}h_u)\cdot(\phi^u_1,\phi^u_2)]\right| \leq K_3|\phi^u_1|\,|\phi^u_2|, \quad \phi^u_1,\phi^u_2 \in \mathcal{C}^u, \tag{4.11}$$
$$\left|\frac{d}{d\theta}\left[\left(D_{\phi^u}\frac{d}{d\theta}h_u\right)\cdot\phi^u_1\right]\right| \leq K_3|\phi^u|\,|\phi^u_1|, \quad \phi^u_1 \in \mathcal{C}^u. \tag{4.12}$$

By the smoothness of the stable and unstable manifolds (Theorem 3.2) and following the argument in the proof of Proposition 3.4 in Chow and Deng [CD89], we have the following lemma.

Lemma 4.2. *Let $\phi^u \in \mathcal{C}^u$ with $|\phi^u| < \delta_1, \theta \in [-r, 0]$ and $\epsilon \in [-\epsilon_1, \epsilon_1]$. Then there exists a constant $K_4 > 0$ depending on $\delta_1, \epsilon_1, K_1, K_2$ and K_3 such that*
(i) $\frac{d}{dt}T(t)h_u|_{t=0^+} = \frac{d}{d\theta}h_u - X_0 g(h_u + \phi^u, \epsilon)$;
(ii) $\left|T(t)\frac{d}{d\tau}T(\tau)h_u|_{\tau=0^+}\right| \le K_4 e^{-\mu t}|\phi^u|^2, \quad t \ge 0.$

Define $\bar{\phi} = H(\phi, \epsilon), \phi \in B(\delta_1), |\epsilon| \le \epsilon_1$, by

$$\bar{\phi}^s = \phi^s - h_u(\phi^u, \epsilon), \quad \bar{\phi}^u = \phi^u. \tag{4.13}$$

In terms of the new variable $\bar{\phi}$, we have

$$W^u_{\text{loc}}(\epsilon) = \{\bar{\phi} : \bar{\phi} \in H(B(\delta_1), \epsilon), \quad \bar{\phi}^s = 0\}.$$

The inverse H^{-1} of H is given by

$$\phi^s = \bar{\phi}^s + h_u(\bar{\phi}^u, \epsilon), \quad \phi^u = \bar{\phi}^u.$$

The variation of constants formula (2.8) becomes

$$\begin{aligned} u^s_t(\bar{\phi}, \epsilon) &= T(t)\bar{\phi}^s + \int_0^t T(t-\alpha)\bar{f}(\bar{u}_\alpha, \epsilon)d\alpha \\ u^u_t(\bar{\phi}, \epsilon) &= e^{\lambda t}\bar{\phi}^u + \int_0^t e^{\lambda(t-\alpha)}X^u_0 \bar{g}(\bar{u}_\alpha, \epsilon)d\alpha \end{aligned} \tag{4.14}$$

where

$$\bar{u}_t = \bar{u}_t(\bar{\phi}, \epsilon) = H(u_t(\phi, \epsilon), \epsilon), \quad \bar{g}(\bar{\phi}, \epsilon) = g(H^{-1}(\bar{\phi}, \epsilon), \epsilon),$$

and

$$\bar{f}(\bar{\phi}, \epsilon) = X^s_0 \bar{g}(\bar{\phi}, \epsilon) - \frac{d}{dt}T(t)h_u(\bar{\phi}^u, \epsilon)|_{t=0^+} - D_{\bar{\phi}^u}h_u(\bar{\phi}^u, \epsilon)\cdot[\lambda_{\bar{\phi}^u} + X^u_0 \bar{f}(\bar{\phi}, \epsilon)]. \tag{4.15}$$

Lemma 4.3. *There exist constants $0 < \delta_3 < \delta_2$ and $0 < \epsilon_3 < \epsilon_2$ and a map $F : B(\delta_3) \times [-\epsilon_3, \epsilon_3] \to L(\mathcal{C}^s, L^\infty)$, $L^\infty = L^\infty([-r, 0], X)$, such that $\bar{f}(\bar{\phi}, \epsilon) = F(\bar{\phi}, \epsilon)\cdot\bar{\phi}^s$ for $\bar{\phi} \in B(\delta_3)$ and $|\epsilon| < \epsilon_3$, where $\bar{\phi}^s = P^s\bar{\phi}$. Furthermore*
(i) F is C^1;
(ii) if $\bar{\psi} \in \mathcal{C}^s$, then $(F(\bar{\phi}, \epsilon)\cdot\bar{\psi}^s)(\theta)$ is C^1 in $\theta \in [-r, 0]$;
(iii) there exists a constant $K_5 > 0$ depending on δ_3, ϵ_3 and $K_i (i = 1, 2, 3, 4)$ such that for every $\phi \in B(\delta_3)$, $\epsilon \in [-\epsilon_3, \epsilon_3]$,

$$|F(\bar{\phi}, \epsilon)\cdot\bar{\psi}^s|_{L^\infty} \le K_5|\bar{\phi}|\,|\bar{\psi}^s|,$$
$$\sup_{-r\le\theta\le0}\left|\frac{d}{d\theta}(F(\bar{\phi}, \epsilon)\cdot\bar{\psi}^s)(\theta)\right| \le K_5|\bar{\psi}^s|.$$

Proof. Claim: There exist $0 < \delta_3 < \delta_2$ and $0 < \epsilon_3 < \epsilon_2$ such that if $|\bar{\phi}| < \delta_3, \bar{\phi} \in \mathcal{C}^u$ and $|\epsilon| \le \epsilon_3$, then $\bar{f}(\bar{\phi}, \epsilon)(\theta) = 0$ for $\theta \in [-r, 0]$.

Suppose $|\bar{\phi}| < \delta_2$ and $|\epsilon| \le \epsilon_2$. Define

$$t_0 = t_0(\phi, \epsilon) = \sup\{t \ge 0 : u_t(\phi, \epsilon) \in H^{-1}(B(\delta_2), \epsilon)\}.$$

If $\delta_2 > 0$ is sufficiently small, then $t_0 > 2r$ for all $|\phi| \le \bar{\delta}_2$ and $|\epsilon| \le \epsilon_2$. Since H is near the identity map, there exists $0 < \epsilon_3 < \epsilon_2$ and $0 < \delta_3 < \delta_2$ such that

$$B(\delta_3) \subset H(B(\bar{\delta}_2), \epsilon), \quad |\epsilon| \le \epsilon_3.$$

To prove the claim, we suppose that there exist $\bar{\phi}_0^u \in \mathcal{C}^u$ with $|\bar{\phi}_0^u| < \delta_3$, $|\epsilon_0| \le \epsilon_3$ and $\theta_0 \in [-r, 0]$, such that

$$\bar{f}(\bar{\phi}_0^u, \epsilon_0)(\theta_0) \ne 0. \tag{4.16}$$

Let $\phi_0 \in H^{-1}(\bar{\phi}_0^u, \epsilon)$. Then $\phi_0 \in W^u_{\rm loc}(\epsilon_0)$ and $|\phi_0| \le \bar{\delta}_2$. Let $\bar{u}_t = H(u_t(\phi_0, \epsilon_0), \epsilon_0)$, $-\theta_0 \le t \le t_0 - \theta_0$. Since $W^u_{\rm loc}(\epsilon_0) \subset \{\bar{\phi} : \bar{\phi}^s = P^s\bar{\phi} = 0\}$ and $\bar{u}^s_{t+\theta_0} = 0$ for $-\theta_0 \le t \le t_0 - \theta_0$, $\bar{u}_0 = H(\phi_0, \epsilon) = \bar{\phi}_0^u$. By (4.14)

$$0 = \bar{u}^s_{t+\theta_0} = \int_{-\theta_0}^{t} T(t-\alpha)\bar{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)d\alpha.$$

Let $t = \theta_0 + \sigma$ and $\theta < \sigma \le t_0$ with $\sigma < -\theta_0$ if $\theta_0 \ne 0$. By (2.5), we have

$$\begin{aligned} 0 &= \int_{-\theta_0}^{-\theta_0+\sigma} [T(-\theta_0+\sigma-\alpha)\bar{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)](\theta_0)d\alpha \\ &= \begin{cases} \int_{-\theta_0}^{-\theta_0+\sigma} \bar{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)(\sigma-\alpha)d\alpha & \text{if } -\theta_0 > \sigma > 0 \\ \int_0^\sigma T(\sigma-\alpha)[\bar{f}(\bar{u}_\alpha, \epsilon_0)(0)]d\alpha & \text{if } \theta_0 = 0. \end{cases} \end{aligned} \tag{4.17}$$

Dividing (4.17) by σ and letting $\sigma \to 0^+$, we have

$$\begin{aligned} 0 &= \lim_{\sigma\to 0^+} \frac{1}{\sigma}\int_{-\theta_0}^{-\theta_0+\sigma} [T(-\theta_0+\sigma-\alpha)\bar{f}(\bar{u}_{\alpha+\theta_0}, \epsilon_0)](\theta_0)d\alpha \\ &= \bar{f}(\bar{\phi}_0^u, \epsilon_0)(\theta_0), \quad \theta_0 \in [-r, 0], \end{aligned}$$

which contradicts (4.16). This proves the claim.

Hence

$$\bar{f}(\bar{\phi}, \epsilon) = \bar{f}(\bar{\phi}^s + \bar{\phi}^u, \epsilon) - \bar{f}(\bar{\phi}^u, \epsilon) = \left(\int_0^1 D_{\bar{\phi}^s}\bar{f}(\alpha\bar{\phi}^s + \bar{\phi}^u, \epsilon)d\alpha\right)\cdot\bar{\phi}^s.$$

Define

$$F(\bar{\phi}, \epsilon) = \int_0^1 D_{\bar{\phi}^s}\bar{f}(\alpha\bar{\phi}^s + \bar{\phi}^u, \epsilon)d\alpha. \tag{4.18}$$

By (4.15), we have

$$D_{\bar{\phi}^s}\bar{f}(\alpha\bar{\phi}^s + \bar{\phi}^u, \epsilon) = X_0^u(\bar{L}_1 + \bar{g}_1) - (D_{\bar{\phi}^u}h)\cdot(X_0^u\bar{g}_1) + X_0\bar{g}_1, \tag{4.19}$$

where $\bar{L}_1 = D_{\bar{\phi}^s}\bar{L}(\alpha\bar{\phi}^s + \bar{\phi}^u)$ and $\bar{g}_1 = D_{\bar{\phi}^s}\bar{g}(\alpha\bar{\phi}^s + \bar{\phi}^u, \epsilon)$. Thus $F(\bar{\phi}, \epsilon) : B(\delta_3) \times [-\epsilon_3, \epsilon_3] \to L(\mathcal{C}^s, L^\infty)$ is C^1 and $[F(\bar{\phi}, \epsilon)\cdot\bar{\psi}^s](\theta)$ is C^1 in $\theta \in [-r, 0]$ for all $\bar{\psi}^s \in \mathcal{C}^s$.

