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J. Differential Equations 194 (2003) 237–286

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Stable phase-locked periodic solutions in a delay differential system

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Received May 31, 2002; revised February 18, 2003

Abstract

For a delay differential system where the nonlinearity is motivated by applications of neural networks to spatiotemporal pattern association and can be regarded as a perturbation of a step function, we obtain the existence, stability and limiting profile of a phase-locked periodic solution using an approach very much similar to the asymptotic expansion of inner and outer layers in the analytic method of singular perturbation theory.

MSC: 34K15; 34K20; 34C25

Keywords: Periodic solutions; Stability; Phase-locking; Delay; Neural networks

1. Introduction

The purpose of this paper is to develop a new approach, elementary and constructive, for the existence of stable periodic solutions of general systems of delay differential equations whose nonlinearities are motivated by additive models of neural networks.

Much has been achieved for the study of periodic solutions of delay differential equations. Among various developed methods are local/global Hopf bifurcation theory, fixed point arguments, and general geometric approaches. Each of the above methods has its own advantages and drawback: the local Hopf bifurcation (and normal formal reduction) approach provides information about the existence,

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doi:10.1016/S0022-0396(03)00187-6

asymptotic form, direction and stability of the periodic solutions but the conclusion remains valid only when the parameter is in a small neighborhood of the critical value and the obtained periodic solutions are usually of small amplitudes; the global Hopf bifurcation theory/fixed point theoretical argument may yield important information about the global continuation of a branch of periodic solutions and thus obtain the existence of periodic solutions of large amplitudes for a large range of parameter values, but this approach seems to generate little information for the stability; the general geometric approach does yield useful details of the periodic solutions including the existence, domain of attraction and limiting profiles but the nonlinearities are quite restrictive (much of the work available requires monotonicity, for example). Listing even just a reasonable portion of related references is clearly a challenging task, we refer interested readers to the two standard references by Diekmann et al. [10] and Hale and Verduyn Lunnel [12], as well as some recent work reported in [21,23,24,29].

In this paper, we develop an alternative method for the existence and stability of periodic solutions for delay differential systems, motivated by the previous success of Walther [36–38] for scalar delay differential equations with negative feedback. The approach seems to be theoretically simple and straightforward, although the detailed analysis may be complicated and is very much similar to the asymptotic expansion of inner and outer layers in the analytic method of singular perturbation theory involving slow and fast motions. This approach seems to have great potential for applications as its requirements on the nonlinearity are minimal. This relaxation on the requirement of the nonlinearity is particularly important for applications to spatiotemporal pattern storage and recognition by delayed neural networks due to the different choices of the signal function by different scientific communities and due to the presence of noise in the signal transmission.

To be more precise, we consider the following bi-directional system:

$$\dot{x}_i(t) = -\mu x_i(t) + af(x_i(t-\tau)) + b[f(x_{i-1}(t-\tau)) + f(x_{i+1}(t-\tau))], \quad i\,(\mathrm{mod})\,3 \tag{1.1}$$

as a special case of the general Hopfield network of neurons [17] describing the computational performance of a network of neurons, where the positive constant τ was added by Marcus and Westervelt [26] to account for the finite switching speed of neurons and for the finite propagation velocity of signals, $\mu > 0$ is the internal decay rate, (a, b) describes the strength and characteristics of the self-feedback and neighborhood interaction, and $f : R \to R$ is the so-called signal function. Commonly used signal functions include step functions (describing the on-or-off characteristics in the McCullon–Pitts model, see [27]), piecewise linear functions (widely used in cellular neural networks, see [7,8]), and hyperbolic tangent and other sigmoid functions (popular functions used in the literature with some physiological and biological justifications, see [19,22,30]). The existence of noise adds further complication (see [41]) and it is thus essential to know whether the qualitative behaviours under investigation are independent of the choice of the signal function.

We should also mention that in all of the aforementioned cases, the signal functions do approach the step function when the neural gain $(f'(0))$ is large.

It is a simple corollary of a general convergence theorem due to Cohen and Grossberg [9] and Hopfield [17] that, for a wide variety of signal functions, all solutions of system (1.1) with instantaneous feedback $(\tau = 0)$ are convergent to the set of equilibria. This convergence result holds because the synaptic connection is symmetric, and this convergence is essential for the network's application to associative memory and optimization. It is also well-known now that (see, for example, [2,3,25,26,40,41]):

(i) in electronic implementations of analog neural networks, time delays are present in the communication and response of neurons due to the finite switching speed of amplifiers (neurons);
(ii) designing a network to operate more quickly increases the relative size of the intrinsic delay; and
(iii) the interaction of this intrinsic delay with the neuron gain $(f'(0))$ and the size and connection topology of the network has significant impact on the existence of oscillatory modes in continuous-time analog neural networks.

While this may impose significant challenging for circuit designers wishing to build fast analog electronic networks, since the maximal operating speed of an electronic network will be limited by the onset of delayed-induced instability [26], we also note that Herz [14] argued that time delay, omnipresent in the brain, does not induce a loss of the associative capabilities of neural networks as one might fear. On the contrary, if properly included in the learning process, they provide a physical structure to perform spatiotemporal computation at low architecture cost. See also [11,13,15,16,20,33,35] for applications of delayed neural networks to spatiotemporal association.

Our work here is related to the application of delayed neural networks to spatiotemporal pattern storage and retrieval, where it is important to know when the network does have periodic solutions, whether these periodic solutions are stable and what the patterns of these periodic solutions and their domains of attraction are. To answer the aforementioned questions, we first note that system (1.1) has some special solutions which are described by certain sub-systems. For example, the so-called synchronized solutions, those satisfying $x_1 = x_2 = x_3$, are clearly described by the scalar delay differential equation

$$\dot{x}_1(t) = -\mu x_1(t) + (a + 2b)f(x_1(t - \tau))$$

whose global dynamics is one of lasting interests, and the mirror-reflecting symmetric solutions satisfying $x_2 = x_3$ are characterized by the system of two coupled delay differential equations

$$\begin{cases} \dot{x}_1(t) = -\mu x_1(t) + af(x_1(t-\tau)) + 2bf(x_2(t-\tau)), \\ \dot{x}_2(t) = -\mu x_2(t) + bf(x_1(t-\tau)) + (a+b)f(x_2(t-\tau)), \end{cases}$$

some special forms of the above sub-system have been recently investigated in [1,4–6,31,34]. It seems however that the dynamics of the full system (1.1) is much richer than that of a scalar equation or a system of two coupled equations. For example, the work of Wu et al. [42] shows the coexistence of 8 branches of periodic solutions simultaneously bifurcated from an equilibrium, among which two (phased-locked periodic solutions) can be stable. Their work is based on the local Hopf bifurcation theory and normal form calculations, and thus the obtained stable periodic solutions are of small amplitudes. In the recent work of Huang and Wu [18], the global symmetric Hopf bifurcation theory developed in [40] was applied to investigate the global continuations of the above 8 branches and to obtain the global existence of periodic solutions of large amplitudes. Unfortunately, the approach in [18] does not yield any information about the stability properties of the periodic solutions.

In this paper, we are constructing stable phase-locked periodic solutions from a completely different point of view. Namely, we first start with the explicit construction of a phase-locked periodic solution for system (1.1) with step signal function given by

$$f(x) = \begin{cases} 1 & \text{if } x>0, \\ -1 & \text{if } x<0, \end{cases} \tag{1.2}$$

and secondly we try to construct a completely continuous returning map defined in a convex closed set near the constructed phase-locked periodic orbit and then we show stable fixed points of the returning map exist and give rise to stable phase-locked periodic orbits for system (1.1) with a general nonlinear signal function sufficiently close to (1.2). The periodic solutions obtained in this way are stable and with large amplitudes and thus can be easily observed in numerical simulations. This approach also has great advantage in applications as the nonlinearity can be a very general Lipschitz map close to the step function (1.2).

We can now describe our main results in details. First of all, we assume that $a<0$ and $b>0$, and thus the network has the feature of inhibitory self-feedback and excitatory interaction. System (1.1) seems to be the smallest size of a network with the aforementioned structure, and it is hoped that our detailed study of system (1.1) can shed some light on the global dynamics of a general large scale network of neurons (see [28]).

Letting $\hat{t}=\frac{t}{\tau}$, $\hat{\tau}=\mu\tau$, $\hat{a}=\frac{a}{\mu}$, $\hat{b}=\frac{b}{\mu}$ and then dropping the hat, we can rewrite (1.1) as

$$\begin{cases} \dot{x}_1(t) = -\tau x_1(t) + a\tau f(x_1(t-1)) + b\tau[f(x_3(t-1)) + f(x_2(t-1))], \\ \dot{x}_2(t) = -\tau x_2(t) + a\tau f(x_2(t-1)) + b\tau[f(x_1(t-1)) + f(x_3(t-1))], \\ \dot{x}_3(t) = -\tau x_3(t) + a\tau f(x_3(t-1)) + b\tau[f(x_2(t-1)) + f(x_1(t-1))]. \end{cases} \tag{1.3}$$

Let

$$\Sigma = \{\Phi = (\phi_1, \phi_2, \phi_3);\ \phi_i \in C([-1,0]),\ i=1,2,3\}$$

be equipped with the usual super-norm $\|\cdot\|$. For any given $\Phi\in\Sigma$ and $t\geqslant -1$, define $X(t,\Phi)=(x_1(t,\Phi),x_2(t,\Phi),x_2(t,\Phi))$ as the solution of (1.3) such that $x_i(t,\Phi)=\phi_i(t)$ for $t\in[-1,0]$ and $i=1,2,3$. Furthermore, for any given $t\geqslant 0$, we define the mapping $X_t:\Sigma\to\Sigma$ by

$$X_t(\Phi)(\theta) := X_t(\theta,\Phi) := X(t+\theta,\Phi) \quad \text{for } \Phi\in\Sigma \text{ and } \theta\in[-1,0].$$

We start with the signal function (1.2). We need to introduce the candidates for each component of the initial functions whose solutions will be eventually periodic. For this purpose, for given constants α, c_3, c_2 with $\alpha>0$, $c_2>0$ and $c_3<0$, we define subsets $\Omega_1,\Omega_2,\Omega_3\subset C([-1,0])$ as follows:

$$\Omega_1=\{\phi\in C([-1,0]);\phi(\theta)>0 \text{ for } \theta\in[-1,0) \text{ and } \phi(0)=0\},$$

$$\Omega_2=\{\phi\in C([-1,0]);\phi(\theta)<0,\ \theta\in[-1,-\alpha),\phi(-\alpha)=0,$$
$$\phi(\theta)>0,\ \theta\in(-\alpha,0),\phi(0)=c_2\},$$

$$\Omega_3=\{\phi\in C([-1,0]);\phi(\theta)<0,\ \theta\in[-1,0),\phi(0)=c_3\}.$$

Let

$$K_3=\{(i,j,k);1\leqslant i,j,k\leqslant 3 \text{ and } i,j,k \text{ are distinct integers}\}.$$

For any given $(i,j,k)\in K_3$, define

$$\Sigma(i,j,k)=\{\Phi=(\phi_1,\phi_2,\phi_3)\in\Sigma;\ \phi_i\in\Omega_1,\ \phi_j\in\Omega_2,\ \phi_k\in\Omega_3\},$$

and let

$$\Sigma^-(i,j,k)=\{\Phi\in\Sigma;\ -\Phi\in\Sigma(i,j,k)\}.$$

Much of the calculations in Section 2 is about finding α, c_2 and c_3 so that for any given $(i,j,k)\in K_3$, we have

$$X_\alpha:\Sigma(i,j,k)\to\Sigma^-(k,i,j),\quad X_\alpha:\Sigma^-(i,j,k)\to\Sigma(k,i,j). \tag{1.4}$$

It turns out that if

$$k=\frac{-a+2b}{-a}\geqslant e^\tau-e^{\frac{\tau}{2}}, \tag{1.5}$$

then (1.4) holds provided (α,c_2,c_3) are chosen so that

$$c_3=-e^{-\tau\alpha}[2b(e^{2\tau\alpha}-e^\tau)-ae^{\tau\alpha}(e^{\tau\alpha}-1)],$$
$$c_2=e^{-2\tau\alpha}[2b(e^\tau-e^{\tau\alpha})+ae^{\tau\alpha}(1-e^{\tau\alpha})]$$

and $\alpha \in (1/2, 1)$ is the unique number such that

$$h(\alpha; \tau, a, b) := c_3 + c_2 e^{-\tau\alpha} + a[1 + e^{-\tau\alpha} - 2e^{\tau(1-2\alpha)}] = 0.$$

For the above given c_2, c_3 and α, we then have

$$X_{2\alpha} : \Sigma(i,j,k) \to \Sigma(j,k,i),$$

and hence

$$X_{6\alpha} : \Sigma(i,j,k) \to \Sigma(i,j,k). \tag{1.6}$$

Since $f(x)$ depends only on the sign of x, we can easily see that if $\Phi, \Psi \in \Sigma(1,2,3)$, then $X(t,\Phi) = X(t,\Psi)$ for all $t \geqslant 0$. This, together with (1.6), then yields

Theorem A. *Assume that $a<0$ and $b>0$ satisfy* (1.5). *Then*

(i) *For any $\Phi, \Psi \in \Sigma(1,2,3)$, $X(t,\Phi) = X(t,\Psi)$ for all $t \geqslant 0$.*
(ii) *Let $P(t) = X(t,\Phi)$ for $t \geqslant 0$ and for a given $\Phi \in \Sigma(1,2,3)$. Then $P(t+6\alpha) = P(t)$ for all $t \geqslant 0$, and P is phased-locked in the sense*

$$p_1(t) = p_2(t+2\alpha), \quad p_2(t) = p_3(t+2\alpha), \quad p_3(t) = p_1(t+2\alpha), \quad t \geqslant 0,$$

and satisfies the following additional symmetry property:

$$p_1(t) = -p_3(t+\alpha), \quad p_2(t) = -p_1(t+\alpha), \quad p_3(t) = -p_2(t+\alpha), \quad t \geqslant 0.$$

Our next step is to construct phase-locked periodic solutions in a small neighborhood of $P|_{[-1,0]}$ for f close to the step function. We first introduce the restriction for the nonlinearity. Let

$$N(M,\beta,\varepsilon) = \{f : \mathbb{R} \to \mathbb{R};\ f \text{ is continuous and odd, } |f(x)| \leqslant M$$
$$\text{for } x \in \mathbb{R},\ |f(x) - 1| \leqslant \varepsilon \text{ if } x \geqslant \beta\}.$$

Note that f converges to the step function, except at zero, when $\beta \to 0$ and $\varepsilon \to 0$.

We also need to restrict the initial functions to a certain convex and closed set containing $P|_{[-1,0]}$ of the phase-locked periodic solution P constructed above. Let $(\alpha_0, \delta_2, \delta_3, D, \beta)$ be given positive constants with $0 < \alpha_0 \leqslant \frac{1}{2}$ and define

$$A(\alpha_0, \delta_2, \delta_3, D, \beta)$$
$$= \{\Phi = (\phi_1, \phi_2, \phi_3)^T \in \Sigma;$$
$$\phi_1(s) \geqslant \beta \text{ for } s \in [-1,0], \phi_1(0) = \beta,$$

ϕ_1 is nonincreasing on $[-2\alpha_0, 0]$ and

$\phi_1(t) - \phi_1(s) \leqslant -D(t-s)$ for $-\alpha_0 \leqslant s \leqslant t \leqslant 0$;

$\phi_2(s) \leqslant -\beta$ for $s \in [-1, -\alpha - \alpha_0], \phi_2(s) \geqslant \beta$ for $s \in [-\alpha + \alpha_0, 0]$,

ϕ_2 is nondecreasing on $[-\alpha - \alpha_0, -\alpha + \alpha_0]$ and

$\phi_2(t) - \phi_2(s) \geqslant D(t-s)$ for $-\alpha - \alpha_0 \leqslant s \leqslant t \leqslant -\alpha + \alpha_0$,

$|\phi_2(0) - c_2| \leqslant \delta_2$;

$\phi_3(s) \leqslant -\beta$ for $s \in [-1, 0], |\phi_3(0) - c_3| \leqslant \delta_3\}$.

