PERGAMON Nonlinear Analysis: Real World Applications 4 (2003) 173–188

Nonlinear Analysis
Real World Applications

www.elsevier.com/locate/na

Travelling wavefronts of scalar reaction–diffusion equations with and without delays☆

K.Q. Lan[1], J.H. Wu*

Department of Mathematics and Statistics, York University, Toronto, Ont., Canada M3J 1P3

Received 14 April 2001; accepted 15 November 2001

Abstract

This paper deals with the existence of travelling wavefronts for scalar nonlinear reaction–diffusion equations with and without delays in one-dimensional space. New iterative techniques for a class of integral operators of Hammerstein type are established and applied to tackle the existence of travelling wavefronts in a unified way. Our results without delays only require the functions involved to be continuous and satisfy a suitable monotonicity condition. Our results with multiple delays employ the usual C^1-assumption but generalize the well-known results.

MSC: 35K57; 47H10; 47H30

Keywords: Travelling wavefronts; Reaction–diffusion equations; Hammerstein integral equations; Iterative techniques

1. Introduction

The scalar nonlinear reaction–diffusion equation of the form

$$-D\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{\partial u(t,x)}{\partial t}=f(u(t,x)),\quad t\geqslant 0,\ x\in\mathbb{R} \tag{1.1}$$

has been widely used to model the propagation phenomena which appear in biology, chemistry and physics. For example, the Fisher equation, where $f(u)=ku(1-u)$,

☆ Research partially supported by National Sciences and Engineering Research Council of Canada, and by Network of Centers of Excellence.

* Corresponding author. Tel.: +1-416-736-5250; fax: +1-416-736-5757.
E-mail addresses: klan@ryerson.ca (K.Q. Lan), wujh@mathstat.yorku.ca (J.H. Wu).

[1] Present address: Department of Mathematics, Physics and Computer Science, Ryerson University, 350 Victoria Street Toronto, Ont., Canada M5B 2K3.

1468-1218/02/$ - see front matter
PII: S1468-1218(02)00020-2

was proposed by Fisher [11] to model the advance of a mutant gene in an infinite one-dimensional habitat. Eq. (1.1) with $f(u) = ku^p(1-u^q)$ is also used in the density-dependent diffusion–reaction models, see [24, Section 11.4]. Eq. (1.1) also arises in logistic population growth models [24], autocatalytic chemical reaction [2,10], branching Brownian motion processes [5], neurophysiology [32] and nuclear reactor theory [8]. Recently, some generalizations of Eq. (1.1) are also used as mathematical models for tumor encapsulation [26,30].

The existence of travelling wavefronts for Eq. (1.1) has been widely studied by using phase-plane techniques, for example, in [2,7,9,34], where f is continuously differentiable. Berestycki and Nirenberg [4] obtained results on the existence of travelling wavefronts even for multidimensional problems, where f is Lipschitz continuous and satisfies some other conditions. There are also results on a system of reaction–diffusion equations (see [12,34,36]).

On the other hand, Schaaf [29] studied the existence of travelling wavefronts for the reaction–diffusion equations with single delay of the form

$$-\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial u(t,x)}{\partial t} = F(u(t,x), u(t-r,x)), \quad t \geqslant 0, \ x \in \mathbb{R}. \tag{1.2}$$

The main tool is the sub- and supersolution technique due to Atkinson and Reuter [3]. Zou and Wu [35] studied the existence of travelling wavefronts for a system of reaction–diffusion equations with single delay, where the well-known monotone iteration techniques for elliptic systems with advanced arguments [21,25] are used.

In this paper, we study the existence of travelling wavefronts for Eq. (1.1) and for the following reaction–diffusion equations with multiple delays:

$$-D\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial u(t,x)}{\partial t} = F(u(t,x), u(t-r_2,x), \ldots, u(t-r_n,x)). \tag{1.3}$$

For Eq. (1.1), we only require f to be continuous and satisfy a suitable monotonicity condition. For Eq. (1.3), the usual C^1-assumption together with other conditions are imposed on F, but these conditions are weaker than those used in [29] even when $n=2$. Mallet–Paret [22,23] studied the existence of travelling wavefronts for Eq. (1.3) with $D=0$ and $r_i \in \mathbb{R}$. Some lattice differential equations can be changed into such equations.

The main idea is to change the problems on travelling waves for both Eqs. (1.1) and (1.3) into a fixed point problem for a Hammerstein integral operator of the form

$$z(t) = \int_{-\infty}^{\infty} k(t,s)((\mathscr{F}z)(s) + \gamma z(s))\,\mathrm{d}s \equiv Az(t), \quad t \in \mathbb{R}, \tag{1.4}$$

where $\mathscr{F} : [u,v] \subset BC(\mathbb{R}) \to L^\infty$ is a suitable map. (The symbols and the precise definitions of concepts mentioned in the Introduction will be given later in this paper.) A similar idea was first used by Zou and Wu [36]. In applications, $\mathscr{F}$ is defined by f or F. We establish a new iterative technique for the map A, which can be used to treat the existence of travelling waves for Eqs. (1.1) and (1.3) in a unified way. The main difficulty in establishing the theory is that the map A may not be continuous and it is not clear if $\overline{A(Q)}$ is compact in $BC(\mathbb{R})$ for all bounded sets Q in $[u,v]$. Hence, the classical iterative techniques for compact maps in ordered Banach spaces (see [1,13]) cannot be used. To overcome the difficulty, we introduce and employ the so-called M-continuity for $\mathscr{F}$ and show that the closure of $P_a^b A(Q)$ is compact in $C[a,b]$ for each bounded subset Q, where P_a^b maps each element in $BC(\mathbb{R})$ to its restriction to $[a,b]$. These, together with Lebsegue's dominated convergence theorem, enable us to prove that the iterative sequences involved are convergent in some sense (see (3.5)). To apply the theory to treat Eqs. (1.1) and (1.3), we construct two functions u and v which satisfy $u \leqslant Au$ and $Av \leqslant v$ and prove that the solutions of the map A are the required travelling wavefronts.

In Section 2, we introduce a class of bounded, M-continuous and γ-increasing maps and provide examples of such maps. In Section 3, we establish the iterative techniques and apply them to obtain the existence of solutions for second order functional differential equations. In Section 4, we apply the new iterative techniques to treat the existence of travelling wavefronts for Eq. (1.1). In the last section, we treat the existence of travelling wavefronts for Eq. (1.3) again using our new iterative techniques.

2. A class of nonlinear maps

We introduce a class of maps which are bounded, M-continuous and γ-increasing and provide several examples of such maps.

We denote by $C(\mathbb{R})$ the space of all continuous real-valued functions defined on $\mathbb{R}$. Let $BC(\mathbb{R})=\{x\in C(\mathbb{R})\colon \sup\{|x(t)|: t\in\mathbb{R}\} < \infty\}$ and $BC^2(\mathbb{R})=\{x\in BC(\mathbb{R})\colon x', x'' \in BC(\mathbb{R})\}$. Then $BC(\mathbb{R})$ and $BC^2(\mathbb{R})$ are Banach spaces with the norms $\|x\| = \sup\{|x(t)|: t\in\mathbb{R}\}$ and $\|x\|_{BC^2(\mathbb{R})} = \max\{\|x\|, \|x'\|, \|x''\|\}$, respectively. We write $L^p = L^p(\mathbb{R})$.