To prove (iii), we notice from (4.18) and (4.19) that $F(0, \epsilon) = 0$ for $\epsilon \in [-\epsilon_3, \epsilon_3]$. Thus, there exists $\tilde{K}_5 > 0$ depending on δ_2, ϵ_3 and $K_i (i = 1, 2, 3, 4)$ such that for $|\bar{\phi}| < \delta_3$, $|\epsilon| < \epsilon_3$ and $\bar{\psi}^s \in \mathcal{C}^s$,

$$|F(\bar{\phi}, \epsilon)\cdot\bar{\psi}^s| \le \tilde{K}_5|\bar{\phi}||\bar{\psi}^s|.$$

Also, by Lemma 4.1, there exists $\tilde{\tilde{K}}_5 > 0$ depending on δ_3, ϵ_3 and $K_i (i = 1, 2, 3, 4)$ such that for $|\bar{\phi}| < \delta_3, |\epsilon| < \epsilon_3$, and $\bar{\psi} \in \mathcal{C}^s$.

$$\left| \frac{d}{d\theta}[F(\bar{\phi}, \epsilon) \cdot \bar{\psi}^s](\theta) \right| \leq \tilde{\tilde{K}}_5 |\bar{\psi}^s|, \quad -r \leq \theta \leq \theta.$$

Choose $K_5 = \max\{\tilde{K}_5, \tilde{\tilde{K}}_5\}$, we prove (iii). □

From (4.15), we have

$$T(t)F(\bar{\phi}, \epsilon) = T(t)X_0^s \bar{g} - T(t)\frac{d}{d\tau}T(\tau)h_u|_{\tau=0^+} - T(t)D_{\bar{\phi}^u} h_u \cdot (\lambda \bar{\phi}^u + X_0^u \bar{g}), \quad (4.20)$$

where $\bar{g} = \bar{g}(\bar{\phi}, \epsilon)$ and $h_u = h_u(\bar{\phi}^u, \epsilon)$. Since $\mathcal{C}^s$ is closed, we have $T(t)X_0^s(\bar{L}+\bar{g}) \in \mathcal{C}^s$ for $t > r$ and $(D_{\bar{\phi}^u} h_u) \cdot (\lambda \bar{\phi}^u + X_0^u \bar{g}) \in \mathcal{C}^s$. Thus, by (2.7), (4.20) and Lemma 4.2, there exists a constant $K_6 > 0$ depending on δ_3, ϵ_3 and $K_i (i = 1, \dots, 5)$ such that for $\bar{\phi} \in B(\delta_3)$ and $|\epsilon| \leq \epsilon_3$,

$$|T(t)\bar{f}(\bar{\phi}, \epsilon)| \leq K_6 e^{-\mu t} |\bar{\phi}||\bar{\phi}^s|, \quad t \geq 0. \quad (4.21)$$

By Lemma 4.3, we can rewrite (4.14) as follows

$$\begin{aligned} \bar{u}_t^s(\bar{\phi}, \epsilon) &= T(t)\bar{\phi}^s + \int_0^t T(t-\alpha)F(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon) \cdot \bar{u}_\alpha^s(\bar{\phi}, \epsilon) d\alpha \\ \bar{u}_t^u(\bar{\phi}, \epsilon) &= e^{\lambda t}\bar{\phi}^u + \int_0^t e^{\lambda(t-\alpha)} \bar{g}(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon) d\alpha. \end{aligned} \quad (4.22)$$

Denote $\bar{v}_t = D_{\bar{\phi}} \bar{u}_t(\bar{\phi}, \epsilon)$. Denote $\bar{v}_t = D_{\bar{\phi}} \bar{u}_t(\bar{\phi}, \epsilon)$. Differentiating (4.22) with respect to $\bar{\phi}$ in L^∞, we can see that $\bar{v}_t^s = D_{\bar{\phi}} \bar{u}_t^s(\bar{\phi}, \epsilon)$ and $\bar{v}_t^u = D_{\bar{\phi}} \bar{u}_t^u(\bar{\phi}, \epsilon)$ satisfy the following variational equations:

$$\begin{aligned} \bar{v}_t^s &= T(t)P^s + \int_0^t T(t-\alpha)[D_{\bar{\phi}} F(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon) \cdot (\bar{v}_\alpha, \bar{u}_\alpha^s(\bar{\phi}, \epsilon)) \\ &\quad + F(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon) \cdot \bar{v}_\alpha^s] d\alpha, \quad 0 \leq t \leq t_0, \\ \bar{v}_t^u &= e^{\lambda t} P^u + \int_0^t e^{\lambda(t-\alpha)} D_{\bar{\phi}} \bar{g}(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon) \cdot \bar{v}_\alpha d\alpha, \quad 0 \leq t \leq t_0. \end{aligned} \quad (4.23)$$

Lemma 4.4. *There is a constant $K_7 > 0$ depending on δ_3, ϵ_3 and $K_i (i = 1, \dots 6)$ such that if $\bar{u}_t(\bar{\phi}, \epsilon) \in B(\delta_3)$ for all $0 \leq t \leq t_0$ where $t_0 > 0$ is any given constant, then*

$$|\bar{u}_t^s(\bar{\phi}, \epsilon)| \leq K_7 |\bar{\phi}^s| e^{-\mu t}, \quad 0 \leq t \leq t_0.$$

Proof. From (4.14), we have

$$|\bar{u}_t^s(\bar{\phi}, \epsilon)| \leq |T(t)\bar{\phi}^s| + \int_0^t |T(t-\alpha)\bar{f}(\bar{u}_\alpha(\bar{\phi}, \epsilon), \epsilon)| d\alpha.$$

By (2.7) and (4.21) and the assumption that $\bar{u}_t \in B(\delta_3)$, we obtain

$$|\bar{u}_t^s(\bar{\phi}, \epsilon)| \leq K_1 e^{-\mu t} |\bar{\phi}^s| + \delta_3 K_6 \int_0^t e^{-\mu(t-\alpha)} |\bar{u}_\alpha^s(\bar{\phi}, \epsilon)| d\alpha.$$

Let $x(t) = e^{\mu t}|\bar{u}_t^s(\bar{\phi}, \epsilon)|$. Then

$$x(t) \le K_1|\bar{\phi}^s| + \delta_3 K_6 \int_0^t x(\alpha)d\alpha.$$

The Gronwall's Inequality implies that

$$x(t) \le K_1|\bar{\phi}^s|e^{\delta_3 K_6 t_0} \quad \text{for} \quad 0 \le t \le t_0.$$

Choosing $K_7 = K_1 e^{\delta_3 K_6 t_0}$, we obtain the desired inequality. $\square$

Denote $\tilde{\mu} = \frac{\mu-\lambda}{4}$, $\tilde{\lambda} = \frac{\mu+\lambda}{2}$. Then $\tilde{\mu} < \mu$, $\tilde{\lambda} > \lambda$, and $-\mu + 2\tilde{\mu} + \tilde{\lambda} = 0$. Let $\delta_4 > 0$ be a small constant such that

$$\delta_4 \le \min\left\{\delta_3, \frac{1}{4K_1(K_2K_7(1/\tilde{\mu} + 1/\tilde{\lambda}) + K_5/(\mu - \tilde{\mu}))}, \frac{1}{2[K_2(1/(\lambda + \tilde{\mu}) + 1/(\tilde{\lambda} - \lambda))]^{\frac{1}{2}}}\right\}.$$

Lemma 4.5. *Let $\tilde{\lambda}, \tilde{\mu}$, and δ_4 be as above and $\tilde{\tilde{\lambda}} \in (0, \lambda)$. If $\bar{u}_t(\bar{\phi}, \epsilon) \in B(\delta_4)$ for $0 \le t \le t_0$ where t_0 is any given constant, then*

$$|D_{\bar{\phi}}\bar{u}_t^s(\bar{\phi}, \epsilon)| \le 2e^{-\tilde{\mu}t}, \quad t \in [0, t_0], \ \epsilon \in [-\epsilon_3, \epsilon_3],$$
$$|D_{\bar{\phi}}\bar{u}_t^u(\bar{\phi}, \epsilon)| \le 2e^{\tilde{\lambda}t}, \quad t \in [0, t_0], \ \epsilon \in [-\epsilon_3, \epsilon_3],$$
$$|D_{\bar{\phi}^u}\bar{u}_t^u(\bar{\phi}, \epsilon)| \le \frac{1}{2}e^{\tilde{\tilde{\lambda}}t}, \quad t \in [0, t_0], \ \epsilon \in [-\epsilon_3, \epsilon_3].$$

Proof. Let $\tilde{V}$ be a subset of $\mathcal{C}$ with a metric

$$d(w_{1t}, w_{2t}) = \max_{0 \le t \le t_0} |w_{1t} - w_{2t}|.$$

Define a subset V in $\tilde{V}$ as follows:

$$V = \{w_t : w_t = w_t^s + w_t^u \in \tilde{V}, w_0 = w_0^s + w_0^u = \text{ the identity map in } \mathcal{C},$$
$$|w_t^s| \le 2e^{-\tilde{\mu}t}, \ |w_t^u| \le 2e^{\tilde{\lambda}t}, \ 0 \le t \le t_0\}.$$

Clearly V is a closed subset of $\tilde{V}$. Let $\Phi : V \to \tilde{V}$, $\overline{w}_t = \Phi(w_t)$, be defined by

$$\overline{w}_t^s = T(t)P^s + \int_0^t T(t-\alpha)[D_{\overline{\phi}}F(\overline{u}_\alpha(\overline{\phi}, \epsilon), \epsilon) \cdot (\overline{w}_\alpha, \overline{u}_\alpha^s(\overline{\phi}, \epsilon)) + F(\overline{u}_\alpha(\overline{\phi}, \epsilon), \epsilon) \cdot \overline{w}_\alpha^s]d\alpha,$$
$$\overline{w}_t^u = e^{\lambda t}P^u + \int_0^t e^{\lambda(t-\alpha)}X_0^u D_{\overline{\phi}}\overline{g}(\overline{u}_\alpha(\overline{\phi}, \epsilon), \epsilon) \cdot \overline{w}_\alpha d\alpha.$$

By using (2.7), (4.18), (4.19), (4.2), Lemmas 4.3 and 4.4, we have

$$|\overline{w}_t^s| \le e^{-\mu t} + 2K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t e^{-\tilde{\mu}\alpha}d\alpha + 2K_1K_5\delta_4 e^{-\mu t}\int_0^t e^{(\mu-\tilde{\mu})\alpha}d\alpha$$
$$+ 2K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t e^{\tilde{\lambda}\alpha}d\alpha.$$

Since $-\mu < -\tilde{\mu}, -\mu + \tilde{\lambda} = -2\tilde{\mu} < -\tilde{\mu}$, we have

$$|\overline{w}_t^s| \le e^{-\tilde{\mu}t}\left(1 + \frac{2K_1K_2K_7\delta_4}{\tilde{\mu}} + \frac{2K_1K_5\delta_4}{\mu - \tilde{\mu}} + \frac{2K_1K_2K_7\delta_4}{\tilde{\lambda}}\right).$$

Since $\delta_4 \le 1/[4K_1(K_2K_7(1/\tilde{\mu} + 1/\tilde{\lambda}) + K_5/(\mu - \tilde{\mu}))]$, it follows that

$$|\overline{w}_t^s| \le 2e^{-\tilde{\mu}t}.$$

Similarly, since $\tilde{\lambda} > \lambda$, we have

$$\begin{aligned}
|\overline{w}_t^u| &\le e^{\lambda t} + 2K_2\delta_4^2 e^{\lambda t}\int_0^t e^{-(\lambda+\tilde{\mu})\alpha}d\alpha + 2K_2\delta_4^2 e^{\lambda t}\int_0^t e^{(\tilde{\lambda}-\lambda)\alpha}d\alpha \\
&\le e^{\tilde{\lambda}t}\left(1 + \frac{2K_2\delta_4^2}{\lambda + \tilde{\mu}} + \frac{2K_2\delta_4^2}{\tilde{\lambda} - \lambda}\right) \\
&\le 2e^{\tilde{\lambda}t}
\end{aligned}$$

provided $\delta_4 \le 1/2[K_2(1/(\lambda+\tilde{\mu}) + 1/(\tilde{\mu}-\lambda))]^{\frac{1}{2}}$. It follows that Φ maps V into itself. For $w_{1t}, w_{2t} \in V$, define another metric $\overline{d}$ as follows

$$\overline{d}(w_{1t}, w_{2t}) = \max_{0 \le t \le t_0}(e^{\tilde{\mu}t}|w_{1t}^s - w_{2t}^s| + e^{-\tilde{\lambda}t}|w_{1t}^u - w_{2t}^u|).$$