It is easy to prove that the set $A(\alpha_0, \delta_2, \delta_3, D, \beta)$ is a closed and convex subset of the Banach space Σ.

For a fixed $\Phi \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$, if we let $-\alpha_1 := -\alpha_1(\Phi)$ be the smallest α in $[-\alpha - \alpha_0, -\alpha + \alpha_0]$ so that $\phi_2(-\alpha_1) = -\beta$ and let $-\alpha_2 := -\alpha_2(\Phi)$ be the largest α in $[-\alpha - \alpha_0, -\alpha + \alpha_0]$ so that $\phi_2(-\alpha_2) = \beta$. Then $\alpha_1 > \alpha_2$ and

$$\alpha_1 - \alpha_2 \leqslant \frac{2\beta}{D}.$$

This observation turns out to be essential: while we cannot control the locations where ϕ_2 crosses $\pm\beta$, we can control the distance of $\alpha_1 - \alpha_2$ by β.

Fix $\Phi \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$ and let $X = X(t, \Phi) = (x_1, x_2, x_3)$ be the solution of (1.3) with a fixed $f \in N(M, \beta, \varepsilon)$. Let $T = T(\Phi) \in (1 - \alpha - \alpha_0, 1)$ be the first time where $x_3(T) = \beta$ (the existence will be established in Section 4). Then we can show that

$$\begin{cases} |\alpha - T| \leqslant a_{32}\delta_3 + a_{33}\alpha_0 + O(\beta, \varepsilon) + o(\delta_3, \alpha_0), \\ |x_1(T) + c_2| \leqslant |a_{12}|\delta_3 + |a_{13}|\alpha_0 + O(\beta, \varepsilon) + o(\delta_3, \alpha_0), \\ |x_2(T) + c_3| \leqslant a_{12}\delta_2 + a_{22}\delta_3 + a_{23}\alpha_0 + O(\beta, \varepsilon) + o(\delta_2, \delta_3, \alpha_0), \end{cases} \tag{1.7}$$

where a_{ij} are given explicitly in terms of $\{a, b, \tau\}$. To generate a self-returning map by following the solution semiflow of (1.3), we then need to look at the following nonnegative matrix

$$\Theta = \begin{pmatrix} 0 & -a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

for which we can show that there exists $\tau^* > 0$ so that for every $\tau > \tau^*$ there exist a constant $\rho \in (0, 1)$ and a positive vector $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ so that

$$\Theta z = \rho z.$$

To obtain the above result, we need to establish the fact that $\alpha \to 1/2$ as $\tau \to \infty$, and more precisely, we need the following asymptotic expansion, assuming that $A := ke^{-\tau} \in (1,2)$, given by

$$1 - e^{\tau(1-2\alpha)} = A^{-1}e^{-2\tau\alpha}(A-1) + A^{-1}(2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha}).$$

Fix $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ as above. We can then show that there exists $m^* > 0$ such that for each $m \in (0, m^*)$ there exist $\beta^* > 0$ and $\varepsilon^* > 0$ so that if $0 < \beta < \beta^*$ and $0 < \varepsilon < \varepsilon^*$ and if $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$ then the solution $X = X(t, \Phi) = (x_1, x_2, x_3)$ of system (1.3) with $f \in N(M, \beta, \varepsilon)$ and $\Phi \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$ for a properly chosen D satisfies

$$x_i(t) = p_i(t) + O(\alpha_0, \delta_2, \delta_3, \beta, \varepsilon), \quad t \in [0, 1] \tag{1.8}$$

and

$$|\alpha - T| \leqslant \frac{3+\rho}{4}\alpha_0, \quad |x_1(T) + c_2| \leqslant \frac{3+\rho}{4}\delta_2, \quad |x_2(T) + c_3| \leqslant \frac{3+\rho}{4}\delta_3. \tag{1.9}$$

As a consequence, we obtain that the mapping F given by

$$F(\Phi) = (x_3^{\Phi}, x_1^{\Phi}, x_2^{\Phi})|_{[T-1,T]}, \quad \Phi \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$$

satisfies $-F(\Phi) \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$. This, together with the symmetry of the nonlinearity and the fact that $T > 1/2$ shows that F^2 is a completely continuous self-returning map on $A(\alpha_0, \delta_2, \delta_3, D, \beta)$, and hence has a fixed point which gives rise to a phase-locked periodic solution since

$$F^2(\Phi) = (x_2^{\Phi}, x_3^{\Phi}, x_1^{\Phi})|_{[\tilde{T}-1,\tilde{T}]}$$

with some $\tilde{T} > 1$. Therefore, we obtain

Theorem B. *Let $A = ke^{-\tau} \in (1,2)$ and assume that $\tau > \tau^*$ so that there exist a constant $\rho \in (0,1)$ and a positive vector $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ such that $\Theta z = \rho z$. Let $0 < \alpha_0 \leqslant \min\{\alpha - \frac{1}{2}, \frac{1-\alpha}{3}\}$. Then there exist $m^* > 0$ such that for each $m \in (0, m^*)$ there exist positive constants β^* and ε^* so that if $0 < \beta < \beta^*$ and $0 < \varepsilon < \varepsilon^*$ and if $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$ then system* (1.3) *with $f \in N(M, \beta, \varepsilon)$ has a phase-locked $6P$-periodic solution $Q = (q_1, q_2, q_3)$ with $Q|_{[-1,0]} \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$ for a suitable D and*

$$q_i(t + 6P) = q_i(t), \quad q_i(t) = q_{i+1}(t + 2P), \quad i \bmod (3), \; t \geqslant 0.$$

Moreover, $|P - \alpha| \leqslant \alpha_0$ and the periodic solution so obtained satisfies $q_i - p_i \to 0$ uniformly as $\beta, \varepsilon, m \to 0$.

As for the stability of Q, we need an exponential norm: for every $\Phi = (\phi_1, \phi_2, \phi_3) \in \Sigma$ and for a constant $\gamma > 0$ to be specified, let

$$||\Phi||_\tau = \max_{i=1,2,3} \left\{ \max_{s \in [-1, -\alpha+\alpha_0]} \gamma e^{\tau s} |\phi_i(s)|, \max_{s \in [-\alpha+\alpha_0, 0]} e^{\tau s} |\phi_i(s)| \right\}.$$

Clearly, the above-defined norm is equivalent to the super-norm. The goal is to choose $\gamma > 0$ appropriately so that the fixed point of F^2 is asymptotically attractive with respect to the above exponential norm. To be more concrete, let α_2^Q and α_1^Q be as defined above but with the supindex Q to denote the dependence on Q. Let $\omega > 0$ be given and define

$$C_{Q,\omega} := \{\Phi \in \Sigma; ||\Phi - Q_0||_\tau \leqslant \omega\}.$$

It is easy to show that for any $\beta > 0$ and $\eta > 0$ there exists $\omega = \omega(\beta, \eta)$ such that if $\Phi \in C_{Q,\omega}$, then

$$\phi_1(s) \geqslant \beta \quad \text{for } s \in [-1, -\eta],$$

$$\phi_2(s) \leqslant -\beta \quad \text{for } s \in [-1, -\alpha_1^Q - \eta),$$

$$\phi_2(s) \geqslant \beta \quad \text{for } s \in (-\alpha_2^Q + \eta, 0],$$

$$\phi_3(s) \leqslant -\beta \quad \text{for } s \in [-1, 0].$$

We need to impose a certain Lipschitz continuity on the nonlinearity in order to achieve the required attractivity. Namely, for a fixed $L_\infty > 0$, we define

$$N(L_\infty, L_\beta, M, \beta, \varepsilon) := \{f \in N(M, \beta, \varepsilon); |f(x) - f(y)| \leqslant L_\infty |x - y|, x, y, \in \mathbb{R};$$
$$|f(x) - f(y)| \leqslant L_\beta |x - y|, x, y \geqslant \beta\}.$$

We then fix an element f from the above set. Much of the challenging to obtain the aforementioned attractivity of $Q|_{[-1,0]}$ as a fixed point of F^2 lies on the different scales of growth rates of the difference $x_i - q_i$ on the intervals $FM := [0, 1 - \alpha_1^Q - \eta] \cup [1 - \alpha_2^Q + \eta, 1 - \eta]$ and on the intervals $SM := [1 - \alpha_1^Q - \eta, 1 - \alpha_2^Q + \eta]$. On FM, the contraction is easy as $f(x_i(t-1)) - f(q_i(t-1))$ is bounded by a small Lipschitz constant L_β (multiplied by $||\Phi - Q_0||_\tau$) but the contraction on SM is very difficult since $f(x_i(t-1)) - f(q_i(t-1))$ is bounded only by the global Lipschitz constant L_∞ (multiplied by $||\Phi - Q_0||_\tau$) and this global Lipschitz constant L_∞ goes to infinity as $\beta \to 0$. Consequently, we need to choose $\gamma > 0$ very carefully so that the corresponding exponential norm compensates the expansion due to the fast growth of $x_i - q_i$ on FM, while keeping the contraction on SM. In this spirit, our analysis is very much similar to the asymptotic expansion of inner and outer layers in the analytic theory of singularly perturbed systems.

It turns out that $\gamma = \Gamma^{-1}(\tau)e^{\tau T}$, with $T = T(Q|_{[-1,0]})$, suits the above purpose, where $\Gamma(\tau) \in (1, e^{\tau(2T-1)})$. In particular, we can show that if $A = ke^{-\tau} \in (1,2)$, $\tau > \tau^*$ and if $0 < \alpha_0 \leqslant \min\{\alpha - \frac{1}{2}, \frac{1-\alpha}{4}\}$ then there exist $m^{**} > 0$ such that for each $m \in (0, m^{**})$ there exist $\tilde{\beta}^*$ and ε^* so that if $0 < \beta < \tilde{\beta}^*$ and $0 < \varepsilon < \varepsilon^*$ and if $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$ then for a chosen $\eta \in \min\{\frac{\alpha}{4}, -\alpha + \alpha_0 - \alpha_2^Q\}$ and for every fixed $\Phi \in C_{Q,\omega}$, we have

$$||X_T^\Phi - Q_T||_\tau \leqslant \max\{B_1, B_2, \gamma e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}\}||\Phi - Q_0||_\tau \tag{1.10}$$

and the constant

$$K := \max\{B_1, B_2, \gamma e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}\}$$

is less than 1 if τ is sufficiently large and L_β is sufficiently small. As a consequence, we obtain

Theorem C. *Assume all conditions in Theorem B are satisfied, and assume* $0 < \alpha_0 \leqslant \min\{\alpha - \frac{1}{2}, \frac{1-\alpha}{4}\}$. *Then there exists* $\tau^{**} \geqslant \tau^*$ *so that for every fixed* $\tau > \tau^{**}$ *and any* $\Gamma(\tau) \in (1, e^{\tau(2T-1)})$ *there exist* $L_\beta^* > 0$, $\beta^{**} \in (0, \tilde{\beta}^*)$ *and* $\varepsilon^{**} \in (0, \varepsilon^*)$ *so that if* $L_\beta \in [0, L_\beta^*)$, $\beta \in (0, \beta^{**})$, $\varepsilon \in (0, \varepsilon^{**})$ *and* $L_\infty \beta < \Gamma^{-1}(\tau)e^{\tau(2T-1)}$, *then* Q *is asymptotically stable.*

We now comment about conditions so far assumed, listed below:

(H1) $\tau > \tau^{**}$;
(H2) $\frac{-a+2b}{-a} = k = Ae^\tau$, $A \in (1,2)$;
(H3) $0 \leqslant L_\beta < L_\beta^*$;
(H4) $\varepsilon \in [0, \varepsilon^{**})$, $\beta \in [0, \beta^{**})$;
(H5) $L_\infty \beta \leqslant \Gamma^{-1}(\tau)e^{\tau(2T-1)}$.

Conditions (H4) and (H5) are satisfied if the nonlinearity is close to the piecewise linear function used in cellular neural networks, given by $f(x) = 1$ for $x \geqslant \beta$, $f(x) = -1$ for $x \leqslant -\beta$ and $f(x) = x/\beta$ for $x \in [-\beta, \beta]$. Condition (H3) requires that the Lipschitz constant of f to be small outside a neighborhood of the zero. Note that if $f \in N(M, \beta, \varepsilon)$ then $L_\infty \beta \geqslant 1$ and hence, L_∞ is surly large if $\beta \to 0$. This makes our analysis very much similar to the asymptotic expansion of inner and outer layers in the singular perturbation theory. In terms of the aforementioned convergence (to equilibria) of all solutions of (1.1) with $\tau = 0$, to obtain a stable periodic solution, the delay must be sufficiently large, so is condition (H1), though unfortunately our method does not yield information about the minimal value of τ^{**}. Certain conditions on the ratio $\frac{-a+2b}{-a}$ are necessary as well for the occurrence of stable phase-locked periodic solutions. This is because if $a = 0$ then system (1.3) is a cooperative and irreducible functional differential equation in the sense of Smith [32] and thus system (1.3) does not have any stable periodic solution. On the other hand, if b is small, system (1.3) is weakly coupled and hence stable phase-locked solutions are unlikely.

We should mention that condition (H2), though natural as explained above, is different from what one normally would get from the Hopf bifurcation theory. This is because we deal with the existence of a stable phase-locked periodic solution from a new viewpoint: the periodic solution is regarded as a perturbation of the periodic solution of (1.3) with the step function which has an infinite jump (gain $f'(0) = \infty$). It is not a bifurcation problem, but rather a persistence issue. Looking at this condition from a bifurcation point of view, we can regard condition (H2) as the one that guarantees not only the occurrence of a local Hopf bifurcation of periodic solutions, but also the global continuation of this branch to infinity (continued as $\tau \to \infty$) as well as the persistence of stability of the periodic solutions.

We conclude this long introduction with some remarks about potential future development of the method. First of all, according to the work of Wu [40] and Huang and Wu [18], as τ increases, system (1.3) has multiple periodic solutions including phase-locked solutions, mirror-reflecting waves and standing waves. Some preliminary work of Wu [39], using a vector discrete valued Lyapunov functional, shows that the dynamics of (1.3) is still regular and the global attractor is expected to be the set of equilibria, the phase-locked, mirror-reflecting and standing wave periodic solutions and their connecting orbits. The difficulty to obtain this result is to describe the structure of all connecting orbits between various periodic orbits. It should be possible to describe the connecting orbits of various periodic solutions for system (1.3) when the nonlinearity f is a step function, and we hope our approach can be used not only to obtain the mirror-reflecting waves and standing waves, but also to describe the persistence of connecting orbits when f is sufficiently close to the step function.

2. Limiting profiles at infinite gain

In this section, we consider system (1.3) with the nonlinearity given by the step function (1.2), and with $a<0$ and $b>0$. Our goal is to construct explicitly a phased-locked periodic solution.