The following lemma gives relations between $BC(\mathbb{R})$ and $C[a,b]$. Its proof is straightforward and is omitted.

Lemma 2.1. (1) $\|x\| = \sup\{\|x\|_{C[a,b]}: -\infty < a < b < \infty\}$ *for each* $x\in BC(\mathbb{R})$.

(2) *If* $\{x_n\}\cup\{x\}\subset BC(\mathbb{R})$ *and* $\|x_n - x\|\to 0$, *then* $\|x_n - x\|_{C[a,b]}\to 0$ *for* $a,b\in\mathbb{R}$ *with* $a<b$.

(3) *If* $\{x_n\}\cup\{x\}\subset BC(\mathbb{R})$ *and* $\|x_n - x\|_{C[a,b]}\to 0$ *for* $a,b\in\mathbb{R}$ *with* $a<b$, *then* $x_n(t)\to x(t)$ *for each* $t\in\mathbb{R}$.

Remark 2.1. (i) The converse of (2) in Lemma 2.1 is false. For example, if $x_n(t) = (1+\mathrm{e}^{-t})^{-n}$ and $x(t)\equiv 0$, then $\|x_n - x\|_{C[a,b]} = (1+\mathrm{e}^{-b})^{-n}\to 0$ for $a,b\in\mathbb{R}$ with $a<b$. However, $\|x_n - x\| = 1$ for each $n\in\mathbb{N}$ and $\{x_n\}$ does not converge to x in $BC(\mathbb{R})$.

(ii) The converse of (3) in Lemma 2.1 is false. For example, let

$$x_n(t) = \begin{cases} nt & \text{for } t\in[0,1/n],\\ 2-nt & \text{for } t\in[1/n,2/n],\\ 0 & \text{for } t\in(-\infty,0)\cup(2/n,\infty) \end{cases}$$

and $x(t)\equiv 0$. Then $x_n(t)\to x(t)$ for each $t\in\mathbb{R}$. However, $\|x_n - x\|_{C[0,1]} = 1$ for $n\geqslant 3$.

If $u, v \in L^\infty$ and $u(t) \leqslant v(t)$ a.e. on $\mathbb{R}$, we write $u \leqslant v$. Let $[u,v] = \{x \in BC(\mathbb{R}): u \leqslant x \leqslant v\}$. Recall that a map $T:[u,v] \to L^\infty$ is said to be increasing if $Tx \leqslant Ty$ for $x, y \in [u,v]$ with $x \leqslant y$. Now, we introduce the concept of a γ-increasing map.

Definition 2.1. A map $T:[u,v] \to L^\infty$ is said to be γ-increasing if there exists $\gamma > 0$ such that

$$Ty - Tx \geqslant -\gamma(y-x) \quad \text{for } x, y \in [u,v] \text{ with } x \leqslant y.$$

It is easy to see that T is γ-increasing if and only if $T + \gamma I$ is increasing. To establish our iterative techniques, we introduce M-continuity of maps.

Definition 2.2. A map $T:[u,v] \to L^\infty$ is said to be M-continuous on $[u,v]$ if $\{x_n\} \cup \{x\} \subset [u,v]$ and $\|x_n - x\|_{C[a,b]} \to 0$ for all $a, b \in \mathbb{R}$ with $a < b$ imply $(Tx_n)(t) \to (Tx)(t)$ a.e on $\mathbb{R}$.

Notation. Let $\hat{\mathbb{R}} = \{\hat{a}: a \in \mathbb{R}\}$, where $\hat{a}(t) \equiv a$ for $t \in \mathbb{R}$.

Now, we provide bounded, M-continuous and γ-increasing maps.

Theorem 2.1. *Assume that $f:\mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$ is bounded and satisfies Carathéodory conditions. Assume that there exists $\gamma > 0$ such that*

$$f(t,y) - f(t,x) \geqslant -\gamma(y-x) \quad \text{for } x, y \in [\alpha,\beta] \text{ with } x \leqslant y \quad \text{and all } t \in \mathbb{R}.$$

Then the map $\mathscr{F}$ defined by $\mathscr{F}x(t) = f(t,x(t))$ maps $[\hat{\alpha},\hat{\beta}]$ into L^∞ and is bounded, M-continuous and γ-increasing.

Proof. Since f satisfies Carathéodory conditions, $f(t,x(t))$ is measurable for $x \in [\hat{\alpha},\hat{\beta}]$ and $\mathscr{F}$ is M-continuous. It is easy to see that $\mathscr{F}x \in L^\infty$ for $x \in [\hat{\alpha},\hat{\beta}]$ and $\mathscr{F}$ is γ-increasing. □

As special cases of Theorem 2.1, we obtain the following.

Corollary 2.1. *Assume that $f \in C[\alpha,\beta]$ and there exists $\gamma > 0$ such that*

$$f(y) - f(x) \geqslant -\gamma(y-x) \quad \text{for } x, y \in [\alpha,\beta] \quad \text{with } x \leqslant y. \tag{2.1}$$

Then the map $\mathscr{F}$ defined by $\mathscr{F}x(t) = f(x(t))$ maps $[\hat{\alpha},\hat{\beta}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing.

Corollary 2.2. *Assume that $f:[\alpha,\beta] \to \mathbb{R}_+$ is continuous and satisfies* (2.1) *of Corollary* 2.1 *and $g:[\alpha,\beta] \to \mathbb{R}_+$ is increasing and continuous. Then the map $\mathscr{F}$ defined by $\mathscr{F}x(t) = g(x(t))f(x(t))$ maps $[\hat{\alpha},\hat{\beta}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ_1-increasing, where $\gamma_1 = \gamma\omega$ and $\omega = \sup\{g(x): x \in [\alpha,\beta]\}$.*

Proof. Let $h(x) = g(x)f(x)$ for $x \in [\alpha,\beta]$. Then we have for $x, y \in [\alpha,\beta]$ with $x \leqslant y$,

$$h(y) - h(x) = g(y)f(y) - g(x)f(x) \geqslant g(x)(f(y) - f(x)) \geqslant -\gamma\omega(y-x).$$

The result follows from Corollary 2.1. □

Corollary 2.3. *If $f \in C^1[\alpha,\beta]$, then the map $\mathscr{F}$ defined by $\mathscr{F}x(t)=f(x(t))$ maps $[\hat{\alpha},\hat{\beta}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing, where $\gamma=\omega$ if $\omega_1>0$ and $\gamma>0$ if $\omega_1=0$ and $\omega_1=\sup\{|f'(\xi)|\colon \xi\in[\alpha,\beta]\}$.*

Proof. Let $x,y\in[\hat{\alpha},\hat{\beta}]$ with $x\leqslant y$. Then $f(y)-f(x)=f'(\xi)(y-x)$ for some $\xi\in[x,y]$. This implies $f(y)-f(x)\geqslant-\gamma(y-x)$ and the result follows from Corollary 2.1. □

By Corollaries 2.2 and 2.3, we obtain

Example 2.1. *Let* $g:[0,1]\to\mathbb{R}_+$ is continuous and increasing and $f(x)=x^p(1-x^q)g(x)$ for $x\in[0,1]$, where $p\geqslant 1$, $q>0$. Then the map $\mathscr{F}$ defined by $\mathscr{F}x(t)=f(x(t))$ maps $[\hat{0},\hat{1}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing, where $\omega=\sup\{g(x)\colon x\in(0,1)\}$ and

$$\gamma=\begin{cases}\omega\max\{p,q\} & \text{if } p=1,\\ \omega q & \text{if } p>1.\end{cases}$$