Then $(V, \overline{d})$ is a complete space. Let $\overline{w}_{1t} = \Phi(w_{1t}), \overline{w}_{2t} = \Phi(w_{2t})$. Then

$$\begin{aligned}
&|\overline{w}_{1t}^s - \overline{w}_{2t}^s| \\
&\quad\le \int_0^t (K_1K_2K_7\delta_4 e^{-\mu t} + K_1K_5\delta_4 e^{-\mu(t-\alpha)})|\overline{w}_{1\alpha}^s - \overline{w}_{2\alpha}^s|d\alpha \\
&\qquad + K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t |\overline{w}_{1\alpha}^u - \overline{w}_{2\alpha}^u|d\alpha \\
&\quad= \int_0^t (K_1K_2K_7\delta_4 e^{-\mu t} + K_1K_5\delta_4 e^{-\mu(t-\alpha)})e^{-\tilde{\mu}\alpha}(e^{\tilde{\mu}\alpha}|\overline{w}_{1\alpha}^s - \overline{w}_{2\alpha}^s|)d\alpha \\
&\qquad + K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t e^{\tilde{\lambda}\alpha}(e^{-\tilde{\lambda}\alpha}|\overline{w}_{1\alpha}^u - \overline{w}_{2\alpha}^u|)d\alpha \\
&\quad\le (K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t e^{-\tilde{\mu}\alpha}d\alpha + K_1K_5\delta_4 e^{-\mu t}\int_0^t e^{(\mu-\tilde{\mu})\alpha}d\alpha)\cdot\overline{d}(\overline{w}_{1t}, \overline{w}_{2t}) \\
&\qquad + K_1K_2K_7\delta_4 e^{-\mu t}\int_0^t e^{\tilde{\lambda}\alpha}d\alpha\cdot\overline{d}(\overline{w}_{1t}\overline{w}_{2t}) \\
&\quad\le K_1\delta_4\left[K_2K_7\left(\frac{1}{\tilde{\mu}} + \frac{1}{\tilde{\lambda}}\right) + \frac{K_5}{\mu - \tilde{\mu}}\right]e^{-\tilde{\mu}t}\cdot\overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).
\end{aligned}$$

Hence

$$e^{\tilde{\mu}t}|\overline{w}_{1t}^s - \overline{w}_{2t}^s| \le K_1\delta_4\left[K_2K_7\left(\frac{1}{\tilde{\mu}} + \frac{1}{\tilde{\lambda}}\right) + \frac{K_5}{\mu - \tilde{\mu}}\right]\cdot\overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).$$

Similarly,

$$e^{-\tilde{\lambda}t}|\overline{w}_{1t}^u - \overline{w}_{2t}^u| \le K_2\delta_4^2\left(\frac{1}{\lambda + \tilde{\mu}} + \frac{1}{\tilde{\lambda} - \lambda}\right)\cdot\overline{d}(\overline{w}_{1t}, \overline{w}_{2t}).$$

Thus,

$$
\begin{aligned}
&\overline{d}(\overline{w}_{1t},\overline{w}_{2t})\\
&\quad= \max_{0\le t\le t_0}(e^{\tilde{\mu}t}|\overline{w}^s_{1t}-\overline{w}^s_{2t}|+e^{-\tilde{\lambda}t}|\overline{w}^u_{1t}-\overline{w}^u_{2t}|)\\
&\quad\le \left\{K_1\delta_4\left[K_2K_7\left(\frac{1}{\tilde{\mu}}+\frac{1}{\tilde{\mu}}\right)+\frac{K_5}{\mu-\tilde{\mu}}\right]+K_2\delta_4^2\left(\frac{1}{\lambda+\tilde{\mu}}+\frac{1}{\tilde{\lambda}-\lambda}\right)\right\}\cdot\overline{d}(\overline{w}_{1t},\overline{w}_{2t})\\
&\quad\le\left(\frac14+\frac14\right)\overline{d}(\overline{w}_{1t},\overline{w}_{2t})\\
&\quad=\frac12\overline{d}(\overline{w}_{1t},\overline{w}_{2t}),
\end{aligned}
$$

which implies that $\Phi : V \to V$ is a contractive mapping under the new topology $(V,\overline{d})$. Hence, Φ has a unique fixed point, say $\tilde{w}_t\in V$, such that $\tilde{w}_t=\Phi(\tilde{w}_t)$. By the uniqueness of the solution of (4.23), we have

$$\tilde{w}^s_t = D_{\overline{\phi}}\overline{u}^s_t(\overline{\phi},\epsilon),\quad \tilde{w}^u_t = D_{\overline{\phi}}\overline{u}^u_t(\overline{\phi},\epsilon).$$

Thus, we have established the first two estimates. To show the third estimate, we differentiate the second equation in (4.22) and obtain

$$
\begin{aligned}
&D_{\overline{\phi}^u}\overline{u}^u_t(\overline{\phi},\epsilon)=e^{\lambda t}+\int_0^t e^{\lambda(t-\alpha)}X^u_0D_{\overline{\phi}^u}\overline{g}(\overline{u}_\alpha(\overline{\phi},\epsilon),\epsilon)\cdot D_{\overline{\phi}^u}\overline{u}\alpha(\overline{\phi},\epsilon)d\alpha),\\
&D_{\overline{\phi}^u}\overline{u}^u_0(\overline{\phi},\epsilon)=1.
\end{aligned}
$$

Let $t_1=\sup\{t: 0\le t\le t_0, D_{\overline{\phi}^u}\overline{u}^u_t(\overline{\phi},\epsilon)\ge 0\}$. Then $t_1>0$. We will show that $t_1=t_0$. Suppose $t_1<t_0$. Then $D_{\overline{\phi}^u}\overline{u}^u_{t_1}(\overline{\phi},\epsilon)=0$. We have

$$
\begin{aligned}
\frac{dD_{\overline{\phi}^u}\overline{u}^u_t(\overline{\phi},\epsilon)}{dt}&=\lambda D_{\overline{\phi}^u}\overline{u}^u_t(\overline{\phi},\epsilon)+X^u_0D_{\overline{\phi}^u}\overline{g}(\overline{u}_t(\overline{\phi},\epsilon),\epsilon)\cdot D_{\overline{\phi}^u}\overline{u}_t(\overline{\phi},\epsilon)\\
&\ge\tilde{\tilde{\lambda}}D_{\overline{\phi}^u}\overline{u}_t(\overline{\phi},\epsilon)-|X^u_0D_{\overline{\phi}^u}\overline{g}(\overline{u}_t(\overline{\phi},\epsilon),\epsilon)\cdot D_{\overline{\phi}^u}\overline{u}_t(\overline{\phi},\epsilon)|.
\end{aligned}
$$

It follows that

$$D_{\overline{\phi}^u}\overline{u}^u_t(\overline{\phi},\epsilon)\ge\frac12 e^{\tilde{\tilde{\lambda}}t},\quad 0\le t\le t_1.$$

Hence $D_{\overline{\phi}^u}\overline{u}^u_{t_1}(\overline{\phi},\epsilon)\ge\frac12 e^{\tilde{\tilde{\lambda}}t_1}\ne 0$, a contradiction. So $t_1=t_0$ and this completes the proof. $\square$

Note that in terms of the variable $\bar{\phi}$, the local stable and unstable manifolds are given by

$$
\begin{aligned}
W^s_{\rm loc}(\epsilon)&=\{\bar{\phi}:\bar{\phi}^u=\bar{h}_s(\bar{\phi}^s,\epsilon),\,|\bar{\phi}^s|<\delta_4\},\\
W^u_{\rm loc}(\epsilon)&=\{\bar{\phi}:\bar{\phi}^s=0,\,|\bar{\phi}^u|<\delta_4\},
\end{aligned}
$$

where $\bar{h}_s$ is C^3, $\bar{h}_s(0,\epsilon)=0$, $\epsilon\in[-\epsilon_3,\epsilon_3]$ and $D_{\bar{\phi}^s}\bar{h}_s(0,0)=0$. For every $\epsilon\in[-\epsilon_3,\epsilon_3]$, we define

$$
\begin{aligned}
\Omega=\Omega(\delta_4,\rho,\epsilon)&=\left\{\bar{\phi}:|\bar{\phi}^s|<\delta_4/K_7,\ |\langle\phi^*_\lambda,\bar{\phi}^u-\bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|<\rho\right\}\subset B(\delta_4),\\
\Omega^+=\Omega^+(\delta_4,\rho,\epsilon)&=\{\bar{\phi}:\bar{\phi}\in\Omega(\delta_4,\rho,\epsilon),\ 0<\langle\phi^*_\lambda,\bar{\phi}^u-\bar{h}_s(\bar{\phi}^s,\epsilon)\rangle<\rho\},\\
\Omega^-=\Omega^-(\delta_4,\rho,\epsilon)&=\{\bar{\phi}:\bar{\phi}\in\Omega(\delta_4,\rho,\epsilon),\ -\rho<\langle\phi^*_\lambda,\bar{\phi}^u-\bar{h}_s(\bar{\phi}^s,\epsilon)\rangle<0\},
\end{aligned}
$$

where $\langle\cdot\,;\,\cdot\rangle$ is defined in section 2. Note that since $W^s_{\rm loc}(\epsilon)$ has codimension one and $\Omega^+\cap\Omega^-=\emptyset$, $\Omega=\Omega^+\cup W^s_{\rm loc}(\epsilon)\cup\Omega^-$, we must have

Lemma 4.6. *Consider $\bar{u}_t(\bar{\phi},\epsilon)$ satisfying (4.22) in $B(\delta_4)$ and $0<\rho<\delta_4/4$, $\delta_4 < 1/(2K_2)$. If $\phi \in \Omega^+ \cup \Omega^-$, then there exists $\tau = \tau(\bar{\phi},\epsilon) > 0$ such that*

$$\bar{u}^u_\tau(\bar{\phi},\epsilon) - \bar{u}^u_\tau(\bar{\phi}^s + \bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon) = \begin{cases} \delta_4/2 & \text{if } \bar{\phi}\in\Omega^+, \quad |\epsilon|\le\epsilon_3 \\ -\delta_4/2 & \text{if } \bar{\phi}\in\Omega^-, \quad |\epsilon|\le\epsilon_3. \end{cases}$$

Furthermore, if $\bar{\phi}\in\Omega^+\cup\Omega^-$ and $|\epsilon|\le\epsilon_3$, then

$$\frac{1}{\tilde{\lambda}}\ln\frac{\delta_4}{4|\langle\phi^*_\lambda,\bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|} \le \tau \le \frac{1}{\tilde{\tilde{\lambda}}}\ln\frac{\delta_4}{|\langle\phi^*_\lambda,\bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|},$$
$$|D_{\bar{\phi}}\tau(\bar{\phi},\epsilon)| \le \frac{8}{\lambda\delta_4 - 8K_2\delta_4^2}e^{\tilde{\lambda}\tau(\bar{\phi},\epsilon)}.$$

Proof. If $\bar{\phi}\in\Omega^+\cup\Omega^-\subset\Omega\setminus W^s_{\mathrm{loc}}(\epsilon)$, then the solution cannot stay in $B(\delta_4)$ for all $t>0$. If $|\bar{\phi}^s| < \frac{\delta_4}{2K_7}$, then Lemma 4.4 implies that

$$|\bar{u}^s_t(\bar{\phi},\epsilon)| \le K_7\cdot\frac{\delta_4}{2K_7}\cdot e^{-\mu t} \le \frac{\delta_4}{2} \quad \text{for} \quad 0\le t\le t_0,$$

where $t_0>0$ is such that the solution $\bar{u}_t(\bar{\phi},\epsilon)\in B(\delta_4)$ for all $0\le t\le t_0$. Hence, $\bar{u}_t(\bar{\phi},\epsilon)$ has to leave $B(\delta_4)$ through either $\bar{u}^u_t(\bar{\phi},\epsilon)=\delta_4$ or $\bar{u}^u_t(\bar{\phi},\epsilon) = -\delta_4$. Thus,

$$\tilde{\tau} = \inf\{t>0 : |\bar{u}^u_t(\bar{\phi},\epsilon)| = \delta_4\}$$

is well defined.