Let

$$c_3 = -e^{-\tau\alpha}[2b(e^{2\tau\alpha} - e^{\tau}) - ae^{\tau\alpha}(e^{\tau\alpha} - 1)] \tag{2.1}$$

and

$$c_2 = e^{-2\tau\alpha}[2b(e^{\tau} - e^{\tau\alpha}) + ae^{\tau\alpha}(1 - e^{\tau\alpha})]. \tag{2.2}$$

Define

$$h(\alpha; \tau, a, b) = c_3 + c_2 e^{-\tau\alpha} + a[1 + e^{-\tau\alpha} - 2e^{\tau(1-2\alpha)}].$$

Then

$$h(\alpha;\tau,a,b) = a[e^{\tau\alpha} + e^{-2\tau\alpha} - 2e^{\tau(1-2\alpha)}] + 2b[e^{\tau(1-\alpha)} + e^{\tau(1-3\alpha)} - e^{\tau\alpha} - e^{-2\tau\alpha}].$$

Let

$$k = \frac{a-2b}{a}. \tag{2.3}$$

That is, $2b = (1-k)a$. Then

$$\frac{1}{a}h(\alpha;\tau,a,b) = e^{\tau(1-\alpha)} - 2e^{\tau(1-2\alpha)} + e^{\tau(1-3\alpha)} - k[e^{\tau(1-\alpha)} + e^{\tau(1-3\alpha)} - e^{\tau\alpha} - e^{-2\tau\alpha}]. \tag{2.4}$$

So we have

$$\begin{aligned}\frac{1}{a}h(1;\tau,a,b) &= 1 - 2e^{-\tau} + e^{-2\tau} - k(1 + e^{-2\tau} - e^{\tau} - e^{-2\tau})\\ &= (1-e^{-\tau})^2 - k(1-e^{\tau})\\ &= (1-e^{-\tau})^2 + k(e^{\tau}-1) > 0.\end{aligned}$$

Also, we have

$$\begin{aligned}\frac{1}{a}h\left(\frac{1}{2};\tau,a,b\right) &= e^{\frac{\tau}{2}} - 2 + e^{-\frac{\tau}{2}} - k(e^{\frac{\tau}{2}} + e^{-\frac{\tau}{2}} - e^{\frac{\tau}{2}} - e^{-\tau})\\ &= (e^{\frac{\tau}{4}} - e^{-\frac{\tau}{4}})^2 - k(e^{-\frac{\tau}{2}} - e^{-\tau})\\ &= e^{-\frac{\tau}{2}}(e^{\frac{\tau}{2}} - 1)^2 - ke^{-\tau}(e^{\frac{\tau}{2}} - 1)\\ &= (e^{\frac{\tau}{2}} - 1)e^{-\tau}[e^{\frac{\tau}{2}}(e^{\frac{\tau}{2}} - 1) - k].\end{aligned}$$

Hence, $\frac{1}{a}h(\frac{1}{2};\tau,a,b) < 0$ if

$$k > e^{\frac{\tau}{2}}(e^{\frac{\tau}{2}} - 1). \tag{2.5}$$

Note also for $\alpha \in (1/2, 1)$,

$$\begin{aligned}&\frac{1}{\tau}\frac{\partial}{\partial\alpha}e^{-\tau(1-3\alpha)}\frac{1}{a}h(\alpha;\tau,a,b)\\ &\quad = \frac{1}{\tau}\frac{\partial}{\partial\alpha}[e^{2\tau\alpha} - 2e^{\tau\alpha} + 1 - k(e^{2\tau\alpha} + 1 - e^{\tau(4\alpha-1)} - e^{\tau(\alpha-1)})]\\ &\quad = 2e^{2\tau\alpha} - 2e^{\tau\alpha} - k(2e^{2\tau\alpha} - 4e^{\tau(4\alpha-1)} - e^{\tau(\alpha-1)})\end{aligned}$$

$$= 2e^{\tau\alpha}(e^{\tau\alpha} - 1) + k(4e^{\tau(4\alpha-1)} - 2e^{2\tau\alpha} + e^{\tau(\alpha-1)})$$

$$= 2e^{\tau\alpha}(e^{\tau\alpha} - 1) + k[2e^{2\tau\alpha}(2e^{\tau(2\alpha-1)} - 1) + e^{\tau(\alpha-1)}]$$

$$> 0.$$

Consequently, we have shown

Lemma 2.1. *If* (2.5) *holds, then there exists one and only one* $\alpha = \alpha(\tau, k) \in (1/2, 1)$ *such that*

$$h(\alpha; \tau, a, b) = 0. \tag{2.6}$$

We will also need the following technical preparation.

Lemma 2.2. $c_3 < 0 < c_2$.

Proof. Using $2b = (1 - k)a$, we get

$$c_2 = ae^{-2\tau\alpha}[e^{\tau\alpha} - e^{2\tau\alpha} + e^{\tau} - e^{\tau\alpha} - k(e^{\tau} - e^{\tau\alpha})]$$

$$= ae^{-2\tau\alpha}[(e^{\tau} - e^{2\tau\alpha}) + k(e^{\tau\alpha} - e^{\tau})]$$

$$> 0$$

since $a < 0$ and $1/2 < \alpha < 1$. Also, we have

$$c_3 = -e^{-\tau\alpha}a[e^{2\tau\alpha} - e^{\tau} - k(e^{2\tau\alpha} - e^{\tau}) - e^{2\tau\alpha} + e^{\tau\alpha}]$$

$$= -ae^{-\tau\alpha}[e^{\tau\alpha} - e^{\tau} + k(e^{\tau} - e^{2\tau\alpha})]$$

$$< 0.$$

This completes the proof. □

Let

$$\Sigma = \{\Phi = (\phi_1, \phi_2, \phi_3); \phi_i \in C([-1, 0]), i = 1, 2, 3\}.$$

For any given $\Phi \in \Sigma$ and $t \geqslant -1$, define $X(t, \Phi) = (x_1(t, \Phi), x_2(t, \Phi), x_2(t, \Phi))$ as the solution of (1.3) such that $x_i(t, \Phi) = \phi_i(t)$ for $t \in [-1, 0]$ and $i = 1, 2, 3$. Furthermore, for any given $t \geqslant 0$, we define the mapping $X_t : \Sigma \to \Sigma$ by

$$X_t(\Phi)(\theta) := X_t(\theta, \Phi) := X(t + \theta, \Phi) \quad \text{for } \Phi \in \Sigma \text{ and } \theta \in [-1, 0].$$

We denote by $\phi \sim \bar{\phi}$ for given $\phi, \bar{\phi} \in C([-1,0])$ if and only if

(a) $\operatorname{sign} \phi(\theta) = \operatorname{sign} \bar{\phi}(\theta)$ for all $\theta \in [-1,0]$, except finitely many points;
(b) $\phi(0) = \bar{\phi}(0)$.

We write $\Phi \sim \bar{\Phi}$ for $\Phi, \bar{\Phi} \in \Sigma$ with $\Phi = (\phi_1, \phi_2, \phi_3)$ and $\bar{\Phi} = (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$, if $\phi_i \sim \bar{\phi}_i$ for $i = 1, 2, 3$. As $f(x)$ depends on only the sign of $x \in \mathbb{R}$, we can easily show that

Lemma 2.3. *If $\Phi \sim \bar{\Phi}$, then $X(t, \Phi) = X(t, \bar{\Phi})$ for $t \geqslant 0$.*

We now introduce three essential subsets of $C([-1,0])$.

Definition 2.4. For the constants α, c_3, c_2 determined in (2.1), (2.2) and (2.6), we define subsets $\Omega_1, \Omega_2, \Omega_3 \subset C([-1,0])$ as follows:

(i) Ω_1 consists of $\phi \in C([-1,0])$ satisfying

$$\phi(\theta) > 0 \quad \text{for } \theta \in [-1, 0) \text{ and } \phi(0) = 0; \tag{2.7}$$

(ii) Ω_2 consists of $\phi \in C([-1,0])$ satisfying

$$\phi(\theta) \begin{cases} <0, & \theta \in [-1, -\alpha), \\ = 0, & \theta = -\alpha, \\ > 0, & \theta \in (-\alpha, 0), \\ = c_2, & \theta = 0; \end{cases} \tag{2.8}$$

(iii) Ω_3 consists of $\phi \in C([-1,0])$ satisfying

$$\phi(\theta) \begin{cases} <0, & \theta \in [-1, 0), \\ = c_3, & \theta = 0. \end{cases} \tag{2.9}$$

Let

$$\Omega_i^- = \{\phi \in C([-1,0]); \ -\phi \in \Omega_i\} \tag{2.10}$$

and

$$K_3 = \{(i,j,k); \ 1 \leqslant i,j,k \leqslant 3, \text{ and } i,j,k \text{ are distinct integers}\}.$$

For any given $(i,j,k) \in K_3$, define

$$\Sigma(i,j,k) = \{\Phi = (\phi_1, \phi_2, \phi_3) \in \Sigma; \ \phi_i \in \Omega_1, \ \phi_j \in \Omega_2, \ \phi_k \in \Omega_3\}, \tag{2.11}$$

and let

$$\Sigma^{-}(i,j,k) = \{\Phi \in \Sigma; \ -\Phi \in \Sigma(i,j,k)\}. \tag{2.12}$$

It is clear that if $\Phi, \bar{\Phi} \in \Sigma(i,j,k)$, then $\Phi \sim \bar{\Phi}$ and hence, $X(t,\Phi) = X(t,\bar{\Phi})$ for $t \geqslant 0$.

Lemma 2.5. *For any given* $(i,j,k) \in K_3$, *we have*

$$X_\alpha : \Sigma(i,j,k) \to \Sigma^{-}(k,i,j). \tag{2.13}$$

Proof. By symmetry, we only need to show that $X_\alpha(\Sigma(1,2,3)) \subset \Sigma^{-}(3,1,2)$. Choose $\Phi = (\phi_1, \phi_2, \phi_3) \in \Sigma(1,2,3)$, that is,

$$\phi_1 \in \Omega_1, \phi_2 \in \Omega_2, \phi_2 \in \Omega_3. \tag{2.14}$$

Let

$$\phi_i^\alpha(\theta) = x_i(\alpha + \theta, \Phi) \quad \text{for } i = 1,2,3. \tag{2.15}$$

We want to show that $-\phi_3^\alpha \in \Omega_1, -\phi_1^\alpha \in \Omega_2, -\phi_2^\alpha \in \Omega_3$. That is, we want to show that

$$\phi_3^\alpha(\theta) < 0, \ \theta \in [-1,0) \quad \text{and} \quad \phi_3^\alpha(0) = 0 \tag{2.16}$$

and

$$\phi_1^\alpha(\theta) \begin{cases} > 0, & \theta \in [-1,-\alpha), \\ = 0, & \theta = -\alpha, \\ < 0, & \theta \in (-\alpha, 0), \\ = -c_2, & \theta = 0 \end{cases} \tag{2.17}$$

as well as

$$\phi_2^\alpha(\theta) > 0, \ \theta \in [-1,0) \quad \text{and} \quad \phi_2^\alpha(0) = -c_3. \tag{2.18}$$

(i) Verification of (2.16): For $\theta \in [-1,-\alpha]$, $\alpha + \theta \leqslant 0$, and hence it follows from (2.14) and (2.9) that $\phi_3^\alpha(\theta) = \phi_3(\alpha+\theta) < 0$. So, it suffices to prove

$$x_3(t,\Phi) < 0 \quad \text{for } t \in [0,\alpha) \text{ and } x_3(\alpha,\Phi) = 0. \tag{2.19}$$

By (1.3), (2.14) and (2.7)–(2.9), we obtain that

$$\begin{aligned} x_3(t,\Phi) &= c_3 e^{-\tau t} - (a-b)(1-e^{-\tau t}) \\ &\quad + \begin{cases} -b(1-e^{-\tau t}), & \text{for } t \in [0, 1-\alpha], \\ b[1 + e^{-\tau t} - 2e^{-\tau(t-1+\alpha)}], & \text{for } t \in (1-\alpha, 1]. \end{cases} \end{aligned} \tag{2.20}$$

For $t\in[0,1-\alpha)$, $x_3(t,\Phi)=c_3e^{-\tau t}-a(1-e^{-\tau t})$. So, we have

$$x_3(0,\Phi)=c_3<0 \tag{2.21}$$

and

$$\dot{x}_3(t,\Phi)=-(a+c_3)\tau e^{-\tau t}>0. \tag{2.22}$$

For $t\in(1-\alpha,1]$, (2.20) yields

$$\dot{x}_3(t,\Phi)=\tau e^{-\tau t}[2be^{\tau(1-\alpha)}-(a+c_3)].$$

This, together with $b>0$ and (2.22), yields $\dot{x}_3(t,\Phi)>0$ for $t\in[0,1]$. As $x_3(0,\Phi)<0$ by (2.21), to obtain (2.19), we only need to prove $x_3(\alpha,\Phi)=0$. Note that $\alpha>1-\alpha$. By (2.20), $x_3(\alpha,\Phi)=0$ if and only if

$$c_3e^{-\tau\alpha}-(a-b)(1-e^{-\tau\alpha})+b[1+e^{-\tau\alpha}-2e^{-\tau(2\alpha-1)}]=0,$$

which is equivalent to (2.1). This verifies (2.16).

(ii) Verification of (2.17): Note that for $\theta\in[-1,-\alpha)$, $\alpha+\theta\leqslant 0$ and hence, (2.7) gives $\phi_1^\alpha(\theta)=\phi_1(\alpha+\theta)>0$ and $\phi_1^\alpha(-\alpha)=0$. To prove (2.17), it suffices to prove

$$x_1(t,\Phi)<0 \quad \text{for } t\in[0,\alpha) \text{ and } x_1(\alpha,\Phi)=-c_2. \tag{2.23}$$

By (1.3), (2.14) and (2.7)–(2.9), we obtain that

$$x_1(t,\Phi)=(a-b)(1-e^{-\tau t})+\begin{cases}-b(1-e^{-\tau t}), & \text{for } t\in[0,1-\alpha],\\ b[1+e^{-\tau t}-2e^{-\tau(t-1+\alpha)}], & \text{for } t\in(1-\alpha,1].\end{cases} \tag{2.24}$$

For $t\in[0,1-\alpha]$, by (2.24), $x_1(t,\Phi)=(a-2b)(1-e^{-\tau t})$. So, $a<0$ and $b>0$ imply

$$x_1(t,\Phi)<0 \quad \text{for } t\in[0,1-\alpha] \tag{2.25}$$

and

$$\dot{x}_1(t,\Phi)=\tau(a-2b)e^{-\tau t}<0. \tag{2.26}$$

For $t\in(1-\alpha,1]$, by (2.24), we have

$$\dot{x}_1(t,\Phi)=\tau e^{-\tau t}[(a-2b)+2be^{\tau(1-\alpha)}].$$

In the case where $(a-2b)+2be^{\tau(1-\alpha)}\leqslant 0$, $x_1(t,\Phi)$ is decreasing on $[0,1]$; and in the case where $(a-2b)+2\tau e^{\tau(1-\alpha)}>0$, $x_1(t,\Phi)$ is decreasing on $[0,1-\alpha]$ and then increasing on $[1-\alpha,1]$. In both cases, (2.23) holds if $x_1(\alpha,\Phi)=-c_2$ since $c_2>0$ by Lemma 2.2. By (2.24), $x_1(\alpha,\Phi)=-c_2$ if and only if

$$c_2=-[(a-b)(1-e^{-\tau\alpha})+b(1+e^{-\tau\alpha}-2e^{-\tau(2\alpha-1)})],$$

which is equivalent to (2.2). This verifies (2.17).

