Random Oper. and Stoch. Equ., Vol. 11, No. 1, pp. 25–76 (2003)

Limit theorems for difference equations in random media with applications to biological systems

Anatoliy SWISHCHUK and Jianhong WU [1]

Department of Mathematics & Statistics,
York University, Toronto, Ontario, Canada M3J 1P3

Received for *ROSE* September 14, 2002

0. INTRODUCTION

Let $(Y, \mathcal{Y})$ be a measurable space and $g : X \times Y \to X$ be a function which determines the dynamics in a random environment described by a semi-Markov process $y(t)$. Let ϵ be a small positive parameter and X be a linear space. We consider the dynamical system where states of the system are determined by the following iteration:

$$X^{\epsilon}_{\nu(t/\epsilon^i)+1} = X^{\epsilon}_{\nu(t/\epsilon^i)} + \varepsilon g(X^{\epsilon}_{\nu(t/\epsilon^i)}, y_{\nu(t/\epsilon^i)+1}),$$

for $t \in R_+$, where $X^{\epsilon}_0 = X_0 = x$ is given, $\nu(t)$ is a counting process, $i = 1, 2$. In this chapter we study:

(A) *Averaging* $(i = 1)$ *and diffusion approximation* $(i = 2)$ of solutions of the equation as $\varepsilon \to 0$ under various assumptions of the data (see [8]);

(B) *Normal deviations* of the process $X^{\varepsilon}_{\nu(t/\varepsilon)}$ $(i = 1)$ from averaged one $\tilde{x}^{\epsilon}_{\nu(t/\epsilon)}$, namely, the limit

$$Z^{\varepsilon}(t) := [X^{\varepsilon}_{\nu(t/\varepsilon)} - \tilde{x}^{\varepsilon}_{\nu(t/\varepsilon)}]/\sqrt{\varepsilon} \text{ as } \varepsilon \to 0,$$

where $\tilde{x}^{\varepsilon}_n$ is defined by the averaged difference equation

$$\tilde{x}^{\varepsilon}_{n+1} - x^{\varepsilon}_n = \varepsilon \tilde{g}(\tilde{x}^{\varepsilon}_n),$$

with

$$\tilde{g}(x) := \int_Y p(dy) g(x, y)/m,$$

where $(p(A), A \in \mathcal{Y})$ is a stationary distribution of $(y_n)_{n \in Z_+}$, and m is a mean sojourn time (see [9]);

(C) *Merging of solution* of the following equation

$$X^{\varepsilon}_{\nu(t/\varepsilon)+1} - X^{\varepsilon}_{\nu(t/\varepsilon)} = \varepsilon g(X^{\epsilon}_{\nu(t/\varepsilon)}, y^{\epsilon}_{\nu(t/\varepsilon)+1}),$$

where $(y^{\varepsilon}_n)_{n \in Z_+}$ is a perturbed Markov chain in the splitted phase space $Y = \bigcup_{v \in V} Y_v$, of distinct classes Y_v, where $Y_v \cap Y_{v^1} = \phi, v \neq v^1$ (see [9]);

---

[1]Research partially supported by Canada Research Chair Program and by National Sciences and Engineering Research Council of Canada

(D) *Stability properties* for difference equations in random media, described by a semi-Markov process: we consider a family of difference equations labelled by a parameter $\epsilon \to 0$ such that when $\epsilon \to 0$ the limiting equation becomes an averaged or diffusion equations. We assume that the limiting equation has some stability property, and we show that the corresponding stability property holds for the initial difference equation in random media in series scheme when $\epsilon > 0$ is sufficiently small (see [10]).

To obtain the above mentioned averaging, diffusion approximation, normal deviation, and merging results for difference equations in random environment we need the limit theorems for random evolutions in series scheme that will be studied in the first section of this chapter.

We note that the recent book by A. Skorokhod, F. Hoppensteadt and H. Salehi "Random Perturbation Methods with Applications to Science and Engineering", Springer, 2002 [20] is closed to our paper and also contains some applications to biological systems in random media.

1. LIMIT THEOREMS FOR RANDOM EVOLUTIONS

In this section, we consider the general theory of random evolutions. Definitions and classifications of random evolutions will be given. Martingale methods and their applications to the limit theorems for random evolutions will be considered (averaging, merging, diffusion approximation, and normal deviations).

1.1. Definitions and Classifications of Random Evolutions

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a probability space, $t \in R_+$, $(Y, \mathcal{Y})$ be a measurable phase space, and $(\mathbf{B}, \mathcal{B}, \|\cdot\|)$ be a separable Banach space.

We consider a Markov renewal process $(y_n, \theta_n)_{n \in Z_+}, y_n \in Y, \theta_n \in R_+$, with the stochastic kernel

$$\begin{cases} Q(y, A, t) := P(y, A)G_y(t), \\ P(y, A) := \mathbf{P}\ \{y_{n+1} \in A | y_n = y\}, \\ G_y(t) := \mathbf{P}\ \{\theta_{n+1} | y_n = y\}, \end{cases} \tag{1}$$

for all $y \in Y, A \in \mathcal{Y}, t \in R_+$. Recall that the process $y_t := y_{\nu(t)}$ is a semi-Markov process, where

$$\nu(t) := \max\{n : \tau_n \leq t\}, \quad \tau_n := \sum_{k=0}^{n} \theta_k, \quad y_n = y_{\tau_n},$$

$$\mathbf{P}\{\nu(t) < +\infty, \forall t \in R_+\} = 1.$$

Recall also that if $G_y(t) = 1 - e^{-\lambda(y)t}$, where $\lambda(x)$ is a measurable and bounded function on X, then y_t is called a jump Markov process.

Let $\Gamma(y), y \in Y$, be a family of operators on a dense subspace $\mathbf{B}_0 \in \mathbf{B}$, which is the common domain for $\Gamma(y)$, independent of y. $\Gamma(y)$ are noncommuting and unbounded in general, but we assume that the map $\Gamma(y)f : Y \to \mathbf{B}$ is strongly $\mathcal{Y}/\mathcal{B}$- measurable for all $f \in \mathbf{B}$, and $\forall t \in R_+$, and generate the semigroup of operators $(\Gamma_y(t))_{t \in R_+}$ for every $y \in Y$. Also, let $\{D(y); y \in Y\}$ be a family of bounded linear operators on B such that map $D(y)f : Y \to B$ is $\mathcal{Y}/\mathcal{B}$-measurable, for every $f \in \mathbf{B}$.

A random evolution (RE) is defined as the solution of the following stochastic operator integral equation in the separable Banach space B

$$V(t)f = f + \int_0^t \Gamma(y_s)V(s)f ds + \sum_{k=1}^{\nu(t)} [D(y_k) - I]V(\tau_k-)f, \qquad (2)$$

where I is an identity operator on $\mathbf{B}$, $\tau_k- := \tau_k - 0$, $f \in \mathbf{B}$. In the literature, the random evolution $V(t)$ is called a discontinuous RE.

If y_t given above is a Markov (or semi-Markov) process, then $V(t)$ in (2) is called a *Markov or (semi-Markov) RE.*

If $D(y) \equiv I$, for every $y \in Y$, then $V(t)$ in (2) is called *a continuous RE.*

If $\Gamma(y) \equiv 0$, for every $y \in Y$, then $V(t)$ in (2) is called *a jump RE.*

A RE $V_n := V(\tau_n)$ is called *a discrete-time RE* or, shortly, *discrete RE.*

Intuitively, operators $\{\Gamma(y)\}_{y\in Y}$ describe a continuous component $V^c(t)$ of the RE $V(t)$ in (2), and operators $\{D(y)\}_{y\in Y}$ describe a jump component $V^d(t)$ of the RE $V^d(t)$ in (2). In such a way, a RE is described by two objects:

(i) an operator dynamical system $\{V(t)\}_{t\in R^+}$;

(ii) a random process $(y_t)_{t\in R^+}$.

Under the above conditions, the solution $V(t)$ of (2) is unique and has the representation:

$$V(t) = \Gamma_{y_t}(t - \tau_{\nu(t)}) \prod_{k=1}^{\nu(t)} D(y_k)\Gamma y_{k-1}(\theta_k), \qquad (3)$$

where $\{\Gamma_y(t)\}_{t\in R^+}$ are the semigroups of t generated by the operators $\{\Gamma(y)\}_{y\in Y}$.

This can be proved by a constructive method described in [3,5].

Examples of RE. We now provide several examples of random evolutions. First of all, note that if

$$\Gamma(y) := v(y)\frac{d}{dz},$$

$$D(y) \equiv I,$$

$$\mathbf{B} = \mathbf{C}^1(R),$$

then (2) is a transport equation which describes the motion of a particle with random velocity $v(y_t)$. Consequently, various interpretations of the operators $\Gamma(y)$ and $D(y)$ yield random evolutions in many applications.

(E1). Impulse traffic process. Let $\mathbf{B} = \mathbf{C}(R)$ and assume operators $\Gamma(y)$ and $D(y)$ are defined by

$$\Gamma(y)f(z) := v(z,y)\frac{d}{dz}f(z),$$

$$D(y)f(z) := f(z + a(y)), \qquad (4)$$

where functions $v(z,y)$ and $a(y)$ are continuous and bounded on $R \times Y$ and Y respectively, $z \in R$, $y \in Y$ and $f \in \mathbf{C}^1(R) := \mathbf{B}_0$. Then equation (2) takes the form

$$f(z_t) = f(z) + \int_0^t v(z_s, y_s)\frac{d}{dz}f(z_s)ds + \sum_{k=1}^{\nu(t)} [f(z_{\tau_k-} + a(y_k)) - f(z_{\tau_k-})], \qquad (5)$$

and the RE $V(t)$ is defined by the relation

$$V(t)f(z) = f(z_t),$$

$$z_0 = z.$$

Equation (5) is a functional equation for the *impulse traffic process* z_t, which satisfies the equation

$$z_t = z + \int_0^t v(z_s, y_s)ds + \sum_{k=1}^{\nu(t)} a(y_k). \tag{6}$$

We note that the impulse traffic process z_t in (6) is a realization of a discontinuous RE.

(E2). Summation on a Markov chain. Let $v(z,x) \equiv 0, z \in R, x \in X$, in (6). Then the process

$$z_t = z + \sum_{k=1}^{\nu(t)} a(y_k) \tag{7}$$

is a summation on a Markov chain $(y_n)_{n\in Z_+}$ and it is a realization of a jump RE. Let $z_n := z_{\tau_n}$ in (7). Then the discrete process

$$z_n = z + \sum_{k=1}^{n} a(y_k)$$

is a realization of a discrete RE.

(E3). Diffusion process in random media. Let

$$\mathbf{B} = \mathbf{C}(R), \mathbf{B}_0 = C^2(R), P_x(t, z, A)$$

be a Markov continuous distribution function, which respects to the diffusion process $\xi(t)$, that is the solution of the stochastic differential equation in R with semi-Markov switchings:

$$d\xi(t) = \mu(\xi(t), y_t)dt + \sigma(\xi(t), y_t)dw_t,$$

$$\xi(0) = z, \tag{8}$$

where y_t is a semi-Markov process independent on a standard Wiener process w_t, coefficients $\mu(z,y)$ and $\sigma(z,y)$ are bounded and continuous functions on $R \times Y$. Let us define the following contraction semigroups of operators on $\mathbf{B}$:

$$\Gamma y(t)f(z) := \int_R P_y(t, z, dz)f(z), f(z) \in \mathbf{B}, \quad y \in Y. \tag{9}$$

Their infinitesimal operators $\Gamma(y)$ have the following kind:

$$\Gamma(y)f(z) = \mu(z,y)\frac{d}{dz}f(z) + 2^{-1}\sigma^2(z,y)\frac{d^2}{dz^2}f(z),$$

$$f(z) \in \mathbf{B}_0.$$

The process $\xi(t)$ is a continuous one, that is why the operators $\mathcal{D}(y) \equiv I,\ \forall y \in Y$, are identify operators. Then the equation (2) takes the form:

$$f(\xi(t)) = f(z) + \int_0^t \left[\mu(\xi(s), y_s)\frac{d}{dz} + 2^{-1}\sigma^2(\xi(s), y_s)\frac{d^2}{dz^2}\right] f(\xi(s))\, ds, \tag{10}$$

and RE $V(t)$ is defined by the relation

$$V(t)f(z) = \mathbf{E}\,[f(\xi(t))/y_s; 0 \le s \le t; \xi(0) = x].$$

Equation (10) is a functional one for diffusion process $\xi(t)$ in (8) in semi-Markov random media y_t. We note that diffusion process $\xi(t)$ in (8) is a realization of continuous RE.

(E4). Biological systems in random media. Let $\mathbf{B}$ be the same space as in **E1.** Let us define the operators $\Gamma(y)$ and $D(y)$ in the following way: $\Gamma(y) := I$, and $D(y)f(x) := f(x + g(x, y))$, where $g(x, y)$ is bounded and continuous function. Then equation (2) takes the form:

$$f(X_{\nu(t)}) = f(x) + \sum_{k=1}^{\nu(t)} [f(X_{\tau_k-} + g(X_k, y_{k+1})) - f(X_{\tau_k-})],$$

and RE $V(t)$ is defined by the relation for $t = \tau_{\nu(t)}$:

$$V(\tau_{\nu(t)})f(x) = f(X_{\nu(t)}),$$

$$X_0 = x.$$

The equation for $f(X_t)$ is a functional one for many biological systems in random media, which satisfy the equation:

$$X_{\nu(t)+1} = X_{\nu(t)} + g(X_{\nu(t)}, y_{\nu(t)+1}).$$

For example, for logistic growth model

$$g(x, y) := r(y)x(1 - x/K(y)).$$

1.2. Martingale methods in random evolutions

Martingale characterization of random evolutions

The main approach to the study of REs are *martingale methods*.

The main idea is that a process

$$M_n f := V_n f - f - \sum_{k=0}^{n-1} \mathbf{E}\,[(V_{k+1} - V_k)f/\mathcal{F}_k],\ V_0 = I, \tag{11}$$

is an $\mathcal{F}_n$-martingale in $\mathbf{B}$, where

$$\mathcal{F}_n := \sigma\{y_k, \tau_k; 0 \le k \le n\},$$

$$V_n := V(\tau_n),$$

$\mathbf{E}$ is an expectation with respect to probability $\mathbf{P}$. Representation of the martingale M_n (see(4)) in the form of the martingale-difference

$$M_n f = \sum_{k=0}^{n-1} [V_{k+1} f - \mathbf{E}\,(V_{k+1} f / \mathcal{F}_k)] \tag{12}$$

gives us the possibility of calculating the expression

$$< l(M_n f) > := \sum_{k-0}^{n-1} \mathbf{E}\,[l^2((V_{k+1} - V_k) f) / \mathcal{F}_k], \tag{13}$$

where $l \in \mathbf{B}^*$, and $\mathbf{B}^*$ is a dual space to $\mathbf{B}$, dividing points of $\mathbf{B}$.

The martingale method obtaining of the limit theorems for the sequence of REs is founded on the solution of the following problems:

(i) *weak compactness* of the family of measures generated by the sequences of REs;

(ii) any limiting point of this family of measures is the *solution of a martingale problem*;

(iii) the solution of the martingale problem is *unique*.

The conditions (i)-(ii) guarantee the existence of a weakly converging subsequence, and condition (iii) gives the uniqueness of the weak limit. It follows from (i)-(iii) that sequence of RE converges weakly to the unique solution of martingale problem. The weak convergence of RE in a series scheme we obtain from the criterion of weakly compactness of the processes with values in separable Banach space. The limit RE we obtain from the solution of some martingale problem in form of some integral operator equations in Banach space $\mathbf{B}$. We also use the representation

$$\begin{aligned} V_{k+1} - V_k &= [\Gamma_{y_k}(\theta_{k+1}) D(y_{k+1}) - I] V_k, \\ V_k : &= V(\tau_k), \end{aligned} \tag{14}$$

and the following expression for semigroups of operators $\Gamma_y(t)$:

$$\begin{aligned} \Gamma_y(t) f &= f + \textstyle\sum_{k=1}^{n-1} \frac{t^k}{k!} \Gamma^k(y) f + ((n-1)!)^{-1} \int_0^t (t-s)^n \Gamma_y(s) \Gamma^n(y) f\, ds, \\ & \forall y \in Y, \quad \forall f \in \textstyle\bigcap_{y \in Y} \mathrm{Dom}\ (\Gamma^n(y)). \end{aligned} \tag{15}$$

Taking into account (11)-(15) we obtain the limit theorems for RE. In the previous subsection we considered the evolution equation associated with random evolutions by using the jump structure of the semi-Markov process or jump Markov process.

In order to deal with more general driving processes and to consider other applications, it is useful to re-formulate the treatment of random evolution in terms of a *martingale problem*. It has been shown by Stroock and Varadhan (1969) that the entire theory of multi-dimensional diffusion processes (and many other continuous parameter Markov processes) can be so formulated.

Suppose that we have an evolution equation of the form:

$$\frac{df}{dt} = Gf. \tag{16}$$

The *martingale problem* is to find a Markov process $(y(t))_{t\in R^+}$ with infinitesimal operator Q, and a RE $V(t)$ so that for all smooth functions

$$V(t)f(y(t)) - \int_0^t V(s)Gf(y(s))\,ds \tag{17}$$

is a martingale. It is immediate that this gives the required solution. Indeed, the operator

$$f \to T(t)f := \mathbf{E}_y[V(t)f(y(t))]$$

defines a semigroup of operators on the Banach space $\mathbf{B}$, whose infinitesimal generator can be computed by taking the expectation:

$$\mathbf{E}_y[V(t)f(y(t))] - f(y) = \mathbf{E}_y\left[\int_0^t V(s)Gf(y(s))\,ds\right],$$

divided both part by t and then going to the limit as $t \to 0$.

The quadratic variation $< m(t) >$ for martingale $m(t)$ in (45) has the following form:

$$< m(t) >= \int_0^t [Gf^2(y(s)) - 2f(y(s))Gf(y(s))]\,ds. \tag{18}$$

The following result solves martingale problem for Markov chain y_n with infinitesimal operator $\mathbf{P} - I$. Let us consider a homogeneous Markov chain $(y_n)_{n\in Z_+}$ on a measurable phase space $(Y, \mathcal{Y})$ with stochastic kernel $P(y, A), y \in Y, A \in \mathcal{Y}$, respected to the operator $\mathbf{P}$ on the Banach space $\mathbf{C}(Y)$:

$$\mathbf{P}f(y) = \int_Y P(y, dz)f(z) = \mathbf{E}\,[f(y_n)/y_{n-1} = y].$$

Since

$$\mathbf{P}f(y_n) - f(y) = \sum_{k=0}^{n}[\mathbf{P} - I]f(y_n),\ y_0 = y,$$

and

$$\mathbf{E}\,[f(y_n) - f(y) - \sum_{k=0}^{n}[\mathbf{P} - I]f(y_k)/y_{n-1} = y] = 0,$$

then the process

$$m_n := f(y_n) - f(y) - \sum_{k=0}^{n}[\mathbf{P} - I]f(y_k) \tag{19}$$

is $\mathcal{F}_k^Y$-martingale, where $\mathcal{F}_n^Y := \sigma\{y_k; 0 \le k \le n\}$.

The inverse result is also true: if we have martingale in (19), then process y_n is Markov chain with infinitesimal operator $\mathbf{P} - I$. We note that the quadratic variation $< m_n >$ for martingale m_n in (19) has the following form:

$$< m_n >= \sum_{k=0}^{n}[\mathbf{P}f^2(y_k) - (\mathbf{P}f(y_k))^2]. \tag{20}$$

Remark. We note that a measurable process $V(t)$ is a solution of martingale problem for operator A if and only if for all $l, l_k \in \mathbf{B}^*$:

$$\mathbf{E}\left[l(V(t_{k+1})f - V(t_k)f - \int_{t_k}^{t_{k+1}} V(s)Gfds)\right] \cdot \prod_{k=1}^{n} l_k(V(t_k)f) = 0, \tag{21}$$

where $0 \le t_1 < t_2 < \cdots < t_{n+1}, \forall f \in \mathrm{Dom}(A)$, and $k = 1, ..., n$. Consequently, the statement that a measurable process is a solution of martingale problem is a statement about its finite-dimensional distributions.

Remark. It is known, that {convergence of finite-dimensional distributions of the process}+ {tightness of sequence of the processes}={weak compactness of the processes}.

In connection with the Remark, and (21) and the previous statement we obtain: {the of martingale problem for the sequence of the processes}+{tightness of sequence of the processes}= {weak compactness of the processes}.

1.3 Limit theorems for random evolutions

The main approach to the investigation of SMRE in the limit theorems is a martingale method.

The martingale method of obtaining of the limit theorems (averaging and diffusion approximation) for the sequence of SMRE is bounded on the solution of the following problems:

(i) weakly compactness of the family of measures generated by the sequence of SMRE;

(ii) any limiting point of this family of measures is the solution of martingale problem;

(iii) the solution of martingale problem is unique.

The conditions (i)-(ii) guarantee the existence of weakly converging subsequence, and condition (iii) gives the uniqueness of a weakly limit.

From (i)-(iii) it follows that consequence of SMRE converges weakly to the unique solution of martingale problem.

Weak convergence of random evolutions

A *weak convergence of SMRE* in series scheme we obtain from the criterion of weakly compactness of the process with values in separable Banach space [4,5]. The *limit SMRE* we obtain from the solution of some martingale problem in kind of some integral operator equations in Banach space B.