Let $[\alpha,\beta]^n=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n\colon x_i\in[\alpha,\beta],\ i=1,\ldots,n\}$

Theorem 2.2. *Assume that $F:[\alpha,\beta]^n\to\mathbb{R}$ satisfies the following conditions:*

(i) *$\partial F(.)/\partial x_i$ is continuous on $[\alpha,\beta]^n$ for $i\in\{1,\ldots,n\}$.*
(ii) *$\partial F(u)/\partial x_i\geqslant 0$ for $u\in[\alpha,\beta]^n$ and $2\leqslant i\leqslant n$.*

Then for each $\{r_2,\ldots,r_n\}\subset\mathbb{R}$, the map $\mathscr{F}$ defined by

$$\mathscr{F}x(t)=F(x(t),x(t+r_2),\ldots,x(t+r_n))$$

maps $[\hat{\alpha},\hat{\beta}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing, where $\gamma=\omega$ if $\omega>0$ and $\gamma>0$ if $\omega=0$ and $\omega=\inf\{\partial F(u)/\partial x_1\colon u\in[\alpha,\beta]^n\}$.

Proof. We only prove that $\mathscr{F}$ is γ-increasing. Let $x,y\in[\hat{\alpha},\hat{\beta}]$ with $x\leqslant y$ and let $z_1(t)=ty(s)+(1-t)x(s)$ for $t\in[0,1]$ and $x^1(s)=(x(s+r_2),\ldots,x(s+r_n))$. By (i) and (ii) we have

$$\begin{aligned}(\mathscr{F}y)(s)-(\mathscr{F}x)(s)&\geqslant F(y(s),x^1(s))-F(x(s),x^1(s))=w(s)(y(s)-x(s))\\ &\geqslant-\gamma(y(s)-x(s)),\end{aligned}$$

where $w(s)=\int_0^1[\partial F(z_1(t),x^1(s))/\partial u_1]\,\mathrm{d}t$. □

As an application of Theorem 2.2, we give

Example 2.2. Let $q,p_i\geqslant 1$, $i=2,\ldots,n$, $k>0$ and $F(x_1,\ldots,x_n)=k(1-x_1^q)x_2^{p_2},\ldots,x_n^{p_n}$ for $x=(x_1,\ldots,x_n)\in[0,1]^n$. Let $r_i\in\mathbb{R}$ and $r_i\neq 0$ for $i=2,\ldots,n$. Then the map

$\mathscr{F}$ defined by $\mathscr{F}x(t) = k(1 - x^q(t))x^{p_2}(t + r_2), \ldots, x^{p_n}(t + r_n)$ maps $[\hat{0}, \hat{1}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing, where $\gamma = kq$.

Theorem 2.3. *Let $c \in \mathbb{R}$ and $\tau > 0$. Define a map $G : [\hat{\alpha}, \hat{\beta}] \times \mathbb{R} \to C[-\tau, 0]$ by*

$$G(x,t)(s) = x(t + cs).$$

Assume that $f : C[-\tau, 0] \to \mathbb{R}$ is bounded and continuous and there exists $\gamma > 0$ such that

$$f(\phi) - f(\psi) \geqslant -\gamma(\phi(0) - \psi(0)) \quad \text{for } \psi, \phi \in C[-\tau, 0] \text{ with } \psi \leqslant \phi. \tag{2.2}$$

Then the map $\mathscr{F}$ defined by $(\mathscr{F}x)(t) = f(G(x,t))$ maps $[\hat{\alpha}, \hat{\beta}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and γ-increasing.

Proof. It is easy to verify that $\mathscr{F} : [\hat{\alpha}, \hat{\beta}] \to \in BC(\mathbb{R})$ is bounded and M-continuous. Let $t \in \mathbb{R}$ and $x, y \in [\hat{\alpha}, \hat{\beta}]$ with $x \leqslant y$. Let $\phi(s) = G(y,t)(s) = y(t + cs)$ and $\psi(s) = G(x,t)(s) = x(t + cs)$ for $s \in [-\tau, 0]$. Then $\phi(s) \leqslant \psi(s)$ for $s \in C[-\tau, 0]$ and $\phi(0) = y(t)$ and $\psi(0) = x(t)$. By (2.2), we have for $t \in \mathbb{R}$,

$$(\mathscr{F}y)(t) - (\mathscr{F}x)(t) = f(\phi) - f(\psi) \geqslant -\gamma(\phi(0) - \psi(0)) = -\gamma(y(t) - x(t)).$$

Hence, $\mathscr{F}$ is γ-increasing. □

Example 2.3. Let $c \in \mathbb{R}$ and $\tau > 0$. Define a map $G : [\hat{0}, \hat{1}] \times \mathbb{R} \to C[-\tau, 0]$ by $G(x,t)(s) = x(t + cs)$. Assume that $f : C[-\tau, 0] \to \mathbb{R}$ is defined by $f(\phi) = (1 - \phi(0))\phi(-\tau)$. Then the map $\mathscr{F}$ defined by $(\mathscr{F}x)(t) = f(G(x,t))$ maps $[\hat{0}, \hat{1}]$ into $BC(\mathbb{R})$ and is bounded, M-continuous and 1-increasing.

Proof. Let $\phi, \psi \in C[-\tau, 0]$ with $0 \leqslant \psi \leqslant \phi \leqslant 1$. Then we have $f(\phi) - f(\psi) \geqslant -(\phi(0) - \psi(0))\phi(-\tau) \geqslant -(\phi(0) - \psi(0))$. The result follows from Theorem 2.3. □

3. Existence of solutions of second order functional differential equations

In this section, we consider the existence of solutions of a second order functional differential equation of the form

$$-Dz''(t) + cz'(t) = (\mathscr{F}z)(t) \quad \text{a.e. on } \mathbb{R}, \tag{3.1}$$

where $D > 0$, $c \in \mathbb{R}$ and $\mathscr{F} : [u, v] \subset BC(\mathbb{R}) \to L^\infty$.

By a solution to Eq. (3.1) we mean a function $z \in Y$ which satisfies Eq. (3.1), where $Y = \{x \in BC(\mathbb{R}) : x', x'' \in L^\infty\}$ is a Banach space with the norm $\|x\|_Y = \max\{\|x\|, \|x'\|_{L^\infty}, \|x''\|_{L^\infty}\}$.

Let $\gamma > 0$. We write

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4\gamma D}}{2D} \quad \text{and} \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4\gamma D}}{2D}.$$

Then $\lambda_1 < 0 < \lambda_2$ and $-D\lambda_i^2 + c\lambda_i + \gamma = 0$, $i = 1, 2$. Let $\rho = D(\lambda_2 - \lambda_1)$. We define a map $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$k(t,s) = \rho^{-1} \begin{cases} e^{\lambda_1(t-s)} & \text{for } s \leqslant t, \\ e^{\lambda_2(t-s)} & \text{for } s \geqslant t. \end{cases}$$

We consider the linear Hammerstein integral operator $\mathscr{L} : L^\infty \to Y$ defined by

$$\mathscr{L}z(t) = \int_{-\infty}^{\infty} k(t,s)z(s)\,\mathrm{d}s. \tag{3.2}$$

The following result shows that $\mathscr{L}$ is a homoeomorphic map from L^∞ to Y and from $BC(\mathbb{R})$ to $BC^2(\mathbb{R})$. Its proof follows from convergence theorem (see [27, Theorem 14, p. 200]) and some calculations. We leave it to the reader.