(iii) Verification of (2.18). For $\theta \in [-1, -\alpha]$, we have $\alpha + \theta \leqslant 0$ and $\alpha + \theta \geqslant \alpha - 1 > -\alpha$ due to $\alpha > 1/2$, and therefore, $\phi_2^\alpha(\theta) = \phi_2(\alpha + \theta) > 0$. So to prove (2.18), it suffices to prove

$$x_2(t, \Phi) > 0 \quad \text{for } t \in [0, \alpha) \text{ and } x_2(\alpha, \Phi) = -c_3. \tag{2.27}$$

By (1.3), (2.14) and (2.7)–(2.9), we obtain

$$x_2(t, \Phi) = c_2 e^{-\tau t} + \begin{cases} -a(1 - e^{-\tau t}), & \text{for } t \in [0, 1 - \alpha], \\ a[1 + e^{-\tau t} - 2e^{-\tau(t-1+\alpha)}], & \text{for } t \in [1 - \alpha, 1]. \end{cases} \tag{2.28}$$

For $t \in [0, 1 - \alpha]$, (2.28) gives

$$x_2(0, \Phi) = c_2 > 0 \tag{2.29}$$

and

$$\dot{x}_2(t, \Phi) = -\tau(a + c_2)e^{-\tau t}. \tag{2.30}$$

For $t \in (1 - \alpha, 1]$, (2.30) gives

$$\dot{x}_2(t, \Phi) = \tau e^{-\tau t}[-(a + c_2) + 2ae^{\tau(1-\alpha)}]. \tag{2.31}$$

There are three possible cases:

(iiia) $a + c_2 < 0$ and $-(a + c_2) + 2ae^{\tau(1-\alpha)} > 0$. In this case, (2.30) and (2.31) yield $\dot{x}_2(t, \Phi) > 0$ for $t \in [0, 1]$. Using (2.29), we conclude that (2.27) holds if

$$x_2(\alpha, \Phi) = -c_3. \tag{2.32}$$

(iiib) $a + c_2 < 0$ and $-(a + c_2) + 2ae^{\tau(1-\alpha)} < 0$. In this case, $x_2(t, \Phi)$ achieves its maximum at $1 - \alpha$, and its minimum at either 0 or α. Thus, again, (2.27) holds if (2.32) holds.

(iiic) $a + c_2 \geqslant 0$. As $a < 0$, we have $-(a + c_2) + 2ae^{\tau(1-\alpha)} < 0$. Therefore, (2.30) and (2.31) yield $\dot{x}_2(t, \Phi) < 0$ for $t \in [0, 1]$. Using (2.29), we again know that (2.32) implies (2.27).

In summary, we need only to verify (2.32). That is, by using (2.28), we need to verify

$$c_3 + c_2 e^{-\tau\alpha} + a[1 + e^{-\tau\alpha} - 2e^{-\tau(2\alpha-1)}] = 0. \tag{2.33}$$

This is exactly $h(\alpha; \tau, a, b) = 0$. Consequently, the choice of α ensures (2.32). This completes the proof. $\square$

Corollary 2.6. *For any* $(i,j,k)\in K_3$ *we have*

$$X_\alpha : \Sigma^-(i,j,k)\to\Sigma(k,i,j). \tag{2.34}$$

Proof. For any given $\Phi\in\Sigma^-(i,j,k)$, we have $-\Phi\in\Sigma(i,j,k)$. Thus, by Lemma 2.5, we have $X_\alpha(-\Phi)\in\Sigma^-(k,i,j)$. That is,

$$-X_\alpha(-\Phi)\in\Sigma(k,i,j). \tag{2.35}$$

As $X(t,-\Phi) = -X(t,\Phi)$ due to the symmetry of f, we obtain

$$X_\alpha(\theta,\Phi) = X(\alpha+\theta,\Phi) = -X(\alpha+\theta,-\Phi) = -X_\alpha(\theta,-\Phi),\quad \theta\in[-1,0].$$

This, together with (2.35), yields $X_\alpha(\Phi)\in\Sigma(k,i,j)$. This completes the proof. □

Corollary 2.7. *We have, for any* $(i,j,k)\in K_3$, *the following*

$$X_{2\alpha} : \Sigma(i,j,k)\to\Sigma(j,k,i) \tag{2.36}$$

and

$$X_{6\alpha} : \Sigma(i,j,k)\to\Sigma(i,j,k). \tag{2.37}$$

Proof. This is an immediate consequence of Lemma 2.5 and Corollary 2.6, using the semigroup property of X. □

We can now state the main result of this section.

Theorem 2.8. *Assume that* $a<0$ *and* $b>0$ *satisfy* (2.5). *Then*

(i) *For any* $\Phi,\Psi\in\Sigma(1,2,3)$, $X(t,\Phi) = X(t,\Psi)$ *for all* $t\geqslant 0$.
(ii) *Let* $P(t) = X(t,\Phi)$ *for* $t\geqslant 0$ *and for a given* $\Phi\in\Sigma(1,2,3)$. *Then* $P(t+6\alpha) = P(t)$ *for all* $t\geqslant 0$, *and P is phased-locked in the sense*

$$p_1(t) = p_2(t+2\alpha),\quad p_2(t) = p_3(t+2\alpha),\quad p_3(t) = p_1(t+2\alpha),\quad t\geqslant 0, \tag{2.38}$$

and satisfies the following additional symmetry property:

$$p_1(t) = -p_3(t+\alpha),\quad p_2(t) = -p_1(t+\alpha),\quad p_3(t) = -p_2(t+\alpha),\quad t\geqslant 0. \tag{2.39}$$

Proof. (i) is obvious since $\Phi\sim\Psi$ if they are both elements in $\Sigma(1,2,3)$, by Lemma 2.3. To prove (ii), we note that if $\Phi\in\Sigma(1,2,3)$ then $X_{6\alpha}(\Phi)\in\Sigma(1,2,3)$ by Corollary 2.7. Therefore, $\Phi\sim X_{6\alpha}(\Phi)$ and thus, $X(t,\Phi) = X(t,X_{6\alpha}(\Phi))$. This shows that $P(t) = X(t,\Phi)$ is 6α-periodic. Note also that $X_\alpha(\Phi)\in\Sigma^-(3,1,2)$ and hence, by the symmetry of f and Lemma 2.3, we get $p_1(t) = -p_3(t+\alpha)$, $p_2(t) = -p_1(t+\alpha)$ and $p_3(t) = -p_2(t+\alpha)$ from which (2.39) follows. This completes the proof. □

3. A 3×3 matrix and a closed convex set: preparation

Throughout the remaining part of this paper, we will use $O(\varepsilon)$ to denote a function or a constant vector bounded by a constant multiple of $|\varepsilon|$, and $o(\varepsilon)$ to denote a function or a constant vector whose norm divided by $|\varepsilon|$ goes to zero as $|\varepsilon| \to 0$, where ε is a vector in a certain Euclidean space.

We need the following asymptotic expansion, which clearly shows that $\alpha \to 1/2$ as $\tau \to \infty$.

Lemma 3.1. *If $A := ke^{-\tau} \in (1, 2)$, then as $\tau \to \infty$, we have the following*:

$$1 - e^{\tau(1-2\alpha)} = A^{-1}e^{-2\tau\alpha}(A-1) + A^{-1}(2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha}). \tag{3.1}$$

Proof. Eq. (2.6) which determines α can, by using (2.4), be written as

$$e^{-\tau\alpha} - 2e^{-2\tau\alpha} + e^{-3\tau\alpha} = Ae^{\tau\alpha}[e^{\tau(1-2\alpha)} - 1 + e^{\tau(1-4\alpha)} - e^{-3\tau\alpha}].$$

Therefore,

$$\begin{aligned} 1 - e^{\tau(1-2\alpha)} &= e^{\tau(1-4\alpha)} - e^{-3\tau\alpha} - A^{-1}e^{-\tau\alpha}[e^{-\tau\alpha} - 2e^{-2\tau\alpha} + e^{-3\tau\alpha}] \\ &= A^{-1}e^{-2\tau\alpha}[Ae^{\tau(1-2\alpha)} - Ae^{-\tau\alpha} - 1 + 2e^{-\tau\alpha} - e^{-2\tau\alpha}] \\ &= A^{-1}e^{-2\tau\alpha}[A(e^{\tau(1-2\alpha)} - 1) + A - 1 + (2-A)e^{-\tau\alpha} - e^{-2\tau\alpha}]. \end{aligned}$$

That is,

$$[1 - e^{\tau(1-2\alpha)}][1 + e^{-2\tau\alpha}] = A^{-1}e^{-2\tau\alpha}[A - 1 + (2-A)e^{-\tau\alpha} - e^{-2\tau\alpha}],$$

from which it follows that

$$\begin{aligned} 1 - e^{\tau(1-2\alpha)} &= A^{-1}e^{-2\tau\alpha}[A - 1 + (2-A)e^{-\tau\alpha} - e^{-2\tau\alpha}][1 - e^{-2\tau\alpha} + O(e^{-4\tau\alpha})] \\ &= A^{-1}e^{-2\tau\alpha}[A - 1 + (2-A)e^{-\tau\alpha} - e^{-2\tau\alpha} \\ &\quad - (A-1)e^{-2\tau\alpha} - (2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha})] \\ &= A^{-1}e^{-2\tau\alpha}(A-1) + A^{-1}(2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha}). \end{aligned}$$

This completes the proof. □

Let

$$
\begin{aligned}
a_{12} &= [-a+2b-2be^{\tau(1-\alpha)}]\frac{e^{-2\tau\alpha}}{-a+2b},\\
a_{13} &= [-a+2b-2be^{\tau(1-\alpha)}]\frac{2b\tau e^{\tau(1-3\alpha)}}{-a+2b}+2b\tau e^{\tau(1-2\alpha)},\\
a_{22} &= (c_2+a)\frac{e^{-2\tau\alpha}}{-a+2b}-\frac{2a}{-a+2b}\,e^{\tau(1-3\alpha)},\\
a_{23} &= (c_2+a)\frac{2b}{-a+2b}\,\tau e^{\tau(1-3\alpha)}-2a\,\frac{2b}{-a+2b}\,\tau e^{\tau(2-4\alpha)}+2\tau a e^{\tau(1-2\alpha)},\\
a_{32} &= \frac{e^{-\tau\alpha}}{\tau(-a+2b)},\\
a_{33} &= \left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}.
\end{aligned}
$$

Recall that $k=(a-2b)/a$ implies that

$$\frac{-a}{-a+2b}=\frac{1}{k},\quad \frac{2b}{-a+2b}=1-\frac{1}{k}.$$

Furthermore, note that

$$c_2+a=ae^{\tau(1-2\alpha)}+(-a+2b)e^{\tau(1-2\alpha)}-(-a+2b)e^{-\tau\alpha}.$$

Consequently,

$$
\begin{aligned}
a_{12} &= \left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-\alpha)}\right]e^{-2\tau\alpha}\\
&= \left[1-e^{\tau(1-\alpha)}+\frac{1}{k}\,e^{\tau(1-\alpha)}\right]e^{-2\tau\alpha}\\
&= [1-e^{\tau(1-\alpha)}+A^{-1}e^{-\tau\alpha}]e^{-2\tau\alpha}.
\end{aligned}
$$

Therefore, since $\alpha\to 1/2$ as $\tau\to\infty$, we have

Lemma 3.2. *If $A=ke^{-\tau}\in(1,2)$ then there exists $\tau_{12}>0$ so that $a_{12}<0$ if $\tau>\tau_{12}$.*

Note that

$$
\begin{aligned}
a_{13} &= \left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-\alpha)}\right]2b\tau e^{\tau(1-3\alpha)}+2b\tau e^{\tau(1-2\alpha)}\\
&= 2b\tau e^{\tau(1-3\alpha)}-\left(1-\frac{1}{k}\right)e^{\tau(2-4\alpha)}2b\tau+2b\tau e^{\tau(1-2\alpha)}
\end{aligned}
$$

$$= 2b\tau e^{\tau(1-3\alpha)} + 2b\tau e^{\tau(1-2\alpha)}\left[1 - \left(1 - \frac{1}{k}\right)e^{\tau(1-2\alpha)}\right]$$
$$> 0,$$

since $2\alpha > 1$.

We also have

$$\begin{aligned} a_{22} &= [ae^{\tau(1-2\alpha)} + (-a+2b)e^{\tau(1-2\alpha)} - (-a+2b)e^{-\tau\alpha}]\,\frac{e^{-2\tau\alpha}}{-a+2b} - \frac{2a}{-a+2b}e^{\tau(1-3\alpha)} \\ &= e^{\tau(1-4\alpha)} - e^{-3\tau\alpha} - \frac{1}{k}e^{\tau(1-4\alpha)} + \frac{2}{k}e^{\tau(1-3\alpha)} \\ &= e^{\tau(1-4\alpha)}\left[1 - e^{\tau(-1+\alpha)} - \frac{1}{k} + \frac{2}{k}e^{\tau\alpha}\right] \\ &> 0, \end{aligned}$$

since $2e^{\tau\alpha} > 1$ and $\alpha < 1$.

We note further, by using Lemma 3.1, that

$$\begin{aligned} a_{23} &= 2b\tau e^{\tau(1-3\alpha)}\left[e^{\tau(1-2\alpha)} - e^{-\tau\alpha} - \frac{1}{k}e^{\tau(1-2\alpha)}\right] \\ &\quad + \frac{2}{k}2b\tau e^{\tau(2-4\alpha)} + 2\tau a e^{\tau(1-2\alpha)} \\ &= 2b\tau e^{\tau(2-5\alpha)}\left[1 - \frac{1}{k} - e^{\tau(-1+\alpha)}\right] \\ &\quad + 2\tau e^{\tau(1-2\alpha)}\left[\frac{2b}{k}e^{\tau(1-2\alpha)} + a\right] \\ &= (-a+2b)\left[\left(1 - \frac{1}{k}\right)\tau e^{\tau(2-5\alpha)}\left(1 - \frac{1}{k} - e^{\tau(-1+\alpha)}\right)\right. \\ &\quad \left. + 2\tau e^{\tau(1-2\alpha)}\left[\left(1 - \frac{1}{k}\right)\frac{1}{k}e^{\tau(1-2\alpha)} - \frac{1}{k}\right]\right]. \end{aligned}$$

That is,

$$\begin{aligned} a_{23} &= (-a+2b)\tau e^{\tau(1-2\alpha)}\left[\left(1 - \frac{1}{k} - e^{\tau(-1+\alpha)}\right)\left(1 - \frac{1}{k}\right)e^{\tau(1-3\alpha)}\right. \\ &\quad \left. + 2\left[\frac{1}{k}\left(1 - \frac{1}{k}\right)e^{\tau(1-2\alpha)} - \frac{1}{k}\right]\right] \\ &= (-a+2b)\tau e^{\tau(1-2\alpha)}[e^{\tau(1-3\alpha)} - e^{-2\tau\alpha} \\ &\quad + \frac{1}{k}(2e^{\tau(1-2\alpha)} - 2e^{\tau(1-3\alpha)} - 2 + e^{-2\tau\alpha}) \end{aligned}$$

$$+ \frac{1}{k^2}(e^{\tau(1-3\alpha)} - 2e^{\tau(1-2\alpha)})]$$

$$= (-a+2b)\tau e^{\tau(1-4\alpha)}[e^{\tau(1-\alpha)} - 1$$

$$+ A^{-1}(2 + e^{-\tau} - 2e^{-\tau\alpha} - 2e^{\tau(-1+2\alpha)})$$

$$+ A^{-2}e^{-\tau}(e^{-\tau\alpha} - 2)]$$

$$= (-a+2b)\tau e^{\tau(1-4\alpha)}[e^{\tau(1-\alpha)} + O(1)] > 0 \quad \text{as } \tau \to \infty,$$

since $\alpha \in (1/2, 1)$ and $-a+2b>0$. Therefore, we have

Lemma 3.3. $a_{13}>0$, $a_{22}>0$. *Moreover, if* $A = ke^{-\tau} \in (1,2)$ *then there exists* $\tau_{23}>0$ *so that* $a_{23}>0$ *if* $\tau>\tau_{23}$.