The main idea is that the process

$$M_n f := V_n f - f - \sum_{k=0}^{n-1} \mathbf{E}\left[(V_{k+1} - V_k)f/\mathcal{F}_k\right],\ V_0 f = f, \tag{22}$$

is an $\mathcal{F}_n - martingale$ in $\mathbf{B}$, where

$$\mathcal{F}_n := \sigma\{y_k; \tau_k; 0 \le k \le n\},\ V_n := V(\tau_n),$$

$\mathbf{E}$ is an expectation by probability $\mathbf{P}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Representation of the martingale M_n in the form of *martingale differences*

$$M_n = \sum_{k=0}^{n-1} [V_{k+1} - \mathbf{E}\,(V_{k+1}/\mathcal{F}_k)] \tag{23}$$

gives us the possibility to calculate the *weak quadratic variation*:

$$< l(M_n f) > := \sum_{k=0}^{n-1} \mathbf{E}\,[l^2((V_{k+1} - V_k)f)/\mathcal{F}_k], \tag{24}$$

where $l \in \mathbf{B}^*$, and $\mathbf{B}^*$ is a dual space to $\mathbf{B}$, dividing points of $\mathbf{B}$.

From (23) it follows that

$$V_{k+1} - V_k = [\Gamma_{y_k}(\theta_{k+1}) D(y_{k+1}) - I] \cdot V_k. \tag{25}$$

We note that the following expression for semigroup of operators $\Gamma_y(t)$ is fulfilled:

$$\Gamma_y(t) f = f + \sum_{k=1}^{n-1} \frac{t^k}{k!} \Gamma^k(y) f + \frac{1}{(n-1)!} \int_0^t (t-s)^n \Gamma_y(s) \Gamma^n(y) f\, ds,$$

$$\forall y \in Y,\ \forall f \in \bigcap_y \mathrm{Dom}\ (\Gamma^n(y)). \tag{26}$$

Taking into account (22)-(25) we obtain the mentioned above results.

Everywhere we suppose that the following conditions be satisfied:

(A) there exists Hilbert spaces $\mathbf{H}$ and $\mathbf{H}^*$ such that compactly imbedded in Banach spaces B and $\mathbf{B}^*$ respectively, $\mathbf{H} \subset \mathbf{B}, \mathbf{H}^* \subset \mathbf{B}^*$, where $\mathbf{B}^*$ is a dual space to $\mathbf{B}$, that divides points of $\mathbf{B}$;

(B) operators $\Gamma(y)$ and $(\Gamma(y))^*$ are dissipative on any Hilbert space $\mathbf{H}$ and $\mathbf{H}^*$ respectively;

(C) operators $D(y)$ and $D^*(y)$ are contractive on any Hilbert space $\mathbf{H}$ and $\mathbf{H}^*$ respectively;

(D) $(y_n)_{n \in Z_+}$ is a uniformly ergodic Markov chain with stationary distribution $p(A), A \in \mathcal{Y}$;

(E) $m_i(y) := \int_0^\infty t^i G_y(dt)$ are uniformly integrable, $\forall i = 1, 2, 3$, where

$$G_y(t) := \mathbf{P}\,\{\omega : \theta_{n+1} \le t / y_n = y\}; \tag{27}$$

(F)

$$\int_Y p(dy) \parallel \Gamma(y) f \parallel^k < +\infty;$$

$$\int_Y p(dy) \parallel P D_j(y) f \parallel^k < +\infty;$$

$$\int_Y p(dy) \parallel \Gamma(y) f \parallel^{k-1} \cdot \parallel P D_j(y) f \parallel^{k-1} < +\infty;$$

$$\forall k = 1, 2, 3, 4, j = 1, 2, f \in B, \tag{28}$$

where P is on operator generated by the transition probabilities $P(y, A)$ of Markov chain $(y_n)_{n \in Z_+}$:

$$P(y, A) := \mathbf{P}\{\omega : y_{n+1} \in A/y_n = y\}, \tag{29}$$

and $\{D_j(y)\}_{y\in Y}$, $j = 1, 2$, is a family of some closed operators, defined by the jumps operators $\{D^\epsilon(y); y \in Y\}$, which, in their turn, define a jump part of the semi-Markov RE in series scheme (see *averaging* and *diffusion approximation* of RE).

If $\mathbf{B} := \mathbf{C}_0(R)$, then $\mathbf{H} := \mathbf{W}^{l,2}(R)$ is a Sobolev space, and $\mathbf{W}^{l,2}(R) \subset \mathbf{C}_0(R)$ and this imbedding is compact. For the spaces $\mathbf{B} := \mathbf{L}_2(R)$ and $\mathbf{H} := \mathbf{W}^{l,2}(R)$ it is the same.

It follows from the conditions (A)-(B) that operators $\Gamma(y)$ and $(\Gamma(y))^*$ generate a strongly continuous contractive semigroup of operators $\Gamma_y(t)$ and $\Gamma^*_y(t), \forall y \in Y$, in $\mathbf{H}$ and $\mathbf{H}^*$ respectively. From the conditions (A)-(C) it follows that the SMRE $V(t)$ in (2) is a contractive operator in $\mathbf{H}, \forall t \in R_+$, and $\| V(t)f \|_{\mathbf{H}}$ is a semimartingale $\forall f \in \mathbf{H}$. In such a way, the conditions (A)-(C) supply the following result:

SMRE $V(t)f$ is a tight process in $\mathbf{B}$, namely, $\forall \triangle > 0$ there exists a compact set $K_\triangle$:

$$\mathbf{P}\{V(t)f \in K_\triangle; 0 \le t \le T\} \ge 1 - \triangle. \tag{30}$$

This result follows from the Kolmogorov-Doob inequality for the semi-martingale

$$\| V(t)f \|_{\mathbf{H}} .$$

Condition (30) is the main step in the providing of limit theorems and rates of convergence for the sequence of SMRE in the series scheme.

Averaging of random evolutions

Let us consider a SMRE in series scheme:

$$V_\varepsilon(t)f = f + \int_0^t \Gamma(y(s/\varepsilon))V_\varepsilon(s)f ds + \sum_{k=1}^{\nu(t/\varepsilon)} [D^\varepsilon(y_k) - I]V_\varepsilon(\varepsilon\tau_k-)f, \tag{31}$$

where

$$D^\varepsilon(y) = I + \varepsilon D_1(y) + 0(\varepsilon), \tag{32}$$

$\{D_1(y)\}_{y\in Y}$ is a family of closed linear operators, $\| 0(\varepsilon)f \| /\varepsilon \to 0$ as $\varepsilon \to 0, \varepsilon$ is a small parameter,

$$f \in \mathbf{B}_0 := \bigcap_{y\in Y} \text{Dom } (\Gamma^2(y)) \cap \text{Dom}(D_1^2(y)). \tag{33}$$

Another form for $V_\varepsilon(t)$ in (31) is:

$$V_\varepsilon(t) = \Gamma_{y(t/\varepsilon)}(t - \varepsilon\tau_{\nu(t/\varepsilon)}) \prod_{k=1}^{\nu(t/\varepsilon)} D^\varepsilon(y_k)\Gamma_{y_{k-1}}(\varepsilon\theta_k). \tag{34}$$

Under conditions (A)-(C) the sequence of SMRE $V_\varepsilon(t)f$ is tight (see (3)) p-a.s..

Under conditions (D), (E), $i = 2$, (F), $k = 2, j = 1$, the sequence of SMRE $V_\varepsilon(t)j$ is weakly compact p-a.s. in $\mathbf{D_B}[0, +\infty)$ with limit points in $\mathbf{C_B}[0, +\infty), f \in \mathbf{B}_0$.

Let's consider the following process in $\mathbf{D_B}[0, +\infty)$:

$$M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon := V^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon - f^\varepsilon - \sum_{k=0}^{\nu(t/\varepsilon)-1} \mathbf{E}_p[V^\varepsilon_{k+1}f^\varepsilon_{k+1} - V^\varepsilon_k f^\varepsilon_k/\mathcal{F}_k], \tag{35}$$

where $V_n^\varepsilon := V_\varepsilon(\varepsilon\tau_n)$ (see (34)),

$$f^\varepsilon := f + \varepsilon f_1(y(t/\varepsilon)),$$

$$f_k^\varepsilon := f^\varepsilon(y_k),$$

function $f_1(x)$ is defined from the equation

$$(\mathbf{P} - I)f_1(y) = [(\hat{\Gamma} + \hat{D}) - (m(y)\Gamma(y) + \mathbf{P}D_1(y))]f,$$

$$\hat{\Gamma} := \int_Y p(dy)m(y)\Gamma(y),$$

$$\hat{D} := \int_Y p(dy)D_1(y),$$

$$m(y) := m_1(y) \tag{36}$$

(see (E)), $f \in \mathbf{B}_0$.

The process $M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon$ is an $\mathcal{F}_t^\varepsilon$- martingale with respect to the σ-algebra $\mathcal{F}_t^\varepsilon := \sigma\{y(s/\varepsilon); 0 \le s \le t\}$.

The martingale $M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon$ in (35) has the asymptotic representation:

$$M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon = V^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon f - f - \varepsilon \sum_{k=0}^{\nu(t/\varepsilon)} (\hat{\Gamma} + \hat{D})V_k^\varepsilon f + 0_f(\varepsilon), \tag{37}$$

where $\hat{\Gamma}, \hat{D}, f, f^\varepsilon$ are defined in (35)-(36) and

$$\| 0_f(\varepsilon) \| /\varepsilon \to \text{const as } \varepsilon \to 0, \ \forall f \in \mathbf{B}_0.$$

We have used (25)-(26) as $n = 2$, and representation (32)and (33) in (37).

The families $l(M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon)$ and

$$(\sum_{k=0}^{\nu(t/\varepsilon)} \mathbf{E}_p[(V^\varepsilon_{k+1}f^\varepsilon_{k+1} - V_k^\varepsilon f_k^\varepsilon)/\mathcal{F}_k])$$

are weakly compact for all $l \in \mathbf{B}_0^*$ is a some dense subset from $\mathbf{B}^*$. Let $V_0(t)$ be a limit process for $V_\varepsilon(t)$ as $\varepsilon \to 0$.

Since (see (34))

$$[V_\varepsilon(t) - V^\varepsilon_{\nu(t/\varepsilon)}] = [\Gamma_{y_{(t/\varepsilon)}}(t - \varepsilon\tau_{\nu(t/\varepsilon)}) - I] \cdot V^\varepsilon_{\nu(t/\varepsilon)} \tag{38}$$

and the right hand side in (38) tends to zero as $\varepsilon \to 0$, then it's clear that the limits for $V_\varepsilon(t)$ and $V^\varepsilon_{\nu(t/\varepsilon)}$ are the same, namely, $V_0(t)$, p-a.s.

The sum

$$\varepsilon \cdot \sum_{k=0}^{\nu(t/\varepsilon)} (\hat{\Gamma} + \hat{D})V_k^\varepsilon f$$

converges strongly as $\varepsilon \to 0$ to the integral

$$m^{-1} \cdot \int_0^t (\hat{\Gamma} + \hat{D})V_0(s)f ds.$$

The quadratic variation of the martingale $l(M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon)$ tends to zero, and, hence

$$M^\varepsilon_{\nu(t/\varepsilon)}f^\varepsilon \to 0 \text{ as } \varepsilon \to 0,\ \forall f \in \mathbf{B}_0,\ \forall l \in \mathbf{B}^*_0.$$

Passing to the limit in (37) as $\varepsilon \to 0$ and taking into account the all previous reasonings we obtain that the limit process $V_0(t)$ satisfies the equation:

$$0 = V_0(t)f - f - m^{-1}\int_0^t (\hat{\Gamma} + \hat{D})V_0(s)f ds, \tag{39}$$

where

$$m := \int_X p(dx)m(x),\ f \in \mathbf{B}_0,\ t \in [0, T].$$

Diffusion approximation of random evolutions

Let us consider SMRE $V_{\varepsilon(t/\varepsilon)}$, where $V_\varepsilon(t)$ is defined in (31) or (34), with the operators

$$D^\varepsilon(y) := I + \varepsilon D_1(y) + \varepsilon^2 D_2(y) + 0(\varepsilon^2), \tag{40}$$

$\{D_i(y); y \in Y, i = 1, 2\}$ are closed linear operators and $\| 0(\varepsilon^2)f \| /\varepsilon^2 \to 0,\ \varepsilon \to 0$

$$\forall f \in \mathbf{B}_0 := \bigcap_{x,y\in X} \text{Dom } (\Gamma^4(y)) \bigcap \text{Dom } (D_2(y)),$$

$$\text{Dom } (D_2(y)) \subseteq \text{Dom}(D_1(y)); D_1(y) \subseteq \text{Dom } (D_1(y)),$$
$$\forall y \in Y, \Gamma^i(y) \subset \text{Dom } (D_2(y)), i = \overline{1, 3}. \tag{41}$$

In this way

$$V_\varepsilon(t/\varepsilon) = \Gamma_{y(t/\varepsilon^2)}(t/\varepsilon - \varepsilon\tau_{\nu(t/\varepsilon^2)}) \prod_{k=1}^{\nu(t/\varepsilon^2)} D^\varepsilon(y_k)\Gamma_{y_{k-1}}(\varepsilon\theta_k), \tag{42}$$

where $D^\varepsilon(y)$ are defined in (40).

Under conditions (A)-(C) the sequence of SMRE $V_\varepsilon(t/\varepsilon)f$ is tight (see (30)) p- a.s.

Under conditions (D), (E), $i = 3, (F), k = 4$, the sequence of SMRE $V_\varepsilon(t/\varepsilon)f, f \in \mathbf{B}_0$.

Let the *balance condition* be satisfied:

$$\int_Y p(dy)[m(y)\Gamma(y) + D_1(y)]f = 0,\ \forall f \in \mathbf{B}_0 \tag{43}$$

Let's consider the following process in $\mathbf{D_B}[0 + \infty)$:

$$M^\varepsilon_{\nu(t/\varepsilon^2)}f^\varepsilon := V^\varepsilon_{\nu(t/\varepsilon^2)}f^\varepsilon - f^\varepsilon - \sum_{k=0}^{\nu(t/\varepsilon^2)} \mathbf{E}_p[V^\varepsilon_{k+1}f^\varepsilon_{k+1} - V^\varepsilon_k f^\varepsilon_k/\mathcal{F}_k], \tag{44}$$

where $f^\varepsilon := f + \varepsilon f_1(y(t/\varepsilon^2)) + \varepsilon^2 f_2(y(t/\varepsilon^2))$, and functions f_1 and f_2 are defined from the following equations:

$$\begin{aligned}
(\mathbf{P} - I)f_1(y) &= -[m(y)\Gamma(y) + D_1(y)]f, \\
(\mathbf{P} - I)f_2(y) &= [\hat{L} - L(x)]f, \\
\hat{L} : &= \textstyle\int_Y p(dy)L(y), \\
L(y) &:= (m(y)\Gamma(y) + D_1(y))(R_0)(m(y)\Gamma(y) + D_1(y)) \\
&\quad + m_2(y)\Gamma^2(y)/2 + m(y)D_1(y)\Gamma(y) + D_2(y),
\end{aligned} \tag{45}$$

$\mathbf{R}_0$ is a potential operator of $(y_n)_{n\in Z_+}$

The balance condition (43) and condition $\prod(\hat{L}-L(y))=0$ give the solvability of the equations in (45).

The process $M_{\nu(t/\varepsilon^2)}f^\varepsilon$ is an $\mathcal{F}_t^\varepsilon$-martingale with respect to the σ-algebra $\mathcal{F}_t^\varepsilon := \sigma\{y(s/\varepsilon^2); 0\le s\le t\}$.

This martingale has the asymptotic representation:

$$M_{\nu(t/\varepsilon^2)}f^\varepsilon = V_{\nu(t/\varepsilon^2)}f^\varepsilon - f - \varepsilon^2 \sum_{k=0}^{\nu(t/\varepsilon^2)} \hat{L}V_k^\varepsilon f + O_f(\varepsilon), \tag{46}$$

where $\hat{L}$ is defined in (45) and

$$\| 0_f(\varepsilon) \| /\varepsilon \to const,$$

as $\varepsilon\to 0$, for all $f\in \mathbf{B}_0$.

We have used (25), (26) as $n=3$, and representation (40) and (45) in (46).

The families $l(M_{\nu(t/\varepsilon^2)}f^\varepsilon)$ and $l(\sum_{k=0}^{\nu(t/\varepsilon^2)} \mathbf{E}_p[(V_{k+1}^\varepsilon f_{k+1}^\varepsilon - V_k^\varepsilon f_k^\varepsilon)/\mathcal{F}_k])$ are weakly compact for all $l\in\mathbf{B}_0^*, f\in\mathbf{B}_0$.

From (34) we obtain that the limits for $V_\varepsilon(t/\varepsilon)$ and $V_{\nu(t/\varepsilon^2)}^\varepsilon$ are the some, namely, $V^0(t)$.

The sum $\varepsilon^2\sum_{k=0}^{\nu(t/\varepsilon^2)} \hat{L}V_k^\varepsilon f$ converges strongly as $\varepsilon\to 0$ to the integral

$$m^{-1}\int_0^t \hat{L}V^0(s)fds.$$

Let $M^0(t)f$ be a limit martingale for $M_{\nu(t/\varepsilon^2)}^\varepsilon f^\varepsilon$ as $\varepsilon\to 0$.

Then, from (44)-(46) and previous reasonings we have as $\varepsilon\to 0$:

$$M^0(t)f = V^0(t)f - f - m^{-1}\cdot\int_0^t \hat{L}V^0(s)fds. \tag{47}$$

The quadratic variation of the martingale $M^0(t)f$ has the form:

$$< l(M^0(t)f) >= \int_0^t\int_Y l^2(\sigma(y)\Gamma(y)V^0(s)f)p(dy)ds, \tag{48}$$

where

$$\sigma^2(y) := [m_2(y) - m^2(y)]/m.$$

The solution of martingale problem for $M^0(t)$ (namely, to find the representation of $M^0(t)$ with quadratic variation (48)) is expressed by the integral over Wiener orthogonal martingale measure $W(dy,ds)$ with quadratic variation $p(dy)\cdot ds$:

$$M^0(t)f = \int_0^t\int_Y \sigma(y)\Gamma(y)V^0(s)fW(dy,ds). \tag{49}$$

In this way, the limit process $V^0(t)$ satisfies the following equation (see (47) and (49)):

$$V^0(t)f = f + m^{-1}\cdot\int_0^t \hat{L}\cdot V^0(s)fds + \int_0^t\int_Y \sigma(y)\Gamma(y)V^0(s)fW(dy,ds). \tag{50}$$

If the operator $\hat{L}$ generates the semigroup $U(t)$ then the process $V^0(t)f$ in (50) satisfied equation:

$$V^0(t)f = U(t)f + \int_0^t \int_Y \sigma(y)U(t-s)\Gamma(y)V^0(s)fW(dy,ds). \tag{51}$$

The *uniqueness* of the limit evolution $V_0(t)f$ in the *averaging* scheme follows from the equation (39) and the fact that if the operator $\hat{\Gamma} + \hat{D}$ generates a semigroup, then $V_0(t)f = \exp\{(\hat{\Gamma} + \hat{D}) \cdot t\}f$ and this representation is unique.

The *uniqueness* of the limit evolution $V^0(t)f$ in *diffusion approximation* scheme follows from the uniqueness of the solution of martingale problem for $V^0(t)f$ (see (47)-(49)). The latter is proved by *dual SMRE* in series scheme by the constructing the limit equation in diffusion approximation and by using a dual identity [4].

Averaging of random evolutions in reducible phase space. Merged random evolutions

Suppose that the following conditions hold true:

(a) *decomposition* of phase space X (*reducible* phase space):

$$Y = \bigcup_{v\in V} Y_v, Y_v \bigcap Y_v' = \phi, v \neq v' : \tag{52}$$

where $(V, \mathcal{V})$ is a some measurable phase space (*merged* phase space);

(b) Markov renewal process $(y_n^\varepsilon), \theta_n;)_{n\in Z_+}$ on $(Y, \mathcal{Y})$ has the *semi-Markov kernel*:

$$Q_\varepsilon(y, A, t) := P_\varepsilon(y, A)G_y(t), \tag{53}$$

where $P_\varepsilon(y, A) = P(y, A) - \varepsilon^1 P_1(y, A), y \in Y, A \in \mathcal{Y}, P(y, A)$ are the transition probabilities of the *supporting nonperturbed* Markov chain $(y_n)_{n\in Z_+}$; P_1 is a some probability measure;

(c) the stochastic kernel $P(y, A)$ is adapted to the decomposition (52) in the following form:

$$P(y, Y_v) = \begin{cases} 1, & x \in Y_v \\ 0, & x \notin Y_v, \qquad v \in V; \end{cases}$$

(d) the Markov chain $(y_n)_{n\in Z_+}$ is uniformly ergodic with stationary distributions $p_v(B)$:

$$p_v(B) = \int_{Y_v} P(y, B)p_v(dy), \forall v \in V, \forall B \in \mathcal{Y}. \tag{54}$$

(e) there is a family $\{p_v^\varepsilon(A); v \in V, A \in \mathcal{Y}, \varepsilon > 0\}$ of stationary distributions of perturbed Markov chain $(y_n^\varepsilon)_{n\in Z_+}$;

(f)

$$b(v) := \int_{Y_v} \rho_v(dx)P_1(x, Y_v) > 0, \qquad \forall v \in V,$$

$$b(v, \triangle) := -\int_{Y_v} \rho_v(dx)P_1(x, Y_\triangle) > 0, \qquad \forall v \notin \triangle, \qquad \triangle \in V; \tag{55}$$

(g) the operators $\Gamma(v) := \int_{Y_v} p_v(dy)m(y)\Gamma(y)$ and

$$\hat{D}(v) := \int_{Y_v} \rho_v(dx) \int_{Y_v} P(x,dy)D_1(y) \tag{56}$$

are closed $\forall v \in V$ with common domain $\mathbf{B}_0$, and operators $\hat{\Gamma}(v) + \hat{D}(v)$ generate the semigroup of operators $\forall v \in V$.