Let

$$\Delta_t(\bar{\phi},\epsilon) = \langle\phi^*_\lambda, \bar{u}^u_t(\bar{\phi},\epsilon) - \bar{u}^u_t(\bar{\phi}^s + \bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon)\rangle, \quad 0\le t\le\tilde{\tau}.$$

Note that Δ_t is C^2. Since $\bar{\phi}\in\Omega$,

$$|\Delta_0(\bar{\phi},\epsilon)| = |\langle\phi^*_\lambda,\bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle| < \rho < \delta_4/4.$$

Note that by (4.1), if $|\bar{\phi}^s| < \delta_4$, then $|\bar{h}_s(\bar{\phi}^s,\epsilon)| \le K_2\delta_4^2 < \delta_4/2$. Since $\bar{u}_t(\bar{\phi}^s + \bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon)\in W^s_{\mathrm{loc}}(\epsilon)$, we have

$$|\Delta_{\tilde{\tau}}(\bar{\phi},\epsilon)| \ge |\langle\phi^*_\lambda,\bar{u}^u_{\tilde{\tau}}(\bar{\phi},\epsilon)\rangle| - |\langle\phi^*_\lambda,\bar{u}^s_{\tilde{\tau}}(\bar{\phi}^s+\bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon)\rangle| > \delta_4 - \delta_4/2 > \delta_4/4.$$

Thus, by the Intermediate Value Theorem,

$$\tau(\bar{\phi},\epsilon) = \inf\left\{t : 0<t<\tilde{t}, |\Delta_t(\bar{\phi},\epsilon)| = \delta_4/4\right\}$$

is well defined for $\bar{\phi}\in\Omega^+\cup\Omega^-$. We shall prove that

$$\Delta_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon) = \begin{cases} \delta_4/2 & \text{if } \bar{\phi}\in\Omega^+ \\ -\delta_4/2 & \text{if } \bar{\phi}\in\Omega^-. \end{cases} \tag{4.24}$$

We only prove $\Delta_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon) = \delta_4/2$ for $\bar{\phi}\in\Omega^+$, the other case can be treated similarly.

By the way of contradiction, suppose $\Delta_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon) = -\delta_4/2$. We have

$$\langle\phi^*_\lambda,\bar{u}^u_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon)\rangle = -\delta_4/2 + \langle\phi^*_\lambda,\bar{u}^u_{\tau(\bar{\phi},\epsilon)}(\bar{\phi}^s+\bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon)\rangle < -\delta_4/4. \tag{4.25}$$

Define

$$d_t(\bar{\phi},\epsilon) = \langle \phi_\lambda^*, \bar{u}_t^u(\bar{\phi},\epsilon) - \bar{h}_s(\bar{u}_t^s(\bar{\phi},\epsilon),\epsilon)\rangle.$$

Since $\bar{\phi} \in \Omega^+$, we have

$$d_0(\bar{\phi},\epsilon) = \langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle = \Delta_0(\bar{\phi},\epsilon) > 0.$$

On the other hand, by (4.25), we have

$$d_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon) < -\delta_4/4 - \langle \phi_\lambda^*, \bar{h}_s(\bar{u}_t^s(\bar{\phi},\epsilon),\epsilon)\rangle < 0.$$

Thus the Intermediate Value Theorem implies that there exists $\tau_0, 0 < \tau_0 < \tau(\bar{\phi},\epsilon)$, such that

$$d_{\tau_0}(\bar{\phi},\epsilon) = 0$$

which implies that $\bar{u}_{\tau_0}(\bar{\phi},\epsilon) \in W^s_{\text{loc}}(\epsilon)$. Therefore,

$$\bar{u}_t(\bar{\phi},\epsilon) \in W^s_{\text{loc}}(\epsilon)$$

which contradicts $\bar{\phi} \in \Omega \setminus W^s_{\text{loc}}(\epsilon)$. This proves (4.24).

Since $\Delta_t(\bar{\phi},\epsilon)$ is C^2 and

$$\frac{\partial}{\partial t}\Delta_t(\bar{\phi},\epsilon) = \lambda\Delta_t(\bar{\phi},\epsilon) + \bar{g}(\bar{u}_t(\bar{\phi},\epsilon),\epsilon) - \bar{g}(\bar{u}_t(\bar{\phi}^s + \bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon),\epsilon),$$

we have

$$\frac{\partial}{\partial t}\Delta_t(\bar{\phi},\epsilon)|_{t=\tau(\bar{\phi},\epsilon)} \geq \frac{\lambda\delta_4}{2} - 4K_2\delta_4^2 > 0, \quad \bar{\phi} \in \Omega^+. \tag{4.26}$$

Applying the Implicit Function Theorem to

$$\Delta_t(\bar{\phi},\epsilon) = \delta_4/2, \;\; \bar{\phi} \in \Omega, \;\; \epsilon \in [-\epsilon_2, \epsilon_2],$$

it follows that $\tau(\bar{\phi},\epsilon)$ is C^2. Moreover, by Lemma 4.5, we have

$$\begin{aligned}\delta_4/2 &= \langle \phi_\lambda^*, \bar{u}^u_{\tau(\bar{\phi},\epsilon)}(\bar{\phi},\epsilon) - \bar{u}^u_{\tau(\bar{\phi},\epsilon)}(\bar{\phi}^s + \bar{h}_s(\bar{\phi}^s,\epsilon),\epsilon)\rangle| \\ &\leq 2e^{\tilde{\lambda}\tau(\bar{\phi},\epsilon)}|\langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|.\end{aligned}$$

Therefore,

$$\tau(\bar{\phi},\epsilon) \geq \frac{1}{\tilde{\lambda}} \ln \frac{\delta_4}{4|\langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|}.$$

Similarly we can show that

$$\tau(\bar{\phi},\epsilon) \leq \frac{1}{\tilde{\tilde{\lambda}}} \ln \frac{\delta_4}{|\langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s,\epsilon)\rangle|}.$$

Let $\bar{\phi} \in \Omega^+ \cup \Omega^-$. Differentiating (4.24) and using the Chain Rule, we have

$$\frac{\partial}{\partial t}\Delta_t(\bar{\phi},\epsilon)|_{t=\tau(\bar{\phi},\epsilon)} \cdot D_{\bar{\phi}}\tau(\bar{\phi},\epsilon) + D_{\bar{\phi}}\Delta_t(\bar{\phi},\epsilon)|_{t=\tau(\bar{\phi},\epsilon)} = 0.$$

Therefore, by (4.26) and Lemma 4.4, we obtain

$$|D_{\bar{\phi}}\tau(\bar{\phi},\epsilon)| \leq \frac{8}{\lambda\delta_4 - 8K_2\delta_4^2} e^{\tilde{\lambda}\tau(\bar{\phi},\epsilon)}.$$

This completes the proof. $\square$

Lemma 4.7. *Let $\bar{\phi} \in \Omega$, $|\epsilon| \leq \epsilon_3$ and $t_0 > 2r$ be as in the proof Lemma 4.3. Then there exists a constant $K_8 > 0$ depending on δ_3, ϵ_3 and $K_i (i = 1, \dots 7)$ such that if a solution $\bar{u}_t(\bar{\phi}, \epsilon)$ of (4.22) is in $B(\delta_4)$ for $0 \leq t \leq t_0$, then $\bar{u}^s_t(\bar{\phi}, \epsilon)$ is differentiable in $t \in (r, t_0)$ and satisfies*

$$\left| \frac{d}{dt} \bar{u}^s_t(\bar{\phi}, \epsilon) \right| \leq K_8 |\bar{\phi}| e^{-\mu t}, \quad r < t < t_0,$$

where $\frac{d}{dt}$ is taken in L^∞.

Proof. Let $t > r$. By (4.22) we have

$$\begin{aligned} \bar{u}^s_t(\theta) = T(t+\theta)\bar{\phi}^s(0) + \int_0^{t+\theta} T(t+\theta-\alpha)[F(\bar{u}_\alpha, \epsilon) \cdot \bar{u}^s_\alpha](0) d_\alpha \\ + \int_{t+\theta}^t [F(\bar{u}_\alpha, \epsilon) \cdot \bar{u}^s_\alpha](t+\theta-\alpha) d\alpha, \quad \theta \in [-r, 0]. \end{aligned}$$

By (4.18) and (4.19), we have

$$\int_{t+\theta}^t [F(\bar{u}_\alpha, \epsilon) \cdot \bar{u}^s_\alpha](t+\theta-\alpha) d\alpha = \int_{t+\theta}^t \tilde{F}(\alpha, t+\theta-\alpha) \cdot \bar{u}^s_\alpha d_\alpha,$$

where

$$\tilde{F}(\alpha, \theta) \cdot \bar{\psi}^s = [F(\bar{u}_\alpha, \epsilon) \cdot \bar{\psi}^s](\theta) - X_0(\theta) \int_0^1 \bar{g}_1(\bar{u}_\alpha, \epsilon, \beta) d\beta \cdot \bar{\psi}^s.$$

Lemma 4.3 implies that $\tilde{F}(\alpha, \theta)$ is C^1 in $\theta \in [-r, 0]$ and

$$\left| \frac{\partial}{\partial \theta} \tilde{F}(\alpha, \theta) \cdot \bar{\psi}^s \right| \leq K_5 |\bar{\psi}^s|, \quad |\tilde{F}(\alpha, \theta) \cdot \bar{\psi}^s| \leq K_5 |\bar{\psi}^s| \tag{4.27}$$

for $\quad 0 \leq \alpha \leq t_0, \theta \in [-r, 0]$ and $\bar{\psi}^s \in \mathcal{C}^s$. Thus

$$\begin{aligned} \frac{d}{dt} \bar{u}^s_t(\theta) = & L(T(t(\theta)\bar{\phi}^s) + [F(\bar{u}_{t+\theta}, \epsilon) \cdot \bar{u}^s_{t+\theta}](0) + \int_0^{t+\theta} L[T(t+\theta-\alpha) F(\bar{u}_\alpha, \epsilon) \cdot \bar{u}^s_\theta] d\alpha \\ & + \tilde{F}(t, \theta) \cdot \bar{u}^s_t - \tilde{F}(t+\theta, 0) \cdot \bar{u}^s_{t+\theta} - \int_{t+\theta}^t \frac{\partial}{\partial \theta} \tilde{F}(\alpha, t+\theta-\alpha) \cdot \bar{u}^s_\alpha d\alpha. \end{aligned}$$

This, together with (2.4), (4.27), Lemmas 4.1 and 4.4, implies the desired estimate and completes the proof. □

Define a function $\ell : B(\delta_4) \times [-\epsilon_3, \epsilon_3] \to R$ by

$$\ell(\bar{\phi}_1, \bar{\phi}_2, \epsilon) = \max_{i=1,2} \{ \langle \phi^*_\lambda, \bar{\phi}^u_i - \bar{h}_s(\bar{\phi}^s_i, \epsilon) \rangle \}.$$