In what follows, we assume that $\tau > \max\{\tau_{12}, \tau_{23}\}$ and $A = ke^{-\tau} \in (1,2)$. So the following is a nonnegative matrix:

$$\Theta = \begin{pmatrix} 0 & -a_{12} & a_{13} \\ e^{-\tau\alpha} & a_{22} & a_{23} \\ 0 & \frac{e^{-\tau\alpha}}{\tau(-a+2b)} & (1-\frac{1}{k})e^{\tau(1-2\alpha)} \end{pmatrix}.$$

Lemma 3.4. *There exists* $\tau^* > \max\{\tau_{12}, \tau_{23}\}$ *so that for every* $\tau > \tau^*$ *there exist a constant* $\rho = \rho(\tau) \in (0,1)$ *and a positive vector* $z = z(\tau) = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ *so that*

$$\Theta z = \rho z. \tag{3.2}$$

Proof. Let $h(\lambda) = \det(\lambda I - \Theta)$, $\lambda \in \mathbb{R}$. Then

$$0 < \lambda_1 := \left(1 - \frac{1}{k}\right)e^{\tau(1-2\alpha)} < 1$$

and

$$h\left(\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right) = \frac{e^{-\tau\alpha}}{\tau(-a+2b)} \det\begin{pmatrix} (1-\frac{1}{k})e^{\tau(1-2\alpha)} & -a_{13} \\ -e^{-\tau\alpha} & -a_{23} \end{pmatrix} < 0.$$

Moreover,

$$h(1) = \det\begin{pmatrix} 1 & a_{12} & -a_{13} \\ -e^{-\tau\alpha} & 1-a_{22} & -a_{23} \\ 0 & -\frac{e^{-\tau\alpha}}{\tau(-a+2b)} & 1-(1-\frac{1}{k})e^{\tau(1-2\alpha)} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & a_{12} & -a_{13} \\ 0 & 1-a_{22}+a_{12}e^{-\tau\alpha} & -a_{23}-a_{13}e^{-\tau\alpha} \\ 0 & -\frac{e^{-\tau\alpha}}{\tau(-a+2b)} & 1-(1-\frac{1}{k})e^{\tau(1-2\alpha)} \end{pmatrix}$$

$$= (1-a_{22}+a_{12}e^{-\tau\alpha})\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right]$$
$$-\frac{e^{-\tau\alpha}}{\tau(-a+2b)}(a_{23}+a_{13}e^{-\tau\alpha})$$
$$= (1-a_{22}-a_{12}e^{-\tau\alpha})\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right]$$
$$-\frac{e^{-\tau\alpha}}{\tau(-a+2b)}(a_{23}+a_{13}e^{-\tau\alpha})$$
$$+\ 2a_{12}e^{-\tau\alpha}\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right].$$

Hence,

$$h(1) = \left\{1-e^{\tau(1-4\alpha)}\left[1-e^{\tau(-1+\alpha)}-\frac{1}{k}+\frac{2}{k}e^{\tau\alpha}\right]\right.$$
$$\left.-\left[1-e^{\tau(1-\alpha)}+\frac{1}{k}e^{\tau(1-\alpha)}\right]e^{-3\tau\alpha}\right\}\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right]$$
$$-\ e^{-\tau\alpha}\left\{e^{\tau(1-2\alpha)}\left(1-\frac{1}{k}-e^{\tau(-1+\alpha)}\right)\left(1-\frac{1}{k}\right)e^{\tau(1-3\alpha)}\right.$$
$$+\ 2e^{\tau(1-2\alpha)}\left[\frac{1}{k}\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}-\frac{1}{k}\right]$$
$$\left.+\left(1-\frac{1}{k}\right)e^{\tau(1-4\alpha)}+\left(1-\frac{1}{k}\right)e^{\tau(1-3\alpha)}\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right]\right\}$$
$$+\ 2a_{12}e^{-\tau\alpha}\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right].$$

With straightforward calculations, we get

$$h(1) = 1-e^{\tau(1-2\alpha)}-e^{\tau(1-4\alpha)}+\frac{1}{k}[e^{\tau(1-2\alpha)}+e^{\tau(1-4\alpha)}]$$
$$+\ 2a_{12}e^{-\tau\alpha}\left[1-\left(1-\frac{1}{k}\right)e^{\tau(1-2\alpha)}\right].$$

Using $k = Ae^{\tau}$ and (3.1), we get

$$\begin{aligned}
h(1) &= 1 - e^{\tau(1-2\alpha)} - e^{\tau(1-4\alpha)} + A^{-1}e^{-2\tau\alpha} + A^{-1}e^{-4\tau\alpha}] \\
&\quad + 2e^{-3\tau\alpha}[1 - e^{\tau(1-\alpha)} + A^{-1}e^{-\tau\alpha}][1 - e^{\tau(1-2\alpha)} + A^{-1}e^{-2\tau\alpha}] \\
&= A^{-1}(A-1)e^{-2\tau\alpha} + A^{-1}(2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha}) \\
&\quad - e^{\tau(1-4\alpha)} + A^{-1}e^{-2\tau\alpha} + A^{-1}e^{-4\tau\alpha} \\
&\quad + 2e^{\tau(1-4\alpha)}[-1 + e^{\tau(-1+\alpha)} + A^{-1}e^{-\tau}][1 - e^{\tau(1-2\alpha)} + A^{-1}e^{-2\tau\alpha}] \\
&= e^{-2\tau\alpha} - e^{\tau(1-4\alpha)} + A^{-1}(2-A)e^{-3\tau\alpha} + O(e^{-4\tau\alpha}) \\
&\quad + 2e^{\tau(1-4\alpha)}[-1 + e^{\tau(-1+\alpha)} + A^{-1}e^{-\tau}][1 - e^{\tau(1-2\alpha)} + A^{-1}e^{-2\tau\alpha}] \\
&= e^{-2\tau\alpha}[1 - e^{\tau(1-2\alpha)}] + \frac{2-A}{A}e^{-3\tau\alpha} + O(e^{-4\tau\alpha}) \\
&\quad + 2e^{\tau(1-4\alpha)}[-1 + e^{\tau(-1+\alpha)} + A^{-1}e^{-\tau}][1 - e^{\tau(1-2\alpha)} + A^{-1}e^{-2\tau\alpha}] \\
&= \frac{2-A}{A}e^{-3\tau\alpha} + O(e^{-4\tau\alpha}).
\end{aligned}$$

In the last equality, we used the fact that $1 - e^{\tau(1-2\alpha)} = O(e^{-2\tau\alpha})$ from (3.1) and the fact that $1 - 4\alpha < -2\alpha$. Therefore, $h(1) > 0$ if $1 < A < 2$ and if τ is sufficiently large.

Consequently, there exists $\tau^* > 0$ so that if $A \in (1, 2)$ and $\tau > \tau^*$ then there exists $\rho \in (\lambda_1, 1)$ so that $h(\rho) = 0$. Let $z = (z_1, z_2, z_3)^T$ be an eigenvector associated with the eigenvalue ρ of Θ with $z_3 > 0$. Substituting this to the third equation of $\Theta z = \rho z$ and noting that $\rho > \lambda_1$, we get $z_2 > 0$. Substituting this result to the first equation of $\Theta z = \rho z$, we get $z_1 > 0$. This completes the proof. $\square$

We now introduce the restriction for the nonlinearity. Let

$$\begin{aligned}
N(M, \beta, \varepsilon) = \{ f : \mathbb{R} \to \mathbb{R};\ & f \text{ is continuous and odd}, \\
& |f(x)| \leqslant M \text{ for } x \in \mathbb{R},\ |f(x) - 1| \leqslant \varepsilon \text{ if } x \geqslant \beta \}.
\end{aligned}$$

We also need to restrict the initial functions to a certain closed and convex set containing the phase-locked periodic solution constructed in the last section. Let $(\alpha_0, \delta_2, \delta_3, D, \beta)$ be given positive constants with $0 < \alpha_0 \leqslant \frac{1}{2}$

and define

$$
\begin{aligned}
A(\alpha_0,\delta_2,\delta_3,D,\beta) = \{\Phi = (\phi_1,\phi_2,\phi_3)\in\Sigma;\;& \\
&\phi_1(s)\geqslant\beta \text{ for } s\in[-1,0], \phi_1(0)=\beta,\\
&\phi_1 \text{ is nonincreasing on } [-2\alpha_0,0] \text{ and}\\
&\phi_1(t)-\phi_1(s)\leqslant -D(t-s) \text{ for } -2\alpha_0\leqslant s\leqslant t\leqslant 0;\\
&\phi_2(s)\leqslant -\beta \text{ for } s\in[-1,-\alpha-\alpha_0], \phi_2(s)\geqslant\beta \text{ for } s\in[-\alpha+\alpha_0,0],\\
&\phi_2 \text{ is nondecreasing on } [-\alpha-\alpha_0,-\alpha+\alpha_0] \text{ and}\\
&\phi_2(t)-\phi_2(s)\geqslant D(t-s) \text{ for } -\alpha-\alpha_0\leqslant s\leqslant t\leqslant -\alpha+\alpha_0,\\
&|\phi_2(0)-c_2|\leqslant\delta_2;\\
&\phi_3(s)\leqslant -\beta, \text{ for } s\in[-1,0], |\phi_3(0)-c_3|\leqslant\delta_3\}.
\end{aligned}
$$

It is easy to prove that

Lemma 3.5. *For fixed positive constants* $(\alpha_0,\delta_2,\delta_3,D,\beta)$ *with* $0<\alpha_0\leqslant\frac{1}{2}$, *the set* $A(\alpha_0,\delta_2,\delta_3,D,\beta)$ *is a closed and convex subset of the Banach space* Σ *equipped with the usual super-norm.*

4. Existence: singular perturbation

In this section, we consider the existence of phase-locked periodic solutions of (1.3) with a general nonlinearity close to the step function (1.2). Our approach is to choose sufficiently small $(\alpha_0,\delta_2,\delta_3)$ along the eigenvector z of Θ associated with $\rho\in(0,1)$, and then choose appropriate D and sufficiently small β and ε so that the solution of (1.3) through a given $\Phi\in A(\alpha_0,\delta_2,\delta_3,D,\beta)$ will return to $A(\alpha_0,\delta_2,\delta_3,D,\beta)$, and thus the fixed points of such a returning map give rise the phase-locked periodic solutions.

In what follows, we are going to fix $f\in N(M,\beta,\varepsilon)$ and $\Phi\in A(\alpha_0,\delta_2,\delta_3,D,\beta)$ with $0<\alpha_0\leqslant\frac{1}{2}$. We also let $-\alpha_1 := -\alpha_1(\Phi)$ be the smallest number in $[-\alpha-\alpha_0,-\alpha+\alpha_0]$ so that $\phi_2(-\alpha_1)=-\beta$ and let $-\alpha_2 := -\alpha_2(\Phi)$ be the largest number in $[-\alpha-\alpha_0,-\alpha+\alpha_0]$ so that $\phi_2(-\alpha_2)=\beta$.

Then $\alpha_1>\alpha_2$ and, using $\phi_2(t)-\phi_2(s)\geqslant D(t-s)$ for all $-\alpha-\alpha_0\leqslant s\leqslant t\leqslant -\alpha+\alpha_0$ we get $2\beta\geqslant D(\alpha_1-\alpha_2)$ from which it follows that

$$\alpha_1-\alpha_2\leqslant\frac{2\beta}{D}. \tag{4.1}$$

This observation turns out to be essential: while we cannot control the locations where ϕ_2 crosses $\pm\beta$, we can control the distance of $\alpha_1 - \alpha_2$ by β.

We now estimate x_3 for the corresponding solution $X(\cdot, \Phi)$. On $[0, 1-\alpha_1]$, we have

$$\dot{x}_3 = -\tau x_3 - a\tau + r_3^1(t),$$

$$|r_3^1(t)| \leqslant (-a+2b)\tau\varepsilon,$$

and thus,

$$x_3(t) = e^{-\tau t}x_3(0) - a[1-e^{-\tau t}] + O(\varepsilon) \tag{4.2}$$

with

$$x_3(1-\alpha_1) = e^{-\tau(1-\alpha_1)}x_3(0) - a[1-e^{-\tau(1-\alpha_1)}] + O(\varepsilon).$$

On $[1-\alpha_1, 1-\alpha_2]$, we have

$$\dot{x}_3 = -\tau x_3 + (-a+b)\tau + r_3^2(t),$$

$$|r_3^2(t)| \leqslant (-a+b)\tau\varepsilon + b\tau M,$$

$$x_3(t) = e^{-\tau(t-1+\alpha_1)}x_3(1-\alpha_1) + (-a+b)[1-e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_3^*(t),$$

$$|r_3^*(t)| \leqslant bM[1-e^{-\tau(t-1+\alpha_1)}].$$

Using the above expression for $x_3(1-\alpha_1)$, we get

$$\begin{aligned} x_3(t) &= e^{-\tau t}x_3(0) - ae^{-\tau(t-1+\alpha_1)}[1-e^{-\tau(1-\alpha_1)}] \\ &\quad + (-a+b)[1-e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_3^*(t) \\ &= e^{-\tau t}[x_3(0)+a] - ae^{-\tau(t-1+\alpha_1)} \\ &\quad + (-a+b)[1-e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_3^*(t). \end{aligned} \tag{4.3}$$

In particular,

$$\begin{aligned} x_3(1-\alpha_2) &= e^{-\tau(1-\alpha_2)}[x_3(0)+a] - ae^{-\tau(\alpha_1-\alpha_2)} \\ &\quad + (-a+b)[1-e^{-\tau(\alpha_1-\alpha_2)}] + O(\varepsilon) + r_3^*(1-\alpha_2). \end{aligned}$$

On $[1-\alpha_2, 1]$, we have

$$\dot{x}_3 = -\tau x_3 + (-a+2b)\tau + r_3^3(t),$$

$$|r_3^3(t)| \leqslant (-a+2b)\tau\varepsilon = O(\varepsilon),$$

$$x_3(t) = e^{-\tau(t-1+\alpha_2)} x_3(1-\alpha_2) + (-a+2b)[1-e^{-\tau(t-1+\alpha_2)}] + O(\varepsilon).$$

Using the above expression for $x_3(1-\alpha_2)$, we get

$$\begin{aligned} x_3(t) =& e^{-\tau t}[x_3(0)+a] - ae^{-\tau(t-1+\alpha_1)} \\ &+ (-a+b)e^{-\tau(t-1+\alpha_2)}[1-e^{-\tau(\alpha_1-\alpha_2)}] + O(\varepsilon) \\ &+ e^{-\tau(t-1+\alpha_2)} r_3^*(1-\alpha_2) + (-a+2b)[1-e^{-\tau(t-1+\alpha_2)}] \\ =& e^{-\tau t}[x_3(0)+a] - ae^{-\tau(t-1+\alpha_1)} + (-a+2b)[1-e^{-\tau(t-1+\alpha_2)}] \\ &+ (-a+b)[e^{-\tau(t-1+\alpha_2)} - e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_3(t). \end{aligned} \tag{4.4}$$

In particular,

$$|r_3(t)| \leqslant bM[1-e^{-\tau(\alpha_1-\alpha_2)}]e^{-\tau(t-1+\alpha_2)}.$$

We need to determine the first $T \in (1-\alpha_2, 1)$ such that

$$x_3(T) = -\beta.$$

That is,

$$\begin{aligned} & e^{-\tau T}[x_3(0)+a] - ae^{-\tau(T-1+\alpha_1)} + (-a+2b) - (-a+2b)e^{-\tau(T-1+\alpha_2)} \\ & \quad + (-a+b)[e^{-\tau(T-1+\alpha_2)} - e^{-\tau(T-1+\alpha_1)}] + O(\varepsilon) + r_3(T) \\ & = -\beta. \end{aligned}$$