Decomposition (52) in a) defines the *merging* function

$$v(y) = v \qquad \forall y \in Y_v, \qquad v \in V. \tag{57}$$

We note that σ-algebras $\mathcal{Y}$ and $\mathcal{V}$ are coordinated such that

$$Y_\triangle = \bigcup_{v\in\triangle} Y_v, \qquad \forall v \in V, \qquad \triangle \in \mathcal{V}. \tag{58}$$

We set

$$\pi_v f(v) := \int_{Y_v} p_v(dy)f(y) \;\text{ and }\; y^\varepsilon(t) := y^\varepsilon_{\nu(t/\varepsilon)}.$$

SMRE in reducible phase space X is defined by the solution of the equation:

$$V_\varepsilon(t) = I + \int_0^t \Gamma(y^\varepsilon(s/\varepsilon))V_\varepsilon(s)\,ds + \sum_{k=0}^{\nu(t/\varepsilon)} [D^\varepsilon(y^\varepsilon_k) - I]V_\varepsilon(\varepsilon\tau_k^-), \tag{59}$$

where $D^\varepsilon(y)$ are defined in (32).

Let's consider the martingale

$$\begin{aligned} M^\varepsilon_{\nu(t/\varepsilon)} f^\varepsilon(y^\varepsilon(t/\varepsilon)) \;&:=\; V^\varepsilon_{\nu(t/\varepsilon)} f^\varepsilon(y^\varepsilon(t/\varepsilon) - f^\varepsilon(y) \\ &\quad - \textstyle\sum_{k=0}^{\nu(t/\varepsilon)-1} \mathbf{E}^{\,\varepsilon}_{p_u}[V^\varepsilon_{k+1} f^\varepsilon_{k+1} - V^\varepsilon_k f^\varepsilon_k / \mathcal{F}^\varepsilon_k], \end{aligned} \tag{60}$$

where

$$\mathcal{F}^\varepsilon_n := \sigma\{y^\varepsilon_k, \theta_k; 0 \le k \le n\},$$

$$f^\varepsilon(y) := \hat{f}(v(y)) + \varepsilon f^1(y), \hat{f}(v) := \int_{Y_v} p_v(dy)f(y), \tag{61}$$

$$(\mathbf{P} - I)f_1(y) = [-(m(y)\Gamma(y) + D_1(y)) + \hat{\Gamma}(v) + \hat{D}(v) + (\Pi_v - I)P_1]\hat{f}(v), \tag{62}$$

$$f^\varepsilon_k := f^\varepsilon(y^\varepsilon_k), V^\varepsilon_n := V_\varepsilon(\varepsilon\tau_n),$$

and $V_\varepsilon(t)$ is defined in (59), P_1 is an operator generated by $P_1(y, A)$ (see (53)).

The following representation is true [4]:

$$\Pi^\varepsilon_u = \Pi_u - \varepsilon^r \Pi_u P_1 R_0 + \varepsilon^{2r} \Pi^\varepsilon_u (P_1 R_0)^2, \;\; r = 1, 2, \tag{63}$$

where $\Pi^\varepsilon_v, \Pi_v, P_1$ are the operators generated by p^ε_v, p_v and $P_1(y, A)$ respectively, $y \in Y, A \in \mathcal{Y}, v \in V$.

It follows from (63) that for any continuous and bounded function $f(x)$

$$\mathbf{E}^{\,\varepsilon}_{p_v} f(x) \underset{\varepsilon\to 0}{\to} \mathbf{E}_{p_v} f(x), \qquad \forall_v \in V.$$

We use the calculations like for averaging RE's section replacing $\mathbf{E}_{p_u}$ by $\mathbf{E}_{p_u^\varepsilon}$ that is reduced to the calculations $\mathbf{E}_{p_u}$ with respect to the presentation (63).

Under conditions (A)-(C) the sequence of SMRE $V_\varepsilon(t)f$ in ((17), $f \in \mathbf{B}_0$ (see (27)), is tight (see (30)) $p_u -$ a.s.,$\forall v \in V$.

Under conditions (D), (E), $i = 2$, and (F), $k = 2, j = 1$, the sequence of SMRE $V_\varepsilon(t)f$ is weakly compact $p_v -$ a.s.,$\forall v \in V$, in $\mathbf{D_B}[0, +\infty)$ with limit points in $\mathbf{C_B}[0, +\infty)$.

We note that $u(y^\varepsilon(t/\varepsilon)) \to \hat{y}(t)$ as $\varepsilon \to 0$, where $\hat{y}(t)$ is a *merged* jump Markov process in $(V, \mathcal{V})$ with *infinitesimal operator* $\Lambda(\hat{\mathbf{P}} - I)$,

$$\Lambda \hat{f}(v) := [b(v)/m(v)]\hat{f}(v),$$

$$\hat{\mathbf{P}}\hat{f}(v) := \int_V [b(v, dv')/b(v)]\hat{f}(v'),$$

$$m(v) := \int_{Y_v} p_v(dx)m(x), \tag{64}$$

$b(v)$ and $b(v, \triangle)$ are defined in (55). We also note that

$$\Pi_v P_1 = \Lambda(\hat{\mathbf{P}} - I), \tag{65}$$

where Π_v is defined in (58), P_1-in (53), Λ and $\hat{\mathbf{P}}$-in (64).

Using (25), (26) as $n = 2$, and (61)-(62), (63) as $r = 1$, (65), we obtain the following representation:

$$\begin{aligned} &M^\varepsilon_{\nu(t/\varepsilon} f^\varepsilon(y^\varepsilon(t/\varepsilon))) = V^\epsilon_{\nu(t/\epsilon)}\hat{f}(u(y^\epsilon(t/\epsilon))) - \hat{f}(u(x)) \\ &-\varepsilon \textstyle\sum_{k=0}^{\nu(t/\varepsilon)}[m(u)\hat{\Gamma}(u) + m(u)\hat{D}(u) + m(u)\Lambda(\hat{P} - I)V^\varepsilon_k \hat{f}(u(x^\varepsilon_k)) + 0_f(\varepsilon), \end{aligned} \tag{66}$$

where $\| 0_f(\varepsilon) \| /\varepsilon \to$ *const* as $\varepsilon \to 0, \forall f \in \mathbf{B}_0$. Since the third term in (66) tends to the integral

$$\int_0^t [\Lambda(\hat{\mathbf{P}} - I) + \hat{\Gamma}(y(s)) + \hat{D}(\hat{y}(s))]\hat{V}_0(s)\hat{f}(\hat{y}(s))ds$$

and the quadratic variation of the martingale $l(M^\varepsilon_{\nu(t/\varepsilon)} f^\varepsilon(y^\varepsilon(t/\varepsilon)))$ tends to zero as $\varepsilon \to 0$ (and, hence $M^\varepsilon_{\nu(t/\varepsilon)} f^\varepsilon(y^\varepsilon(t/\varepsilon))) \to 0,\ \varepsilon \to 0$), $\forall l \in \mathbf{B}_0^*$, then we obtain from (51) that the limit evolution $\hat{V}_0(t)$ satisfies equation:

$$\hat{V}_0(t)\hat{f}(\hat{x}(t)) = \hat{f}(u) + \int_0^t [\Lambda(\hat{\mathbf{P}} - I) + \hat{\Gamma}(\hat{x}(s)) + \hat{D}(\hat{x}(s))]\hat{V}_0(s)\hat{f}(\hat{x}(s))\, ds. \tag{67}$$

Normal deviations of random evolutions

The averaged evolution obtained in averaging and merging schemes can be considered as the first approximation to the initial evolution. The diffusion approximation of the SMRE determine the second approximation to the initial evolution, since the first approximation under balance condition–the averaged evolution-appears to be trivial.

Here we consider the *double approximation* to the SMRE-the averaged and the diffusion approximation-provided that the balance conditions bails.

We introduce the *deviation process* as the normalized difference between the initial and averaged evolutions. In the limit we obtain the *normal deviations* of the initial SMRE from the averaged one.

Let us consider the SMRE $V_\varepsilon(t)$ in (31) and the averaged evolution $V_0(t)$ in (39). Let's also consider the deviation of the initial evolution $V_\varepsilon(t)f$ from the averaged one $V_0(t)f$:

$$W_\varepsilon(t)f := \varepsilon^{-1/2}\cdot[V_\varepsilon(t)-V_0(t)]f, \quad \forall f\in \mathbf{B}_0 \tag{68}$$

(see (33)). We assume that transition probabilities satisfy a *strong mixing condition* described below:

$$\sum_{k=1}^{+\infty}\sup_{y\in Y, A\in\mathcal{Y}}|P_k(y,A)-p(A)|<+\infty.$$

Taking into account the equations (31) and (30) we obtain the relation for $W_\varepsilon(t)$:

$$\begin{aligned} W_\varepsilon(t)f \;&=\; \varepsilon^{-1/2}\textstyle\int_0^t(\Gamma(y(s/\varepsilon))-\hat\Gamma)V_\varepsilon(s)f\,ds+\int_0^t\hat\Gamma W_\varepsilon(s)f\,ds\\ &\quad+\varepsilon^{-1/2}[V_\varepsilon^d(t)-\textstyle\int_0^t\hat D\cdot V_0(s)\,ds]f,\ \forall f\in B_0, \end{aligned} \tag{69}$$

where

$$V_\varepsilon^d(t)f := \sum_{k=1}^{\nu(t/\varepsilon)}[D^\varepsilon(y_k)-I]V_\varepsilon(\varepsilon\tau_k^-)f,$$

and $\hat\Gamma, \hat D$ are defined in (36).

If the process $W_\varepsilon(t)f$ has the weak limit $W_0(t)f$ as $\varepsilon\to 0$ then we obtain:

$$\int_0^t\hat\Gamma W_\varepsilon(s)f ds\to\int_0^t\hat\Gamma W_0(s)f ds, \varepsilon\to 0. \tag{70}$$

Since the operator $(\Gamma(y)-\hat\Gamma)$ satisfies to the balance condition

$$\Pi(\Gamma(y)-\hat\Gamma)f=0,$$

then the diffusion approximation of the first term in the righthand side of (69) gives:

$$\varepsilon^{-1/2}\int_0^t l(\Gamma(y(s/\varepsilon))-\hat\Gamma)f)ds\to l(\sigma_1 f)w(t),\varepsilon\to 0 \tag{71}$$

where

$$l^2\sigma_1 f)=\int_Y\rho(dx)[m(x)l((\Gamma(x)-\hat\Gamma)f)\mathbf{R}_0\Gamma(x)-\hat\Gamma)f)+2^{-1}\cdot m_2(x)l^2((\Gamma(x)-\hat\Gamma)f)]/m,$$

$\forall l\in\mathbf{B}_0, w(t)$ is a standard Wiener process.

Since $\Pi(\mathbf{P}D_1(y)-\hat D)f=0$, then the diffusion approximation of the third term in the right-hand side of (69) gives the following limit:

$$\varepsilon^{-1/2}\cdot l(V_\varepsilon^d(t)f-\int_0^t\hat D V_0(s)f ds)\to l(\sigma_2 f)\cdot w(t),\varepsilon\to 0, \tag{72}$$

where

$$l^2(\sigma_2 f):=\int_Y\rho(dx)l((D_1(y)-\hat D)f)\mathbf{R}_0\cdot l((D_1(y)-\hat D)f).$$

The passage to the limit as $\varepsilon \to 0$ in the representation (68) by encountering (69)-(72) arrives at the equation for $W_0(t)f$:

$$W_0(t)f = \int_0^t \hat{\Gamma} W_0(s) f ds + \sigma f w(t), \tag{73}$$

where the variance operator σ is determined from the relation:

$$l^2(\sigma f) := l^2(\sigma_1 f) + l^2(\sigma_2 f), \qquad \forall l \in \mathbf{B}_0, \qquad \forall l \in \mathbf{B}_0^*, \tag{74}$$

and operators σ_1 and σ_2 are defined in (71) and (72) respectively.

Double approximation of the SMRE has the form:

$$V_\varepsilon(t)f \approx V_0(t)f + \sqrt{\varepsilon} W_0(t)f$$

for small ε, which perfectly fits the standard form of the CLT with non-zero limiting mean value.

2. AVERAGING OF DIFFERENCE EQUATIONS IN RANDOM MEDIA

In this section we consider two types of difference equations: a) difference equation with Markov random perturbations as a random media and discrete parameter to explain how the method of random evolutions works; b) difference equation with semi-Markov random perturbations as a random media and continuous parameter using the scheme from a).

2.1. Averaging in Markov random media

We consider a system in a linear phase space X with discrete time $n \epsilon Z_+$ which is perturbed by a Markov chain $(y_n)_{n \epsilon Z_+}$ defined on a measurable space $(Y, \mathcal{Y})$. The system depends on a small parameter $\epsilon > 0$. Let $X_n^\epsilon \varepsilon X$ denote the state of the system at time n. We suppose that X_n^ϵ is determined by the recurrence relations:

$$X_{n+1}^\epsilon - X_n^\epsilon = \varepsilon g(X_n^\epsilon, y_{n+1}), \qquad X_n^\epsilon = X_0, \tag{75}$$

where $X_0 = x$ is given initial value; $g : X \times Y \to X$ is given function.

We consider system (75) in the following phase spaces:

$$X = R^d, \ d \geq 1.$$

Function $g(x, y)$ is measurable in y and continuous in $x \in X$.

The problem is to investigate the asymptotic behavior of the system as $\varepsilon \to 0$ and $n \to \infty$,

Let's rewrite equation (75) in the form:

$$X_{n+1}^\epsilon = X_0 + \varepsilon \sum_{k=1}^n g(X_k^\epsilon, y_{k+1}). \tag{76}$$

Let's consider the following family of operators $D^{\epsilon}(y)$ on $\mathbf{B}=\mathbf{C}_0^1(x)$ (space of differentiable functions on x vanishing on infinity):

$$D^{\epsilon}(y)f(x):=f(x+\varepsilon g(x,y)),\ f(x)\epsilon\mathbf{C}_0^1(X). \tag{77}$$

Taking into account the representation (76) and (77) we obtain that $f(X_{n+1}^{\epsilon})$ my be represented on the form:

$$f(X_{n+1}^{\epsilon})=f(X_0+\varepsilon\sum_{k=1}^{n}g(X_k^{\epsilon},y_{k+1}))=\prod_{k=0}^{n}D^{\epsilon}(y_{k+1})f(x)=:V_n^{\epsilon}f(x),\ X_0=x. \tag{78}$$

We note that operators $D^{\epsilon}(y)$ are linear contractive and admits the representation:

$$D^{\epsilon}(y)f(x)=f(x)+\ \varepsilon g(x,y)\frac{d}{dx}f(x)+\varepsilon O_{\epsilon}(1)f(x) \tag{79}$$

as $\varepsilon\to 0$, where $\|\ 0_{\epsilon}(1)f(x)\ \|\to 0,\varepsilon\to O$. $\|\cdot\|$ is a norm in the space $\mathbf{C}_0^1(x)$. Namely,

$$D^{\epsilon}(y)f(x)=f(x)+\varepsilon D_1(y)f(x)+\varepsilon O_{\epsilon}(1)f(x), \tag{80}$$

where

$$D_1(y)f(x):=g(x,y)\frac{d}{dx}f(x). \tag{81}$$

We suppose here that the noise process $(y_n)_{n\in Z_+}$ is a stationary ergodic process with ergodic distribution $p(dy)$, namely, for any function $f(y):Y\to R$, for which

$$\int_y |f(y)|p(dy)\ <\ +\infty,$$

we have

$$\mathbf{P}\{\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}f(y_k)=\int_Y f(y)p(dy)\}=1. \tag{82}$$

Put

$$\hat{D}f(x):=\int_Y D_1(y)f(y)p(dy), \tag{83}$$

and consider the following equation:

$$\hat{V}_tf(x)-f(x)-\int_0^t\hat{D}\hat{V}_sf(x)ds=0,\ \forall\ f\epsilon\mathbf{C}_0^1(X). \tag{84}$$

From the theory of random evolution (see Section 4.1.3, Averaging of RE) it follows that if

$$\int_Y p(dy)||D_1(y)f(x)||^2\ <\ +\infty \tag{85}$$

and there exists a compact set $K_T^{\triangle}\subset\mathbf{C}^1(x)$ such that

$$\lim_{\varepsilon\to 0}\inf\mathbf{P}\{V_{[t/\epsilon]}^{\epsilon}f\varepsilon K_T^{\triangle};0\le t\le T\}\ge 1-\triangle, \tag{86}$$

$\forall\ \triangle\ >\ 0,T\ >\ 0$, then the sequence $V_{[t/\epsilon]}^{\epsilon}$ is relatively compact in $\mathbf{D}_B[O,+\infty]$ with limit points in $\mathbf{C}[0,+\infty]$.

Moreover, if operator $\hat{D}$ in (83) generates semigroup, then $V^{\epsilon}_{[t/\epsilon]}$ converges weakly as $\epsilon \to o$ to the process $\hat{V}_t$ in (10), $\forall\ t\epsilon[0,T]$.

The family $V^{\epsilon}_{[t/\epsilon]}$, corresponds to the following process (see (4)) $X^{\epsilon}_{[t/\epsilon]+1}$, since

$$V^{\epsilon}_{[t/\epsilon]+1}f(x) = \prod_{k=0}^{[t/\epsilon]+1} D^{\epsilon}(y_{k+1})f(x) = f(X_0 + \varepsilon \sum_{k=o}^{[t/\epsilon]+1} g(X^{\epsilon}_k, y_{k+1})) = f(X^{\epsilon}_{[t/\epsilon]+1}). \quad (87)$$

Suppose that the following conditions are satisfied:

$$\int_Y |g(x,y)|^2 p(dy)\ <\ +\infty \text{ for all fixed x}\epsilon\text{X}, \quad (88)$$

and $g^l_x(x,y)$ is bounded and continuous on x and y, where g^l_x is a l-th derivative by $x, l \geq 1$.

To obtain compact condition (12) for our process $X^{\epsilon}_{[t/\epsilon]}$, it's need to construct a compact set in Banach space $\mathbf{B}$, for it's need to construct Hilbert space $\mathbf{H}$ compactly embedded in $\mathbf{B} := \mathbf{C}^1_0(X)$.

The Sobolev imbedding theorem [7] states that the bounded sets in $\mathbf{W}^{l,2}(R^d)$ are compacts in $\mathbf{C}^1_0(R^d)$ provided $2l \geq d$.

In this case we have:

$$\mathbf{B} = \mathbf{C}^1_0(R^d), \mathbf{H} = \mathbf{W}^{l,2}(R^d), 2\ \ell \geq d, \qquad ||f||_{\mathbf{W}^{l,2}(R^d)} := \{\sum_{|\alpha|\leq \ell} ||D^{\alpha}f||^2_{\mathbf{L}^2(R^d)}\}^{1/2}.$$

We note that if $d = 1$ (then $X = R$), then it sufficient to take $l = 1$. From conditions (88) it follows that conditions (85) and (86) are fulfilled.

It means that family of measures, generated by process $X^{\epsilon}_{[t/\epsilon]}$ is relatively compact and there exists a unique limiting for $X^{\epsilon}_{[t/\epsilon]}$ process $\hat{X}_t$ as $\varepsilon \to o$, in the sense of weak convergence.

From (85) and (86) it follows that

$$V^{\epsilon}_{[t/\epsilon]}f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \hat{V}_t f(x), \quad (89)$$

and from (87) and (8) we obtain:

$$f(X^{\epsilon}_{[t/\epsilon]}) = V^{\epsilon}_{[t/\epsilon]}f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \hat{V}_t f(x) = f(\hat{x}_t),$$

namely,

$$f(X^{\epsilon}_{[t/\epsilon]}) \underset{\varepsilon\to 0}{\Longrightarrow} f(\hat{x}_t) \quad (90)$$

Moreover, from (84), (87) and (89)-(90) we obtain that

$$f(\hat{x}_t) - f(x) - \int_0^t \hat{D}f(\hat{x}_t)ds = 0, \quad (91)$$

We note that in our case (see (81), (83)):

$$\hat{D}f(x)=\int_Y p(dy)g(x,y)\frac{d}{dx}f(x)=\hat{g}(x)\frac{d}{dx}f(x). \tag{92}$$

Taking into account (92) we obtain:

$$f(\hat{x})-f(x)-\int_0^t \hat{g}(\hat{x}_s)\frac{d}{dx}f(\hat{x}_s)ds=0. \tag{93}$$

It means, that $f(\hat{x}_t)$ satisfies the equation:

$$\begin{cases} \frac{df(\hat{x}_t)}{dt}=\hat{g}(\hat{x})\frac{d}{dx}f(\hat{x}_t) \\ f(\hat{x}_0)=f(x), \end{cases}$$

and $\hat{x}_t$ satisfies the equation:

$$\begin{cases} \frac{d\hat{x}_t}{dt}=\hat{g}(\hat{x}_t) \\ \hat{x}_0=x. \end{cases}$$

In this way, we obtain the following result:

Theorem 1. *Under conditions (82), (88) and (89) process $X^\epsilon_{[t/\epsilon]}$ in (75) converges weakly to the process $\hat{x}_t$ in (93) as $\epsilon\to 0$, where $\hat{g}(x):=\int_Y p(dy)g(x,y)$.*

Remark 1. Consider the following process:

$$X^\epsilon_t=X^\epsilon_{[t/\epsilon]}+(t/\epsilon-[t/\epsilon])(X^\epsilon_{[t/\epsilon]+1}-X^\epsilon_{[t/\epsilon]}=X^\epsilon_{[t/\epsilon]})+\epsilon(t/\epsilon-[t/\epsilon])g(X^\epsilon_{t/\epsilon]},y_{[t/\epsilon]+1}). \tag{94}$$

Then we obtain from here and Theorem 1 then process X^ϵ_t also converges weakly to the process $\hat{x}(t)$ as $\epsilon\to 0$.