Lemma 4.8. *Let $\bar{\phi} \in \Omega^+(\delta_4, \rho, \epsilon), \rho < \frac{\delta_4}{2}$ and $\epsilon \in [-\epsilon_3, \epsilon_3]$. There exist constants $K_9 > 0$ and $a > 0$ depending on δ_4, ϵ_3 and $K_i (i = 1, \dots, 8)$ such that if $\bar{\phi}_1, \bar{\phi}_2 \in \Omega^+(\delta_4, \rho, \epsilon)$ and $\epsilon \in [-\epsilon_3, \epsilon_3]$, then*

$$|\bar{u}_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}_1, \epsilon) - \bar{u}_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}_2, \epsilon)| \le K_9[\ell(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a |\bar{\phi}_1 - \bar{\phi}_2|. \tag{4.28}$$

Proof. For $\bar{\phi}_i = \bar{\phi}_i^s + \bar{\phi}_i^u \in \Omega^+(\delta_4, \rho, \epsilon),\ i = 1, 2$, define

$$\begin{aligned}
&\tilde{\phi}^s(\theta, \epsilon) = (1-\theta)\bar{\phi}_1^s + \theta\bar{\phi}_2^s,\\
&\tilde{\phi}^u(\theta, \epsilon) = \bar{\phi}_1^u - \bar{h}_s(\bar{\phi}_1^s, \epsilon) + \bar{h}_s(\tilde{\phi}^s(\theta, \epsilon), \epsilon),\ \ 0 \le \theta \le 1
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\tilde{\phi}}^s(\theta, \epsilon) = \tilde{\phi}_2^s,\\
&\tilde{\tilde{\phi}}^u(\theta, \epsilon) = (1-\theta)\tilde{\phi}^u(1, \epsilon) + \theta\bar{\phi}_2^u,\ \ 0 \le \theta \le 1.
\end{aligned}$$

Then

$$\tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon) \in \Omega^+(\delta_4, \rho, \epsilon), \quad \tilde{\tilde{\phi}}^s(\theta, \epsilon) + \tilde{\tilde{\phi}}^u(\theta, \epsilon) \in \Omega^+(\delta_4, \rho, \epsilon), \quad 0 \le \theta \le 1.$$

By the Chain Rule, Lemmas 4.5, 4.6, and 4.7, it follows that

$$\begin{aligned}
|D_\phi \bar{u}^s_{\tau(\bar{\phi},\epsilon)}(\bar{\phi}, \epsilon)| &= \left|\frac{d}{dt}\bar{u}^s_{\tau(\bar{\phi},\epsilon)}(\bar{\phi}, \epsilon) \cdot D_\phi \tau(\bar{\phi}, \epsilon)\right| + \left|D_\phi \bar{u}_t^s(\bar{\phi}, \epsilon)|_{t=\tau(\bar{\phi},\epsilon)}\right|\\
&\le \left|K_8|\bar{\phi}|e^{-\mu\tau(\bar{\phi},\epsilon)} \cdot \frac{8}{\lambda\delta_4 - 8K_2\delta_4^2} e^{\tilde{\lambda}\tau(\bar{\phi},\epsilon)}\right| + 2|e^{-\tilde{\mu}\tau(\bar{\phi},\epsilon)}|\\
&\le \frac{8K_8(\frac{\delta_4}{4})^{(-\mu+\tilde{\lambda})/\tilde{\lambda}}}{\lambda - 8K_2\delta_4} |\langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s, \epsilon)\rangle|^{(\tilde{\lambda}-\mu)/\tilde{\mu}}\\
&\quad + 2\left(\frac{\delta_4}{4}\right)^{-\tilde{\mu}/\tilde{\mu}} |\langle \phi_\lambda^*, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s, \epsilon)\rangle|^{\tilde{\mu}/\tilde{\lambda}}.
\end{aligned}$$

Also, by (4.1) and (4.2), we have

$$\begin{aligned}
\left|\frac{d}{d\theta}(\tilde{\phi}^s(\theta, \epsilon) + \tilde{\phi}^u(\theta, \epsilon))\right| &= |\bar{\phi}_1^s - \bar{\phi}_2^s| + \left|D_{\phi^s}\bar{h}_s(\tilde{\phi}(\theta, \epsilon), \epsilon) \cdot \frac{d}{d\theta}\tilde{\phi}(\theta, \epsilon)\right|\\
&\le (1 + K_2\delta_4^2)|\bar{\phi}_1^s - \bar{\phi}_2^s|\\
&\le (1 + K_2\delta_4^2)|\bar{\phi}_1 - \bar{\phi}_2|
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{d}{d\theta}(\tilde{\tilde{\phi}}^s(\theta, \epsilon) + \tilde{\tilde{\phi}}^u(\theta, \epsilon))\right| &= |\bar{\phi}_2^u - \tilde{\phi}^u(1, \epsilon)|\\
&\le |\bar{\phi}_2^u - \bar{\phi}_1^u| + |\bar{h}_s(\bar{\phi}_1^s, \epsilon) - \bar{h}_s(\phi_2^s, \epsilon)|\\
&\le (1 + K_2\delta_4^2)|\bar{\phi}_1 - \bar{\phi}_2|.
\end{aligned}$$

Denote

$$\tilde{K}_9(\delta_4) = \left[\frac{8K_8(\frac{\delta_4}{4})^{(-\mu+\tilde{\lambda})/\tilde{\lambda}}}{\lambda - 8K_2\delta_4} + 2\left(\frac{\delta_4}{4}\right)^{-\tilde{\mu}/\tilde{\lambda}}\right](1 + K_2\delta_4^2), \quad a = \min\left\{\frac{\mu - \tilde{\lambda}}{\tilde{\lambda}}, \frac{\tilde{\mu}}{\tilde{\lambda}}\right\}.$$

Thus

$$\begin{aligned}
&|\bar{u}^s_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}_1,\epsilon) - \bar{u}^s_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}_2,\epsilon)| \\
&\quad = |\bar{u}^s_{\tau(\bar{\phi}^s_1+\tilde{\phi}^u_1,\epsilon)}(\bar{\phi}^s_1+\tilde{\phi}^u_1,\epsilon) - \bar{u}^s_{\tau(\bar{\phi}^s_2+\tilde{\phi}^u(1,\epsilon),\epsilon)}(\bar{\phi}^s_2+\tilde{\phi}^u(1,\epsilon),\epsilon) \\
&\qquad + \bar{u}^s_{\tau(\bar{\phi}^s_2+\tilde{\phi}^u(1,\epsilon),\epsilon)}(\bar{\phi}^s_2+\tilde{\phi}^u(1,\epsilon),\epsilon) - \bar{u}^s_{\tau(\bar{\phi}^s_2+\bar{\phi}^u_s,\epsilon)}(\bar{\phi}^s_2+\bar{\phi}^u_2,\epsilon)| \\
&\quad = \Big|\int_0^1 \frac{d}{d\theta}\bar{u}^s_{\tau(\tilde{\phi}^s(\theta,\epsilon)+\tilde{\phi}^u(\theta,\epsilon),\epsilon)}(\tilde{\phi}^s(\theta,\epsilon)+\tilde{\phi}^u(\theta,\epsilon),\epsilon)d\theta \\
&\qquad + \frac{d}{d\theta}\bar{u}^s_{\tau(\tilde{\tilde{\phi}}^s(\theta,\epsilon)+\tilde{\tilde{\phi}}^u(\theta,\epsilon),\epsilon)}(\tilde{\tilde{\phi}}^s(\theta,\epsilon)+\tilde{\tilde{\phi}}^u(\theta,\epsilon),\epsilon)d\theta\Big| \\
&\quad \le \Big|\int_0^1 D_\phi\bar{u}^s_{\tau(\tilde{\phi}^s(\theta,\epsilon)+\tilde{\phi}^u(\theta,\epsilon),\epsilon)}(\tilde{\phi}^s(\theta,\epsilon)+\tilde{\phi}^u(\theta,\epsilon),\epsilon)\cdot\frac{d}{d\theta}(\tilde{\phi}^s(\theta,\epsilon)+\tilde{\phi}^u(\theta,\epsilon))d\theta\Big| \\
&\qquad + \Big|\int_0^1 D_\phi\bar{u}^s_{\tau(\tilde{\tilde{\phi}}^s(\theta,\epsilon)+\tilde{\tilde{\phi}}^u(\theta,\epsilon),\epsilon)}(\tilde{\tilde{\phi}}^s(\theta,\epsilon)+\tilde{\tilde{\phi}}^u(\theta,\epsilon),\epsilon)\cdot\frac{d}{d\theta}(\tilde{\tilde{\phi}}^s(\theta,\epsilon)+\tilde{\tilde{\phi}}^u(\theta,\epsilon))d\theta\Big| \\
&\quad \le \tilde{K}_9(\delta_4)\int_0^1 |\langle\tilde{\phi}^s_\lambda(\theta,\epsilon),\tilde{\phi}^u(\theta,\epsilon)-\bar{h}_s(\tilde{\phi}^s(\theta,\epsilon),\epsilon)\rangle|^a d\theta\cdot|\bar{\phi}_1-\bar{\phi}_2| \\
&\qquad + \tilde{K}_9(\delta_4)\int_0^1 |\langle\tilde{\tilde{\phi}}^s_\lambda(\theta,\epsilon),\tilde{\tilde{\phi}}^u(\theta,\epsilon)-\bar{h}_s(\tilde{\tilde{\phi}}^s(\theta,\epsilon),\epsilon)\rangle|^a d\theta\cdot|\bar{\phi}_1-\bar{\phi}_2| \\
&\quad = \tilde{K}_9(\delta_4)(|\langle\phi^*_\lambda,\bar{\phi}^u_1-\bar{h}_s(\bar{\phi}^s_1,\epsilon)\rangle|^a + |\langle\phi^*_\lambda,\bar{\phi}^u_2-\bar{h}_s(\bar{\phi}^s_2,\epsilon)\rangle|^a)\cdot|\bar{\phi}_1-\bar{\phi}_2| \\
&\quad \le 2\tilde{K}_9(\delta_4)[\ell(\bar{\phi}_1,\bar{\phi}_2,\epsilon)]^a|\bar{\phi}_1-\bar{\phi}_2|.
\end{aligned}$$

Next, by (4.1) and (4.24),

$$\begin{aligned}
&|\bar{u}^u_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}_1,\epsilon) - \bar{u}^u_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}_2,\epsilon)| \\
&\quad = \Big|\delta_4/2 + \bar{u}^u_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}_1+\bar{h}_s(\bar{\phi}^s_1,\epsilon),\epsilon) - \delta_4/2 + \bar{u}^u_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}^s_2+\bar{h}_s(\bar{\phi}^s_2,\epsilon),\epsilon)\Big| \\
&\quad = |h_s(\bar{u}^s_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}^s_1+\bar{h}_s(\bar{\phi}^s_1,\epsilon),\epsilon) - h_s(\bar{u}^s_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}^s_2+\bar{h}_s(\bar{\phi}^s_2,\epsilon),\epsilon)| \\
&\quad \le K_2\delta_4^2|\bar{u}^s_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}^s_1+\bar{h}_s(\bar{\phi}^s_1,\epsilon),\epsilon) - \bar{u}^s_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}^s_2+\bar{h}_s(\bar{\phi}^s_2,\epsilon),\epsilon)| \\
&\quad \le 2K_2\delta_4^2\tilde{K}_9(\delta_4)[\ell(\bar{\phi}_1,\bar{\phi}_2,\epsilon)]^a|\bar{\phi}_1-\bar{\phi}_2|.
\end{aligned}$$

Let $K_9 = 2(1+K_2\delta_4^2)\tilde{K}_9(\delta_4)$. Then we obtain

$$|\bar{u}_{\tau(\bar{\phi}_1,\epsilon)}(\bar{\phi}_1,\epsilon) - \bar{u}_{\tau(\bar{\phi}_2,\epsilon)}(\bar{\phi}_2,\epsilon)| \le K_9[\ell(\bar{\phi}_1,\bar{\phi}_2,\epsilon)]^a|\bar{\phi}_1-\bar{\phi}_2|.$$

This completes the proof. □

5. The Šil'nikov Map. In this section, we shall define a map in a small neighborhood of the hyperbolic equilibrium which is closely related to the Poincaré map but is somehow different. The idea of construction of the map is due to Šil'nikov [Si68], so we call it a *Šil'nikov map.* We shall show that the Šil'nikov map is Lipschitzian with a small Lipschitz constant.