Theorem 3.1. *The map $\mathscr{L}$ defined in* (3.2) *maps L^∞ onto Y and is linear, bounded and one to one. Its inverse $\mathscr{L}^{-1} : Y \to L^\infty$ is defined by*

$$\mathscr{L}^{-1}y(t) = -Dy''(t) + cy'(t) + \gamma y(t). \tag{3.3}$$

Moreover, $\mathscr{L}$ maps $BC(\mathbb{R})$ onto $BC^2(\mathbb{R})$ and is linear, bounded and one to one.

It is known that the linear integral operator $Lz = \int_a^b k(t,s)z(s)\,\mathrm{d}s$ with a suitable kernel k is compact in $C[a,b]$. We refer to [16–18] for such results. However, we do not know if the map $\mathscr{L}$ defined in (3.2) is compact in $BC(\mathbb{R})$. We introduce the concept of G-compactness and show that the map $\mathscr{L}$ is G-compact. The concept of G-compactness is sufficient for us to establish our iterative techniques.

We define a map $P_a^b : BC(\mathbb{R}) \to C[a,b]$ by $P_a^b(x) = x|_{[a,b]}$, where $x|_{[a,b]}$ denotes the restriction of x to $[a,b]$.

Definition 3.1. A map $T : L^\infty \to BC(\mathbb{R})$ is said to be G-compact if $\overline{P_a^b T(B)}$ is compact in $C[a,b]$ for all $a, b \in \mathbb{R}$ with $a < b$ and for every bounded $B \subset L^\infty$.

Theorem 3.2. *The map $\mathscr{L}$ defined in* (3.2) *maps L^∞ into $BC(\mathbb{R})$ and is G-compact.*

Proof. By Theorem 3.1 $\mathscr{L}$ maps L^∞ into $BC(\mathbb{R})$. Let B be a bounded subset in L^∞, that is, there exists $m > 0$ such that $\|x\|_{L^\infty} \leqslant m$ for all $x \in B$. Then $P_a^b\mathscr{L}(B)$ is uniformly bounded in $C[a,b]$ since $\|P_a^b\mathscr{L}z\|_{C[a,b]} \leqslant m\gamma^{-1}$. By [27, Theorem 14, p. 200], we obtain $\lim_{t\to\tau} \int_{-\infty}^{\infty} |k(t,s) - k(\tau,s)|\,\mathrm{d}s \to 0$ for each $\tau \in [a,b]$. Using finite cover theorem, we can show that for $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $t, \tau \in [a,b]$ with $|t - \tau| < \delta$,

$$\int_{-\infty}^{\infty} |k(t,s) - k(\tau,s)|\,\mathrm{d}s < \varepsilon/m.$$

This implies $|P_a^b\mathscr{L}z(t) - P_a^b\mathscr{L}z(\tau)| \leqslant m\int_{-\infty}^{\infty} |k(t,s) - k(\tau,s)|\,\mathrm{d}s < \varepsilon$. Hence, $P_a^b\mathscr{L}(B)$ is equicontinuous and $\overline{P_a^b\mathscr{L}(B)}$ is compact in $C[a,b]$. The result follows from Definition 3.1. □

Remark 3.1. Since $P_a^b: BC(\mathbb{R}) \to C[a,b]$ and $\mathscr{L}: L^\infty \to BC(\mathbb{R})$ are continuous, it follows from Theorem 3.2 that $P_a^b\mathscr{L}: L^\infty \to C[a,b]$ is compact.

Eq. (3.1) can be changed into a Hammerstein integral equation of the form

$$z(t) = \int_{-\infty}^{\infty} k(t,s)((\mathscr{F}z)(s) + \gamma z(s))\,\mathrm{d}s \equiv Az(t), \quad t \in \mathbb{R}. \tag{3.4}$$

The following result gives relations between Eqs. (3.1) and (3.4). Its proof follows from Theorem 3.1 and omitted.

Lemma 3.1. (1) *Let $y \in Y$ and $z \in [u,v]$. Then $y = Az$ if and only if*

$$-Dy'' + cy' + \gamma y = \mathscr{F}z + \gamma z.$$

(2) *z is a solution of Eq.* (3.1) *if and only if $z \in Y$ and $z = Az$.*

Now, we are in a position to give our main result in this section.

Theorem 3.3. *Assume that $\mathscr{F}: [u,v] \to L^\infty$ is bounded, M-continuous and γ-increasing and A satisfies*

(H) $u \leqslant Au$ *and* $Av \leqslant v$.

Then Eq. (3.1) *has a maximal solution v^* in $[u,v]$ and a minimal solution u_* in $[u,v]$. Moreover, for $a, b \in \mathbb{R}$ with $a < b$.*

$$\|u_n - u_*\|_{C[a,b]} \to 0 \quad \text{and} \quad \|v_n - v^*\|_{C[a,b]} \to 0, \tag{3.5}$$

where $u_n = Au_{n-1}$, $v_n = Av_{n-1}$ and

$$u = u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0 = v. \tag{3.6}$$

Proof. Let $\mathscr{T} = \mathscr{F} + \gamma I$. Then $\mathscr{T}: [u,v] \to L^\infty$ is bounded, M-continuous and increasing. By Theorem 3.2 and condition (H), A maps $[u,v]$ into $[u,v]$ and is increasing. By (3.6), there exist $u_*, v^* \in L^\infty$ such that $u_n(t) \to u_*(t)$ and $v_n(t) \to v^*(t)$ for each $t \in \mathbb{R}$. By Theorem 3.2, $\overline{P_a^b\mathscr{L}\mathscr{T}\{u_n\}} = \overline{P_a^bA\{u_n\}}$ is compact in $C[a,b]$ for $a, b \in \mathbb{R}$ with $a < b$. It follows that there exists $y \in C[a,b]$ such that $\|Au_n - y\|_{C[a,b]} \to 0$. Hence, we have $u_*(t) = y(t)$ for $t \in [a,b]$ and thus, $u_* \in BC(\mathbb{R})$. Since $\mathscr{T}$ is M-continuous and bounded, it follows that for each $t \in \mathbb{R}$,

$$k(t,s)\mathscr{T}u_n(s) \to k(t,s)\mathscr{T}u_*(s) \quad \text{for } s \in \mathbb{R}$$

and $|k(t,s))\mathscr{T}u_n(s)| \leqslant mk(t,s)$ for $s \in \mathbb{R}$ and some $m > 0$. It follows from Lebsegue's dominated theorem ([27, Theorem 5, p. 160]) that $Au_n(t) \to Au_*(t)$ for each $t \in \mathbb{R}$ and $u_* = Au_*$. A similar argument shows that $v^* = Av^*$. Let $x \in [u,v]$ be such that $x = Ax$. Since A is increasing on $[u,v]$, it follows from $u_* \leqslant x \leqslant v^*$, that is, u_* is a minimal solution and v^* is a maximal solution. □

Remark 3.2. We remark that the map A need not be continuous. Moreover, we do not know if the set $\overline{A(Q)}$ is compact for each bounded subset Q. Although the closure of

$P_a^b A(Q)$ is compact, the map $P_a^b A$ may not be continuous, so $P_a^b A$ may not be compact. Hence, the classical iterative techniques for compact maps in ordered Banach spaces (see, for example, [1,13]) cannot be applied to treat Theorem 3.3.