Remark 2. From Theorem 1 it follows that $\sup_{\mathrm{n}\epsilon\leq\mathrm{t}}||\mathrm{X}^\epsilon_\mathrm{n}-\hat{\mathrm{x}}(\epsilon\mathrm{n})||\underset{\varepsilon\to 0}{\Longrightarrow}0$ for any $t>0$, where X^ϵ_n and $\hat{x}(t)$ are defined in (75) and (93), respectively.

Remark 3. The result analogical to the Theorem 1 was obtained by Hoppensteadt, Salehi and Skorohod [5, p. 466, Theorem 1].

2.2 Averaging in semi-Markov random media

Let us consider Markov renewal process [3] $(y_n;\theta_n)_{n\in Z_+}$ with stochastic kernel

$$Q(y,dz,dt):=\mathbf{P}\left\{y_{n+1}\in dz,\theta_{n+1}\leq dt/y_0=y\right\}=P(y,dz)G_y(dt), \tag{95}$$

and let

$$\nu(t)=\max\left\{\mathrm{n}:\tau_\mathrm{n}\leq\mathrm{t}\right\} \tag{96}$$

being a counting process, where

$$\tau_u=\sum_{k=1}^n\theta_k,\theta_0=0.$$

We consider the following difference equation in semi-Markov random media:

$$\begin{cases} X^\epsilon_{\nu(t/\epsilon)+1}-X^\epsilon_{\nu(t/\epsilon)}=\epsilon\ g(X^\epsilon_{\nu(t/\epsilon)},y_{\nu(t/\epsilon)+1}) \\ X^\epsilon_0=X_0=x\in X, \end{cases} \tag{97}$$

where $y_{\nu(t)}$ is a semi-Markov process, $\nu(t)$ is defined in (96).

We note, that if $t \in [\varepsilon\tau_n, \varepsilon\tau_{n+1})$, where τ_n are defined in (97), then $X^{\epsilon}_{\nu(t/\epsilon)}$ satisfies the equation:

$$X^{\varepsilon}_{n+1} - X^{\varepsilon}_{n} = \varepsilon g(X^{\epsilon}_{n}, y_{n+1}),$$

(see (75)). We consider regular semi-Markov process, namely

$$\mathbf{P}\{\nu(t) < +\infty\} = 1, \qquad \forall t \in R_+ \tag{98}.$$

The problem is to investigate the asymptotic behavior of the system (25) as $\epsilon \to 0$.

Let us rewrite equation (97) in the form:

$$X^{\epsilon}_{\nu(t/\epsilon)+1} = X_0 + \varepsilon \sum_{k=0}^{\nu(t/\epsilon)} g(X^{\epsilon}_{k}, y_{k+1}). \tag{99}$$

Taking into account the representation (99) and (77) we obtain that $f(X_{\nu(t/\epsilon)+1})$ may be expressed in the form

$$f(X_{\nu(t/\epsilon)+1}) = f(X_0+\varepsilon \sum_{k=0}^{\nu(t/\epsilon)} g\,(x^{\epsilon}_{k}, y_{k+1})) = \prod_{k=0}^{\nu(t/\epsilon)} D^{\epsilon}(y_{k+1})f(x) =: V^{\epsilon}_{\nu(t/\epsilon)}f(x), \tag{100}$$

where operators $D^{\epsilon}(y)$ are defined in (77), and V^{ϵ}_{u} in (78). We note that produce in (100) is finite as condition (98) is satisfied. We note that operators $D^{\epsilon}(y)$ are linear and contractive uniformly by y and admits the representation (79), or equivalently, (80) with operator $D_1(y)$ in (81).

We suppose that the following condition is satisfied:

$$m_2(y) := \int_{o}^{\infty} t^2 G_y(dt) \tag{101}$$

is uniformly integrable, where $G_y(dt) := \mathbf{P}\{\theta_{n+1} \le dt/y_0 = y\}$. Put

$$\tilde{D}f(x) = \int_Y p(dy)D_1(y)f(y)/m = \int_Y p(dy)g(x,y)\frac{d}{dx}f(x)/m =: \tilde{g}(x)\frac{d}{dx}f(x), \tag{102}$$

where

$$m := \int_Y m(y)p(dy), m(y) := \int_0^{\infty} tG_y(dt) \tag{103}$$

Let us consider the following equation:

$$\tilde{V}_t f(x) - f(x) - \int_o^t \tilde{D}\tilde{V}_s f(x)ds = 0,\ \forall f\varepsilon \mathbf{C}^1_0(X). \tag{104}$$

From the theory of semi-Markov random evolutions (see Section 4.1.3, Averaging of RE) it follows that if conditions (82), (85), (93), (101) are satisfied and there exists a compact set $K^{\triangle}_T \subset \mathbf{C}^1_0(X)$ such that

$$\lim_{\varepsilon\to o} \inf \mathbf{P}\{V_{\nu(t/\epsilon)}f(x)\epsilon K^{\triangle}_T; 0 \le t \le T\} \ge 1 - \triangle, \tag{105}$$

$\forall \triangle > 0, \forall T > 0$, then the sequence $V^{\varepsilon}_{\nu(t/\epsilon)}$ is relatively compact in $\mathbf{D_B}[0, +\infty]$ with limit points in $\mathbf{C}[0, +\infty)]$. Moreover, if operator $\tilde{D}$ in (102) generates semigroup, then $V^{\varepsilon}_{\nu(t/\epsilon)}$ converges weakly as $\epsilon \to o$ to the process $\tilde{V}_t$ in (104), $\forall\ t\epsilon[0, T]$.

The family $V^{\epsilon}_{\nu(t/\epsilon)}$ corresponds to the following process (see (100)) $X^{\epsilon}_{\nu(t/\epsilon)}$, as (100) is satisfied.

From condition (88), (89), (98), (101) it follows that conditions (85) and (105) are satisfied. It means that family of measures, generated by processes $X^{\epsilon}_{\nu(t/\epsilon)}$, is relatively compact and there exists a unique limiting for $X^{\epsilon}_{\nu(t/\epsilon)}$ process $\tilde{x}_t$ as $\epsilon \to 0$, in sense of weak convergence.

From (100) and (104) it follows that

$$f^{\epsilon}(X_{\nu(t/\epsilon)}) = V^{\epsilon}_{\nu(t/\epsilon)} f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \tilde{V}_t f(x) = f(\tilde{x}_t), \tag{106}$$

namely,

$$f^{\epsilon}(X_{\nu(t/\epsilon)}) \underset{\varepsilon\to 0}{\Longrightarrow} f(\tilde{x}_t).$$

Moreover, from (102), (104), (106), (107) we obtain that

$$f(\tilde{x}_t) - f(x) - \int_0^t \tilde{D} f(\tilde{x}_s) ds = 0, \tag{107}$$

where $\tilde{D}$ is defined in (102), and taking into account (107) we have that $f(\tilde{x}_t)$ satisfies the equation:

$$f(\tilde{x}_t) - f(x) - \int_o^t \tilde{g}(\tilde{x}_s) \frac{d}{dx} f(\tilde{x}_s) ds = 0, \forall f(x) \in \mathbf{C}^1_0(X).$$

It means that $\tilde{x}_t$ satisfies the equation:

$$\frac{d\tilde{x}_t}{dt} = \tilde{g}(\tilde{x}_t), \tilde{x}_0 = x_0 = x. \tag{108}$$

Hence, we obtain the following result.

Theorem 2. *Under above mentioned conditions (82), (88), (89), (98), (101), the process $X^{\epsilon}_{\nu(t/\epsilon)}$ in (25) converges weakly to the process $\tilde{x}_t$ in (36) as $\epsilon \to o$, where*

$$\tilde{g}(x) = \int_Y p(dy) g(x,y)/m,$$

m is defined in (108).

Remark 4. Let us consider the following process

$$\begin{aligned} X(t)^{\epsilon} : &= X^{\epsilon}_{\nu(t/\epsilon)} + (t/\epsilon - \tau_{\nu(t/\epsilon)})(X^{\epsilon}_{\nu(t/\epsilon)+1} - X^{\epsilon}_{\nu(t/\epsilon)}) \\ &= X^{\epsilon}_{\nu(t/\epsilon)} + \varepsilon(t/\epsilon - \tau_{\nu(t/\epsilon)}) g(X^{\epsilon}_{\nu(t/\epsilon)}, y^{\epsilon}_{\nu(t/\epsilon)+1}), \end{aligned} \tag{109}$$

then this process also converges weakly to the process $\tilde{x}_t$ in (108) as $\epsilon \to o$, that follows from Theorem 2, representation (109) and the following result [2, p. 163]: for every bounded and continuous by y function $f(y)$ and $\forall t \epsilon [0,T]$

$$\lim_{\epsilon\to o} \mathbf{E}_y[(t/\epsilon - \tau_{\nu(t/\epsilon)}) f(y_{\nu(t/\epsilon)}) = t \int_Y p(dy) m_2(y) f(y)/2m,$$

where m and $m_2(y)$ are defined in (103) and (101), respectively.

Remark 5. The respective for process $X(t)^{\epsilon}$ in (37) semi-Markov random evolution $V^{\epsilon}(t)$ has a form:

$$V^{\epsilon}(t) f(x) = V^{\epsilon}_{\nu(t/\epsilon)} + (t/\epsilon - \tau_{\nu(t/\epsilon)})(V^{\epsilon}_{\nu(t/\epsilon)+1} - V^{\epsilon}_{\nu(t/\epsilon)}).$$

3. DIFFUSION APPROXIMATION OF DIFFERENCE EQUATIONS IN RANDOM MEDIA

In this section we also consider two types of equations as in Section 1. Under *balance condition*, using the method of random evolutions in diffusion approximation scheme, we obtain diffusion approximation of difference equations in Markov and semi-Markov random media.

3.1. Diffusion approximation in Markov media

We suppose in this section that the *balance condition* is fulfilled:

$$\hat{g}(x) := \int_Y p(dy)g(x,y) = 0, \quad \forall x \epsilon X. \tag{110}$$

Under condition (110) $\hat{x}_t = \tilde{x}_t = x_0$, where $\hat{x}_t$ and $\tilde{x}_t$ are defined in (93) and (108), respectively, and the solutions of these equations don't exit off neighborhood of initial value x_0.

In this case we can study the behavior of solutions of the equations (93) and (108) making a change of time $n^1 = \epsilon^2 n$ and $t^1 = \epsilon^2 t$, respectively.

Let us consider the following difference equation:

$$X^\epsilon_{n+1} = X^\epsilon_n = \epsilon g(X^\epsilon_n, y_{n+1}), X^\epsilon_0 = X_0 = x\epsilon X, \tag{111}$$

where $(y_n)_{n\in Z_+}$ and $g(x,y)$ are defined in Section 4.2.

Also, consider the operators $D^\epsilon(y)$ in (77) and representation (111). We note that operators $D^\epsilon(y)$ have the expansion:

$$D^\epsilon(y)f(x) = f(x) + \epsilon g(x,y)f_x(x) + \frac{\varepsilon^2}{2}(ggf^2_x + g^2 f_{xx}) + \varepsilon^2 O_\epsilon(1)f(x), \tag{112}$$

where $||O_\epsilon(1)f(x)|| \underset{\varepsilon\to 0}{\Longrightarrow} 0$, for $f(x)\epsilon\mathbf{C}^2(X)$.

Namely,

$$D^\epsilon(y)f(x) = f(x) + \epsilon D_1(y)f(x) + \epsilon^2 D_2(y)f(x) + \epsilon^3 O_\epsilon(1)f(x), \tag{113}$$

where

$$D_1(y)f(x) := g(x,y)\frac{d}{dx}f(x), D_2(y)f(x) := 1/2 \ g^2\frac{d^2 f}{dx^2}. \tag{114}$$

Also, we define discrete random evolutions as follows:

$$V^\epsilon_n f(x) = \prod_{k=0}^{n} D^\epsilon(y_{k+1})f(x) = f(X^\epsilon_{n+1}), \tag{115}$$

where X^ϵ_{n+1} and $D^\epsilon(y)$ are defined in (111) and (113), respectively.

Put

$$\bar{L}f(x) := \int_Y p(dy)[D_1(y)\mathbf{R}_0 D_1(y) + D_2(y)]f(x), \tag{116}$$

and consider the following expression:

$$\bar{M}_t f(x) := \bar{V}f(x) - f(x) - \int_o^t \bar{L}\bar{V}_s f ds, \quad \forall f(x)\epsilon B. \tag{117}$$

From the theory of random evolutions (see Section 4.1.3, Diffusion Approximation of RE) it follows that if the conditions (82) is satisfied, the third moment of $G_y(dt)$

$$m_3(y) := \int_0^{\infty} t^3 G_y(dt) \tag{118}$$

is uniformly integrable, and

$$\int_Y p(dy)||D_1(y)f||.||D_2(y)f|| \ < \ +\infty; \quad \int_Y p(dy)||D_1(y)f||^3 \ < \ \infty;$$

$$\int_Y p(dy)||D_2(y)f||^2 \ < \ +\infty, \quad \forall f \in Dom(D_2(y)), \tag{119}$$

and there exists a compact set $K_T^{\triangle} \subset \mathbf{B}$ such that

$$\lim_{\epsilon \to 0} \inf \mathbf{P}\{V^{\epsilon}_{[t/\epsilon^2]} f \in K_T^{\triangle}; 0 \leq t \leq T\} \geq 1 - \triangle, \tag{120}$$

$\forall \triangle > 0, \quad \forall T > 0$, then the sequence $V^{\epsilon}_{[t/\epsilon^2]}$ is relatively compact in $\mathbf{D_B}[0, +\infty]$ with limit points in $\mathbf{C_B}[0, +\infty]$.

Moreover, if operator $\bar{L}$ in (115) generates semigroup, then $V^{\epsilon}_{[t/\epsilon^2]}$ converges weakly as $\epsilon \to o$ to the process $\bar{V}_t$ in (117) such that $\bar{M}_t f(x)$ being a continuous $\mathcal{F}_t$-martingale, where $\mathcal{F}_t := \sigma\{y(s) := y_{\nu(s)}; 0 \leq s \leq t\}$.

The family $V^{\epsilon}_{[t/\epsilon^2]}$ corresponds to the following process (see (111) and (115)) $X^{\epsilon}_{[t/\epsilon^2]}$, as

$$V^{\epsilon}_{[t/\epsilon^2]} f(x) = \prod_{k=o}^{[t/\epsilon^2]} D^{\epsilon}(y_{k+1}) f(x) = f(x + \epsilon \sum_{k=0}^{[t/\epsilon^2]} g(X^{\epsilon}_k, y_{k+1})) = f(X^{\epsilon}_{[t/\epsilon^2]}). \tag{121}$$

Suppose that

$$\begin{cases} \int_Y p(dy)|g(x,y)|^4 < +\infty \\ \int_Y p(dy)|g'_x(x,y)|^2 < +\infty, \text{for all } x \in X \end{cases} \tag{122}$$

From conditions (122) it follows that condition (119) is satisfied.

To obtain compact condition (120) for our process $X^{\epsilon}_{[t/\epsilon^2]}$, it's need again to construct a compact set in Banach space $\mathbf{B}$, namely, to construct Hilbert space $\mathbf{H}$ compactly embedded in $\mathbf{B} = \mathbf{C}^2(X)$.

The Sobolev embedding theorem [7] states that the bounded sets in $\mathbf{W}^{l,2}(R^d)$ are compacts in $\mathbf{C}^2_0(R^d)$ provided $2l \geq d$. In our case we have:

$$\mathbf{B} = \mathbf{C}^2(R^d), \quad \mathbf{H} = \mathbf{W}^{l,2}(R^d), \quad 2l \geq d.$$

From conditions (122) and (88) it follows that conditions (119) and (120) are satisfied.

It means that family of measures generated by process $X^{\epsilon}_{[t/\epsilon^2]}$ is relatively compact and there exists a unique limiting for $X^{\epsilon}_{[t/\epsilon]}$ process $\bar{x}_t$ as $\epsilon \to o$, in the sense of weak convergence.

From (119) and (120) it follows that

$$V^{\epsilon}_{[t/\epsilon^2]} f(x) \underset{\varepsilon \to 0}{\Longrightarrow} \hat{V}_t f(x), \tag{123}$$

and from (121) and (123) we obtain

$$f(X^{\epsilon}_{[t/\epsilon^2]}) = V^{\epsilon}_{[t/\epsilon^2]} f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \hat{V}_t f(x) = f(\bar{X}_t),$$

namely,

$$f(X^{\epsilon}_{[t/\epsilon^2]}) \underset{\varepsilon\to 0}{\Longrightarrow} f(\bar{X}_t). \tag{124}$$

Moreover, from (117), (119) and (124) we obtain that

$$f(\bar{X}_t) - f(x) - \int_0^t \bar{L} f(\bar{X}_s)\, ds \tag{125}$$

is a continuous $\mathcal{F}_t$ martingale.

Let us calculate the operator $\bar{L}$ in (116) with $D_1(y)$ and $D_2(y)$ in (114):

$$\begin{aligned} \bar{L} f(x) &= \int_Y p(dy)[g f_x \mathbf{R}_0 g f_x + 1/2 g^2 f_{xx} \\ &= \int_y p(dy)[g \mathbf{R}_0 g_x f_x + g \mathbf{R}_0 g f_{xx} + 1/2 g^2 f_{xx}] \\ &= \alpha(x) f_x + 1/2 \beta^2(x) f_{xx}, \end{aligned} \tag{126}$$

where

$$\alpha(x) := \int_Y p(dy)[g \mathbf{R}_0 g_x]$$

$$\beta^2(x) := 2 \int_Y p(dy)[g \mathbf{R}_0 g + 1/2 g^2], \tag{127}$$

and $\mathbf{R}_0$ is a potential of Markov chain $(y_n)_{n\in Z_+}[1-3]$.

From (125)-(127) it follows that process $\bar{X}_t$ is a diffusion process with infinitesimal operator $\bar{L}$ in (126), namely, with drift coefficient $\alpha(x)$ and diffusion coefficient $\beta(x)$ in (127).

Hence, process $\bar{X}_t$ satisfies the following stochastic differential equation:

$$d\bar{X}_t = \alpha(\bar{X}_t)dt + \beta(\bar{X}_t)dw_t, \tag{128}$$

where w_t is a standard Wiener process, and coefficients $\alpha(x)$ and $\beta(x)$ are defined in (127).

Theorem 3. *Under conditions (82), (88), (110), (118), (122), the process* $(X^{\epsilon}_{[t/\epsilon^2]})$ *converges weakly to the process* $\bar{X}_t$ *in (128) as* $\epsilon \to 0$ *with coefficients* $\alpha(x)$ *and* $\beta(x)$ *in (127).*

3.2. Diffusion Approximation in Semi-Markov Random Media

Let us consider Markov renewal process $(y_n, \theta_n)_{n\in Z_+}$ with stochastic kernel $Q(y, dz, dt)$ in (95), counting process in (96) with regular condition (98). We suppose also that *balance condition* (110) is satisfied.

In this section we study the following difference equation:

$$\begin{cases} X^{\epsilon}_{\nu(t/\epsilon^2)+1} - X^{\epsilon}_{\nu(t/\epsilon^2)} = \epsilon g(X^{\epsilon}_{\nu(t/\epsilon^2)}, y^{\epsilon}_{\nu(t/\epsilon^2)+1}) \\ X^{\epsilon}_0 = X_0 = x \in X \end{cases} \tag{129}$$

It means that if $t \in [\epsilon^2\tau_n, \epsilon^2\tau_{n+1})$, where τ_n are defined in (97), then $X^\epsilon_{\nu(t/\epsilon^2)+1}$ satisfies the equation (111).

The problem is to study the asymptotic behavior of the system (129) as $\epsilon \to 0$.

Let us rewrite down the equation (129) in the form:

$$X^\epsilon_{\nu(t/\epsilon^2)+1} = X_0 + \epsilon \sum_{k=1}^{\nu(t/\epsilon^2)} g(X^\epsilon_k, y_{k+1}). \tag{130}$$

Taking into account (115) we may express $f(X^\epsilon_{\nu(t/\epsilon^2)})$ in the following form:

$$f(X^\epsilon_{\nu(t/\epsilon^2)}) = f(X_0+\epsilon \sum_{k=1}^{\nu(t/\epsilon^2)} g(X^\epsilon_k, y_{k+1})) = \prod_{k=1}^{\nu(t/\epsilon^2)} D^\epsilon(y_{k+1})f(x) = V^\epsilon_{\nu(t/\epsilon^2)}f(x), \tag{131}$$

where operators $D^\epsilon(y)$ are defined in (77), and V^ϵ_n in (78).

Put

$$\tilde{L}f := \bar{L}f/m, \tag{132}$$

where operator $\bar{L}$ is defined in (116) and m - in (103).

Let us consider the following equation

$$\tilde{V}(t)f(x) = f(x) + \int_0^t \tilde{L}\tilde{V}(s)f(x)ds + \tilde{M}(t)f(x), \tag{133}$$

in Banach space $\mathbf{B}$, $f \in \mathbf{B}$, with process $\tilde{M}(t)f(x)$ being a continuous $\mathcal{F}_t$- martingale.

From the theory of random evolutions (see Section 4.1.3, Diffusion Approximation of RE) it follows, that under conditions of Subsection 2.2 and condition (98), the process $V^\epsilon_{\nu(t/\epsilon)^2}$ is relatively compact in $\mathbf{D_B}[0, +\infty]$ with limit points in $\mathbf{C_B}[0, +\infty]$.

Moreover, if operator $\tilde{L}$ in (132) generates semigroup, then $V^\epsilon_{\nu(t/\epsilon^2)}$ converges weakly (as $\epsilon \to 0$) to the process $\tilde{V}(t)$ in (133). From here we obtain that

$$V^\epsilon_{\nu(t/\epsilon^2)} \underset{\varepsilon\to 0}{\Longrightarrow} \tilde{V}(t)f, \tag{134}$$

and from (131) we have:

$$f(X^\epsilon_{\nu(t/\epsilon^2)}) = V^\epsilon_{\nu(t/\epsilon)^2}f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \tilde{V}(t)f(x) := f(\tilde{X}(t)). \tag{135}$$

$$f(X^\epsilon_{\nu(t/\epsilon^2)}) \underset{\varepsilon\to 0}{\Longrightarrow} f(\tilde{X}(t)) \tag{136}$$

The family of measures generated by processes $X^\epsilon_{\nu(t/\epsilon^2)}$ is relatively compact and there exists a unique limiting for $X^\epsilon_{\nu(t/\epsilon^2)}$ process $\tilde{X}(t)$ as $\epsilon \to 0$ (due to (134)–(136)), in the sense of weak convergence.