5.1. Construction of the map π^1. Let δ_4 and $\phi < \delta_4/4$ be fixed. Denote

$$\rho_0 \le \min\{[2^{1+a}(1+K_2\delta_4^2)K_9(\delta_4)]^{-1/a}, \rho/2\}, \tag{5.1}$$

$$B(\rho_0) = \{\bar{\phi} : |\bar{\phi}^s| < \rho_0, |\bar{\phi}^u| < \rho_0\}, \tag{5.2}$$

$$S(\delta_4,\epsilon) = \{\bar{\phi} : \langle\phi^*_\lambda,\bar{\phi}^u - h_s(\bar{\phi}^s,\epsilon)\rangle = \delta_4/4, |\bar{\phi}^s| < \delta_4/2\}. \tag{5.3}$$

Since $W^s_{\mathrm{loc}}(\epsilon)$ and $\Omega(\delta_4, \rho, \epsilon)$ continuously depend on $\epsilon \in [-\epsilon_3, \epsilon_3]$, there exist a small $0 < \epsilon_4 < \epsilon_3$ such that $B(\rho_0) \cap W^s_{\mathrm{loc}}(\epsilon) \neq \emptyset$ and $B(\rho_0) \subset \Omega(\delta_4, \rho, \epsilon)$ for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Define a map $\tilde{\pi}^1 : B(\rho_0) \times [-\epsilon_4, \epsilon_4] \to C$ by

$$\tilde{\pi}^1(\bar{\phi}, \epsilon) = \begin{cases} \bar{u}_{\tau(\bar{\phi},\epsilon)}(\bar{\phi}, \epsilon) & \text{if } \bar{\phi} \in \Omega^+(\delta_4, \rho, \epsilon) \cap B(\rho_0), \\ \delta_4/2 & \text{if } \bar{\phi} \in \{\Omega - \Omega^+\} \cap B(\rho_0). \end{cases} \tag{5.4}$$

Notice that for each $\epsilon \in [-\epsilon_4, \epsilon_4]$, $\tilde{\pi}^1$ maps $B(\rho_0)$ into $S(\delta_4, \epsilon)$.

Lemma 5.1. *$\tilde{\pi}^1(\bar{\phi}, \epsilon)$ is continuous in $(\bar{\phi}, \epsilon) \in B(\rho_0) \times [-\epsilon_4, \epsilon_4]$ and is Lipschitzian continuous in $\bar{\phi} \in B(\rho_0)$.*

Proof. The continuity of $\tilde{\pi}^1$ follows from Lemmas 4.5 and 4.6. To show $\tilde{\pi}^1$ is Lipschitzian, denote

$$\Delta_1 = \langle \phi^*_\lambda, \bar{\phi}^u_1 - \bar{h}_s(\bar{\phi}^s_1, \epsilon)\rangle > 0, \quad \Delta_2 = \langle \phi^*_\lambda, \bar{\phi}^u_2 - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle \leq 0$$

and let

$$\tilde{\phi}_2(\theta) = \theta[\bar{h}_s(\bar{\phi}^s_2, \epsilon) + \bar{\phi}^u_1 - \bar{h}_s(\bar{\phi}^s_1, \epsilon)] + (1-\theta)\bar{\phi}^u_2, \quad 0 \leq \theta \leq 1.$$

We have

$$\begin{aligned} |\tilde{\pi}^1(\bar{\phi}_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}_2, \epsilon)| &\leq |\tilde{\pi}^1(\bar{\phi}^s_1 + \bar{\phi}^u_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(1), \epsilon)| \\ &\quad + |\tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \bar{\phi}^u_2, \epsilon)|. \end{aligned} \tag{5.5}$$

Since $\langle \phi^*_\lambda, \tilde{\phi}_2(1) - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle = \langle \phi^*_\lambda, \bar{\phi}^u_1 - \bar{h}_s(\bar{\phi}^s_1, \epsilon)\rangle = \Delta_1 > 0$, $\bar{\phi}^s_2 + \tilde{\phi}_s(1) \in \Omega^+(\delta_4, \rho, \epsilon)$. Lemma 4.8 implies that

$$\begin{aligned} &|\tilde{\pi}^1(\bar{\phi}^s_1 + \bar{\phi}^u_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_s(1), \epsilon)| \\ &\quad \leq K_9(\delta_4)[l(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a(|\bar{\phi}^s_1 - \bar{\phi}^s_2| + |\bar{\phi}^u_1 - \tilde{\phi}_2(1)|) \\ &\quad \leq (1 + K_2\delta_4^2)K_9(\delta_4)[(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a|\bar{\phi}^s_1 - \bar{\phi}^s_2|. \end{aligned} \tag{5.6}$$

Note that $\langle \phi^*_\lambda, \tilde{\phi}_2(1) - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle = \Delta_1 > 0$, but $\langle \phi^*_\lambda, \tilde{\phi}_2(0) - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle = \langle \phi^*_\lambda, \bar{\phi}^u_2 - \bar{h}^s(\bar{\phi}^s_2, \epsilon)\rangle = \Delta_2 \leq 0$, there must exist $0 \leq \tilde{\theta} \leq 1$ such that $\langle \phi^*_\lambda, \tilde{\phi}_2(\theta) - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle = 0$ and $\langle \phi^*_\lambda, \tilde{\phi}_2(\theta) - \bar{h}_s(\bar{\phi}^s_2, \epsilon)\rangle > 0$ for $\tilde{\theta} < \theta \leq 1$, which implies that $\bar{\phi}^s_2 + \tilde{\phi}_s(\theta) \in \Omega^+(\delta_4, \rho, \epsilon)$ for $\tilde{\theta} < \theta \leq 1$. By the definition of $\tilde{\pi}^1$, $\tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(\tilde{\theta}), \epsilon) = \tilde{\pi}^1(\bar{\phi}^s_2 + \bar{\phi}^u_2, \epsilon) = \delta_4/2$. Thus, Lemma 4.8 implies that

$$\begin{aligned} &|\tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \bar{\phi}^u_2, \epsilon)| \\ &\quad = |\tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(1), \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \tilde{\phi}_2(\tilde{\theta}), \epsilon)| \\ &\quad \leq K_9(\delta_4)[l(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a|\tilde{\phi}_2(1) - \tilde{\phi}_2(\tilde{\theta})| \\ &\quad = K_9(\delta_4)[l(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a|(1 - \tilde{\theta})[\bar{h}_s(\bar{\phi}^s_2, \epsilon) - \bar{h}_s(\bar{\phi}^s_1, \epsilon)] + (\bar{\phi}^u_1 - \bar{\phi}^u_2)| \\ &\quad \leq (1 + K_2\delta_4^2)K_9(\delta_4)[l(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a(|\bar{\phi}^s_1 - \bar{\phi}^s_2| + |\bar{\phi}^u_1 - \bar{\phi}^u_2|). \end{aligned} \tag{5.7}$$

Therefore, by (5.6) and (5.7), we have

$$\begin{aligned} &|\tilde{\pi}^1(\bar{\phi}^s_1 + \bar{\phi}^u_1, \epsilon) - \tilde{\pi}^1(\bar{\phi}^s_2 + \bar{\phi}^u_2, \epsilon)| \\ &\quad \leq 2(1 + K_2\delta_4^2)K_9(\delta_4)[l(\bar{\phi}_1, \bar{\phi}_2, \epsilon)]^a(|\bar{\phi}^s_1 - \bar{\phi}^s_2| + |\bar{\phi}^u_1 - \bar{\phi}^u_2|), \end{aligned}$$

which means that $\tilde{\pi}^1$ is Lipschitzian. $\square$

For $\epsilon \in [-\epsilon_4, \epsilon_4]$, define

$$\begin{aligned} W^u_+(\epsilon) = \{\phi :\ & \text{there exists } t > 0 \text{ and } \psi \in W^u_{\text{loc}}(\epsilon) \text{ with} \\ & \langle \phi^*_\lambda, \psi^u - h_s(\psi^s, \epsilon)\rangle > 0 \text{ and } \phi = u_t(\psi, \epsilon)\}, \end{aligned} \tag{5.8}$$

$$\phi_1(\epsilon) = W^u_+(\epsilon) \cap \Sigma(\delta_4/2, \epsilon), \tag{5.9}$$

where

$$\Sigma(\delta_4/2, \epsilon) = \{\phi :\ \langle \phi^*_\lambda, \bar{\phi}^u - \bar{h}_s(\bar{\phi}^s, \epsilon)\rangle = \delta_4/2,\ \bar{\phi} = H(\phi, \epsilon)\}. \tag{5.10}$$

Fix $0 < \rho_0 < \delta_4/2$. Since H is near the identity map, there exists $0 < \rho_1 < \rho_0$ and $0 < \epsilon_5 < \epsilon_4$ such that

$$B(\rho_1) \subset H^{-1}(\Omega(\delta_4, \rho_0, \epsilon), \epsilon), \quad \epsilon \in [-\epsilon_5, \epsilon_5]. \tag{5.11}$$

Let $\phi_0 \in W^s_{\text{loc}}(0)$ with $|\phi_0| < \rho_1$ be fixed. By the continuity property of $W^s_{\text{loc}}(\epsilon)$ in ϵ, for every $0 < \rho <$ dist $(\partial B(\rho_1), \phi_0)$, there exists $0 < \epsilon_6(\rho) < \epsilon_5$ such that

$$B(\phi_0, \rho) \cap W^s_{\text{loc}}(\epsilon) \neq \emptyset, \quad \epsilon \in [-\epsilon_6, \epsilon_6], \tag{5.12}$$

where $B(\phi_0, \rho) = \{\bar{\phi} \in \mathcal{C} : |\bar{\phi}^s - \phi^s_0| < \rho,\ |\bar{\phi}^u - \phi^u_0| < \rho\}$. Define $H \times I : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to \mathcal{C} \times R$ by

$$(H \times I)(\bar{\phi}, \epsilon) = (H(\bar{\phi}, \epsilon), \epsilon)$$

and $\tilde{\pi}^1 \times I : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to \mathcal{C} \times R$ by

$$(\tilde{\pi}^1 \times I)(\bar{\phi}, \epsilon) = (\tilde{\pi}^1(\bar{\phi}, \epsilon), \epsilon).$$

Thus $\pi^1 : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to \mathcal{C}$ given by

$$\pi^1(\bar{\phi}, \epsilon) = H^{-1}((\tilde{\pi}^1 \times I) \circ (H \times I)(\bar{\phi}, \epsilon), \epsilon) \tag{5.13}$$

is well-defined. Notice that by (4.2) and (4.13),

$$|D_{\bar{\phi}} H(\bar{\phi}, \epsilon)| \leq 2 + K_2 \delta_4^2,$$
$$|D_{\bar{\phi}} H^{-1}(\bar{\phi}, \epsilon)| \leq 2 + K_2 \delta_4^2$$

for $(\bar{\phi}, \epsilon) \in B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6]$. Thus, by Lemma 4.8, π^1 is also continuous and Lipschitzian in $\bar{\phi}$ and $\phi_1(\epsilon)$ is the unique intersection point of $W^u_+(\epsilon)$ and $\Sigma(\delta_4/2, \epsilon)$.