Corollary 3.1. *Let $u, v \in Y$. Assume that $\mathscr{F} : [u,v] \to L^\infty$ is bounded, M-continuous and γ-increasing and satisfies*

$$(\mathrm{H}') \quad -Du'' + cu' \leqslant \mathscr{F}u \quad \text{and} \quad \mathscr{F}v \leqslant -Dv'' + cv'.$$

Then the results of Theorem 3.3 *hold.*

Proof. By (H′) we have $\mathscr{L}^{-1}u \leqslant \mathscr{F}u + \gamma u$ and $\mathscr{F}v + \gamma v \leqslant \mathscr{L}^{-1}v$. Note that $\mathscr{L}y \geqslant 0$ for $y \geqslant 0$. By Theorem 3.1, we obtain

$$u = \mathscr{L}(\mathscr{L}^{-1}u) \leqslant \mathscr{L}(\mathscr{F}u + \gamma u) = Au \quad \text{and} \quad Av = \mathscr{L}(\mathscr{F}v + \gamma v) \leqslant \mathscr{L}(\mathscr{L}^{-1}v) = v.$$

The results follow from Theorem 3.3. □

4. Travelling wavefronts of reaction–diffusion equations

In this section we consider the existence of travelling wavefronts for a reaction–diffusion equation of the form

$$-D\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial u(t,x)}{\partial t} = f(u(t,x)), \quad t \geqslant 0, \; x \in \mathbb{R}, \tag{4.1}$$

where $D > 0$ is a fixed constant.

We list the following conditions:

(C_1) $f : [0,1] \to \mathbb{R}$ is continuous:

(C_2) There exists $\gamma > 0$ such that

$$f(y) - f(x) \geqslant -\gamma(y - x) \quad \text{for } x, y \in [0,1] \text{ with } x \leqslant y.$$

(C_3) $f(\mu) > 0$ for $\mu \in (0,1)$.

(C_4) $f(0) = f(1) = 0$.

Eq. (4.1) has been widely studied when $f \in C^1[0,1]$ (see [2,7,9,34]) and f is Lipschitz continuous (see [4]). Here, we employ conditions (C_1) and (C_2). In addition to the references mentioned above, condition (C_3) was also employed, for example, in [15,19,20,31,33]. Condition (C_4) is a necessary condition for Eq. (4.1) to have travelling wavefronts (see Lemma 4.1). Under conditions (C_1)–(C_4), we shall use our theory developed in the above section to prove the existence of travelling wavefronts and provide iterations to compute the travelling wavefronts.

Since we consider the existence of travelling wavefronts $u(t,x) = z(x + ct)$ for Eq. (4.1), we can write Eq. (4.1) in the form

$$-Dz''(t) + cz'(t) = f(z(t)), \quad t \in \mathbb{R} \tag{4.2}$$

subject to the following boundary condition:

$$z(-\infty) = 0 \quad \text{and} \quad z(\infty) = 1. \tag{4.3}$$

By a travelling wavefront (with a wave speed c) to Eq. (4.1) we mean an increasing function $z \in BC^2(\mathbb{R})$ and a number $c \in \mathbb{R}$ which satisfy (4.2)–(4.3):

Eq. (4.2) can be changed into the following Hammerstein integral equation:

$$z(t) = \int_{-\infty}^{\infty} k(t,s)(f(z(s)) + \gamma z(s))\, \mathrm{d}s \equiv Az(t), \quad t \in \mathbb{R}, \tag{4.4}$$

where k is same as in Section 3.

Notation. Let $\mu^- = \lim_{t\to-\infty} z(t)$ and $\mu^+ = \lim_{t\to\infty} z(t)$.

Lemma 4.1. *Under* (C_1) *and* (C_2), *if* $z \in [\hat{0}, \hat{1}]$ *is increasing and* $z = Az$, *then*

$$f(\mu^-) = f(\mu^+) = 0.$$

Proof. Assume that $f(\mu^+) > 0$. Since z is increasing, it follows from (C_1) and (C_2) that there exists $t_1 \in \mathbb{R}$ such that $f(z(t)) + \gamma z(t) \geqslant f(z(t_1)) + \gamma z(t_1) > 0$ for $t \geqslant t_1$. This implies

$$\int_{t_1}^{t} \mathrm{e}^{-\lambda_1 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s \geqslant (f(z(t_1)) + \gamma z(t_1)) \int_{t_1}^{t} \mathrm{e}^{-\lambda_1 s}\, \mathrm{d}s \to \infty.$$

Noting that $\int_{-\infty}^{t_1} \mathrm{e}^{-\lambda_1 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s < \infty$, we have

$$\lim_{t\to\infty} \int_{-\infty}^{t} \mathrm{e}^{-\lambda_1 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s = \infty.$$

Applying L'Hospital's rule and (C_1) we obtain

$$\lim_{t\to\infty} \frac{\int_{-\infty}^{t} \mathrm{e}^{-\lambda_1 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s}{\mathrm{e}^{-\lambda_1 t}} = -\lambda_1^{-1}(f(\mu^+) + \gamma\mu^+).$$

Note that $\lim_{t\to\infty} \int_t^{\infty} \mathrm{e}^{-\lambda_2 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s = 0$. Again using L'Hospital's rule and (C_1), we have

$$\lim_{t\to\infty} \frac{\int_{t}^{\infty} \mathrm{e}^{-\lambda_2 s}(f(z(s)) + \gamma z(s))\, \mathrm{d}s}{\mathrm{e}^{-\lambda_2 t}} = \lambda_2^{-1}(f(\mu^+) + \gamma\mu^+).$$

Since $z = Az$, we obtain

$$\mu^+ = \lim_{t\to\infty} A(z(t)) = \rho^{-1}(\lambda_2^{-1} - \lambda_1^{-1})(f(\mu^+) + \gamma\mu^+) = \gamma^{-1} f(\mu^+) + \mu^+.$$

This implies $f(\mu^+) = 0$, which contradicts the hypothesis $f(\mu^+) > 0$. Hence, we must have $f(\mu^+) = 0$. Similarly, we can prove that $f(\mu^-) = 0$. □

Let $\Gamma = \{z \in [\hat{0}, \hat{1}]$: z is increasing and satisfies $\mu^- < 1$ and $\mu^+ > 0\}$.