Moreover, from (134)–(136) and (133) we obtain that

$$f(\tilde{X}(t)) - f(x) - \int_0^t \tilde{L}f(\tilde{X}(s)ds \tag{137}$$

is a continuous $\mathcal{F}_t$ - martingale.

We note that operator $\tilde{L}$ is equal to $\bar{L}/m$ (see (132) and $\bar{L}$ has been already calculated in (126). In this way

$$\tilde{L} = \tilde{\alpha}(x)\frac{d}{dx} + 1/2\tilde{\beta}^2(x)\frac{d^2}{dx^2}, \tag{138}$$

where

$$\tilde{\alpha}(x) := \frac{\alpha(x)}{m}, \tilde{\beta}^2(x) := \frac{\beta^2(x)}{m}, \tag{139}$$

and $\alpha(x), \beta^2(x)$ are defined in (127). From (137)-(139) we obtain that process $\tilde{X}(t)$ is diffusion process with infinitesimal operator $\tilde{L}$ in (138) and with drift coefficients $\tilde{\alpha}(x)$ and diffusion $\tilde{\beta}^2(x)$ in (139).

Hence, process $\tilde{X}(t)$ satisfies the following stochastic differential equation:

$$d\tilde{X}(t) = \tilde{\alpha}(\tilde{X}(t))dt + \tilde{\beta}(\tilde{X}(t))dw(t), \tag{140}$$

where $w(t)$ is a standard Wiener process, and $\tilde{\alpha}(x), \tilde{\beta}(x)$ are defined in (139).

We obtain the following result.

Theorem 4. *Under conditions of Theorem 3 and (98) the process $X^{\epsilon}_{\nu(t/\epsilon^2)}$ in (129) converges weakly as $\epsilon \to o$ to the diffusion process $\tilde{X}(t)$ in (140) with infinitesimal operator $\tilde{L}$ in (138) and coefficients drift $\tilde{\alpha}(x)$ and diffusion $\tilde{\beta}(x)$ in (139).*

4. NORMAL DEVIATIONS OF DIFFERENCE EQUATIONS IN RANDOM MEDIA

Let's consider a system in a linear phase space X with discrete parameter of time $n\varepsilon Z_+$ which is perturbed by a Markov chain $(y_n)_{n\in Z_+}$ defined on a measurable space $(Y, \mathcal{Y})$. The system depends on a small parameter $\varepsilon > 0$.

Let $X^\varepsilon_n \in X$ denote the state of the system at time n. We suppose that X^ε_n is determined by the recurrence relations:

$$X^\varepsilon_{n+1} - X^\varepsilon_n = \epsilon g(X^\epsilon_n, y_{n+1}), X^\varepsilon_0 = x_0, \tag{141}$$

where x_0 is given; $g : X \times Y \to X$ is given function. We consider system (141) in the phase space $X = R^d, d \geq 1$.

Function $g(x, y)$ is measurable in y and continuous in x. We suppose that the process $(y_n)_{n\in Z_+}$ is a stationary ergodic Markov chain in $(Y, \mathcal{Y})$ with ergodic distribution $p(A), A \in \mathcal{Y}$, function $g(x, y)$ has continuous second derivative by $x \in X$, and $\int_Y ||g(x, y)||^2 p(dy) < +\infty, \forall x \in X$. Also, we suppose that transition probabilities satisfy a *strong mixing condition*:

$$\sum_{k=1}^{+\infty} \sup_{y\in Y, A\in\mathcal{Y}} |P_k(y, A) - p(A)| < +\infty.$$

In Section 4.2 we proved that under above mentioned conditions process $X^\varepsilon_{[t/\varepsilon]}$ converges as $\varepsilon > 0$ to the process $\hat{x}_t$ such that

$$\frac{d\hat{x}_t}{dt} = \hat{g}(\hat{x}_t), \hat{x}_0 = x, \tag{142}$$

where

$$\hat{g}(x) := \int_Y p(dy)g(x, y). \tag{143}$$

If $\mathbf{P}\{\nu(t) < +\infty)\} = 1, \forall t\epsilon R+$, where $\nu(t)$ is a counting process, $\nu(t) := \max\{u : \tau_n \leq t\}, \tau_n := \sum_{k=1}^{u} \theta_k$, and $\{\theta_n; n \geq 0\}$ is a sojourn time with distribution function $G_y(dt)$ such that $\int_0^\infty t^2 f_y(dt) := m_2(y)$ is uniformly integrable, then process $X^\varepsilon_{\nu(t/\varepsilon)+1}$ converges weakly as $\varepsilon > 0$ to the process $\hat{x}(t)$ such that:

$$\frac{d\tilde{x}(t)}{dt} = \tilde{g}(\tilde{x}(y)),\ \tilde{x}(0) = x_0, \tag{144}$$

where

$$\tilde{g}(x) := \hat{g}(x)/m, m := \int_Y p(dy)m(y),$$

$$m(y) := \int_0^\infty tG_y(dt), \tag{145}$$

and $\hat{g}(x)$ is defined in (143).

Let us define the following non-random sequence $\tilde{X}^\varepsilon_n$:

$$\hat{X}^\varepsilon_{n+1} - \hat{X}^\varepsilon_n = \epsilon\hat{g}(\tilde{X}^\varepsilon_n), \tag{146}$$

where $\hat{g}(x)$ is defined in (143). We suppose that $\hat{g}(x) \neq 0, \forall x \varepsilon X$.

In this section, we study normal deviations of the solution of the perturbed system (141) from the solution of the averaged system (146). Let

$$Z^\varepsilon_n := [X^\varepsilon_n - \hat{X}^\varepsilon_n]/\sqrt{\varepsilon}, \tag{147}$$

where X^ε_n is defined in (141) and $\hat{X}^\varepsilon_n$ is defined in (146).

We show that under natural conditions Z^ε_n converges weakly to a diffusion process.

Let the following condition be satisfied:

(**C**) there exists a measurable function $h(y) : y \to R+$ for which

$$\int_Y h(y)p(dy) := \hat{h} < +\infty \text{ and for } y \in Y,\ x, x^1 \in R^d : \tag{148}$$

$$||g(x, y) - g(x^1, y)|| \leq h(y)||x - x^1||. \tag{149}$$

We note that Z^ε_n in (147) satisfies the relation:

$$\begin{aligned} Z^\varepsilon_n \quad = \quad & \sqrt{\varepsilon} \cdot \textstyle\sum_{k=0}^{n}[g(X^\varepsilon_k, y_{k+1}) - \hat{g}(\hat{X}^\varepsilon_k)] = \sqrt{\varepsilon} \cdot \textstyle\sum_{k=0}^{u}[g(X^\varepsilon_k, y_{k+1}) - g(\hat{X}^\varepsilon_n, y_{k+1})] \\ & + \sqrt{\varepsilon} \cdot \textstyle\sum_{k=0}^{u}[g(\hat{X}^\varepsilon_k, y_{k+1}) - \hat{g}(\hat{X}^\varepsilon_k)]. \end{aligned} \tag{150}$$

From condition (**C**) it follows that:

$$||Z^\varepsilon_n|| \leq\ \epsilon \sum_{k=0}^{n} h(y_{k+1})||Z^\epsilon_k|| + \sqrt{\varepsilon} \cdot \sum_{k=0}^{n}[g(\hat{X}^\varepsilon_k, y_{k+1}) - \hat{g}(\hat{X}^\varepsilon_k)],$$

and

$$\sup_{k \leq n} ||Z^\varepsilon_k|| \leq \sup_{k \leq n} \sqrt{\varepsilon} \cdot ||S^\varepsilon_k|| \exp\{\varepsilon \cdot \sum_{k=0}^{n} h(y_{k+1})\}, \tag{151}$$

where

$$S^\varepsilon_n := \sum_{k=0}^{n}[g(\hat{X}^\varepsilon_k, y_{k+1}) - \hat{g}(\hat{X}^\varepsilon_k)]. \tag{152}$$

We note that,

$$\lim_{\varepsilon\to 0}\varepsilon\sum_{\varepsilon n\le t_0}h(y_{n+1})=t_0\int_Y h(y)p(y), \tag{153}$$

due to ergodicity of $(y_n)_{n\in Z_+}$, and

$$\lim_{\varepsilon\to 0}\varepsilon\sum_{n\le\nu(t/\varepsilon)}h(y_{n+1})=t_0\int_Y h(y)p(dy)/m, \tag{154}$$

due to (153) and renewal theorem [9, Chapter X1, section 1] where m is defined in (145).

From (151)-(154) it follows that $\sup_{n\le\nu(t/\varepsilon)}||Z_n^\varepsilon||$ is bounded in probability as $\varepsilon\to 0$, if only $\sup_{n\le\nu(t/\varepsilon)}\sqrt{\varepsilon}||S_n^\varepsilon||$ converges. Let's consider firstly the expression $\sqrt{\varepsilon}S_n^\varepsilon$ (with S_n^ε being defined in (152)), which is the second term in the righthand side of (150).

We note that function $G(x,y):=(g(x,y)-\hat g(x))$ satisfies balance condition:

$$\int_Y p(dy)G(x,y)=\int_Y p(dy)(g(x,y)-\hat g(x))=0. \tag{155}$$

From the theory of random evolutions (see Section 4.1.3, Normal Deviations of RE) it follows that under condition (155) the process $\sqrt{\varepsilon}\sum_{k=0}^{\nu(t/\varepsilon)}[g(\hat X_k^\varepsilon,y_{k+1})-\hat g(\hat X_k^\varepsilon]$ converges weakly as $\varepsilon\to 0$ to the stochastic Ito integral with diffusion coefficient $\sigma(x)$ such that:

$$\sigma^2(x):=\int_Y p(dy)[(g(x,y)-\hat g(x))\mathbf{R}_0(g(x,y)-\hat g(x))+(g(x,y)-\hat g(x))^2/2]/m, \tag{156}$$

namely, to the integral

$$\int_0^t\sigma(\hat x_s)dw(s), \tag{157}$$

where $w(s)$ is a standard Wiener process and $\mathbf{R}_0$ is a potential of Markov chain $(y_n)_{n\in Z_+}$.

It means that the second term in the righthand side of (150) converges weakly as $\varepsilon\to 0$ to the integral (157).

Let us consider the first term in the righthand side of (150). By Taylor formula we obtain:

$$g(X_k^\varepsilon,y_{k+1})-g(\hat X_k^\varepsilon,y_{k+1})=\sqrt{\varepsilon}g_x(\hat X_k^\varepsilon,y_{k+1})\cdot Z_n^\varepsilon+1/2\cdot\varepsilon(Z_n^\varepsilon)^2g_{xx}(\hat X_k^\varepsilon+\sqrt{\varepsilon}\theta Z_n^\varepsilon,y_{k+1}), \tag{158}$$

where $0<\theta<1$. It means that the first term in the righthand side of (150) is equal to:

$$\varepsilon\sum_{k=0}^n g_x(\hat X_k^\varepsilon,y_{k+1})Z_k^\varepsilon+1/2(\varepsilon)^{3/2}\sum_{k=0}^n g_{xx}(\hat X_k^\varepsilon+\sqrt{\varepsilon}\theta Z_k^\varepsilon,y_{k+1})Z_n^{\varepsilon^2} \tag{159}$$

The second term in (159) converges weakly to zero as $\varepsilon\to 0$ due to ergodicity $(y_n)_{n\in Z_+}$ and continuosity of g_{xx} by x.

Let us define the following process:

$$Z_t^\varepsilon:=\sum_{k=1}^\infty Z_k^\varepsilon\mathbf{1}\{\tau_k\le t/\varepsilon<\tau_{k+1}\}. \tag{160}$$

This process is tight in $\mathbf{D}[0,T]$ [1-3], since

$$\mathbf{E}\,||Z^\varepsilon_{t_1} - Z^\varepsilon_{t_2}||^4 \leq C\cdot |t_1 - t_2|^4,$$

that follows from a strong mixing condition of $(y_n)_{n\in Z_+}$ (see [19]), where C does not depend on n and ϵ. Hence, sequence Z^ε_t converges weakly in $\mathbf{D}[O,T]$ to some process $\tilde{z}_t$ in $\mathbf{C}[O,T]$.

For the first term in (159) we obtain:

$$\varepsilon\sum_{k=0}^{\nu(t/\varepsilon)} g_x(\hat{X}^\varepsilon_k, y_{k+1})Z^\varepsilon_k \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t \tilde{g}_x(\tilde{X}_x)\tilde{z}_s ds, \tag{161}$$

where

$$\tilde{g}_x(x) := \int_Y p(dy)g_x(x,y)/m. \tag{162}$$

From (159) and (161) we obtain that the first term in the righthand side of (150) converges weakly as $\varepsilon\to 0$ to the limit in (161).

From (155)-(157) and (158)-(161) we finally obtain the following result.

Theorem 5. *Under mentioned above conditions process Z^ε_t in (160) converges weakly as $\varepsilon\to 0$ to the process $\tilde{z}_t$ which satisfies the following stochastic differential equation:*

$$\tilde{z}_t = \int_0^t \tilde{g}_x(\tilde{x}_s)\tilde{z}_s ds + \int_0^t \sigma(\tilde{x}_s)dw(s), \tag{163}$$

where $\tilde{g}_x^{'}(x)$ is defined in (162), and $\sigma(x)$ is defined in (156), $w(t)$ is a standard Wiener process.

5. MERGING OF DIFFERENCE EQUATIONS IN RANDOM MEDIA

Let $(Y,\mathcal{Y})$ be a measurable space, $(X,\mathcal{X})$ be a linear space, and $g(x,y): X\times Y\to X$ be a function, which determines our equation with semi-Markov process $y(t)$ as a random media and a small parameter $\varepsilon>0$. States of the system are determined by the following iteration relation:

$$X^\varepsilon_{\nu(t/\varepsilon)+1} = X^\varepsilon_{\nu(t/\varepsilon)} + \epsilon g(X^\varepsilon_{\nu(t/\varepsilon)}, y_{\nu(t/\varepsilon)+1}), \qquad \forall t\epsilon R_+,$$

where $X^\varepsilon_0 = X_0 = x$ is given initial value, $\nu(t)$ is a counting process.

In the Sections 4.2 and 4.3 we've studied averaging and diffusion approximation of the mentioned above equation as $\varepsilon > 0$ under various conditions on the data. In particular, it was obtained that $X^\varepsilon_{\nu(t/\varepsilon)}$ converges weakly to the process $\tilde{x}_t$ as $\varepsilon>0$ such that

$$\frac{d\tilde{x}_t}{dt} = \tilde{g}(\tilde{x}_t), \tilde{X}_0 = X_0 = x,$$

where $\tilde{g}(x) := \int_Y p(dy)g(x,y)/m, \{p(A); A\epsilon\mathcal{Y}\}$ is a stationary distribution of imbedded Markov chain $(y_n)_{n\in Z_+}$, m is a mean time of staying Markov chain $(y_n)_{n\in Z_+}$ in the state $y\epsilon\mathcal{Y}$.

Let us consider a Markov renewal process $(y^\varepsilon_n, \theta_n)_{n\in Z_+}$ in phase space $(Y,\mathcal{Y})$ and semi-markov kernel

$$Q_\varepsilon(y,dz,t) := P_\varepsilon(y,dz)G_y(t), \tag{164}$$

where stochastic kernel $P_\varepsilon(y, dz)$, which defines the transition probabilities of perturbed Markov chain $(y_n^\varepsilon)_{n\in Z_+}$, is represented in the form:

$$P_\varepsilon(y, B) := P(y, B) - \varepsilon P_1(y, B), y \epsilon Y, B \epsilon \mathcal{Y}. \tag{165}$$

Here: $P(y, B)$ is a transition probabilities of basic non-perturbed Markov chain $(y_n)_{n\in Z_+}$; $P_1(y, B)$ is a some probability measure.

Basic assumption consists of the proposition that stochastic kernel $P(y, B)$ is co-ordinated with given decomposition of phase space $(Y, \mathcal{Y})$:

$$Y = \bigcup_{v\epsilon V} Y_v, Y_v \bigcap Y_{v^1} = \phi, v \neq v^1, \tag{166}$$

by the following way:

$$P(y, Y_v) = \mathbf{1}_v(y) := \begin{cases} 1, & y \in Y_v, \\ 0, & y \notin Y_v. \end{cases} \tag{167}$$

In each class $Y_v, v\epsilon V$, basic non-perturbed Markov chain is uniformly ergodic by $v\epsilon V$ with stationary distribution $p_v(A), v\epsilon V, A\epsilon Y$: $p_v(B) = \int_{Y_v} p_v(dy)P(y, B), B \subset Y_v, p_v(Y_v) = 1$. The decomposition (166) defines a merging function:

$$v(y) = v, \text{if } y \ \epsilon Y_v, \ v\varepsilon V.$$

Here $(V, \mathcal{V})$ is a measurable merged phase space (see Section 1.9.4, Chapter 1).

Let us introduce the following notations:

$$m_v := \int_{Y_v} p_v(dy)m(y),$$

$$\hat{P}_1(v, H) := \int_{Y_v} p_v(dy)P_1(y, Y_H), v\epsilon V, H\epsilon V,$$

$$Y_H := \bigcup_{v\epsilon H} Y_v \epsilon Y, H \subset V. \tag{168}$$

It is known [5], that kernel

$$\hat{Q}(v, H) := \hat{P}_1(v, H)/m_v, \notin H, \tag{169}$$

with function $q(v)$, such that

$$0 < g(v) := -\hat{P}_1(v)/m_v = \int_{Y_v} p_v(dy)P_1(y, Y_v)/m_v, v \notin H, \tag{170}$$

defines a jump Markov process $\hat{y}(t)$ in phase space (V, v) with stochastic kernel

$$\hat{Q}(v, H, t) := \hat{P}(v, H) \cdot (1 - e^{-q(v)t}), \hat{P}(v, H) := \hat{Q}(v, H)/g(v), \tag{171}$$

where $\hat{Q}(v, H)$ and $q(v)$ are defined in (169) and (170), respectively.

Namely, semi-Markov process $y_{\nu(t/\varepsilon)}^\varepsilon$ converges weakly under conditions (164)-(171) as $\varepsilon \to 0$ to the jump Markov process $\hat{y}(t)$.

This Markov process $\hat{y}(t)$ is called a *merged Markov process* and phase space $(V, \mathcal{V})$ is called a *merged phase space.*

Infinitesimal operator $\hat{Q}$ of the merged Markov process $\hat{y}(t)$ acts by rule:

$$\hat{Q}\hat{f}(v) := q(v) \cdot \int_V \hat{P}(v, dv') \cdot [\hat{f}(v') - \hat{f}(v)],$$

$$\hat{f}(v) := \int_{Y_v} p_v(dy) f(y). \tag{172}$$

Our problem here is to study the behavior of the solution of the following difference equation as $\varepsilon \to 0$;

$$X^\varepsilon_{\nu(t/\varepsilon)+1} - X^\varepsilon_{\nu(t/\varepsilon)} = \epsilon \cdot g(X^\varepsilon_{\nu(t/\varepsilon)}, y^\varepsilon_{\nu(t/\varepsilon)+1}), \tag{173}$$

where $(y^\varepsilon_n)_{n\in Z_+}$ is a perturbed Markov chain in phase space $(Y, \mathcal{Y})$ with transition probabilities $P_\varepsilon(y, B)$ in (165).

Let us consider the following family of operators $D^\varepsilon(y)$ in $\mathbf{B} := \mathbf{C}^1(R^d)$:

$$D^\varepsilon(y) f(x) := f(x + \varepsilon g(x, y)). \tag{174}$$

We note that operators $D^\varepsilon(y)$ are linear contractive uniformly by y and admit the representation:

$$D^\varepsilon(y) f(x) = f(x) + \varepsilon g(x, y) \frac{d}{dx} f(x) + \varepsilon_\varepsilon(1) f(x) \text{as } \varepsilon \to 0, \tag{175}$$

where $||0_1(\varepsilon) f|| \underset{\varepsilon\to 0}{\Longrightarrow} 0, || \cdot ||$ is a norm in $\mathbf{C}^1(R^d)$.

Put

$$D_1(y) f(x) := g(x, y) \frac{d}{dx} f(x), \quad \forall f(x) \epsilon \mathbf{C}^1(R^d). \tag{176}$$

Also, consider the operator

$$\hat{D}(v) := \int_Y p_v(dy) D_1(y)/m_v, \tag{177}$$

where $D_1(y)$ and m_v are defined in (176) and (168), respectively, and define the operator $\hat{V}(t)$ as a solution of the following equation:

$$\hat{V}(t) f - f - \int_0^t \hat{D}(\hat{y}(s)) \hat{V}(s) f ds = 0, \quad \forall f \epsilon \mathbf{B}, \tag{178}$$

where operator $\hat{D}(y)$ is defined in (177).

From (173) we can find that

$$X^\varepsilon_{\nu(t/\varepsilon)+1} = X_0 + \varepsilon \sum_{k=0}^{\nu(t/\varepsilon)} g(X^\varepsilon_k, y^\varepsilon_{k+1}). \tag{179}$$

We note that for $\forall f \epsilon \mathbf{C}^1(R^d)$:

$$\begin{aligned} f(X^\varepsilon_{\nu(t/\varepsilon)}) &= f(x_0 + \varepsilon \cdot \textstyle\sum_{k=0}^{\nu(t/\varepsilon)} g(X^\varepsilon_k, y^\varepsilon_{k+1})) \\ &= \textstyle\prod_{k=0}^{\nu(t/\varepsilon)} D^\varepsilon(y^\varepsilon_{k+1}) f(x) := V^\varepsilon_{\nu(t/\varepsilon)} f(x), \end{aligned} \tag{180}$$

(with $D^\varepsilon(y)$ defined in (174)) that follows from (173) and (174).