Note also that (2.1) implies that

$$-ae^{\tau(1-2\alpha)} + (a+c_3)e^{-\tau\alpha} + (a-2b)e^{\tau(1-2\alpha)} = a-2b.$$

Therefore,

$$e^{-\tau\alpha} = \frac{a-2b}{a+c_3-ae^{\tau(1-\alpha)}-(-a+2b)e^{\tau(1-\alpha)}} := \frac{a-2b}{\Gamma},$$

where

$$\Gamma = a + c_3 - ae^{\tau(1-\alpha)} - (-a+2b)e^{\tau(1-\alpha)}.$$

Then

$$\begin{aligned} & e^{-\tau T}\{\Gamma + x_3(0) - c_3 + a[e^{\tau(1-\alpha)} - e^{\tau(1-\alpha_1)}] \\ & \quad + (-a+2b)[e^{\tau(1-\alpha)} - e^{\tau(1-\alpha_2)}] + (-a+b)[e^{\tau(1-\alpha_2)} - e^{\tau(1-\alpha_1)}] + e^{\tau T} r_3(T)\} \\ & = -(-a+2b) - \beta + O(\varepsilon). \end{aligned}$$

That is,

$$e^{-\tau T}[\Gamma + x_3(0) - c_3 + e^{\tau T} r_3(T) + b\Delta] = -\beta - (-a+2b) + O(\varepsilon),$$

where

$$\begin{aligned} \Delta &= 2e^{\tau(1-\alpha)} - e^{\tau(1-\alpha_1)} - e^{\tau(1-\alpha_2)} \\ &= 2e^{\tau(1-\alpha)} - e^{\tau(1-\alpha-\tilde{\alpha}_0-\delta)} - e^{\tau(1-\alpha-\tilde{\alpha}_0)} \end{aligned}$$

with

$$\alpha_2 = \alpha + \tilde{\alpha}_0, \quad \alpha_1 = \alpha + \tilde{\alpha}_0 + \delta.$$

Therefore, by the choice of α_1 and α_2 and by (4.1), we have

$$|\tilde{\alpha}_0| \leqslant \alpha_0, \quad 0 < \delta \leqslant \frac{2\beta}{D}. \tag{4.5}$$

Note that

$$\begin{aligned} \left.\frac{\partial \Delta}{\partial \tilde{\alpha}_0}\right|_{\tilde{\alpha}_0=\delta=0} &= 2\tau e^{\tau(1-\alpha)}, \\ \left.\frac{\partial \Delta}{\partial \delta}\right|_{\tilde{\alpha}_0=\delta=0} &= \tau e^{\tau(1-\alpha)}. \end{aligned}$$

We get

$$\Delta = 2\tau e^{\tau(1-\alpha)}\tilde{\alpha}_0 + \tau e^{\tau(1-\alpha)}\delta + o(\alpha_0, \beta), \tag{4.6}$$

where we used (4.5) to write $o(\alpha_0, \beta)$ for $o(\tilde{\alpha}_0, \delta)$. Then,

$$\begin{aligned} e^{-\tau T} - e^{-\tau\alpha} &= \frac{-\beta - (-a+2b) + O(\varepsilon)}{\Gamma + x_3(0) - c_3 + e^{\tau T} r_3(T) + b\Delta} - \frac{-a+2b}{\Gamma} \\ &= \frac{-\beta + O(\varepsilon)}{\Gamma + x_3(0) - c_3 + e^{\tau T} r_3(T) + b\Delta} \\ &\quad - \frac{(-a+2b)[x_3(0) - c_3 + b\Delta + e^{\tau T} r_3(T)]}{\Gamma[\Gamma + x_3(0) - c_3 + e^{\tau T} r_3(T) + b\Delta]} \end{aligned}$$

and

$$|r_3(T)| \leqslant bM[1 - e^{-\tau(\alpha_1-\alpha_2)}]e^{-\tau(T-1+\alpha_2)}.$$

Hence,

$$e^{-\tau T} - e^{-\tau\alpha} = O(\beta,\varepsilon) + \frac{-a+2b}{\Gamma^2}[x_3(0) - c_3] + \frac{b(-a+2b)}{\Gamma^2}\Delta + r_3^* + o(\delta_3,\delta,\alpha_0)$$

and

$$\begin{aligned}|r_3^*| &\leqslant \frac{-a+2b}{\Gamma^2} bM[1 - e^{-\tau(\alpha_1-\alpha_2)}]e^{-\tau(-1+\alpha_2)}\\ &= \frac{-a+2b}{\Gamma^2} bM[1 - e^{-\tau\delta}]e^{\tau(1-\alpha-\tilde{\alpha}_0)}\\ &= \frac{-a+2b}{\Gamma^2} bM(\tau\delta)e^{\tau(1-\alpha)} + o(\alpha_0,\delta).\end{aligned}$$

Consequently, using (4.1) and (4.6), we get

$$\begin{aligned}e^{-\tau T} - e^{-\tau\alpha} = O(\beta,\varepsilon) &+ \frac{-a+2b}{\Gamma^2}[x_3(0) - c_3]\\ &+ \frac{b(-a+2b)}{\Gamma^2}[2\tau e^{\tau(1-\alpha)}\tilde{\alpha}_0 + \tau e^{\tau(1-\alpha)}\delta]\\ &+ \tilde{r}_3^* + o(\delta_3,\delta,\alpha_0)\end{aligned}$$

with

$$|\tilde{r}_3^*| \leqslant \frac{-a+2b}{\Gamma^2} bM(\tau\delta)e^{\tau(1-\alpha)}.$$

Recall that

$$e^{-\tau\alpha} = \frac{-(-a+2b)}{\Gamma},$$

we have

$$\frac{-a+2b}{\Gamma^2} = \frac{-a+2b}{(-a+2b)^2 e^{2\tau\alpha}} = \frac{e^{-2\tau\alpha}}{-a+2b},$$

from which it follows that

$$
\begin{aligned}
e^{-\tau T} - e^{-\tau\alpha} = {} & O(\beta, \varepsilon) + o(\delta_3, \delta, \alpha_0) \\
& + \frac{e^{-2\tau\alpha}}{-a+2b}[x_3(0) - c_3] + \frac{2b\tau e^{\tau(1-3\alpha)}}{-a+2b}\tilde{\alpha}_0 \\
& + \frac{b\tau e^{\tau(1-3\alpha)}}{-a+2b}\delta + \tilde{r}_3^* \qquad (4.7)
\end{aligned}
$$

with

$$|\tilde{r}_3^*| \leqslant \frac{bM\tau\delta}{-a+2b} e^{\tau(1-3\alpha)}.$$

As there exists χ between $-\tau\alpha$ and $-\tau T$ such that

$$
\begin{aligned}
e^{-\tau T} - e^{-\tau\alpha} &= e^{\chi}(-\tau T + \tau\alpha) \\
&= e^{-\tau\alpha}\tau(\alpha - T) + O(\beta, \varepsilon, \delta_3, \alpha_0, \delta)\tau(\alpha - T),
\end{aligned}
$$

we get

$$
\begin{aligned}
\tau(\alpha - T) = {} & O(\beta, \varepsilon) + o(\delta_3, \delta, \alpha_0) \\
& + \frac{e^{-\tau\alpha}}{-a+2b}[x_3(0) - c_3] + \frac{2b\tau e^{\tau(1-2\alpha)}}{-a+2b}\tilde{\alpha}_0 \\
& + \frac{b\tau e^{\tau(1-2\alpha)}}{-a+2b}\delta + \tilde{r}_3^* e^{\tau\alpha}
\end{aligned}
$$

with

$$|\tilde{r}_3^* e^{\tau\alpha}| \leqslant \frac{bM\tau\delta}{-a+2b} e^{\tau(1-2\alpha)}.$$

In other words, by (4.1) and (4.5) we have

$$|\alpha - T| \leqslant O(\beta, \varepsilon) + o(\delta_3, \alpha_0) + a_{32}\delta_3 + a_{33}\alpha_0. \qquad (4.8)$$

Lemma 4.1. *Assume all conditions in Lemma* 3.4 *hold and fix* $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ *as given in Lemma* 3.4. *Then there exist* $M_3 > 0$ *and* $m_{3,0} > 0$ *such that for each* $m \in (0, m_{3,0})$ *there exist* $\beta_{3,0} = \beta_{3,0}(m)$ *and* $\varepsilon_{3,0} = \varepsilon_{3,0}(m)$ *so that if* $0 < \beta < \beta_{3,0}$ *and* $0 < \varepsilon < \varepsilon_{3,0}$ *and if* $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$, *then*

(i) $|x_3(t)| \leqslant M_3$ *and* $x_3(t) = p_3(t) + O(\beta, \varepsilon, \delta_3, \alpha_0)$ *for all* $t \in [0, 1]$;
(ii) $|\alpha - T| \leqslant \frac{3+\rho}{4}\alpha_0$;
(iii) $\dot{x}_3(t) \geqslant D_1 := -(c_3 + a)\tau e^{-\tau}/2$ *for all* $t \in [0, 1]$.

Proof. Condition (i) follows from (2.20) and (4.2)–(4.4) directly.

Using (4.8), we get

$$|\alpha - T| \leqslant O(\beta, \varepsilon) + o(\delta_3, \alpha_0) + m(a_{32}z_2 + a_{33}z_3)$$
$$\leqslant O(\beta, \varepsilon) + o(m) + m\rho z_3.$$

Choose $m_{3,0} > 0$ so that if $m \in (0, m_{3,0})$ then $o(m) + m\rho z_3 \leqslant \frac{1}{2}m(1+\rho)z_3$. Thus, we can find $\beta_{3,0} = \beta_{3,0}(m)$ and $\varepsilon_{3,0} = \varepsilon_{3,0}(m)$ so that if $0 < \beta < \beta_{3,0}$ and $0 < \varepsilon < \varepsilon_{3,0}$ and if $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$, then

$$|\alpha - T| \leqslant \frac{1}{2}m(1+\rho)z_3 + \frac{1}{4}m(1-\rho)z_3$$
$$= \frac{3+\rho}{4}mz_3 = \frac{3+\rho}{4}\alpha_0.$$

This proves (ii).

To prove (iii), we note that from (4.2)–(4.5), we get

$$\dot{x}_3(t) = \begin{cases} -[x_3(0)+a]\tau e^{-\tau t} + O(\varepsilon), & t \in [0, 1-\alpha_1], \\ -[x_3(0)+a]\tau e^{-\tau t} + b\tau e^{-\tau(t-1+\alpha_1)} + O(\varepsilon, \beta), & t \in [1-\alpha_1, 1-\alpha_2], \\ -[x_3(0)+a]\tau e^{-\tau t} + b\tau e^{-\tau(t-1)}[e^{-\tau\alpha_2} + e^{-\tau\alpha_1}] + O(\varepsilon, \beta), & t \in [1-\alpha_2, 1]. \end{cases}$$

Therefore, we can choose $m_{3,0}$, $\beta_{3,0} = \beta_{3,0}(m)$ and $\varepsilon_{3,0} = \varepsilon_{3,0}(m)$ so small that if $0 < m < m_{3,0}$, $0 < \beta < \beta_{3,0}$ and $0 < \varepsilon < \varepsilon_{3,0}$ then $\dot{x}_1 \geqslant -(c_3 + a)\tau e^{-\tau}/2$. This completes the proof. □

We now consider the x_1-component. On $[0, 1-\alpha_1]$, we have

$$\dot{x}_1 = -\tau x_1 + (a-2b)\tau + r_1^1(t),$$

$$|r_1^1(t)| \leqslant (-a+2b)\tau\varepsilon.$$

Therefore,

$$x_1(t) = e^{-\tau t}x_1(0) + (a-2b)[1-e^{-\tau t}] + O(\varepsilon) \tag{4.9}$$

and in particular,

$$x_1(1-\alpha_1) = e^{-\tau(1-\alpha_1)}x_1(0) + (a-2b)[1-e^{-\tau(1-\alpha_1)}] + O(\varepsilon).$$

On $[1-\alpha_1, 1-\alpha_2]$, we have

$$\dot{x}_1 = -\tau x_1 + (a-b)\tau + r_1^2(t),$$

$$|r_1^2(t)| \leqslant (-a+b)\tau\varepsilon + b\tau M,$$

$$x_1(t) = e^{-\tau(t-1+\alpha_1)}x_1(1-\alpha_1) + (a-b)[1-e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_1^*(t),$$

$$|r_1^*(t)| \leqslant bM[1-e^{-\tau(t-1+\alpha_1)}].$$

Therefore, we have

$$\begin{aligned} x_1(t) = {} & e^{-\tau t}\beta + (a-2b)e^{-\tau(t-1+\alpha_1)}[1-e^{-\tau(1-\alpha_1)}] \\ & + (a-b)[1-e^{-\tau(t-1+\alpha_1)}] + O(\varepsilon) + r_1^*(t) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} x_1(1-\alpha_2) = {} & e^{-\tau(1-\alpha_2)}\beta - (-a+2b)e^{-\tau(\alpha_1-\alpha_2)} - (a-2b)e^{-\tau(1-\alpha_2)} \\ & + (a-b)[1-e^{-\tau(\alpha_1-\alpha_2)}] + O(\varepsilon) + r_1^*(1-\alpha_2) \\ = {} & e^{-\tau(1-\alpha_2)}[\beta + (-a+2b)] + a - b - be^{-\tau(\alpha_1-\alpha_2)} + O(\varepsilon) + \tilde{r}_1^2 \end{aligned}$$

with

$$|\tilde{r}_1^2| \leqslant bM[1-e^{-\tau(\alpha_1-\alpha_2)}].$$

On $[1-\alpha_2, 1]$, we have

$$\dot{x}_1 = -\tau x_1 + a\tau + r_1^3(t),$$

$$|r_1^3(t)| \leqslant (-a+2b)\tau\varepsilon = O(\varepsilon),$$

$$\begin{aligned} x_1(t) = {} & e^{-\tau(t-1+\alpha_2)}x_1(1-\alpha_2) + a[1-e^{-\tau(t-1+\alpha_2)}] + O(\varepsilon) \\ = {} & e^{-\tau t}[\beta + (-a+2b)] + (a-b)e^{-\tau(t-1+\alpha_2)} - be^{-\tau(t-1+\alpha_1)} \\ & + a[1-e^{-\tau(t-1+\alpha_2)}] + O(\varepsilon) + e^{-\tau(t-1+\alpha_2)}\tilde{r}_1^2, \end{aligned} \tag{4.11}$$

and hence

$$\begin{aligned} x_1(t) = {} & e^{-\tau t}[\beta + (-a+2b)] + a - 2be^{-\tau(t-1+\alpha)} \\ & + b[2e^{-\tau(t-1+\alpha)} - e^{-\tau(t-1+\alpha_2)} - e^{-\tau(t-1+\alpha_1)}] \\ & + O(\varepsilon) + e^{-\tau(t-1+\alpha_2)}\tilde{r}_1^2. \end{aligned}$$

Therefore,

$$\begin{aligned} x_1(T) &= e^{-\tau T}[\beta + (-a+2b)] + a - 2be^{-\tau(T-1+\alpha)} \\ &\quad + be^{-\tau T}[2e^{\tau(1-\alpha)} - e^{\tau(1-\alpha_2)} - e^{\tau(1-\alpha_1)}] \\ &\quad + O(\varepsilon) + e^{-\tau(T-1+\alpha_2)}\tilde{r}_1^2 \\ &= e^{-\tau\alpha}(-a+2b) + a - 2be^{\tau(1-2\alpha)} \\ &\quad + e^{-\tau T}\beta + (e^{-\tau T} - e^{-\tau\alpha})(-a+2b) \\ &\quad + 2be^{\tau(1-2\alpha)} - 2be^{-\tau(T-1+\alpha)} + be^{-\tau T}\Delta \\ &\quad + O(\varepsilon) + e^{-\tau(T-1+\alpha_2)}\tilde{r}_1^2. \end{aligned}$$