Theorem 4.1. *Assume that f satisfies* (C_1)–(C_4) *and there exist* $u \in [\hat{0}, \hat{1}]$ *with* $u \neq 0$, $v \in \Gamma$ *with* $u \leqslant v$ *and* $c \in \mathbb{R}$ *such that* $u \leqslant Au$ *and* $Av \leqslant v$. *Then Eq.* (4.1) *has a travelling wavefront* v^* *in* $[u, v]$. *Moreover, for* $a, b \in \mathbb{R}$ *with* $a < b$.

$$\|v_n - v^*\|_{C[a,b]} \to 0, \tag{4.5}$$

where $-Dv_n'' + cv_n' + \gamma v_n = f(v_{n-1}) + \gamma v_{n-1}$ ($n \in \mathbb{N}$) *and*

$$u \leqslant v^* \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0 = v. \tag{4.6}$$

Proof. We define a map $\mathscr{F}:[u,v]\to BC(\mathbb{R})$ by $\mathscr{F}z(t)=f(z(t))$. It follows from Corollary 2.1 that $\mathscr{F}:[u,v]\to BC(\mathbb{R})$ is bounded, M-continuous and γ-increasing. By Theorem 3.3 there exists $v^*\in[u,v]$ such that (4.5) and (4.6) hold. Since $\mathscr{F}v^*\in BC(\mathbb{R})$, it follows from Theorem 3.1 that $v^*=Av^*\in BC^2(\mathbb{R})$. We prove that $v_1=Av$ is increasing. Let $r>0$. By a change of variable, we have for $t\in\mathbb{R}$,

$$\begin{aligned} v_1(t+r)-v_1(t) &= \rho^{-1}\left[\int_{-\infty}^{t} e^{\lambda_1(t-s)}(f(v(s+r))+\gamma v(s+r))\,ds \right.\\ &\quad \left. +\int_t^{\infty} e^{\lambda_2(t-s)}(f(v(s+r))+\gamma v(s+r))\,ds\right]-v_1(t)\\ &= \int_{-\infty}^{\infty} k(t,s)(f(v(s+r))-f(v(s))+\gamma(v(s+r)-v(s)))\,ds\\ &\geqslant 0. \end{aligned}$$

This implies that v_1 is increasing. Using similar arguments, we can show that v_n is increasing for each $n\in\mathbb{N}$. It follows that $v^*(t+r)\geqslant v^*(t)$ for $t\in\mathbb{R}$ and $r>0$. Hence, v^* is increasing. By Lemma 4.1, $f(\mu^-)=f(\mu^+)=0$, where $\lim_{t\to-\infty}v^*(t)=\mu^-$ and $\lim_{t\to\infty}v^*(t)=\mu^+$. It follows from (C_3) and (C_4) that $\mu^-,\mu^+\in\{0,1\}$. Since $u\neq 0$ and $u\leqslant v^*$, we have $\mu^+>0$ and $\mu^+=1$. Since $\lim_{t\to-\infty}v(t)<1$ and $v^*\leqslant v$, it follows that $\lim_{t\to-\infty}v^*(t)<1$ and $\lim\limits_{t\to-\infty}v^*(t)=0$. □

By an argument similar to that of Corollary 3.1, we obtain the following useful result.

Corollary 4.1. *Assume that f satisfies* (C_1)–(C_4) *and there exist* $u\in[\hat{0},\hat{1}]\cap Y$ *with* $u\neq 0$, $v\in\Gamma\cap Y$ *with* $u\leqslant v$ *and* $c\in\mathbb{R}$ *such that the following conditions hold*:

(h_1) $-Du''(t)+cu'(t)\leqslant f(u(t))$ *a.e. on* $\mathbb{R}$ *and*
(h_2) $f(v(t))\leqslant -Dv''(t)+cv'(t)$ *a.e. on* $\mathbb{R}$,

then the results of Theorem 4.1 *hold.*

Now, we construct u and v and impose an additional condition on f such that (h_1) and (h_2) in Corollary 4.1 hold for suitable c.

Let $V_c(t)=(1+e^{-ct/2D})^{-1}$ for $t\in\mathbb{R}$ and

$$U_c(t)=\begin{cases} e^{ct/D} & \text{for } t\leqslant t_0,\\ e^{ct_0/D} & \text{for } t\geqslant t_0,\end{cases}$$

where $t_0\in(-\infty,0)$ satisfies $e^{ct_0/D}(1+e^{-ct_0/2D})\leqslant 1$.

Notation. Let $M=\sup\{f(x)/x(1-x)\colon x\in(0,1)\}$.

Theorem 4.2. *Assume that f satisfies* (C_1)–(C_3) *and* $M<\infty$. *Then for each* $c\geqslant 2\sqrt{DM}$, *Eq.* (4.1) *has a travelling wavefront* v^* *in* $[u,v]$, *where* $u=U_c$ *and* $v=V_c$. *Moreover*, (4.5) *and* (4.6) *hold.*

Proof. Since $M<\infty$, it follows that $f(0)=f(1)=0$ and (C_4) holds. It is easy to verify that $u\in[\hat{0},\hat{1}]\cap Y$ with $u\neq 0$ and $v\in\Gamma\cap Y$ with $u\leqslant v$. By calculation, we have $-Du''(t)+cu'(t)=0\leqslant f(u(t))$ for $t\in\mathbb{R}$ with $t\neq t_0$ and

$$\begin{aligned}-Dv''(t)+cv'(t)&=\frac{c}{2D}\left(\frac{c}{2}+\frac{c}{D}v(t)\right)v(t)(1-v(t))\geqslant\frac{c^2}{4D}v(t)(1-v(t))\\&\geqslant Mv(t)(1-v(t))\geqslant f(v(t))\quad\text{for } t\in\mathbb{R}.\end{aligned}$$

The result follows from Corollary 4.1. □

Corollary 4.2. *Assume that* $f\in C^1[0,1]$ *and satisfies* (C_3) *and* (C_4). *Then for each* $c\geqslant 2\sqrt{DM}$, *Eq.* (4.1) *has a travelling wavefront* v^* *in* $[u,v]$, *where* $u=U_c$ *and* $v=V_c$. *Moreover*, (4.5) *and* (4.6) *hold.*

Proof. By the proof of Corollary 2.3, f satisfies (C_2). By (C_4), $f'(0)=\lim_{x\to 0^+}f(x)/x$ and $f'(1)=-\lim_{x\to 1^-}f(x)/(1-x)$. This implies $M<\infty$. The result follows from Theorem 4.2. □

Remark 4.1. When $D=1$, the first part of Corollary 4.2 improves (b) of Theorem 4.15 in [9] (also see [2, Theorem 4.2]; [28, Theorem 1, p. 215]), where f satisfies $f'(0)>0$ and $f'(1)<0$. However, (b) of Theorem 4.15 in [9] obtained a larger interval of wave speeds $[c^*,\infty)$, where $c^*\in[2\sqrt{f'(0)},2\sqrt{v}]$ and $v=\sup\{f(x)/x\colon x\in(0,1)\}$. Our method is completely different from the phase-plane techniques used in [9]. The second part of Corollary 4.2 is new and provides an iteration to compute the travelling wavefronts.

As applications of Theorem 4.2, we consider the existence of travelling wavefronts for the following reaction–diffusion equation:

$$-D\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{\partial u(t,x)}{\partial t}=u^p(t,x)(1-u^q(t,x))g(u(t,x))\quad t\geqslant 0,\ x\in\mathbb{R},\tag{4.7}$$

where $D>0$, $p\geqslant 1$, $q>0$ and $g:[0,1]\to\mathbb{R}$ is a function.

When $g\equiv 1$, travelling wavefronts for Eq. (4.7) was studied in [24, Sections 11.1–11.3]. Moreover, some exact travelling wavefronts can be obtained (also see [6]).

Let

$$M_1=\begin{cases}g(1) & \text{if } q\in(0,1],\\ g(1)q & \text{if } q\geqslant 1.\end{cases}$$

Example 4.1. Assume that $g:[0,1]\to\mathbb{R}_+$ is increasing and continuous and satisfies $g(x)>0$ for $x\neq 0$. Then for each $c\geqslant 2\sqrt{DM_1}$, Eq. (4.7) has a travelling wavefront v^* in $[u,v]$. Moreover, (4.5) and (4.6) hold, where $u=U_c$, $v=V_c$, γ is the same as in Example 2.1 and $f(x)=x^p(1-x^q)g(x)$.