Operator process $V^{\varepsilon}_{\nu(t/\varepsilon)}$ is a semi-Markov random evolution.

From the theory of semi-Markov random evolutions (see Section 4.1.3, Merging of RE) it follows that under conditions of Theorem 1 and conditions (164)-(171), family of random evolutions $V^{\varepsilon}_{\nu(t/\varepsilon)}$ converges weakly as $\varepsilon \to 0$ to the *merged Markov random evolution* $\hat{V}(t)$ which is defined by the solution of the equation (178):

$$V^{\varepsilon}_{\nu(t/\varepsilon)} \underset{\varepsilon\to 0}{\Longrightarrow} \hat{V}(t). \tag{181}$$

Let us calculate the operator $\hat{D}(v)$ in (177) using (175)-(176):

$$\begin{aligned}\hat{D}(v)f(x) = \int_{Y_v} p_v(dy)D(y)/m_v \quad &= \quad \int_{Y_v} p_v(dy)g(x,y)\tfrac{d}{dx}f(x)/m_v \\ &:= \quad \hat{g}(x,v)\cdot \tfrac{d}{dx}f(x).\end{aligned}$$

Namely,

$$\hat{D}(v)f(x) = \hat{g}(x,v)\frac{d}{dx}f(x) = \int_{Y_v} p_v(dy)g(x,y)\frac{d}{dx}f(x)/m(v). \tag{182}$$

From (180) and (181) we obtain:

$$f(X^{\varepsilon}_{\nu(t/\varepsilon)}) = V^{\varepsilon}_{\nu(t/\varepsilon)}f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \hat{V}(t)f(x) := f(\hat{x}(t)), \tag{183}$$

where $\hat{x}(t)$ is a limiting for $X^{\varepsilon}_{\nu(t/\varepsilon)}$ process

$$f(\hat{x}(t)) - f(x) - \int_0^t \hat{g}(\hat{x}(s),\hat{y}(s))\frac{d}{dx}f(\hat{x}(s))ds = 0, \tag{184}$$

which follows directly from (178), (182) and (183).

The representation (184) means that $\hat{x}(t)$ satisfies the equation:

$$\begin{cases} \frac{d\tilde{x}(t)}{dt} = \hat{g}(\hat{x}(t),\hat{y}(t)) \\ \hat{x}(0) = X_0 = x. \end{cases} \tag{185}$$

Hence, we obtain the following result.

Theorem 6. *Under conditions of Theorem 1 and (164)-(171) random process $X^{\varepsilon}_{\nu(t/\varepsilon)}$ converges weakly as $\varepsilon \to 0$ to the process $\hat{X}(t)$, which satisfies equation (185) with function $\hat{g}(x,v)$ in (182) and merged Markov process $\hat{y}(f)$ in merged phase space $(V,\mathcal{V})$ with generator $\hat{Q}$ in (172).*

6. STOCHASTIC STABILITY OF DIFFERENCE EQUATIONS IN AVERAGING AND DIFFUSION APPROXIMATION SCHEMES

6.1. Stability of difference equations in averaging scheme

In Section 4.2 we've studied that the following difference equation

$$X^{\epsilon}_{\nu(t/\epsilon)+1} = X^{\epsilon}_{\nu(t/\epsilon)} + \varepsilon g(X^{\epsilon}_{\nu(t/\epsilon)}, y_{\nu(t/\epsilon)+1}),\ X^{\epsilon}_0 = X_0 = x,\ t\in R_+, \tag{186}$$

has in the limit as $\epsilon \to 0$ the following form:

$$\frac{d\tilde{x}_t}{dt} = \tilde{g}(\tilde{x}_t),\ \tilde{x}_0 = x, \tag{187}$$

where $(y_n)_{n\in Z_+}$ is a Markov chain, $\nu_{(t)}$ is a counting process $g(x,y)$ is a bounded and continuous function on $R\times Y, Y$ is a phase space of $(y_n)_{n\in Z_+}$

$$\tilde{g}(x) := \int_Y p(dy)g(x,y)/m,$$

$$m := \int_Y p(dy)m(y),$$

$\{p(A); A\in\mathcal{Y}\}$ are stationary distribution of $(y_n)_{n\in Z_+}$, Y is a Borel σ-algebra on Y [6,7].

Our first purpose in this section is to state the stability property for difference equation (186) under the stability property of averaged equation (187). Namely, we state that if there exists a Lyapurnov function $V(x)$ such that $\tilde{g}(x)\cdot V_x^1(x)\le -\beta V(x)$, for some $\beta>0, \forall x\in R$, then process $X^\epsilon_{\nu(t/\epsilon)}$ in (186) is asymptotically stochastically stable process as $0\le\epsilon\le\epsilon_0, \epsilon$ is fixed, ϵ_0 is a small number.

Let's consider the following difference equation in semi-Markov random media:

$$\begin{cases} X^\epsilon_{\nu(t/\epsilon)+1} - X^\epsilon_{\nu(t/\epsilon)} = \epsilon g(X^\epsilon_{\nu(t/\epsilon)}, y_{\nu(t/\epsilon)+1}) \\ X_0^\epsilon = X_0 = x\in R. \end{cases} \tag{188}$$

Here: $(y_n;\theta_n)_{n\in Z_+}$ is a Markov renewal process (see Section 1.4, Chapter 1) with stochastic kernel

$$Q(y,dz,dt) := \mathbf{P}\{y_{n+1}\in dz, \theta_{n+1}\le dt/y_0=y\} = P(y,dz)\cdot G_y(dt); \tag{189}$$

$\nu(t) := \sum_{k=1}^n \theta_k, \theta_0=0; y_{\nu(t)}$ is a semi-Markov process, regular one, i.e.,

$$\mathbf{P}\{\nu(t)<+\infty\}=1,\ \forall t\in R_+,$$

$g(x,y)$ is a given function on $R\times Y, (Y,\mathcal{Y})$ is a phase space of Markov chain $(y_n)_{n\in Z_+}$ with σ-algebra $\mathcal{Y}$ of Borel sets from Y.

We suppose that:

(i)$(y_n)_{n\in Z_+}$ is a stationary ergodic process with ergodic distribution $p(dy)$;

(ii)

$$\int_Y |g(x,y)|^2 p(dy) < +\infty,\ \forall x\in R; \tag{190}$$

(iii) $g_x^{'}(x,y)$ is bounded and continuous function of x and y;

(iv) second moment $m_2(y) := \int_0^\infty t^2 G_y(dt)$ is uniformly integrable, where $G_y(dt)$ is defined in (189).

Under conditions (i) -(iv) in (190) process $X^\epsilon_{\nu(t/\epsilon)}$ converges weakly as $\varepsilon\to 0$, to the averaged process $\tilde{x}_t$ such that:

$$\frac{d\tilde{x}_t}{dt} = \tilde{g}(\tilde{x}_t), \tilde{x}_0 = x_0 = x, \tag{191}$$

$$\tilde{g}(x) := \int_Y p(dy)g(x_1y)/m, m := \int_Y p(dy)m(y), m(y) := \int_0^\infty tG_y(dt).$$

In the next Theorem 7 we study the stability property of difference equation (188) under the stability conditions of the equation (191).

We study the stability of zero state of the equation (188) using the stability of zero state of the equation (191).

Definition 1. The zero state of the process $X^{\epsilon}_{\nu(t/\epsilon)}$ is *stochastically exponentially stable*, if for any $\triangle_1 > 0$ and $\triangle_2 > 0$, there exist $\delta > 0$ and $\gamma > 0$ such that the following inequality

$$\mathbf{P}_{x,y}\{|X^{\epsilon}_{\nu(t/\epsilon)}| \le \triangle_2 \cdot e^{-\gamma t}; t \ge 0\} \ge 1 - \triangle_1, \tag{192}$$

is fulfilled provided $y \in Y$, $|X^{\epsilon}_0| = |x| < \delta$, $0 \le \epsilon \le \epsilon_0$, where $\mathbf{P}_{x,y}\{\} := \mathbf{P}\{|X_0 = x, y_0 = y\}$.

Definition 2. The zero state of the process $X^{\epsilon}_{\nu(t/\epsilon)}$ is *asymptotically stochastically stable* if there exists $\delta > 0$ such that

$$\mathbf{P}_{x,y}\{\lim_{t\to+\infty} |X^{\epsilon}_{\nu(t/\epsilon)}| = 0\} = 1, \quad \forall y \in Y, \tag{193}$$

where $|X^{\epsilon}_0| = |x| < \delta$.

Process $X^{\epsilon}_{\nu(t/\epsilon)}$ is *stochastically exponentially stable* one, if the following inequality

$$\mathbf{P}_{x,y}\{|X^{\epsilon}_{\nu(t/\epsilon)}| \le \triangle_2 \cdot e^{-\gamma t}; t \ge 0\} \ge 1 - \triangle_1, \tag{192}$$

is fulfilled for any $\triangle_1 > 0, \triangle_2 > 0$, and some $\delta > 0 : |X0^{\epsilon}| = |x| < \lambda, \forall x \in R, y \in Y$, some contant $\gamma > 0$, and $0 \le \epsilon \le \epsilon_0, \epsilon$ is fixed, ϵ_0 is a small number, $\epsilon_0 \ll 1$.

Definition 2. Process $X^{\epsilon}_{\nu(t/\epsilon)}$ is *asymptotically stochastically stable* if

$$\mathbf{P}_{x,y}\{\lim_{t\to+\infty} |X^{\epsilon}_{\nu(t/\epsilon)}| = 0\} = 1, \ \forall X \in R, \ y \in Y. \tag{193}$$

Stochastic stability is L. Arnold's terminology (1974) [11], *stable in probability* is used by R. Khasminskii [3], and *stable with probability 1*- by H. Kushner (1967) [18].

Theorem 7. *Let the conditions (i)-(iv) in (190) be satisfied and the following conditions are true:*

(v) there exists a smooth function $V(x)$ on R (polynomial like) such that: $V(x) \to +\infty$ as $|x| \to +\infty$, and $V(x)$ is positively defined; $V(x) = 0 \Rightarrow x = 0$;

(vi) $g(0, y) = 0, \forall\ y \in Y$;

(vii) function $V(x)$ satisfies the inequality:

$$\tilde{g}(x).V^{'}_x(x) \le -\beta V(x) \tag{194}$$

for some $\beta > 0, \forall x \in R$.

Then process $X^{\epsilon}_{\nu(t/\epsilon)}$ in (188) is stochastically exponentially stable. Moreover, it is asymptotically stochastically stable.

Proof. We note that process $(X^{\epsilon}_{\nu(t/\epsilon)}, y^{\epsilon}_{\nu(t/\epsilon)}, \gamma(t/\epsilon))$ on $R \times Y \times R_+$ is a Markov one with infinitesimal operator

$$\begin{aligned} L_\varepsilon f(t,x,y) &= \tfrac{1}{\varepsilon} Q f(t,x,y) + \tfrac{1}{\varepsilon} P[f(t, x + \varepsilon g(x,y), y) - f(t,x,y)], \\ & \quad f(t,x,y) \in \mathbf{C}^1(R_+) \times \mathbf{C}(R) \times \mathbf{C}(Y), \end{aligned} \tag{195}$$

where $\gamma(t) := t - \tau_{\nu(t)}, Pg(x,y) := \int_Y P(y,dz)g(x,z),\ P(y,A)$ is defined in (189), $y \in Y, \forall A \in \mathcal{Y}$,

$$Qf(t,x,y) := \frac{d}{dt}f(t,x,y) + \frac{h_y(t)}{\bar{G}_y(t)}[Pf(0,,x,y) - f(t,x,y)], \tag{196}$$

$$\bar{G}_y(t) := 1 - G_y(t), h_y(t) := \frac{dG_y(t)}{dt}, \tag{197}$$

$G_y(t)$ is defined in (189).

Let's introduce the family of functions:

$$V_\varepsilon(t,x,y) := V(x) + \epsilon V_1(t,x,y), \tag{198}$$

where V_1 is defined by the solution of the equation:

$$QV_1(t,x,y) + \mathbf{P}g(x,y)V_x^{'}(x) - AV(x) = 0, \tag{199}$$

$$\text{where } AV(x) := \tilde{g}(x)V_x^{'}(x). \tag{200}$$

From (195)-(200) we obtain that:

$$L_\varepsilon V_\varepsilon = AV(x) + \mathbf{P}[V_1(t, x + \epsilon(x,y), y) - V_1(t,x,y)] + 0(\epsilon), \tag{201}$$

where $0(\epsilon) = 1/2\varepsilon g^2(x,y)V_{xx}^{''}(x) + 0(1), 0(1) \underset{\varepsilon\to 0}{\Longrightarrow} 0$.

Let's define the process:

$$\begin{aligned} m_\epsilon(t) \quad := \quad & V_\varepsilon(\gamma(t/\epsilon), X^\epsilon_{\nu(t/\epsilon)}, y_{\nu(t/\epsilon)}) - V_\varepsilon(0,x,y) \\ & - \int_0^t L_\varepsilon V_\varepsilon(\gamma(s/\epsilon), X^\epsilon_{\nu(s/\epsilon)}, y_{\nu(s/\epsilon)})ds. \end{aligned} \tag{202}$$

It's a right-continuous integrable $\mathcal{F}_t^\epsilon$-martingale with zero mean, where $\mathcal{F}_t^\epsilon := \sigma\{y_{\nu(s/\epsilon)}; 0 \le \epsilon \le t\}$.

From the representations (198)-(202) we obtain:

$$\begin{aligned} & V(X^\epsilon_{\nu(t/\epsilon)}) - V(x) - \int_0^t AV(X^\epsilon_{\nu(s/\epsilon)})ds = M_\epsilon(t) + \\ & + \in V_1(t,x,y) + \epsilon V_1(\gamma(t/\epsilon), X^\epsilon_{\nu(t/\epsilon)}, y_{\nu(t/\epsilon)}) + \\ & + \int_0^t P[V_1(s, x + \epsilon g(X(^\epsilon_{\nu(s/\epsilon}), y_{\nu(s/\epsilon)}) - V_1(s,x,y)]ds + 0(\epsilon). \end{aligned}$$

It means that $X^\epsilon_{\nu(t/\epsilon)}$ in (188) approximates the averaged process $\tilde{x}_t$ in (191).

We note that from condition (v) it follows that there exists $0 < \epsilon < \epsilon_0$ such that:

$$\ell_1 V(x) \le V_\epsilon \le \ell_2 \cdot V(x) \tag{203}$$

for some positive constants ℓ_1, ℓ_2, and as $g(x,y)$ is bounded.

Let $\tilde{\beta} > 0$ be a constant. It follows from (198) that

$$(L_\epsilon + \hat{\beta})V_\epsilon \le (\hat{\beta} \cdot \ell_2 + A + \epsilon_0 \ell_3)V(x), \tag{204}$$

for some constant $\ell_3 > 0, \ell_2$ is defined in (203).

Let us take $\hat{\beta}$ in such a way

$$\hat{\beta} \le \frac{\beta - \epsilon_0 \cdot \ell_3}{\ell_2}, \tag{205}$$

where β is defined in (194). Then it follows from (204)-(205) that:

$$(L_\epsilon + \hat{\beta})V_\epsilon \leq 0 \tag{206}$$

Rewriting (202) with $e^{\hat{\beta}\cdot t}V_\epsilon(t,x,y)$ one can obtain:

$$e^{\hat{\beta}\cdot t}V_\epsilon(\gamma(t/\epsilon), X^\epsilon_{\nu(t/\epsilon)}y_{\nu(t/\epsilon)}) = V_\epsilon(0,x,y) + \int_0^t e^{\hat{\beta}\cdot s}(L_\epsilon + \hat{\beta})V_\epsilon \cdot ds + \tilde{M}_\epsilon(t), \tag{207}$$

where $\tilde{M}_\epsilon(t)$ is a right-continues integrable $\mathcal{F}^\epsilon_t$ - martingale with mean zero.
Using (203)-(204) and (206) in (207) we obtain:

$$0 \leq e^{\hat{\beta}\cdot t}\cdot \ell_1 V(X^\epsilon_{\nu(t/\epsilon)}) \leq e^{\hat{\beta}\cdot t}V_\epsilon(\gamma(t/\epsilon), X^\epsilon_{\nu(t/\epsilon)}, y^\epsilon_{\nu(t/\epsilon)}) \leq \ell_2 \cdot V(x) + \tilde{M}_\epsilon(t), \tag{208}$$

In this way, $\ell_2 V(x) + \tilde{M}_\epsilon(t)$ is nonnegative martingale.
Applying Kolmogorov-Doob's inequality we have for any $\tilde{\triangle}_2 > 0$:

$$\begin{aligned} &\mathbf{P}_{x,y}\{\sup_{0\leq t\leq T} e^{\hat{\beta}\cdot t}\ell_1 V((X^\epsilon_{\nu(t/\epsilon)}) \geq \tilde{\triangle}\} \\ &\leq \mathbf{P}_{x,y}\{\sup_{0\leq t\leq T}(\ell_2 \cdot V(x)) + \tilde{M}_\epsilon(t)) \geq \tilde{\triangle}_2\} \leq \tfrac{\ell_2\cdot V(x)}{\tilde{\triangle}_2}. \end{aligned} \tag{209}$$

Taking into account the existence of constants $b_1 > 0$ and $b_2 > 0$ and positive numbers n_1 and n_2 such that (see condition (v)):

$$b_1|x|^{n_1} \leq V(x) \leq b_2|x|^{n_2}, \tag{210}$$

we have

$$\{b_1|X^\epsilon_{\nu(t/\epsilon)}|^{n_1} \leq e^{-\hat{\beta}\cdot t}\frac{\tilde{\triangle}_2}{\ell_1}; t \geq 0\} \supset \{V(X^\epsilon_{\nu(t/\epsilon)}) \leq e^{-\hat{\beta}\cdot t}\frac{\tilde{\triangle}_2}{\ell_1}; t \geq 0\}. \tag{211}$$

From (211) we obtain:

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon)}| \leq e^{\hat{\beta}/n_1}\cdot(\frac{\tilde{\triangle}_2}{b_1\ell_1})^{1/n_1}; t \geq 0\} \geq 1 - \frac{\ell_2 \cdot V(x)}{\tilde{\triangle}_2}. \tag{212}$$

Now let $\triangle_1 > 0$ and $\triangle_2 > 0$ are fixed. We choose $\tilde{\triangle}_2$ to satisfy:

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon)}| \leq e^{-\frac{\hat{\beta}}{n_1}}\triangle_2; t \geq 0\} \geq 1 - \frac{\ell_2 \cdot V(x)}{\tilde{\triangle}_2}. \tag{213}$$

Hence, $\tilde{\triangle}_2 := \triangle_2^{n_1}\cdot \ell_i\ell_1$.
Take $\delta_1 > 0; |x| < \delta_1$ and $V(x) < \ell_2^{-1}\tilde{\triangle}_2 \cdot \triangle_1$, then

$$\delta_1 = (\frac{\tilde{\triangle}_2 \cdot \triangle_1}{b_2 \cdot \ell_2})^{1/n^2}$$

(see (210)). From (212) and (213) we obtain finally:

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon)}| \geq e^{-\frac{\hat{\beta}\cdot t}{n_1}}\triangle_2; t > 0\} \geq 1 - \triangle_1.$$

It means that (192) is proved with $\gamma = \frac{\hat{\beta}}{n_1}$, and first assertion of the Theorem 1 is proved.

To prove (193) and, hence, the second assertion of Theorem 7, we note that:

$$\{\lim_{t\to+\infty} |X^{\epsilon}_{\nu(t/\epsilon)}| = 0\} = \{\lim_{t\to+\infty} V(X^{\epsilon}_{\nu(t/\epsilon)}) = 0\} \supset$$

$$\supset \{\sup_{t\geq o} e^{\hat{\beta}\cdot t}\ell_1 V(X^{\epsilon}_{\nu(t/\epsilon)}) \leq D\}, \tag{214}$$

for source constant D. Then we have from (209) and (214):

$$\mathbf{P}_{\,x,y}\{\lim_{t\to+\infty} |X^{\epsilon}_{\nu(t/\epsilon)}| = 0\} = 1,$$

as $D \to \infty$, and Theorem 7 is completely proved $\triangle$.

6.2. Stability of difference equations in diffusion approximation scheme

Also, we've studied in Section 4.3 that if $\tilde{g}(x) = 0, \forall x \in R$ (the balance condition is fulfilled), than the following difference equation

$$X^{\epsilon}_{\nu(t/\epsilon^2)+1} = X^{\epsilon}_{\nu(t/\epsilon^2)} + \varepsilon g(X_{\nu(t/\epsilon^2)}, y^{\epsilon}_{\nu(t/\epsilon^2)+1}) \tag{215}$$

has in the limit as $\epsilon \to o$ the following diffusion model:

$$d\tilde{x}(t) = \tilde{\alpha}(\tilde{x}(t)dt + \tilde{\beta}(\tilde{x}(t))dw(t), \tag{216}$$

where

$$\tilde{\alpha}(x) := \int_Y p(dy)[gRg^{'}_x + 1/2g\cdot g^{'}_x]/m,$$

$$\tilde{\beta}^2(x) := 2\int_Y p(dy)[gR_0g + 1/2g^2]/m, \tag{217}$$

$\mathbf{R}_0$ is a potential of Markov chain $(y_u)_{n\in Z_+}$ (see Section 1.2, Chapter 1).