Using (2.2), we get

$$\begin{aligned} x_1(T) + c_2 &= e^{-\tau T}\beta + (e^{-\tau T} - e^{-\tau\alpha})[-a+2b-2be^{\tau(1-\alpha)}] \\ &\quad + be^{-\tau T}\Delta + O(\varepsilon) + e^{-\tau(T-1+\alpha_2)}\tilde{r}_1^2. \end{aligned}$$

Note that

$$|e^{-\tau(T-1+\alpha_2)}\tilde{r}_1^2| \leqslant e^{\tau(1-2\alpha)}bM\tau\delta + bM\tau\delta|e^{\tau(1-2\alpha)} - e^{\tau(1-T+\alpha_2)}|.$$

Therefore, we obtain

$$\begin{aligned} x_1(T) + c_2 &= O(\beta,\varepsilon) + r_3^* + [-a+2b-2be^{\tau(1-\alpha)}][e^{-\tau T} - e^{-\tau\alpha}] \\ &\quad + be^{-\tau T}\Delta + bM\tau\delta|e^{\tau(1-2\alpha)} - e^{\tau(1-T-\alpha_2)}| \end{aligned}$$

and

$$|r_3^*| \leqslant bM\tau\delta e^{\tau(1-2\alpha)}.$$

Using (4.6) and (4.7), we get

$$\begin{aligned} x_1(T) + c_2 &= O(\beta,\varepsilon) + o(\delta_3,\delta,\alpha_0) + \tilde{r}_3 \\ &\quad + (-a+2b-2be^{\tau(1-\alpha)})\left[\frac{e^{-2\tau\alpha}}{-a+2b}[x_3(0) - c_3] + \frac{2b\tau e^{\tau(1-3\alpha)}}{-a+2b}\tilde{\alpha}_0\right] \\ &\quad + be^{-\tau\alpha}[2\tau e^{\tau(1-\alpha)}\tilde{\alpha}_0] \end{aligned}$$

with

$$\tilde{r}_3 = O(\delta).$$

Therefore, we have from (4.5) that

$$|x_1(T)+c_2| \leqslant O(\beta,\varepsilon)+o(\delta_3,\alpha_0)+|a_{12}|\delta_3+|a_{13}|\alpha_0. \tag{4.12}$$

Lemma 4.2. *Assume all conditions in Lemma* 3.4 *hold and fix* $z=(z_1,z_2,z_3)^T\in\mathbb{R}^3$ *as given in Lemma* 3.4. *Then there exist* $M_1>0$ *and* $m_{1,0}\in(0,m_{3,0})$ *such that for each* $m\in(0,m_{3,0})$ *there exist* $\beta_{1,0}=\beta_{1,0}(m)\in(0,\beta_{3,0})$ *and* $\varepsilon_{1,0}=\varepsilon_{1,0}(m)\in(0,\varepsilon_{3,0})$ *so that if* $0<\beta<\beta_{1,0}$ *and* $0<\varepsilon<\varepsilon_{1,0}$ *and if* $(\delta_2,\delta_3,\alpha_0)^T=m(z_1,z_2,z_3)^T$ *then*

(i) $|x_1(t)|\leqslant M_1$ *and* $x_1(t)=p_1(t)+O(\alpha_0,\delta_3,\beta,\varepsilon)$ *for all* $t\in[0,1]$;
(ii) $|x_1(T)+c_2|\leqslant\frac{3+\rho}{4}\delta_2$;
(iii) *for the first* $d\in(0,1-\alpha_1)$ *such that* $x_1(d)=-\beta$, *we have*

$$d=\frac{2\beta}{\tau(-a+2b)}+\beta O(\beta,\varepsilon), \tag{4.13}$$

and $x_1(t)<-\beta$ *for all* $t\in[d,1]$;
(iv) $\dot{x}_1(t)\leqslant-\frac{1}{2}(-a+2b)e^{-\tau}:=-D_2$ *on* $[0,1-\alpha-\alpha_0]$.

Proof. Conclusion (i) follows from (2.24) and (4.9)–(4.11). The proof for (ii) is the same as that for (ii) of Lemma 4.1, using (4.12).

To prove (iii) and (iv), we first note that on $[0,1-\alpha_1]$, (4.9) implies that

$$\dot{x}_1=-(-a+2b)e^{-\tau t}-\tau\beta e^{-\tau t}+O(\varepsilon),$$

and hence if $\varepsilon_{1,0}$ and $\beta_{1,0}$ are small, then $\dot{x}_1\leqslant-\frac{1}{2}(-a+2b)e^{-\tau}$ and hence x_1 is decreasing on $[0,1-\alpha_1]$. Using (4.9), we also get

$$x_1(t)=e^{-\tau t}\beta-(-a+2b)+(-a+2b)e^{-\tau t}+O(\varepsilon)$$

from which it follows that

$$-\beta=e^{-\tau d}\beta-(-a+2b)+(-a+2b)e^{-\tau d}+O(\varepsilon).$$

That is,

$$e^{-\tau d}=\frac{-a+2b-\beta+O(\varepsilon)}{-a+2b+\beta},$$

from which it follows that

$$\begin{aligned} e^{\tau d} &= \frac{-a+2b+\beta}{-a+2b-\beta+O(\varepsilon)} \\ &= 1+\frac{2\beta}{-a+2b}+\beta O(\beta,\varepsilon). \end{aligned}$$

Hence

$$d = \frac{1}{\tau}\ln\left(1 + \frac{2\beta}{-a+2b} + \beta O(\beta,\varepsilon)\right) = \frac{2\beta}{\tau(-a+2b)} + \beta O(\beta,\varepsilon).$$

Clearly, $x_1(t) < -\beta$ for $t \in (d, 1-\alpha_1)$. On $[1-\alpha_1, 1-\alpha_2]$, we have from (4.5) and(4.10) that

$$\begin{aligned} x_1(t) &= e^{-\tau t}\beta - (-a+2b)e^{-\tau(t-1+\alpha_1)}[1 - e^{-\tau(1-\alpha_1)}] + O(\varepsilon,\beta) \\ &= -\ (-a+2b)e^{-\tau(t-1+\alpha_1)}[1 - e^{-\tau(1-\alpha_1)}] + O(\varepsilon,\beta) \end{aligned}$$

since $|\tilde{r}_2| \leqslant bM\tau\delta$ and $1 - e^{-\tau(t-1+\alpha_1)} \leqslant 1 - e^{-\tau(\alpha_1-\alpha_2)} \leqslant \tau\delta$. Finally, on $[1-\alpha_2, 1]$, we have from the expression below (4.11) and (4.5) that

$$x_1(t) = a(1 - e^{-\tau t}) + 2be^{-\tau t}(1 - e^{\tau(1-\alpha)}) + O(\beta,\varepsilon,\alpha_0).$$

Therefore, if $m_{1,0}, \varepsilon_{1,0}$ and $\beta_{1,0}$ are small, then $x_1(t) < \beta$ on $[1-\alpha_1, 1]$. This completes the proof. □

Finally, we consider the x_2-component. On $[0, 1-\alpha_1]$, we have

$$\dot{x}_2 = -\tau x_2 - a\tau + r_2^1(t),$$

$$|r_2^1(t)| \leqslant (-a+2b)\tau\varepsilon.$$

Therefore,

$$\begin{aligned} x_2(t) &= e^{-\tau t}x_2(0) - a[1 - e^{-\tau t}] + O(\varepsilon) \\ &= [x_2(0) + a]e^{-\tau t} - a + O(\varepsilon), \end{aligned} \tag{4.14}$$

and hence,

$$x_2(1-\alpha_1) = [x_2(0) + a]e^{-\tau(1-\alpha_1)} - a + O(\varepsilon).$$

On $[1-\alpha_1, 1-\alpha_2]$, we have

$$\dot{x}_2 = -\tau x_2 + r_2^2(t),$$

$$|r_2^2(t)| \leqslant O(\varepsilon) + |a|\tau M.$$

Therefore,

$$\begin{aligned} x_2(t) &= e^{-\tau(t-1+\alpha_1)}x_2(1-\alpha_1) + r_2(t) + O(\varepsilon) \\ &= [x_2(0) + a]e^{-\tau t} - ae^{-\tau(t-1+\alpha_1)} + O(\varepsilon) + r_2(t) \end{aligned} \tag{4.15}$$

with

$$|r_2(t)| \leqslant |a| M[1 - e^{-\tau(t-1+\alpha_1)}].$$

In particular, we have

$$x_2(1-\alpha_2) = [x_2(0) + a]e^{-\tau(1-\alpha_2)} - ae^{\tau(\alpha_2-\alpha_1)} + O(\varepsilon) + r_2^*,$$

$$|r_2^*| \leqslant |a| M[1 - e^{\tau(\alpha_2-\alpha_1)}].$$

On $[1-\alpha_2, 1]$, we have

$$\dot{x}_2 = -\tau x_2 + a\tau + r_2^3(t),$$

$$|r_2^3(t)| \leqslant (-a + 2b)\tau\varepsilon = O(\varepsilon).$$

Hence,

$$\begin{aligned} x_2(t) &= e^{-\tau(t-1+\alpha_2)} x_2(1-\alpha_2) + a[1 - e^{-\tau(t-1+\alpha_2)}] + O(\varepsilon) \\ &= e^{-\tau t}[x_2(0) + a] - ae^{-\tau(t-1+\alpha_1)} + a[1 - e^{-\tau(t-1+\alpha_2)}] \\ &\quad + O(\varepsilon) + e^{-\tau(t-1+\alpha_2)} r_2^* \\ &= e^{-\tau t}[x_2(0) + a] + a - 2ae^{\tau(1-2\alpha)} \\ &\quad + a[2e^{\tau(1-2\alpha)} - e^{-\tau(t-1+\alpha_1)} - e^{-\tau(t-1+\alpha_2)}] + O(\varepsilon) \\ &\quad + e^{-\tau(t-1+\alpha_2)} r_2^*. \end{aligned} \quad (4.16)$$

Using (2.1), (2.2) and (2.6), we obtain

$$c_3 = 2ae^{\tau(1-2\alpha)} - a - (a + c_2)e^{-\tau\alpha}$$

and hence (4.16) implies

$$\begin{aligned} x_2(T) + c_3 &= (c_2 + a)(e^{-\tau T} - e^{-\tau\alpha}) + (x_2(0) - c_2)e^{-\tau T} \\ &\quad + a[2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha_1-T)} - e^{\tau(1-\alpha_2-T)}] \\ &\quad + e^{-\tau(T-1+\alpha_2)} r_2^* + O(\varepsilon) = (c_2 + a)[e^{-\tau T} - e^{-\tau\alpha}] + e^{-\tau\alpha}[x_2(0) - c_2] \end{aligned}$$

$$+ \ a[2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha_1-T)} - e^{\tau(1-\alpha_2-T)}]$$

$$+ \ \tilde{r}_2^* + O(\varepsilon) + o(\delta_2, \alpha_0, \beta, \varepsilon, \delta_3, \delta)$$

with

$$\tilde{r}_2^* = e^{\tau(1-2\alpha)}|a|M\tau\delta.$$

Note that

$$\begin{aligned} &2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha_1-T)} - e^{\tau(1-\alpha_2-T)} \\ &\quad = 2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha-\tilde{\alpha}_0-T)} - e^{\tau(1-\alpha-\tilde{\alpha}_0-T-\delta)} \\ &\quad = 2e^{\tau(1-2\alpha)} - e^{\tau(1-2\alpha-\tilde{\alpha}_0+\alpha-T)} - e^{\tau(1-2\alpha-\tilde{\alpha}_0-\delta+\alpha-T)} \\ &\quad = e^{\tau(1-2\alpha)}h(\tilde{\alpha}_0, \delta, \alpha - T), \end{aligned}$$

where

$$h(\tilde{\alpha}_0, \delta, \alpha - T) = 2 - e^{\tau(-\tilde{\alpha}_0+\alpha-T)} - e^{\tau(-\tilde{\alpha}_0-\delta+\alpha-T)}.$$

Therefore,

$$\frac{\partial}{\partial \tilde{\alpha}_0} h(0,0,0) = 2\tau,$$

$$\frac{\partial}{\partial \delta} h(0,0,0) = \tau,$$

$$\frac{\partial}{\partial (\alpha - T)} h(0,0,0) = -2\tau$$

and hence

$$\begin{aligned} &2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha_1-T)} - e^{\tau(1-\alpha_2-T)} \\ &\quad = 2\tau e^{\tau(1-2\alpha)}\tilde{\alpha}_0 + \tau e^{\tau(1-2\alpha)}\delta - 2\tau e^{\tau(1-2\alpha)}(\alpha - T) + o(\alpha_0, \delta, \alpha - T). \end{aligned}$$

By (4.8), we get $\alpha - T = O(\beta, \varepsilon, \delta_3, \alpha_0)$ and, from the expression of $\tau(\alpha - T)$ above (4.8), we get

$$\begin{aligned} &2e^{\tau(1-2\alpha)} - e^{\tau(1-\alpha_1-T)} - e^{\tau(1-\alpha_2-T)} \\ &\quad = 2\tau e^{\tau(1-2\alpha)}\tilde{\alpha}_0 + \tau e^{\tau(1-2\alpha)}\delta \\ &\qquad - 2\tau e^{\tau(1-2\alpha)}\left[\frac{e^{-\tau\alpha}}{-a+2b}(x_3(0) - c_3) + \frac{2b\tau e^{\tau(1-2\alpha)}}{-a+2b}\tilde{\alpha}_0 + \frac{b\tau e^{\tau(1-2\alpha)}}{-a+2b}\delta + \tilde{r}_3^* e^{\tau\alpha}\right] \\ &\qquad + o(\beta, \varepsilon, \delta_3, \delta, \alpha_0). \end{aligned}$$

This, together with (4.7), implies that

$$\begin{aligned} x_2(T) + c_3 &= (c_2 + a)\left[\frac{e^{-2\tau\alpha}}{-a+2b}(x_3(0) - c_3) + \frac{2b\tau e^{\tau(1-3\alpha)}}{-a+2b}\tilde{\alpha}_0\right] \\ &\quad + e^{-\tau\alpha}[x_2(0) - c_2] + 2\tau a e^{\tau(1-2\alpha)}\tilde{\alpha}_0 \\ &\quad - 2ae^{\tau(1-2\alpha)}\left[\frac{e^{-\tau\alpha}}{-a+2b}(x_3(0) - c_3) + \frac{2b\tau e^{\tau(1-2\alpha)}}{-a+2b}\tilde{\alpha}_0\right] \\ &\quad + O(\beta, \varepsilon, \delta) + o(\delta_2, \delta_3, \alpha_0) \\ &= a_{22}[x_3(0) - c_3] + e^{-\tau\alpha}[x_2(0) - c_2] + a_{23}\tilde{\alpha}_0 + O(\beta, \varepsilon) \\ &\quad + o(\delta_2, \delta_3, \alpha_0). \end{aligned} \tag{4.17}$$

Lemma 4.3. *Assume all conditions in Lemma* 3.4 *hold and fix* $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ *as given in Lemma* 3.4. *Then there exist* $M_2 > 0$ *and* $m_{2,0} \in (0, m_{1,0})$ *such that for each* $m \in (0, m_{2,0})$ *there exist* $\beta_{2,0} = \beta_{2,0}(m) \in (0, \beta_{1,0})$ *and* $\varepsilon_{2,0} = \varepsilon_{2,0}(m) \in (0, \varepsilon_{1,0})$ *so that if* $0 < \beta < \beta_{2,0}$ *and* $0 < \varepsilon < \varepsilon_{2,0}$ *and if* $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$ *then*

(i) $|x_2(t)| \leqslant M_2$ *and* $x_2(t) = p_2(t) + O(\alpha_0, \delta_2, \delta_3, \beta, \varepsilon)$ *for all* $t \in [0, 1]$;
(ii) $|x_2(T) + c_3| \leqslant \frac{3+\rho}{4}\delta_3$;
(iii) $x_2(t) \geqslant \beta$ *for all* $t \in [0, T]$.