Proof. It is clear that f satisfies (C_1) and (C_3). By Corollaries 2.2 and 2.3, f satisfies (C_2) with γ given in Example 2.1. Let $h(x)=(1-x^q)(1-x)^{-1}$ for $x\in(0,1)$. If $q\in(0,1]$, then h is decreasing on $(0,1)$. Hence, $h(x)\leqslant \lim_{x\to 0^+} h(x)=1$ for $x\in(0,1)$. If $q>1$, then h is increasing. Hence, $h(x)\leqslant \lim_{x\to 1^-} h(x)=q$ for $x\in(0,1)$. This implies $M\leqslant M_1$ and the result follows from Theorem 4.2. □

5. Travelling wavefronts of reaction–diffusion equations with multiple delays

In this section we consider the existence of travelling wavefronts of a reaction–diffusion equation with multiple delays of the form

$$-D\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{\partial u(t,x)}{\partial t}=F(u(t,x),u(t-r_2,x),\dots,u(t-r_n,x)), \tag{5.1}$$

where $D>0$, $r_i>0$ for $2\leqslant i\leqslant n$ and $n\geqslant 2$.

We always assume that $F:[0,1]^n\to\mathbb{R}_+$ satisfies the following conditions.

(H_1) $\partial F(.)/\partial x_i$ is continuous on $[0,1]^n$ for each $i\in\{1,\dots,n\}$.
(H_2) $\partial F(u)/\partial x_i\geqslant 0$ for $u\in[\alpha,\beta]^n$ and $2\leqslant i\leqslant n$.
(H_3) $F(\bar{x})>0$ for $x\in(0,1)$, where $\bar{x}=(x,\dots,x)\in\mathbb{R}^n$.
(H_4) $F(\bar{0})=F(\bar{1})=0$.

When $n=2$, the existence of travelling wavefronts for Eq. (5.1) was studied in [29], where F satisfies some additional assumptions. We also refer to [14,35] for similar results. Here, even under weaker conditions than those used in [29] and even when $n\geqslant 2$, we obtain that the existence of travelling wavefronts and provide iteration schemes to compute the travelling wavefronts.

Since we consider the existence of travelling wavefronts $u(t,x)=z(x+ct)$ for Eq. (5.1), we can write Eq. (5.1) in the form

$$-Dz''(t)+cz'(t)=F(z(t),z(t-cr_2),\dots,z(t-cr_n)),\quad t\in\mathbb{R} \tag{5.2}$$

subject to the boundary condition (4.3).

By a travelling wavefront (with a wave speed c) to Eq. (5.1) we mean an increasing function $z\in BC^2(\mathbb{R})$ and a number $c\in\mathbb{R}$ which satisfy (5.2) and (4.3).

We refer to Mallet–Paret [22,23] for the study of the existence of travelling wavefronts for Eq. (5.2) with $D=0$ and $r_i\in\mathbb{R}$.

Eq. (5.2) can be changed into the following Hammerstein integral equation

$$z(t)=\int_{-\infty}^{\infty}k(t,s)((\mathscr{F}z)(s)+\gamma z(s))\,\mathrm{d}s\equiv Az(t),\quad t\in\mathbb{R}, \tag{5.3}$$

where k is same as in Section 3 and $\mathscr{F}:[\hat{0},\hat{1}]\to BC(\mathbb{R})$ is defined by

$$\mathscr{F}z(t)=F(z(t),z(t-cr_2),\dots,z(t-cr_n))\quad \text{for } t\in\mathbb{R}.$$

As before, we write $\mu^-=\lim_{t\to-\infty}z(t)$ and $\mu^+=\lim_{t\to\infty}z(t)$.

By a similar argument to that in Lemma 4.1, we obtain

Lemma 5.1. *Under* (H_1) *and* (H_2), *if* $z \in [\hat{0}, \hat{1}]$ *is increasing and* $z = Az$, *then*

$$F(\overline{\mu^-}) = F(\overline{\mu^+}) = 0.$$

Proof. We outline the proof. Assume that $F(\overline{\mu^+}) > 0$. Since z is increasing and F is continuous, there exists $t_1 \in \mathbb{R}$ such that $F(\overline{z(t_1)}) + \gamma z(t_1) > 0$. Let $t_0 = \max\{t_1, t_1 + cr_2, \ldots, t_1 + cr_n\}$. Then $z(t - cr_i) \geqslant z(t_1)$ for $t \geqslant t_0$. By the proof of Theorem 2.2, we have for $t \geqslant t_0$,

$$(\mathscr{F}z)(t) - F(\overline{z(t_1)}) \geqslant F(z(t), z(t_1), \ldots, z(t_1)) - F(\overline{z(t_1)}) \geqslant -\gamma(z(t) - z(t_1)).$$

This implies $(\mathscr{F}z)(t) + \gamma z(t) \geqslant F(\overline{z(t_1)}) + \gamma z(t_1)$ for $t \geqslant t_0$. Hence, we obtain

$$\int_{t_0}^{t} e^{-\lambda_1 s}((\mathscr{F}z)(s)) + \gamma z(s))\, ds \geqslant (F(\overline{z(t_1)}) + \gamma z(t_1)) \int_{t_0}^{t} e^{-\lambda_1 s}\, ds \to \infty.$$

By a similar argument to that of Lemma 4.1, we must $F(\overline{\mu^+}) = 0$ and $F(\overline{\mu^-}) = 0$ and $F(\overline{\mu^-}) = 0$. □

By an argument similar to that of Theorem 4.1, we obtain

Theorem 5.1. *Assume that F satisfies* (H_1)–(H_4) *and there exist* $u \in [\hat{0}, \hat{1}]$ *with* $u \neq 0$ $v \in \Gamma$ *with* $u \leqslant v$ *and* $c \in \mathbb{R}$ *such that* $u \leqslant Au$ *and* $Av \leqslant v$. *Then Eq.* (5.1) *has a travelling wavefront* v^* *in* $[u, v]$. *Moreover, for* $a, b \in \mathbb{R}$ *with* $a < b$.

$$\|v_n - v^*\|_{C[a,b]} \to 0, \tag{5.4}$$

where $-Dv_n'' + cv_n' + \gamma v_n = \mathscr{F}(v_{n-1}) + \gamma v_{n-1}$ $(n \in \mathbb{N})$ *and*

$$u \leqslant v^* \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0 = v. \tag{5.5}$$

As a special case of Theorem 5.1, we have

Corollary 5.1. *Assume that* (H_1)–(H_4) *hold and there exist* $u \in [\hat{0}, \hat{1}] \cap Y$ *with* $u \neq 0$ *and* $v \in \Gamma \cap Y$ *with* $u \leqslant v$ *and* $c \in \mathbb{R}$ *such that the following conditions hold*:

(i) $-Du''(t) + cu'(t) \leqslant F(u(t), u(t - cr_2), \ldots, u(t - cr_n))$ *a.e. on* $\mathbb{R}$ *and*
(ii) $F(v(t), v(t - cr_2), \ldots, v(t - cr_n)) \leqslant -Dv''(t) + cv'(t)$ *a.e. on* $\mathbb{R}$.

Then the results of Theorem 4.1 *hold.*

Applying Corollary 5.1, we obtain the following result.