Our second purpose in this section is to state the stability property for difference equation (215) under the stability property of diffusion model (216). Namely, we state that if there exists a Lyapunov function $W(x)$ such that

$$\tilde{\alpha}(x)\cdot W^{'}_x + 1/2\ \tilde{\beta}(x)\cdot W^{''}_{xx} \leq -\ \gamma W(x),$$

for some $\gamma > 0,\ \forall x \in R$, then process $X^{\epsilon}_{\nu(t/\epsilon^2)}$ is asypmtotically stochastically stable process as $0 \leq \epsilon \leq \epsilon_0, \epsilon$ is fixed, ε_0 is a small paramater, $\tilde{\alpha}$ and $\tilde{\beta}$ are defined in (217).

Let the balance condition be fulfilled:

$$\tilde{g}(x) = 0, \quad \forall x \in R. \tag{218}$$

Then we have to apply a diffusion approximation scheme to the difference equation (6) in scale of time t/ϵ^2 :

$$\begin{cases} X^{\epsilon}_{\nu(t/\epsilon^2)+1} - X^{\epsilon}_{\nu(t/\epsilon^2)} = \ \epsilon g(X^{\epsilon}_{\nu(t/\epsilon^2)}, y_{\nu(t/\epsilon)+1}) \\ X^{\epsilon}_0 = X_0 = x. \end{cases} \tag{219}$$

where all the items in (219) are defined in Section 4.6.1.

We suppose that the following conditions be satisfied: condition (i) from Section 4.6.1;

$$(ii)^{'} \qquad \int_Y p(dy)|g(x,y)|^3 < +\infty; \quad \int_Y p(dy)|g^{'}_x(x,y)|^2 < +\infty, \quad \forall x \in R; \tag{220}$$

$(iii)^{'}$ $g^{'}_x$and $g^{''}_{xx}(x,y)$ are bounded and continuous;
$(iv)^{'}$ the third moment $m_3(y) := \int_0^\infty t^3 G_y(dt)$ is uniformly integrable.

Under conditions (218) and (i), $(ii)^{'} - (iv)^{'})$ in (220) the process $X^\epsilon_{\nu(t/\epsilon^2)}$ in (219) converges weakly as $\epsilon \to o$ to the diffusion process $\tilde{x}(t)$ such that

$$d\tilde{x}(t) = \tilde{\alpha}(\tilde{x}(t)dt + \tilde{\beta}(\tilde{x}(t))dw(t), \tag{221}$$

where drift and diffusion coefficients $\alpha(x)$ and $\tilde{\beta}(x)$ are defined in (217),

$$m := \int_Y p(dy)m(y), m(y) := \int_0^\infty tG_y(dt),$$

and $w(t)$ is a standard Wiener process.

In the next Theorem 8 we study the stability property of difference equation (219) under the stability conditions of the equation (221).

Theorem 8. *Let the conditions (218) and* $(i), (ii)^{'} - (iv)^{'}$ *in (220) and the following conditions be satisfied:*

(a) there exists a smooth function (polynomial like) on R such that: $W(x) \to +\infty$ as $|x| \to +\infty, W(x)$ is positively-defined; $W(x) = 0 \Rightarrow x = 0$;

(b) $g(0,y) = 0, \forall y \in Y$;

(c) $$\tilde{\alpha}(x)W^{'}_x(x) + 1/2\tilde{\beta}^2(x)W^{''}_{xx}(x) \le -\gamma W(x), \ \gamma > 0. \tag{222}$$

Then process $X^\epsilon_{\nu(t/\epsilon^2)}$ in (219) is stochastically exponentially stable. Moreover, it is asymptotically stochastically stable.

Proof. Let's consider the process $(X^\epsilon_{\nu(t/\epsilon^2)}, y^\epsilon_{\nu(t/\epsilon^2)}, \gamma_{(t/\epsilon^2)})$ on $R \times Y \times R_+$. It's Markov process with infinitesimal operator

$$L^\epsilon f(t,x,y) = \frac{1}{\epsilon^2}Qf(t,x,y) + \frac{1}{\epsilon^2}P[f(t,x+\epsilon g(x,y),y) - f(t,x,y)], \tag{223}$$

where Q and P are defined in (195)-(196), $f \in \mathbf{C}^1(R_+) \times \mathbf{C}(R) \times \mathbf{C}(Y)$.

Let us introduce the following family of functions:

$$W^\epsilon(t,x,y) = W(x) + \epsilon W_1(t,x,y) + \epsilon^2 W_2(t,x,y), \tag{224}$$

where V_1 is defined by the solution of the equation:

$$QW_1 + Pg(x,y)W^{'}_x(x) = 0, \tag{225}$$

and V_2 is defined by the solution of the equation:

$$QW_2 + Pg(x,y)dW_1(x)/dx + 1/2Pg^2(x,y)W(x) - LW(x) = 0, \tag{226}$$

where

$$\bar{L} := \tilde{\alpha}(x)\frac{d}{dx} + 1/2\tilde{\beta}^2(x)\frac{d^2}{dx^2}, \tag{227}$$

$W(x)$ is defined in a). It follows from the equations (225)-(226) that

$$L^\epsilon W^\epsilon = LW(x) + 0(\epsilon)W(x), \tag{228}$$

where $||0(\epsilon)W(x)|| \to 0$, as $\epsilon \to 0$, and L is defined in (227).

Let's define the following process:

$$m^\epsilon(t) := W^\epsilon(\gamma(t/\epsilon^2)), X^\epsilon_{\nu(t/\epsilon^2)}, y_{\nu(t/\epsilon^2)}) - W^\epsilon(0,x,y)$$

$$- \int_0^t L^\epsilon W^\epsilon(\gamma(s/\epsilon^2), X^\epsilon_{\nu(s/\epsilon^2)} y_{\nu(s/\epsilon^2)})ds. \tag{229}$$

The process $m^\epsilon(t)$ is a right-continuous integrable $\mathcal{F}^\epsilon_t$-martingale:

$$\mathcal{F}^\epsilon_t := \sigma\{\gamma_{(s/\epsilon^2)}, y_{\nu(s/e^2)}; 0 \le s \le t\}.$$

From the representations (223)-(229) it follows that the expression (229) may be written down in the form:

$$W(X^\epsilon_{\nu(t/\epsilon^2)}) - W(x) - \textstyle\int_0^t LW(X^\epsilon_{\nu(s/\epsilon^2)} ds$$

$$= m^\epsilon(t) + \epsilon \cdot W_1(0,x,y) + \epsilon^2 W_2((0,x,y) - \epsilon W_1(\gamma(s/\epsilon^2), X^\epsilon_{\nu(t/\epsilon^2)}, y_{\nu(t/\epsilon^2)}) \tag{230}$$

$$-\epsilon^2 \cdot W_2(\gamma_{(t/\epsilon^2)}, X^\epsilon_{\nu(t/\epsilon^2)}, y_{\nu(t/\epsilon^2)}) + 0(\epsilon).$$

It's also easy to see from (230) that the process $X^\epsilon_{\nu(t/\epsilon^2)}$ really approximates the diffusion process $\tilde{x}(t)$ in (221) with infinitesimal operator $\bar{L}$.

Taking into account the conditions of Theorem 2, we conclude that there exists a small number $\epsilon_0 : 0 \le \epsilon < \epsilon_0$:

$$C_1 W(x) \le W^\epsilon(t,x,y) \le C_2 W(x) \tag{231}$$

for some positive constants C_1 and C_2.

Let $\hat{\gamma} > 0$. It follows from (228) that

$$(L^\epsilon + \hat{\gamma})W^\epsilon = \hat{\gamma} \cdot W^\epsilon + LW + 0(\epsilon). \tag{232}$$

Under conditions of Theorem 2 and using the inequality (231) we obtain from (232):

$$(L^\epsilon + \hat{\gamma})W^\epsilon \le (\hat{\gamma} \cdot C_2 + L + \epsilon_0 \cdot C_3)W(x),$$

where C_3 is some positive constant. If we choose $\hat{\gamma}$ in such a way that $C_2\hat{\gamma} + \epsilon_0 \cdot C_3 \le \gamma$, where γ is defined in (222), then we have:

$$(L^\epsilon + \hat{\gamma})W^\epsilon \le 0. \tag{233}$$

Now let's rewrite (229) with $e^{\hat{\gamma}\cdot t}W^\epsilon$:

$$\begin{aligned} &e^{\hat{\gamma}\cdot t}W^\epsilon(\gamma(t/\epsilon^2), X^\epsilon_{\nu(t/\epsilon^2)}, y_{\nu(t/\epsilon^2)}) \\ &= W^\epsilon(0,x,y) + \textstyle\int_0^t e^{\hat{\gamma}\cdot s}(L^\epsilon + \hat{\gamma})W^\epsilon ds + \tilde{m}^\epsilon(t), \end{aligned} \tag{234}$$

where $\tilde{m}^\epsilon(t)$ is a right-continuous integrable $\mathcal{F}_t^\epsilon$ - martingale with zero mean.

Taking into account (231) and (233) we obtain from (224) and (234):

$$O \leq C_1 \cdot e^{\hat{\gamma}\cdot t} W(x^\epsilon_{\nu(t/\epsilon^2)}) \leq e^{\hat{\gamma}\cdot t} W^\epsilon \leq C_2 \cdot W(x) + \tilde{m}^\epsilon(t). \tag{235}$$

The inequality (235) means that $C_2 \cdot W(x) + \tilde{m}^\epsilon(t)$ is a non-negative $\Im_t^\epsilon$ - martingale. From Kolmogorov-Doob inequality we obtain that for every $\tilde{N}_2 > 0$:

$$\mathbf{P}_{x,y}\{\sup_{0\leq t\leq T} C_1 e^{\hat{\gamma}\cdot t} W(X^\epsilon_{\nu(t/\epsilon^2)}) > \tilde{N}_2\} \leq \frac{C_2 \cdot W(x)}{\tilde{N}_2}, \tag{236}$$

and we have as $T \to +\infty$:

$$\mathbf{P}_{x,y}\{\sup_{0\leq t\leq +\infty} C_1 e^{\hat{\gamma}\cdot t} W(X^\epsilon_{\nu(t/\epsilon^2)}) > \tilde{N}_2\} \leq \frac{C_2 \cdot W(x)}{\tilde{N}_2}.$$

By the positive definiteness and smoothness $W(x)$, there exists constants $K_1 > 0$ and $K_2 > 0$ and positive integers p_1 and p_2 such that

$$K_1 |x|^{p_1} \leq W(x) \leq K_2 |x|^{p_2}$$

for $|x|$ small.

In such a way,

$$\{K_1 \cdot |X^\epsilon_{\nu(t/\epsilon^2)}|^{p_1} \leq e^{\hat{\gamma}\cdot t} \frac{\tilde{N}_2}{C_1}; t \geq 0\} \supset \{W(X^\epsilon_{\nu(t/\epsilon^2)}) \leq e^{\hat{\gamma}\cdot t} \frac{\tilde{N}_2}{C_1}; t \geq 0\} \tag{237}$$

and

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon^2)}| \leq e^{\bar{\gamma}\cdot t} (\frac{\tilde{N}_2}{C_1 \cdot K_1})^{1/p_1}; t \geq 0\} \geq 1 - \frac{C_2 \cdot W(x)}{\tilde{N}_2}, \tag{238}$$

where $\bar{\gamma} := \hat{\gamma}/p_1$. Let $N_1 > 0$ and $N_2 > 0$ be given.

Let take $\tilde{N}_2$ so small that (238) yields the inequality.

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon^2)}| \leq e^{\bar{\gamma}\cdot t} N_2; t \geq 0\} \geq 1 - \frac{C_2 \cdot W(x)}{\tilde{N}_2}, \tag{239}$$

namely, $\tilde{N}_2 = N_2^{p_1} \cdot C_1 \cdot K_1$, Then we take $\delta > 0$ in such a way that for $|x| < \delta$ we obtain:

$$W(x) < C_2^{-1} \cdot \tilde{N}_2 \cdot N_1, \tag{240}$$

namely, $\delta := (\frac{\tilde{N}_2 \cdot N_1}{C_2 \cdot K_2})^{1/p_2}$.

Finally, from (239)–(240) we obtain the inequality:

$$\mathbf{P}_{x,y}\{|X^\epsilon_{\nu(t/\epsilon^2)}| \leq e^{\bar{\gamma}\cdot t} N_2; t \geq 0\} \geq 1 - N_1,$$

and stochastic exponential stability is proved. (See Definition 1).

To prove asymptotic stochastic stability (see Definition 2), we note that

$$\{\lim_{t\to+\infty} |X^\epsilon_{\nu(t/\epsilon^2)}| = 0\} \supset \{\lim_{t\to+\infty} W(X^\epsilon_{\nu(t/\epsilon^2)}) = 0\} \supset$$

$$\supset \{\sup_{t\geq 0} C_1 \cdot e^{\bar{\gamma}\cdot t} W(X^\epsilon_{\nu(t/\epsilon^2)}) \leq C\}, \tag{241}$$

where C is a same positive constant.

From (236) and (241) we obtain:

$$\mathbf{P}_{x,y}\{\lim_{t\to+\infty}|X^{\epsilon}_{\nu(t/\epsilon^2)}|=0\}\geq \mathbf{P}_{x,y}\{\sup_{t\geq 0} C_1\cdot e^{\bar{\gamma}\cdot t}W(X^{\epsilon}_{\nu(t/\epsilon^2)})\leq C\}\geq$$

$$\geq 1-\frac{C_2W(x)}{C}.$$

And, finally, we obtain as $C\to+\infty$ that

$$\mathbf{P}_{x,y}\{\lim_{t\to+\infty}|X^{\epsilon}_{\nu(t/\epsilon^2)}|=0\}=1$$

and asymptotic stochastical stability is proved and Theorem 2 is also completely proved. $\triangle$

Remark 6. Stochastic stability of stochastic differential equations ((SDE) has been studied in [15], including asymptotic and global stability. Asymptotic stability of linear stochastic systems was studied in [14]. Asymptotic stability of SDE with jumps has been studied in [13, p.325]. Asymptotic stochastic stability of stochastic systems with wide-band noise disturbances using martingale approach was studied in [16].

7. APPLICATIONS TO SOME BIOLOGICAL SYSTEMS

In this section we apply averaging, diffusion approximation, normal deviations and merging theorems from Sections 1-6 to some biological population systems in semi-Markov random environment, namely, logistic growth model and branching process in semi-Markov random environment.

Logistic growth model (LGM) and branching processes (BP) in semi-Markov random media (RM)

7.1.1. LGM

Let $N(t)$ be the population of the species at time t. Verhulst (1836) proposed that a self-limiting process should operate when a population becomes too large. He suggested

$$\frac{dN(t)}{dt}=rN(1-N/K), \tag{242}$$

where r and K are positive constants [4]. This is called logistic growth in a population. In this model the per capita birth rate is $r(1-N/K)$, that is, it's dependent on N. The constant K is the carrying capacity of the environment.

Let r and K depend on semi-Markov process: $r\equiv r(y_{\nu(t)}), K\equiv K(y_{\nu(t)})$.

We will consider the equation (242) in semi-Markov random environment in series scheme.

7.1.2. Branching process

The generating function $\Phi(t)$ of a homogeneous Markov branching process with a single type of particle in semi-Markov random environment $y_{\nu(t/\epsilon)}$ is defined as a solution of the Cauchy problem [6]:

$$d\Phi(t)/dt=g(\Phi(t),y_{\nu(t)}),\Phi(0)=u, \tag{243}$$

where

$$g(u,y) := a(y)[b(u,y) - u],$$

$$b(u,y) := \sum_{k=0}^{+\infty} u^k p_k(y), |u| \le 1, \tag{244}$$

where $(a(y))_{y\in Y}$ are intensities of lifetimes of particles, $p_k(y)$ are probability distributions of the number of direct descendants. Functions $a(y)$ and $p_k(y)$ are bounded and measurable by y.

Function $g(u,y)$ satisfies all the conditions as function $g(x,y)$.

7.2. Averaging and diffusion approximation of LGM and BP in random media

Let $N^\epsilon(t)$ be the solution of the following difference equation:

$$N^\epsilon_{\nu(t/\epsilon)+1} - N_{\nu(t/\epsilon)} = \epsilon r(y_{\nu(t/\epsilon)+1})N^\epsilon(1 - N^\epsilon/K(y^\epsilon_{\nu(t/\epsilon)+1})), \tag{245}$$

where $r(y)$ and $K(y)$ are positive bounded measurable functions on Y.

The equation (245) is the same as (75) with function

$$g(x,y) = r(y)x(1 - x/K(y)). \tag{246}$$

This function satisfies all the conditions of Theorem 2. Applying the Theorem 2 to the solution $N^\epsilon_{\nu(t/\epsilon)}$ of the equation (245) we obtain the following result.

Averaging of LGM

Process $N^\epsilon_{\nu(t/\epsilon)}$ in (69) converges weakly as $\epsilon \to o$ to the process $\tilde N_t$ such that:

$$\begin{cases} \frac{d\tilde N_t}{dt} = \int_Y p(dy)\frac{r(y)}{K(y)}\tilde N_t \cdot (K(y) - \tilde N_t)/m :=:= \tilde R(\tilde N_t), \\ \tilde N_0 = N_0 \end{cases} \tag{247}$$

Diffusion approximation of LGM

Let

$$\tilde N_0 = N_0 = \frac{\int_Y p(dy)r(y)}{\int_y p(dy)\frac{r(y)}{K(y)}} \tag{248}$$

The condition (72) is a balance condition for LGM.

Consider the following difference equation:

$$N^\epsilon_{\nu(t/\epsilon^2+1} - N^\epsilon_{\nu(t/\epsilon^2} = \epsilon r(y_{\nu(t/\epsilon^2+1})N^\epsilon_{\nu(t/\epsilon^2}(1 - \frac{N^\epsilon_{\nu(t/\epsilon^2}}{K(y_{\nu(t/\epsilon^2+1})}) \tag{249}$$

The function $g(x,y)$ in (246) satisfies all the conditions of Theorem 4 with (248) as a balance condition. Applying the Theorem 4 to the solution $N^\epsilon_{\nu(t/\epsilon)^2}$ of the equation (249) we obtain the following result.

Process $N^\epsilon_{\nu(t/\epsilon^2}$ in (247) converges weakly as $\epsilon \to o$ to the diffusion process $\tilde N(t)$:

$$d\tilde N(t) = \tilde\alpha(\tilde N(t))dt + \beta(\tilde N(t))dw(t),$$

where

$$\begin{aligned}\tilde{\alpha}(u): \quad &= \int_Y p(dy)[(r(y)\mathbf{R}_0 r(y))u- \\ &-(r(y)\mathbf{R}_0\tfrac{r(y)}{K(y)}+\tfrac{r(y)}{K(y)}\mathbf{R}_0 r(y))u^2+ \\ &+(\tfrac{r(y)}{K(y)}\mathbf{R}_0\tfrac{r(y)}{K(y)})u^3]/m,\end{aligned}$$

$$\begin{aligned}\tilde{\beta}^2(u): \quad &= 2\int_Y p(dy)[(r(y)\mathbf{R}_0 r(y)+1/2\tfrac{r^2(y)}{K^2(y)})U^2+ \\ &+(r(y)\mathbf{R}_0\tfrac{r(y)}{K(y)}+\tfrac{r(y)}{K(y)}\mathbf{R}_0 r(y)+\tfrac{r(y)}{K(y)}u^3+ \\ &+\tfrac{r(y)}{K(y)}\mathbf{R}_0\tfrac{r(y)}{K(y)}+1/2\tfrac{r^2(y)}{K^2(y)})u^4]/m.\end{aligned}$$

Averaging of BP in RM

Let us consider the following difference equation:

$$\Phi^\epsilon_{\nu(t/\epsilon)+1}-\Phi^\epsilon_{\nu(t/\epsilon)}=\epsilon g(\Phi^\epsilon_{\nu(t/\epsilon)},y_{\nu(t/\epsilon)+1})^\epsilon,\Phi^\epsilon_0=u. \tag{250}$$

This equation is the same as (93) with function $g(u,y)$ in place of $g(x,y)$.

Applying the Theorem 2 to the function $\Phi^\epsilon_{\nu(t/\epsilon)}$ we obtain the following result.

Process $\Phi^\epsilon_{\nu(t/\epsilon)}$ converges weakly as $\epsilon\to o$ to the process $\tilde{\Phi}_t$ such that:

$$\frac{d\tilde{\Phi}_t}{dt}=\tilde{g}(\tilde{\Phi}_t),\tilde{\Phi}_0=u,$$

where

$$\tilde{g}(u):=\int_Y p(dy)g(u,y)/m:=\int_Y p(dy)a(y)[b(u,y)-u]/m. \tag{251}$$

Diffusion approximation of BP in RM

Let

$$\tilde{g}(u)=0,\forall u, \tag{252}$$

where $\tilde{g}(u)$ is defined in (251).

Condition (252) is a balance condition for function $g(u,y)$.

We are under conditions of Theorem 4.

Let us consider the following difference equation:

$$\Phi^\epsilon_{\nu(t/\epsilon^2)+1}-\Phi^\epsilon_{\nu(t/\epsilon^2)}=\epsilon g(\Phi^\epsilon_{\nu(t/\epsilon^2)},y_{\nu(t/\epsilon^2)+1}),\Phi^\epsilon_0=u. \tag{253}$$

Applying the Theorem 4 to the process $\Phi^\epsilon_{\nu(t/\epsilon^2)}$ we obtain the following result.

Process $\Phi^\epsilon_{\nu(t/\epsilon^2)}$ converges weakly as $\epsilon\to 0$ to the diffusion process $\tilde{\Phi}(t)$:

$$d\tilde{\Phi}(t)=\tilde{\alpha}(\tilde{\Phi}(t))dt+\tilde{\beta}(\tilde{\Phi}(t))dw(t),$$

where

$$\tilde{\alpha}(u):=\int_Y p(dy)[g(u,y)\mathbf{R}_0 g'_u(u,y)]/m,$$

$$\tilde{\beta}^2(u):=2\int_Y p(dy)[g(u,y)\mathbf{R}_0 g(u,y)+1/2g^2(u,y)]/m,$$

and $w(t)$ is a standard Wiener process.