The above discussion can be summarized as a lemma.

Lemma 5.2. *The map $\pi^1 : B(\phi_0, \rho) \times [-\epsilon_6, \epsilon_6] \to \mathcal{C}$ defined in (5.6) satisfies:*

(a) If $|\epsilon| < \epsilon_6(\rho), |\phi - \phi_0| < \rho$ and $\langle \phi^*_\lambda, \phi^u - h_s(\bar{\phi}^s, \epsilon)\rangle > 0$, then $\pi^1(\phi, \epsilon)$ is the intersection point of $\Sigma(\delta_4/2, \epsilon)$ and the solution orbit of (2.1) with the initial value ϕ and parameter ϵ. If $\langle \phi^*_\lambda, \phi^s - h_s(\phi^s, \epsilon)\rangle \leq 0$, then $\pi^1(\phi, \epsilon) = \phi_1(\epsilon)$.

(b) $\pi^1(\phi, \epsilon)$ is continuous in (ϕ, ϵ) and is Lipschitzian in ϕ for each fixed ϵ.

5.2. Construction of the map π^2. Let $t_0 > 0$ be the time such that $u_{t_0}(\phi_1(0), 0) = \phi_0$ and $\tilde{\Gamma}_0 = \{\phi : \phi = u_t(\phi_1(0), 0), 0 \leq t \leq t_0\} \subset \Gamma_0$. Define $\pi^2 : B(\phi_1(0), \rho_1) \times [-\epsilon_6, \epsilon_6] \to B(\rho_1)$ by

$$\pi^2(\phi, \epsilon) = u_{t_0}(\phi, \epsilon). \tag{5.14}$$

By the differentiability of solutions to (2.1) with respect to initial values (see Theorem 3.1), there exist $0 < \rho_2 < \rho_1$ and $0 < \epsilon_7 < \epsilon_6$ such that

$$|D_{(\phi,\epsilon)}\pi^2(\phi_1,\epsilon)| \le |D_{(\phi,\epsilon)}\pi^2(\phi_1(0),0)| + 1$$

for $(\phi_1,\epsilon) \in B(\phi_1(0),\rho_2) \times [-\epsilon_7,\epsilon_7]$. Let

$$K_{10} = |D_{(\phi,\epsilon)}\pi^2(\phi_1(0),0)| + 1. \tag{5.15}$$

Then for every $0 < \rho < \rho_2$ and $0 < \epsilon \le \epsilon_7, \pi^2$ is continuous and Lipschitzian with Lipschitz constant K_{10}.

5.3. Construction of the map π. Since $\phi_1(\epsilon)$ is continuous in $\epsilon \in [-\epsilon_6(\rho),\epsilon_6(\rho)]$ for every $0 < \rho <$ dist $(\phi_0, \overline{\partial B(\rho_1)})$, there exists $0 < \epsilon_8(\rho) < \epsilon_6(\rho)$ such that

$$|\phi_1(\epsilon) - \phi_1(0)| < K_9(\delta_4)\rho^{a+1}, \quad \epsilon \in [-\epsilon_8(\rho),\epsilon_8(\rho)]. \tag{5.16}$$

Let

$$\rho_3 = \min\left\{ \left[\frac{1}{2K_{10}K_9(\delta_4)}\right]^{-\frac{1}{a}},\ \left[\frac{\rho_2}{2K_9(\delta_4)}\right]^{-\frac{1}{a+1}},\ \text{dist } (\partial B(\rho_1),\phi_0) \right\}, \tag{5.17}$$

$$\epsilon_9 = \min\{\epsilon_7, \epsilon_6(\rho_2)\}. \tag{5.18}$$

Then $2K_{10}K_9(\delta_4)\rho_3^a < 1/2$ and the composite map

$$\pi(\cdot,\epsilon) = \pi^2(\pi^1(\cdot,\epsilon),\epsilon) \tag{5.19}$$

mapping $B(\phi_0,\rho_3) \times [-\epsilon_8,\epsilon_8]$ into $B(\phi_0,\rho_3/2)$. By Lemma 5.2 and the definition of π^2, π is continuous and is Lipschitzian in ϕ. The Lipschitz constant is $K_{10}K_9(\delta_4)\rho_3^a < \frac{1}{2}$ by (5.17) and independent of $\epsilon \in [-\epsilon_8,\epsilon_8]$. Thus, π is continuous for every $\epsilon \in [-\epsilon_8,\epsilon_8]$. We now summarize the properties of π in the following result:

Theorem 5.3. *(i) π is continuous and $\pi(\cdot,\epsilon) : B(\phi_0,\rho_3) \to B(\phi_0,\rho_3)$ is a contraction with a contraction constant less than $\frac{1}{2}$ uniformly in $\epsilon \in [-\epsilon_8,\epsilon_8]$.*

(ii) If $\phi \in B(\phi_0,\rho_3)$ and $\langle \phi_\lambda^*, \phi^u - h_s(\phi^s,\epsilon)\rangle > 0$, then $\pi(\phi,\epsilon)$ is on the orbit of (2.1) containing ϕ. If $\langle \phi_\lambda^*, \phi^s - h_s(\phi^s,\epsilon)\rangle \le 0$, then $\pi(\phi,\epsilon)$ is a constant map with $\pi(\phi,\epsilon) = \pi^2(\phi_1(\epsilon),\epsilon)$.

(iii) For every $\epsilon \in [-\epsilon_8,\epsilon_8]$,

$$B(\phi_0,\rho_3) \cap \{\phi : \langle \phi_\lambda^*, \phi^u - h_s(\phi^s,\epsilon)\rangle > 0\} \ne \emptyset,$$
$$B(\phi_0,\rho_3) \cap \{\phi : \langle \phi_\lambda^*, \phi^u - h_s(\phi^s,\epsilon)\rangle \le 0\} \ne \emptyset.$$

6. The Proof of the Main Results. To prove Theorem 2.1, we need the following lemmas.

Lemma 6.1. *There exist $\epsilon_9 > 0$ and neighborhoods N_1 of $\{0\}$ and N_2 of $\tilde{\Gamma}_0$, respectively, such that*

(i) $N(\Gamma_0) = N_1 \cup N_2$ is a neighborhood of the homoclinic orbit Γ_0;

(ii) if γ is an orbit of (2.1) at $\epsilon \in [-\epsilon_9,\epsilon_9]$ satisfying $\gamma \cap N_1 \ne \emptyset$, $\gamma \subseteq N(\Gamma_0)$, then

$$\langle \phi_\lambda^*, \phi^u - h_s(\phi^s,\epsilon)\rangle > 0 \text{ for every } \phi \in \gamma \cap N_1;$$

(iii) if $\phi \in N_2$ and $\epsilon \in [-\epsilon_9, \epsilon_9]$, then $u_t(\phi, \epsilon) \in B(\phi_0, \rho_3)$ for some $t > 0$.

Proof. Let δ_4, ρ and $\Omega(\delta_4, \rho, \epsilon)$ be as in Lemma 4.5, and ϵ_4 and $S(\delta_4, \epsilon)$ (see (5.3)) be as in Lemma 5.1. Define

$$\begin{aligned}\tilde{N}(\epsilon) &= \{\phi : |\phi^s| < \delta_4, -\delta_4/4 < \langle \phi_\lambda^*, \phi^u - h_s, \epsilon)\rangle < \delta_4\},\\ \partial\tilde{N}^-(\epsilon) &= \{\phi : |\phi^s| \le \delta_4, -\delta_4/4 = \langle \phi_\lambda^*, \phi^u - h_s(\phi^s, \epsilon)\rangle\}\end{aligned} \tag{6.1}$$

for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Then

$$\Omega(\delta_4, \rho, \epsilon) \subset \tilde{N}(\epsilon), \quad S(\delta_4, \epsilon) \subset \tilde{N}(\epsilon), \quad \epsilon \in [-\epsilon_4, \epsilon_4]. \tag{6.2}$$

By Lemma 5.1, for every $\phi \in \tilde{N}(\epsilon) \cap \Omega^-(\delta_4, \rho, \epsilon)$, there exists $t > 0$ such that $u_t(\phi, \epsilon) \in \partial\tilde{N}^-(\epsilon)$ for $\epsilon \in [-\epsilon_4, \epsilon_4]$. Denote

$$N_1 = H^{-1}(\tilde{N}(0), 0), \quad \partial N_1^- = H^{-1}(\partial\tilde{N}^-(0), 0). \tag{6.3}$$

Since H is continuous in ϵ, there exists $0 < \epsilon_{10} < \epsilon_9$ such that

$$\begin{aligned}&N_1 \supset H^{-1}(\Omega(\delta_4, \rho, \epsilon), \epsilon) \supset H^{-1}(B(\rho_0), \epsilon) \supset B(\phi_0, \rho_3),\\ &N_1 \supset H^{-1}(S(\delta_4, \epsilon), \epsilon) = \Sigma(\delta_4, \epsilon)\end{aligned} \tag{6.4}$$

for $\epsilon \in [-\epsilon_{10}, \epsilon_{10}]$, where $B(\rho_0)$, $B(\phi_0, \rho_3)$, $S(\delta_4, \epsilon)$ and $\Sigma(\delta_4, \epsilon)$ are defined in section 5. This implies the following properties:

(A) if $\phi \in N_1 \cap H^{-1}(\Omega^-(\delta_4, \rho, \epsilon), \epsilon) \setminus W^s_{loc}(\epsilon)$ and $\epsilon \in [-\epsilon_{10}, \epsilon_{10}]$, then $u_t(\phi, \epsilon)$ will leave N_2 through ∂N_1^-;

(B) ∂N_1^- is closed with $\Gamma_0 \cap \partial N_1^- = \emptyset$.