Theorem 5.2. *Assume that F satisfies* (H_1)–(H_3) *and the following condition*:

$$M' := \sup\left\{\frac{F(\bar{x})}{x(1-x)} : x \in (0,1)\right\} < \infty.$$

Then for each $c \geqslant 2\sqrt{DM'}$, *Eq.* (5.1) *has a travelling wavefront* v^* *in* $[u, v]$, *where* $u = U_c$ *and* $v = V_c$. *Moreover*, (5.4) *and* (5.5) *hold.*

Proof. Since $M' < \infty$, $F(\bar{0}) = F(\bar{1}) = 0$ and (H_4) holds. By calculation, we have for $t \in \mathbb{R}$ with $t \neq t_0$,

$$-Du''(t) + cu'(t) = 0 \leqslant F(u(t), u(t - cr_2), \dots, u(t - cr_n)).$$

Hence, (i) of Corollary 5.1 holds. Note that

$$-Dv''(t) + cv'(t) \geqslant \frac{c^2}{4D} v(t)(1 - v(t)) \geqslant M' v(t)(1 - v(t)) \geqslant F(\overline{v(t)}) \quad \text{for } t \in \mathbb{R}.$$

By (H_2), we have $F(\overline{v(t)}) \geqslant F(v(t), v(t - cr_2), \dots, v(t - cr_n))$ for $t \in \mathbb{R}$. This implies that (ii) of Corollary 5.1 holds. The results follow from Corollary 5.1. □

Remark 5.1. The first part of Theorem 5.2 improves (ii) of Theorem 2.7 in [29], where $D = 1$, $n = 2$ and F satisfies extra conditions: $F \in C^{1,v}(\mathbb{R}^2, \mathbb{R})$ and $\partial F(\bar{0})/\partial x_1 + \partial F(\bar{0})/\partial x_2 > 0$. But, Schaaf [29] obtained a minimal speed c^*. Our method is completely different from the sub- and superlinear techniques due to Atkinson and Reuter [3], used in [29].

As applications of Theorem 5.2, we consider the existence of travelling wavefronts of the following delay reaction–diffusion equation:

$$-D\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial u(t,x)}{\partial t} = ku^{p_2}(t - r_2, x), \dots, u^{p_n}(t - r_n, x)(1 - u^q(t,x)), \qquad (5.6)$$

where $D, k > 0$, $q, p_i \geqslant 1$ and $r_i > 0$ $(i = 2, \dots, n)$.

Example 5.1. For each $c \geqslant 2\sqrt{Dkq}$, Eq. (5.6) has a travelling wavefront v^* in $[u, v]$, where $F(x) = k(1 - x_1^q)x_2^{p_2}, \dots, x_n^{p_n}$, $u = U_c$ and $v = V_c$. Moreover, (5.4) and (5.5) hold.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM. Rev. 18 (1976) 620–709.

[2] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion and nerve propagation, in: Proceedings of the Tulane Program in Partial Differential Equations and Related Topics, Lecture Notes in Mathematics, Vol. 446, Springer, Berlin, 1975, pp. 5–49.

[3] C. Atkinson, G.E. Reuter, Deterministic epidemic waves, Math. Proc. Cambridge Philos. Soc. 80 (1976) 315–330.

[4] H. Berestycki, L. Nirenberg, Travelling fronts in cylinders, Ann. Inst. H. Poincaré 9 (1992) 497–572.

[5] M.D. Bramson, Maximal displacement of branching Brownian motion, Comm. Pure Appl. Math. 31 (1978) 531–581.

[6] P.K. Brazhnik, J.J. Tyson, On traveling wave solutions of Fisher's equation in two spatial dimensions, SIAM J. Appl. Math. 60 (1999) 371–391.

[7] N.F. Britton, Reaction–Diffusion Equations and their Applications to Biology, Academic Press, San Diego, 1986.

[8] J. Canosa, Diffusion in nonlinear multiplicative media, J. Math. Phys. 10 (1969) 1863–1868.

[9] P.C. Fife, Mathematical Aspects of Reacting and Diffusing Systems, Lectures Notes in Biomathematics, Vol. 28, Springer, Berlin, 1979.

[10] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Rational Mech. Anal. 65 (1977) 335–361.

[11] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics 7 (1937) 353–369.

[12] R.A. Gardner, Existence and stability of travelling wave solutions of competition models: a degree theoretical approach, J. Differential Equations 44 (1982) 343–364.
[13] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.
[14] K. Kobayashi, On the semilinear heat equations with time-lag, Hiroshima Math. J. 7 (1977) 59–472.
[15] A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem, Bjul. Moskovskogo Gos. Univ. 1 (1937) 1–25.
[16] K.Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, Differential Equations Dynamic Systems 8 (2) (2000) 175–192.
[17] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc. 63 (2) (2001) 690–704.
[18] K.Q. Lan, J.R.L. Webb, Positive solutions of semilinear differential equations with singularities, J. Differential Equations 148 (1998) 407–421.
[19] D.A. Larson, Transiet bounds and time-asymptotic behavior of solutions to nonlinear equations of Fisher type, SIAM J. Appl. Math. 34 (1978) 93–103.
[20] K.S. Lau, On the nonlinear diffusion equation of Kolmogorov, Petrosky, and Piscounov, J. Differential Equations 59 (1985) 44–70.
[21] A.W. Leung, Systems of Nonlinear Partial Differential Equations with Applications to Biology and Engineering, Kluwer Academic Publishers, Dordrecht, 1989.
[22] J. Mallet-Paret, The Fredolm alternative for functional differential equations of mixed type, J. Dynamics Differential Equations 11 (1999) 1–47.
[23] J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, J. Dynamics Differential Equations 11 (1999) 49–127.
[24] J.D. Murray, Mathematical Biology, Springer, New York, 1989.
[25] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum, New York, 1992.
[26] A.J. Perumpanani, J.A. Sherratt, J. Norbury, Mathematical modelling of capsule formation and multinodularity in benign tumor growth, Nonlinearity 10 (1997) 1599–1614.
[27] M.M. Rao, Measure Theory and Integration, Wiley, New York, 1987.
[28] F. Rothe, Convergence to travelling fronts in semilinear parabolic equations, Proc. Roy. Soc. Edinburgh Sect. A 80 (1978) 213–234.
[29] K.W. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc. 302 (1987) 587–615.
[30] J.A. Sherratt, Traveling wave solutions of a mathematical model for tumor encapsulation, SIAM J. Appl. Math. 60 (1999) 392–407.
[31] A.N. Stokes, On two types of moving front in quasilinear diffusion, Math. Biosci. 31 (1976) 307–315.
[32] Tuchwell, Introduction to Theoretical Neurobiology, Cambridge Studies in Mathematical Biology, Vol. 8, Cambridge University Press, Cambridge, UK, 1988.
[33] K. Uchiyama, The behavior of solutions of some non-linear diffusion equations for large time, J. Math. Kyoto Univ. 18 (1978) 453–508.
[34] A.I. Volpert, V.A. Volpert, V.A. Volpert, Traveling wave solutions of parabolic systems, Translations of Mathematical Monographs, Vol. 140, American Mathematical Society, Providence, RI, 1994.
[35] X.F. Zou, J.H. Wu, Existence of traveling wave fronts in delayed reaction–diffusion systems via the monotone iteration method, Proc. Amer. Math. Soc. 125 (1997) 2589–2598.
[36] X.F. Zou, J.H. Wu, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynamics Differential Equations 13 (2001) 651–687.