7.3. Applications of normal deviations to some biological systems in random media

7.3.1. Normal Deviations of LGM in Semi-Markov RM

Let $N^\varepsilon_{\nu(t/\varepsilon)}$ be the solution of the following difference equation:

$$N^\varepsilon_{\nu(t/\varepsilon)+1} - N^\varepsilon_{\nu(t/\varepsilon)} = \varepsilon \cdot r(y^\varepsilon_{\nu(t/\varepsilon)})N^\varepsilon \cdot (1 - N^\varepsilon / K(y_{\nu(t/\varepsilon)+1})), \tag{254}$$

where $r(y)$ and $K(y)$ are positive measurable bounded function on Y. Here $N^\varepsilon_{\nu(t/\varepsilon)}$ is the population of the species at time t/ε in random environment $\mathcal{Y}$[3].

In Section 7.2 we have proved that under some conditions for function $g(x,y) == r(y)x(1 - x/K(y))$ process $N^\varepsilon_{\nu(t/\varepsilon)}$ converges weakly as $\varepsilon \to 0$ to the process $\tilde{N}_t$ such that:

$$\begin{cases} \frac{d\tilde{N}_t}{dt} = \int_Y p(dy)\frac{r(y)}{K(y)}\tilde{N}_t(K(y) - \tilde{N}_t)/m := \tilde{R}(\tilde{N}_t) \\ \tilde{N}_0 = N_0. \end{cases} \tag{255}$$

Let

$$Z^\varepsilon_n = [N^\varepsilon_n - \tilde{N}^\varepsilon_n]/\sqrt{\varepsilon}, \tag{256}$$

where $\tilde{N}^\varepsilon_n$ is defined in (254) and $\tilde{N}^\varepsilon_n$ is defined from the following equation:

$$\tilde{N}^\varepsilon_{u+1} - \tilde{N}^\varepsilon_n = \varepsilon \tilde{R}(\tilde{N}^\varepsilon_n),$$

and $\tilde{R}(u)$ being defined in (255).

Also,

$$Z^\varepsilon(t) = \sum_{k=1}^\infty Z^\varepsilon_n \mathbf{1}\{\tau_k \le t/\varepsilon < \tau_{k+1}\} \tag{257}$$

Then from Theorem 7 it follows that $Z^\varepsilon(t)$ in (257) converges weakly as $\varepsilon \to 0$ to the process $\tilde{Z}_t$;

$$\tilde{Z}_t = \int_0^t \int_Y p(dy)\frac{r(y)}{K(y)}(K(y) - 2\tilde{N}_s)\tilde{z}_s\ ds + \int_0^t \sigma(\tilde{N}_s)dws,$$

where

$$\sigma^2(u) := \int_Y p(dy)[(R(u,y) - \tilde{R}(u,y))\mathbf{R}_0(R(u,y) - \tilde{R}(u)) + (R(u,y) - \tilde{R}(u))^2/2]/m,$$

where

$$R(u,y) := \frac{r(y)}{K(y)}u(K(y) - u).$$

7.3.2. Normal Deviations of BP in RM

Let us consider a generating function $\Phi^\varepsilon(t)$ of a homogeneous Markov branching process with a single type of particles in the semi-Markov random environment $y_{\nu(t/\varepsilon)}$, which satisfies the following difference equation (see Section 7.1):

$$\Phi^\varepsilon_{\nu(t/\varepsilon)+1} - \Phi^\varepsilon_{\nu(t/\varepsilon)} = \varepsilon g(\Phi^\varepsilon_{\nu(t/\varepsilon)}, y_{\nu(t/\varepsilon)+1}), \Phi^\varepsilon_0 = u. \tag{258}$$

Here:

$$g(u,y) := a(y)[b(u,y) - u],$$

$$b(u,y) := \sum_{k=0}^{+\infty} u^k p_k(y), |u| \le 1, \tag{259}$$

$b(u,y)$ is the generation function of the process [4]. In Section 7.1 we have proved that under some conditions for function $g(u,y)$ process $\Phi^{\varepsilon}_{\nu(t/\varepsilon)}$ converges weakly as $\varepsilon \to 0$ to the process $\tilde{\Phi}_t$ such that

$$\frac{d\tilde{\Phi}_t}{dt} = \tilde{g}(\tilde{\Phi}_t), \tilde{\Phi}_0 = u, \tag{260}$$

where

$$\tilde{b}(u) := \int_Y p(dy)g(u,y)/m. \tag{261}$$

Let

$$Z^{\varepsilon}_n := [\Phi^{\varepsilon}_n - \tilde{\Phi}^{\varepsilon}_n]/\sqrt{\varepsilon},$$

where Φ^{ε}_n is defined in (258) and $\tilde{\Phi}^{\varepsilon}_n$ is defined from the following relation:

$$\tilde{\Phi}^{\varepsilon}_{n+1} - \tilde{\Phi}^{\varepsilon}_n = \varepsilon \tilde{g}(\tilde{\Phi}^{\varepsilon}_n),$$

$\tilde{g}(u)$ being defined in (261).

Also, let

$$z^{\varepsilon}(t) := \sum_{n=1}^{+\infty} Z^{\varepsilon}_n \cdot \mathbf{1}\{\tau_n \le t/\varepsilon < \tau_{n+1}\}.$$

Then from Theorem 5 it follows that $z^{\varepsilon}(t)$ converges weakly as $\varepsilon \to 0$ to the process $\tilde{z}_t$:

$$\tilde{Z}_t = \int_0^t \tilde{b}'_x(\tilde{\Phi}_s)\tilde{Z}_s ds + \int_0^t \sigma(\tilde{\Phi}_s) dws,$$

where

$$\sigma^2(u) := \int_Y p(dy)[(b(u,y) - \tilde{b}(u))R_0(b(u,y) - \tilde{f}(u)) + \\ +(b(u,y) - \tilde{b}(u))^2/2]/m.$$

7.4. Merging of biological systems in random media

7.4.1. Merging of LGM in RM

Let us define the following process (see (254)):

$$N^{\varepsilon}_{\nu(t/\varepsilon)+1} - N^{\varepsilon}_{\nu(t/\varepsilon)} = \varepsilon \cdot r(y_{(\nu(t/\varepsilon)+1})^{\varepsilon} N^{\varepsilon} \cdot (1 - N^{\varepsilon}/K(y^{\varepsilon}_{\nu(t/\varepsilon)+1})), \tag{262}$$

where $(y^{\varepsilon}_n; n \ge 0)$ is a perturbed Markov chain with transition probabilities $P_{\varepsilon}(y,B)$ in (165).

Let the conditions of Theorem 6 be satisfied with function $g(x,y) = r(y)x(1-x/k(y))$ and let the conditions (168)-(172) be satisfied.

Then process $N^{\varepsilon}_{\nu(t/\varepsilon)}$ in (262) converges weakly as $\varepsilon \to 0$ to the process $\hat{N}(t)$ which satisfies the equation:

$$\begin{cases} d\hat{N}/dt = \hat{R}(\hat{N}(t), \hat{y}(t)) \\ \hat{N}(0) = N_0, \end{cases}$$

where

$$\hat{R}(u,v) := \int_{Y_v} p_v(dy)\frac{r(y)}{K(y)}u \cdot (K(y) - u)/m_v,$$

$\hat{y}(t)$ is a merged Markov process.

7.4.2. Merging of BP in RM

Let us define the following process (see (250)):

$$\Phi^{\varepsilon}_{\nu(t/\varepsilon)+1} - \Phi^{\varepsilon}_{\nu(t/\varepsilon)} = \varepsilon g(\Phi^{\varepsilon}_{(\nu(t/\varepsilon)}, y^{\varepsilon}_{\nu(t/\varepsilon)+1}), \tag{263}$$

where $(y^{\varepsilon}_n; n \geq 0)$ is a perturbed Markov chain with transition probabilities $P_{\varepsilon}(y, B)$ in (165).

Let the conditions of Theorem 6 be satisfied with function $g(u, y)$, and let the conditions (168)-(172) be satisfied.

The process $\Phi^{\varepsilon}_{\nu(t/\varepsilon)}$ in (263) converges weakly as $\varepsilon \to 0$ to the process $\hat{\Phi}(t)$ which satisfies the equation:

$$\begin{cases} d\hat{\Phi}(t) = \hat{g}(\hat{\Phi}(t), \hat{y}(t)) \\ \hat{\Phi}(0) = u, \end{cases}$$

where

$$\hat{g}(u,v) := \int_{Y_v} p_v(dy) g(u,y)/m_v,$$

$\hat{y}(t)$ is a merged Markov process.

7.5. Application of stability theorems to LGM in RM

Let $N(t)$ be the population of the species at time t. Verhulst (1836) [8] proposed that a self-limiting process should operate when a population becomes too large. He suggested that

$$\frac{dN(t)}{dt} = rN(t) \cdot (1 - N(t)/K), \tag{264}$$

where r and K are positive constants. This is called logistic growth in a population. In this model the per capita birth rate is $r \cdot (1 - N/K)$, that is, it's dependent upon N. The constant K is the carrying capacity of the environment.

We suggest that r and K depend on a semi-Markov process $y_{\nu(t)} : r \equiv r(y_{\nu(t)})$ and $K \equiv K(y_{\nu(t)})$.

In this way we consider the equation (264) in semi-Markov random environment:

$$\frac{dN(t)}{dt} = r(y_{\nu(t)})N(t) \cdot (1 - \frac{N(t)}{K(y_{\nu(t)})}). \tag{265}$$

7.5.1. Stability of LGM in Averaging Scheme

Let's consider LGM (265) in series scheme in the form of difference equation:

$$N^{\epsilon}_{\nu(t/\epsilon)+1} - N_{\nu(t/\epsilon)} = \epsilon r(y_{\nu(t/\epsilon)+1})N^{\epsilon} \cdot (1 - \frac{N^{\epsilon}}{K(y_{\nu(t/\epsilon)})}), \tag{266}$$

where $r(y)$ and $K(y)$ are positive bounded measurable function on Y.

In Section 6.1 we have stated that under averaging conditions the process $N^{\epsilon}_{\nu(t/\epsilon)}$ in (266) converges weakly as $\epsilon \to o$ to the averaged process $\tilde{N}_t$ such that:

$$\frac{d\tilde{N}_t}{dt} = \tilde{r} \cdot \tilde{N}_t \cdot (1 - \tilde{N}_t/\tilde{K}), \tag{267}$$

where

$$\tilde{r} := \int_Y p(dy)r(y)/m,$$

$$\tilde{K} := \left(\int_Y p(dy)\frac{r(y)}{K(y)}/m \right)^{-1}. \tag{268}$$

Let us study the stability of the average model in (267)-(268).

There are two standard steady states or equilibrium states for (267), namely $\tilde{N} = O$ and $\tilde{N} = \tilde{K}$, that is where $\frac{d\tilde{N}_t}{dt} = O$. $\tilde{N} = O$ is unstable since linearization about it (that is $\tilde{N}^2$ is neglected compared with $\tilde{N}$) gives $\frac{d\tilde{N}}{dt} = \tilde{r} = \tilde{N}$, and so $\tilde{N}$ grows exponentially from any initial value. The other equilibrium k$\tilde{N} = \tilde{K}$ is stable and linearization about it (that is $(\tilde{N} - \tilde{K})^2$ is neglected compared with $|\tilde{N} - \tilde{K}|$ gives $\frac{d(\tilde{N}-\tilde{K})}{dt} \simeq -\tilde{r} \cdot (\tilde{N} - \tilde{K})$ and so $\tilde{N} \to \tilde{K}$ as $t \to +\infty$. The averaged carrying capacity $\tilde{K}$ determines the average size of the stable steady state population, while $\tilde{r}$ is an average measure of the averaged rate at which it is reached, that is, it is an average measure of the dynamics.

If $\tilde{N}_0 = N_0$, the solution of (267) is

$$\tilde{N}_t = \frac{N_0 \cdot \tilde{K} e^{\tilde{r}t}}{[\tilde{K} + N_0(e^{\tilde{r}t} - 1)]} \to \tilde{K} \text{as } t \to +\infty. \tag{269}$$

If $N_0 < \tilde{K}, \tilde{N}_t$ simply increases monotonically to $\tilde{K}$, while $N_o > \tilde{K}$ decreases monotonically to $\tilde{K}$. In the former case, there is a qualitative difference depending on whether $N_o > \tilde{K}/2$ or $N_o < \tilde{K}/2$; with $N_0 < \tilde{K}/2$ the form has a typical sigmoid character, which is commonly observed.

The previous reasonings mainly concerned with stability of averaged model (267). But we are interested in stability of difference equation (266), i.e., LGM in random environment in series scheme.

It means that we are interested in function $V(x)$, satisfying the conditions i)-iii). Let's take the function $V(x) = x^2$. It satisfies all the conditions i)-iii) with $\beta \le \frac{\tilde{r}}{\tilde{K}} \cdot N_o - \tilde{r}$. We note that $N_0 > \tilde{K}$ as β should be positive constant.

By Theorem 7 process $N^{\epsilon}_{\nu(t/\epsilon)}$ in (266) is stochastically asymptotically stable one, namely,

$$\mathbf{P}_{N_0,y}\{\lim_{t\to+\infty} |N^{\epsilon}_{\nu(t/\epsilon)}| = \tilde{K}\} = 1.$$

7.5.2. Stability of LGM in Diffusion Approximation Scheme

As we have seen in section 7.5.1 the equation (267) has two steady states: $\tilde{N} = O$ and $\tilde{N} = \tilde{K}$. Under these conditions the solution of (267) behaves deterministically: grows, exponentially from any initial value or tends to $\tilde{K}$ as $t \to +\infty$, respectively.

There is the third case, where the solution of (266) behaves stochastically as some diffusion process. It is

$$\tilde{N} = \tilde{K}, \tag{270}$$

but we don't neglect the term $(\tilde{N}-\tilde{K})^2$.

Condition (270) is a balance condition and we can apply diffusion approximation scheme for the following difference equation:

$$N^{\epsilon}_{\nu(t/\epsilon^2)+1} - N^{\epsilon}_{\nu(t/\epsilon^2)} = \epsilon r(y_{\nu(t/\epsilon^2)}) \cdot N^{\epsilon} \cdot (1 - N^{\epsilon}_{\nu(t/\epsilon^2)}/K(y_{\nu(t/\epsilon^2)}) \tag{271}$$

The function $g(x,y) = r(y)\cdot X(1-x/K(y))$ satisfies all the conditions of Theorem 8. Process $N^{\epsilon}_{\nu(t/\epsilon^2)}$ converges weakly as $\epsilon \to o$ to the diffusion process $\tilde{N}(t)$:

$$d\tilde{N}(t) = \tilde{\alpha}(\tilde{N}(t))dt + \tilde{\beta}(\tilde{N}(t))dw(t), \tag{272}$$

where

$$\tilde{\alpha}(x) := \tilde{\alpha}_1 x - \tilde{\alpha}_2 x^2 + \tilde{\alpha}_3 x^3, \tag{273}$$

$$\tilde{\alpha}_1 := \int_y p(dy)(r(y)\mathbf{R}_0 r(y))/m,$$

$$\tilde{\alpha}_2 := \int_y p(dy)(r(y)\mathbf{R}_0 \frac{r(y)}{K(y)} + \frac{r(y)}{K(y)}\mathbf{R}_0 \cdot r(y))/m,$$

$$\tilde{\alpha}_3 := \int_y p(dy)(2\frac{r(y)}{K(y)}\mathbf{R}_0 \cdot \frac{r(y)}{K(y)})/m,$$

and

$$\tilde{\beta}^2(x) := 2(\tilde{\beta}_1 x^2 + \tilde{\beta}_2 x^3 + \tilde{\beta}_3 x^4) \tag{274}$$

$$\tilde{\beta}_1 := \int_y p(dy)(r(y)\mathbf{R}_0 r(y) + 1/2\frac{r^2(y)}{K^2(y)})/m,$$

$$\tilde{\beta}_2 := \int_y p(dy)(r(y)\cdot\mathbf{R}_0 \frac{r(y)}{K(y)} + \frac{r(y)}{K(y)}\mathbf{R}_0 \cdot r(y) + \frac{r(y)}{K(y)})/m,$$

$$\tilde{\beta}_3 := \int_y p(dy)(\frac{r(y)}{K(y)}\mathbf{R}_0 \frac{r(y)}{K(y)} + 1/2\frac{r^2(y)}{K^2(y)})/m,$$

$w(t)$ is a standard Wiener process.

Let us take the function $W(x) = X^2$. Then condition (222) takes the following form:

$$\tilde{N}^2 \cdot [(2\tilde{\alpha}_1 + 2\tilde{\beta}_1 + \gamma) + (2\tilde{\beta}_2 - 2\tilde{\alpha}_2)\tilde{N} + (2\tilde{\alpha}_3 + 2\tilde{\beta}_3)\tilde{N}^2] \le 0,$$

or

$$(\tilde{\alpha}_1 + \tilde{\beta}_1 + \gamma/2) + (\tilde{\beta}_2 - \tilde{\alpha}_2)\tilde{N} + (\tilde{\alpha}_3 + \tilde{\beta}_3)\tilde{N}^2 \le 0. \tag{275}$$

Since $\tilde{\alpha}_3 > o$ and $\tilde{\beta}_3$, then $(\tilde{\alpha}_3 + \tilde{\beta}_3) > O$ and the inequality (275) has a solution iff

$$D := (\tilde{\beta}_2 - \tilde{\alpha}_2)^2 - 4(\tilde{\alpha}_3 + \tilde{\beta}_3)(\tilde{\alpha}_1 + \tilde{\beta}_1 + \gamma/2) > 0,$$

or

$$\gamma < \frac{(\tilde{\beta}_2 - \tilde{\alpha}_2)^2 - 4(\tilde{\alpha}_1 + \tilde{\beta}_1)(\tilde{\alpha}_3 + \tilde{\beta}_3)}{2(\tilde{\alpha}_3 + \tilde{\beta}_3)} \tag{276}$$

where $\tilde{\alpha}_i, \tilde{\beta}_i$ are defined in (273) and (274), respectively, i = 1,3.

Let $\tilde{N}_1$ and $\tilde{N}_2$ be the roots of respected quadratic equation in (275), $\tilde{N}_1 \le \tilde{N}_2$.

The inequality (275) is fulfilled if

$$\tilde{N}_1 < \tilde{N} < \tilde{N}_2. \tag{277}$$

Since $\tilde{N}$ depends on γ we have to choose γ from (276) and inequality (277).

It means, by Theorem 8, that process $X^{\epsilon}_{\nu(t/\epsilon)^2}$ in (276) is stochastically asymptotically stable one.

Acknowledgement.

The authors would like to thank very much to anonymous referee for many useful remarks and comments which improved the final version of this paper.

REFERENCES

1. Korolyuk, V.S. and Swishchuk, A.V. *Evolution of Systems in Random Media.* CRC Press, Boca Raton, FL. USA (1995).

2. Korolyuk, V.S. and Swishchuk, A.V. *Semi-Markov Random Evolutions.* Kluwer AP, Dordrecht, The Netherlands (1995).

3. Swishchuk, A.V. *Random Evolutions and Their Applications.* Kluwer AP, Dordrecht, The Netherlands (1997).

4. Hoppensteadt, F.C. *Mathematical Methods in Population Biology.* Cambridge Univ. Press, NY (1982).

5. Hoppensteadt, F.C., Salehi, H. and Skorokhod, A. Discrete time semigroup transformation with random perturbations, *J. of Dynamics Diff. Eq.* **9**, 3 (1997).

6. Harris T.H. *The Theory of Branching Processes.* Springer-Verlag, WY (1963).

7. Sobolev S. *Some applications of functional analysis in mathematical physics.* Nauka, Moscow (1988). (In Russian).

8. Swishchuk, A. and Wu, J.-H. Averaging and diffusion approximation of difference equations in random media with applications to biological systems. 2002, 23p. (submitted to the *Journal of Dynamical Systems and Differential Equations*).

9. Swishchuk, A. and Wu, J.-H. Normal deviations and merging of difference equations in random media with applications to biological systems, LIAM, Department of Mathematics and Statistics, York University, Technical Report No 4 (2002).

10. Swishchuk, A. and Wu, J.-H. Stability of difference equations in random media in averaging and diffusion approximation schemes, LIAM, Department of Mathematics and Statisitcs, York University, Technical Report No 5 (2002).

11. Feller, W. *An Introduction to Probability Theory and its Applications.* J. Wiley & Sons Inc. (1966).

12. Wu, J.-H. *Theory and applications of partial functional differential equations.* Springer-Verlag, Berlin (1996).

13. Gikhman, I. and Skorokhod, A. *Stochastic Differential Equations.* Springer-Verlag, Berlin (1872).

14. Khasminskii, R. Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory Probab. Appl.*, 12, 144–147 (1967).

15. Pinsky, M. Stochastic stability and the Dirichlet problem, *Comm. Res. Appl. Math.*, 27, 311–350 (1974).

16. Blankenship, G. and Papanicolaou, G. Stability and control of stochastic systems with wide-band noise disturbances, I.,*SIAM J. Appl. Math.*, **34**, 3, 437–476 (1978).

17. Kushner, H. *Stochastic Stability and Control.* Academic Press, NY (1967).

18. Arnold, L. *Stochastic Differential Equations: Theory and Applications.* J. Wiley & Sons Inc., 228 p. (1974).

19. Freidlin, M. I. and Wentzel, A. D. *Random Perturbations of Dynamical Systems*, Springer-Verlag, Berlin (1984).

20. Skorokhod, A., Hoppensteadt, F. and Salehi, H. *Random Perturbation Methods with Applications in Science and Engineering.* Springer (2002).