Thus, for t_0 and $\tilde{\Gamma}_0$ defined in section 5 and for every $\tilde{\phi} = u_t(\phi_1(0), 0) \in \tilde{\Gamma}_0$ there exists $\tilde{\rho} = \tilde{\rho}(t) > 0$ and $\tilde{\epsilon} = \tilde{\epsilon}(t) > 0$ such that

$$B(\tilde{\phi}, \tilde{\rho}) \cap \partial N_1^- = \emptyset$$

and if $\phi \in B(\tilde{\phi}, \tilde{\rho}), \epsilon \in [-\tilde{\epsilon}, \tilde{\epsilon}]$, then

$$u_{t_0 - t}(\phi, \epsilon) \in B(\phi_0, \rho_3).$$

Note that $\bigcup_{0 \le t \le t_0} B(\tilde{\phi}, \tilde{\rho})$ is an open cover of $\tilde{\Gamma}_0$. By the compactness of $\tilde{\Gamma}_0$, there exists a finite open cover $N_2 = \bigcup_{i=1}^n B(\tilde{\phi}_i, \tilde{\rho}_i)$, where

$$\tilde{\phi}_i = u_{t_i}(\phi_1(0), 0), \quad \tilde{\rho}_i = \tilde{\rho}_i(t), \quad 0 < t_1 < t_2 < \cdots < t_n \le t_0. \tag{6.5}$$

Define $\epsilon_{10} = \min_{1 \le i \le n}\{\tilde{\epsilon}(t_i)\}$ and $N(\Gamma_0) = N_1 \cup N_2$. Then $N(\Gamma_0)$ is an open neighborhood of $\Gamma_0 \cup \{0\}$, which implies (i). Since $N_2 \cap \partial N_1^- = \emptyset$ and $\partial N_1^- \subset \partial N_1 \subset \partial N(\Gamma_0)$, by property (A), we obtain (ii). (iii) follows from the definition of $B(\tilde{\phi}, \tilde{\rho})$. $\square$

Lemma 6.2. *There exist* $0 < \bar{\epsilon}_0 < \epsilon_{10}$ *and* $\bar{\rho} > 0$ *such that*
(i) if $(\phi, \epsilon) \in B(\phi_1(0), \bar{\rho}) \times [-\bar{\epsilon}_0, \bar{\epsilon}_0]$*, then* $u_t(\phi, \epsilon) \in N_2$ *for all* $0 \le t \le t_0$*;*
(ii) if $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$*, then* $|\phi_1(\epsilon) - \phi_1(0)| < \bar{\rho}/4$*;*
(iii) if $\phi_*(\epsilon)$ *is the unique fixed point of* π *and* $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$*, then*

$$|\langle \phi_\lambda^*, \phi_*^u - h_s(\phi_*^s, \epsilon)\rangle| < \min\left\{\frac{1}{4}K_{10}\bar{\rho}, \rho_3\right\},$$

where K_{10} *and* ρ_3 *are given in (5.15) and (5.17), respectively.*

Proof. By Lemma 6.1, if $\phi \in B(\phi_1(0), \tilde{\rho}(0))$ and $\epsilon \in [-\epsilon_9, \epsilon_9]$, then $u_t(\phi, \epsilon)$ is defined for all $0 \le t \le t_0$. We claim that there exist $0 < \epsilon_{10} < \epsilon_9$ and $0 < \bar{\rho} < \tilde{\rho}(0)$ such that for every $(\phi, \epsilon) \in B(\phi_1(0), \bar{\rho}) \times [-\epsilon_{10}, \epsilon_{10}]$,

$$u_t(\phi, \epsilon) \in N_2, \quad 0 \le t \le t_0.$$

Suppose the contrary. Then there exists a sequence $\{(\phi_k, \epsilon_k, t_k)\}_k$ with $(\phi_k, \epsilon_k) \to (\phi_1(0), 0)$ as $k \to \infty$ and $0 \le t_k \le t_0$ for every $k = 1, 2, \dots$, such that

$$u_{t_k}(\phi_k, \epsilon_k) \in \partial N_2.$$

Since $[0, t_0]$ is closed, without loss of generality, assume that $t_k \to t_0 \in [0, t_0]$ as $k \to \infty$. Thus, $\lim_{k\to\infty} u_{t_k}(\phi_k, \epsilon_k) = u_{t_0}(\phi_1(0), 0) \in \tilde{\Gamma}_0$, which contradicts $\tilde{\Gamma}_0 \subset N_2$. This proves (i). (ii) and (iii) follow from the continuity of $\phi_*(\epsilon)$ and $\phi_1(\epsilon)$ in ϵ. □

Lemma 6.3. *Let* $N(\Gamma_0)$ *be the neighborhood of the homoclinic orbit* Γ_0 *as in Lemma 6.1 and* $\bar{\epsilon}$ *and* $\phi_*(\epsilon)$ *be as in Lemma 6.2. If* γ *is a periodic or homoclinic orbit of equation (2.1) at* $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$ *and* $\gamma \subseteq N(\Gamma_0)$*, then* $\phi_*(\epsilon) \in \gamma$*.*

Proof. If γ is a homoclinic orbit, then $\phi_1(\epsilon) \in \gamma$ and $\phi_2(\epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) \in B(\phi_0, \rho_3)$, where ρ_3 is defined in (5.17) and $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$. Since $\gamma \subset N(\Gamma_0)$, by Lemma 6.1(ii), $\gamma \cap N_1 \cap H^{-1}(\Omega^-(\delta_4, \rho, \epsilon), \epsilon) = \emptyset$. Theorem 5.3(b) then implies that

$$(\pi)^k(\phi_2(\epsilon), \epsilon) \in \gamma, \quad k = 0, 1, 2 \dots,$$

where $(\pi)^k(\phi_2(\epsilon), \epsilon) = \pi((\pi)^{k-1}(\phi_2(\epsilon), \epsilon), \epsilon)$ is the kth iterate of π. Since $\gamma \cap W^s_{\mathrm{loc}}(\epsilon) \ne \emptyset$, there exists $K \ge 0$ such that $(\pi)^K(\phi_2(\epsilon), \epsilon) \in W^s_+(\epsilon)$. Once again by Theorem 5.3,

$$\pi((\pi)^K(\phi_2(\epsilon), \epsilon), \epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) = \phi_2(\epsilon),$$

which means $\phi_2(\epsilon)$ is a fixed point of π^{K+1}, thus $\phi_2(\epsilon) = \phi_*(\epsilon)$. The case that γ is a periodic orbit can be proved similarly. □

Proof of Theorem 2.1. Necessity. Suppose $W^u_+(\epsilon) \subset N(\Gamma_0)$ for some $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$, then Lemma 4.1 (iii) implies that $W^u_+(\epsilon) \cap H^{-1}(\Omega^-(\delta_4, \rho, \epsilon), \epsilon) = \emptyset$. We claim that

$$\phi_*(\epsilon) \notin H^{-1}(\Omega^{-1}(\delta_4, \rho, \epsilon), \epsilon).$$

If not, by Theorem 5.3(b), $\phi_*(\epsilon) = \pi(\phi_*(\epsilon), \epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) \in W^u_+(\epsilon)$, a contradiction. Thus, $\pi(\phi_*(\epsilon), \epsilon) = \phi_*(\epsilon) \in \gamma$, a solution orbit of equation (2.1). It follows that γ is a periodic orbit.

Let $\tau = \tau(\bar{\phi}, \epsilon)$ be as in Lemma 4.6 and

$$\tau_*(\epsilon) = \tau(H(\phi_*(\epsilon), \epsilon), \epsilon).$$

By Lemma 4.6, $u_t(\phi_*(\epsilon), \epsilon) \in N_1$ for $0 \le t \le \tau_*(\epsilon)$. By (5.17), the continuity of π^1 and Lemma 6.2, we have

$$|\pi^1(\phi_*(\epsilon), \epsilon) - \phi_1(0)| \le |\pi^1(\phi_*(\epsilon), \epsilon) - \phi_1(\epsilon)| + |\phi_1(\epsilon) - \phi_1(0)| \le \bar{\rho}/2,$$

where $\bar{\rho}$ is given in Lemma 6.2. Thus, Lemma 6.1 implies that

$$u_{t+\tau_*(\epsilon)}(\phi_*(\epsilon), \epsilon) \in N_2, \quad 0 \le t \le t_0.$$

Therefore, $\gamma \subset N(\Gamma_0)$.

Sufficiency. Let γ be a periodic orbit. Then Lemma 4.4 implies that $\phi_*(\epsilon) \in \gamma$. Let $\phi_2(\epsilon) = \pi^2(\phi_1(\epsilon), \epsilon) \in W^u_+(\epsilon) \cap B(\phi_0, \rho_3)$ be as in the proof of Lemma 6.3 and $\tau = \tau(\bar{\phi}, \epsilon)$ be as in Lemma 4.6. Define

$$\phi_2^k(\epsilon) = (\pi)^k(\phi_2(\epsilon), \epsilon),$$
$$\tau_2^k(\epsilon) = \tau(H(\phi_2^k(\epsilon), \epsilon), \epsilon)$$

for $\epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0]$ and $k = 1, 2, \ldots$

Claim A. $\phi_2^k(\epsilon) \in H^{-1}(\Omega^+(\delta_4, \rho_3, \epsilon), \epsilon),\ \epsilon \in [-\bar{\epsilon}_0, \bar{\epsilon}_0],\ k = 1, 2, \ldots.$

Suppose not, then there exists K such that

$$\phi_2^K(\epsilon) \in H^{-1}(\Omega^-(\delta_4, \rho_3, \epsilon), \epsilon) \cup W^s_{\text{loc}}(\epsilon)$$

and

$$\phi_2^k(\epsilon) \notin H^{-1}(\Omega^{-1}(\delta_4, \rho_3, \epsilon), \epsilon) \cup W^s_{\text{loc}}(\epsilon), \quad k = 1, 2, \ldots K-1.$$

By Theorem 5.3(b), we have

$$\pi(\phi_2^K(\epsilon), \epsilon) = \pi((\pi)^K(\phi_2(\epsilon), \epsilon), \epsilon) = \phi_2(\epsilon),$$

which implies that $\phi_2(\epsilon)$ is the fixed point of π^{K+1}, thus, the fixed point of π. Therefore $\phi_2(\epsilon) = \phi_*(\epsilon) \in \gamma$. This contradiction proves the claim.

Claim B. $\bigcup_{0 \le t \le t_0} \{u_t(\phi_1(\epsilon), \epsilon)\} \subset N(\Gamma_0)$.

By Theorem 5.3(b), $\phi_2^k(\epsilon) \in W^u_+(\epsilon)$ for every $k = 1, 2, \ldots.$ Thus, Lemma 6.3 implies the claim.

Claim C. $\bigcup_{0 \le t \le \tau_2(\epsilon)+t_0} \{u_t(\phi_2^k(\epsilon), \epsilon)\} \subset N(\Gamma_0)$ for $\epsilon \in [-\bar{\epsilon}, \bar{\epsilon}_0], k = 1, 2 \ldots.$

Theorem 5.3(a) and Lemma 6.2 imply that

$$|\phi_2^k(\epsilon) - \phi_*(\epsilon)| \le \left(\frac{1}{2}\right)^k |\tilde{\phi}_*(\epsilon) - \phi_*(\epsilon)|$$
$$\le \left(\frac{1}{2}\right)^k \frac{K_{10}\bar{\rho}}{4},$$

where $\tilde{\phi}_*(\epsilon) = \phi_*^s(\epsilon) + h_s(\phi_*^s(\epsilon), \epsilon) \in W^s_{\text{loc}}(\epsilon)$. Thus, by the continuity of π^1, (5.17) and Claim A, we have

$$|\pi^1(\phi_2^k(\epsilon), \epsilon) - \pi^1(\phi_*(\epsilon), \epsilon)| \le \bar{\rho}/8.$$

Hence

$$|\pi^1(\phi_2^k(\epsilon),\epsilon) - \phi_1(0)| < \bar{\rho}/2 + \bar{\rho}/8 < \bar{\rho},$$

this, together with Lemma 6.2(i), implies that

$$\bigcup_{\tau_2^k(\epsilon)\le t\le \tau_2^k(\epsilon)+t_0} \{u_t(\phi_2^k(\epsilon),\epsilon)\} \subset N_2.$$

By the definition of N_1 and Lemma 4.6, we have

$$\bigcup_{0\le t\le \tau_2^k(\epsilon)} \{u_t(\phi_2^k(\epsilon),\epsilon)\} \subset N_1.$$

Therefore, Claim C is proved.

Given a $\phi \in W_+^u(\epsilon)$, by Claims A, B, and C, ϕ is in either

$$\bigcup_{0\le t\le t_0} \{u_t(\phi_2^k(\epsilon),\epsilon)\} \subset N(\Gamma_0)$$

or

$$\bigcup_{0\le t\le \tau_2^k(\epsilon)+t_0} \{u_t(\phi_2^k(\epsilon),\epsilon)\} \subset N(\Gamma_0)$$

for some $k = 1, 2 \ldots$. Therefore, $W_+^u(\epsilon) \subset N(\Gamma_0)$.

Finally, the exponentially asymptotic stability of γ follows from Theorem 5.3(a). □

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