Proof. Conditions (i) and (iii) follow from (2.28) and (4.14)–(4.16), as well as the fact that $p_2(t)$ attains its minimum on $[0, \alpha]$ at either 0 or α, and $p_2(0) = c_2 > 0$ and $p_2(\alpha) = -c_3 > 0$.

Using (4.17) and as $a_{22} > 0$ and $a_{23} > 0$, we get

$$|x_2(T) + c_3| \leqslant a_{22}\delta_3 + e^{-\tau\alpha}\delta_2 + a_{23}\alpha_0 + O(\beta, \varepsilon) + o(\delta_2, \delta_3, \alpha_0).$$

Therefore, we can get (ii) using the same argument as for (ii) of Lemma 4.1. This completes the proof. □

We can now state the main existence result.

Theorem 4.4. *Let* $A = ke^{-\tau} \in (1, 2)$ *and assume that* $\tau > \tau^*$. *Let* $0 < \alpha_0 \leqslant \min\{\alpha - \frac{1}{2}, \frac{1-\alpha}{3}\}$ *and fix* $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ *as given in Lemma* 3.4. *Then there exist* $m^* > 0$ *such that for each* $m \in (0, m^*)$ *there exist* $\beta^* = \beta^*(m) \in (0, \beta_{2,0}]$ *and* $\varepsilon^* = \varepsilon^*(m) \in (0, \varepsilon_{2,0}]$ *so that if* $0 < \beta < \beta^*$ *and* $0 < \varepsilon < \varepsilon^*$ *and if* $(\delta_2, \delta_3, \alpha_0)^T = m(z_1, z_2, z_3)^T$ *then system* (1.3) *with* $f \in N(M, \beta, \varepsilon)$ *has a phase-locked periodic solution* $Q = (q_1, q_2, q_3)$ *of the minimal period* $6P$ *with* $|P - \alpha| \leqslant \alpha_0$ *and such that* $Q|_{[-1,0]} \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$ *with*

$D=\min\{D_1,D_2\}$ and

$$q_i(t+6P)=q_i(t),\quad q_i(t)=q_{i+1}(t+2P),\quad i \bmod (3),\ t\geqslant 0.$$

Furthermore, $q_i-p_i\to 0$ uniformly as $\beta,\varepsilon,m\to 0$.

Proof. Let $X(\Phi)=(x_1^{\Phi},x_2^{\Phi},x_3^{\Phi})$ be a given solution of (1.3) with the initial condition $\Phi\in\Sigma$. Define a mapping F on $A(\alpha_0,\delta_2,\delta_3,D,\beta)$ by

$$F(\Phi)=(x_3^{\Phi},x_1^{\Phi},x_2^{\Phi})|_{[T-1,T]},\quad \Phi\in A(\alpha_0,\delta_2,\delta_3,D,\beta).$$

We claim that for $D=\min\{D_1,D_2\}$ there exists $\beta_{4,0}=\beta_{4,0}(m)\in(0,\beta_{2,0})$ so that if $0<\beta<\beta_{4,0}$ then $-F(\Phi)\in A(\alpha_0,\delta_2,\delta_3,D,\beta)$.

In fact, as $\alpha_0\leqslant\alpha-\frac{1}{2}$, we have by Lemma 4.1 that $T-1\geqslant\alpha-\alpha_0-1\geqslant-\alpha+\alpha_0$. Hence, by Lemma 4.3 we have $-x_2^{\Phi}(s)\leqslant-\beta$ for all $s\in[T-1,T]$. Moreover,

$$|-x_2^{\Phi}(T)-(-c_3)|=|x_2^{\Phi}(T)+c_3|\leqslant\frac{3+\rho}{4}\delta_3\leqslant\delta_3.$$

Note also (iii) of Lemma 4.1 implies that $x_3^{\Phi}(t)<-\beta$ for all $t\in[0,T]$ and hence, $x_3^{\Phi}(s)\leqslant-\beta$ for all $s\in[T-1,T]$. Furthermore, (iii) of Lemma 4.1 and $\alpha\geqslant 3\alpha_0$ imply $-\dot{x}_3^{\Phi}(t)\leqslant-D_1$ and $T-2\alpha_0\geqslant\alpha-3\alpha_0\geqslant 0$ and hence $-x_3^{\Phi}(T+t)+x_3^{\Phi}(T+s)\leqslant-D_1(t-s)$ for all $-2\alpha_0\leqslant s\leqslant t\leqslant 0$.

Moreover, if $s\in[-1,-\alpha-\alpha_0]$ then by Lemma 4.1,

$$T-1\leqslant T+s\leqslant T-\alpha-\alpha_0\leqslant 0,$$

and hence $-x_1^{\Phi}(T+s)\leqslant-\beta$. If $s\in[-\alpha+\alpha_0,0]$ then by Lemma 4.1, we have

$$T+s\geqslant T-\alpha+\alpha_0\geqslant\alpha_0-\frac{3+\rho}{4}\alpha_0=\frac{1-\rho}{4}\alpha_0.$$

Therefore, there exists $\beta_{4,0}=\beta_{4,0}(m)\in(0,\beta_{2,0})$ such that if $0<\beta<\beta_{4,0}$ then

$$\begin{aligned} T+s\geqslant\frac{1-\rho}{4}\alpha_0 &=\frac{1-\rho}{4}mz_3 \\ &>d=\frac{2\beta}{\tau(-a+2b)}+\beta O(\beta,\varepsilon). \end{aligned}$$

Thus, by (iii) of Lemma 4.2, we have $-x_1^{\Phi}(t+s)\geqslant\beta$. Finally, if $s\in[-\alpha-\alpha_0,-\alpha+\alpha_0]$ then

$$-2\alpha_0=-\alpha-\alpha_0+\alpha-\alpha_0\leqslant T+s\leqslant\alpha+\alpha_0-\alpha+\alpha_0=2\alpha_0\leqslant 1-\alpha-\alpha_0,$$

and hence by the choice of ϕ_1 and by (iv) of Lemma 4.2, we conclude that $-x_1^{\Phi}$ is nondecreasing on $[T-\alpha-\alpha_0,T-\alpha+\alpha_0]$ and $-x_1^{\Phi}(T+t)+x_1^{\Phi}(T+s)\geqslant D(t-s)$ for all $-\alpha-\alpha_0\leqslant s\leqslant t\leqslant-\alpha+\alpha_0$ if $D=\min\{D_1,D_2\}$. This proves the claim.

Using the symmetry property of f, we get $F^2(\Phi) = -F(-F(\Phi)) \in A(\alpha_0, \delta_2, \delta_3, D, \beta)$. By the semigroup property, we have

$$F^2(\Phi) = (x_2^\Phi, x_3^\Phi, x_1^\Phi)|_{[\tilde{T}-1,\tilde{T}]},$$

where we note that $T = T(\phi)$ in the aforementioned arguments depends on Φ and

$$\tilde{T}(\Phi) = T(\Phi) + T(-F(X^\Phi_{T(\Phi)})).$$

Since $\tilde{T} \geqslant 2(\alpha - \alpha_0) \geqslant 1$, by (i) of Lemmas 4.1–4.3, we obtain a completely continuous mapping F^2 from $A(\alpha_0, \delta_2, \delta_3, D, \beta)$ into $A(\alpha_0, \delta_2, \delta_3, D, \beta) \cap \Sigma_{M^*}$ with

$$\Sigma_{M^*} = \left\{ \Phi \in \Sigma; ||\Phi|| = \max_{i=1,2,3, s\in[-1,0]} |\phi_i(s)| \leqslant M^* := \max\{M_1, M_2, M_3\} \right\},$$

and thus F^2 has a fixed point Φ. This gives a phase-locked periodic solution $Q = (q_1, q_2, q_3)$ of the minimal period $6P$, where $P = \frac{1}{2}[T(\Phi) + T(-F(X^\Phi_{T(\Phi)}))]$. The property that $q_i - p_i \to 0$ uniformly as $\beta, \varepsilon, m \to 0$ follows from (i) of Lemmas 4.1–4.3. This completes the proof with $m^* = m_{2,0}$, $\beta^* = \beta_{4,0}$ and $\varepsilon^* = \varepsilon_{2,0}$. □

5. Stability: exponential norm

We start with the following

Lemma 5.1. *Let $Q(t) = (q_1(t), q_2(t), q_3(t))^T$ be a $6P$ periodic solution obtained in Theorem 4.4, and let α_2^Q and α_1^Q be as defined in $A(\alpha_0, \delta_2, \delta_3, D, \beta)$ but with the supindex Q to denote the dependence on Q. Then there exists $\beta_{5,0} \in (0, \beta_{4,0}]$ so that if $\beta \in (0, \beta_{5,0})$ then*

$$\alpha_1^Q - \alpha_2^Q = \frac{2\beta}{(-a+2b)\tau} + \beta O(\beta, \varepsilon),$$

$$-\alpha - \alpha_0 < -\alpha_1^Q < -\alpha_2^Q < -\alpha + \alpha_0.$$

Proof. Let $T = T(Q|_{[-1,0]})$ and recall that $q_3(T) = -\beta$ and

$$-(q_3, q_1, q_2)|_{[T-1,T]} \in A(\alpha_0, \delta_2, \delta_3, D, \beta).$$

Applying (iii) of Lemma 4.1, we conclude that $t = T$ is the unique number in $[0, 1]$ so that $q_3(t) = -\beta$ and there exists a unique $\tilde{\alpha}_2 \in (T, 1)$ so that $q_3(\tilde{\alpha}_2) = \beta$. On the other hand, using the property of phase-locking, we have $q_2(t) = q_3(t + 2P)$ for all $t \in \mathbb{R}$ and with $2P = T(\Phi) + T(-F(X^\Phi_{T(\Phi)}))$, and thus $-\beta = q_2(-\alpha_1^Q) = q_3(-\alpha_1^Q + 2P)$.

Clearly,

$$0 \leqslant -\alpha_1^Q + 2P \leqslant 2(\alpha - \alpha_0) - \alpha + \alpha_0 = \alpha - \alpha_0 \leqslant 1$$

and therefore, $-\alpha_1^Q + 2P = T$ which implies

$$\alpha_1^Q = T(-F(X^{\Phi}_{T(\Phi)})) \leqslant \alpha + \frac{3+\rho}{4}\alpha_0 < \alpha + \alpha_0.$$

This implies that

$$q_2(t) = q_3(t + T(\Phi) + \alpha_1^Q).$$

Using the same argument as that of (iii) of Lemma 4.2 but for $-(q_3, q_2, q_1)|_{[T-1,T]}$ we get

$$\tilde{\alpha}_2 - T = \frac{2\beta}{\tau(-a+2b)} + \beta O(\beta, \varepsilon).$$

Note that $q_3(-\alpha_2^Q + 2P) = q_2(-\alpha_2^Q) = \beta$, we then have $\tilde{\alpha}_2 = 2P - \alpha_2^Q$ and hence

$$\alpha_1^Q - \alpha_2^Q = T(-F(X^{\Phi}_{T(\Phi)})\_2P + \tilde{\alpha}_2 = \tilde{\alpha}_2 - T = \frac{2\beta}{\tau(-a+2b)} + \beta O(\beta, \varepsilon).$$

Furthermore, we have

$$\alpha_2^Q = T - \tilde{\alpha}_2 + T(-F(X^{\Phi}_{T(\Phi)})) \geqslant \alpha - \frac{3+\rho}{4}\alpha_0 - \frac{2\beta}{\tau(-a+2b)} - \beta O(\beta, \varepsilon) > \alpha - \alpha_0$$

as long as β is sufficiently small. This completes the proof. □

For every $\Phi = (\phi_1, \phi_2, \phi_3) \in \Sigma$ and for $\gamma > 0$ to be specified later, let

$$||\Phi||_\tau = \max_{i=1,2,3}\left\{\max_{s\in[-1,-\alpha+\alpha_0]} \gamma e^{\tau s}|\phi_i(s)|, \max_{s\in[-\alpha+\alpha_0,0]} e^{\tau s}|\phi_i(s)|\right\}.$$

Clearly, the above defined norm is equivalent to the super-norm.

Let $\omega > 0$ be given and define

$$C_{Q,\omega} := \{\Phi \in \Sigma; ||\Phi - Q_0||_\tau \leqslant \omega\}.$$

Lemma 5.2. *For any $\beta > 0$ and $\eta > 0$ there exists $\omega = \omega(\beta, \eta)$ such that if $\Phi \in C_{Q,\omega}$, then*

$$\phi_1(s) \geqslant \beta \quad \textit{for } s \in [-1, -\eta],$$

$$\phi_2(s) \leqslant -\beta \quad \textit{for } s \in [-1, -\alpha_1^Q - \eta),$$

$$\phi_2(s) \geqslant \beta \quad \textit{for } s \in (-\alpha_2^Q + \eta, 0),$$

$$\phi_3(s) \leqslant -\beta \quad \textit{for } s \in [-1, 0].$$

Proof. This is obvious from Lemmas 4.1–4.3. □

From now on, we fix $\eta > 0$, $\omega = \omega(\beta, \eta)$ and $\Phi \in C_{Q,\omega}$. We also fix $T = T(Q|_{[-1,0]}) > 0$. We write $X^\Phi = (x_1, x_2, x_3)$.

We also need to impose a certain Lipschitz continuity on the nonlinearity. Namely, for a fixed $L_\infty > 0$ we define

$$N(L_\infty, L_\beta, M, \beta, \varepsilon) := \{ f \in N(M, \beta, \varepsilon); |f(x) - f(y)| \leqslant L_\infty |x - y|, x, y, \in \mathbb{R}$$

$$|f(x) - f(y)| \leqslant L_\beta |x - y|, x, y \geqslant \beta \}.$$

Choose f in the above set. Consider $s \in [-1, -T]$. Since $T \geqslant \alpha - \alpha_0$ by Lemma 4.1, we get

$$s \leqslant -T \leqslant -(\alpha - \alpha_0) = -\alpha + \alpha_0$$

and

$$0 \geqslant T + s \geqslant T - 1 \geqslant \alpha - \alpha_0 - 1 \geqslant -\alpha + \alpha_0$$

as long as

$$\alpha_0 \leqslant \alpha - \tfrac{1}{2}. \tag{5.1}$$

Therefore, if $s \in [-1, -T]$ then

$$\gamma e^{\tau s} |x_i(T+s) - q_i(T+s)| = \gamma e^{-\tau T} e^{\tau(T+s)} |x_i(T+s) - q_i(T+s)|$$

$$\leqslant \gamma e^{-\tau T} ||\Phi - Q_0||_\tau, \quad s \in [-1, -T]. \tag{5.2}$$

We next consider $t \in [0, 1 - \alpha_1^Q - \eta]$. For $i = 1, 2, 3$, we have

$$t - 1 \leqslant -\alpha_1^Q - \eta \leqslant -\alpha + \alpha_0$$

and

$$\dot{x}_i - \dot{q}_i = -\tau(x_i - q_i) + R_1(t),$$

with

$$\begin{aligned}|R_1(t)| &\leqslant (-a+2b)\tau L_\beta \sum_{i=1}^{3} |x_i(t-1)-q_i(t-1)| \\ &= (-a+2b)\tau L_\beta e^{\tau(1-t)} \sum_{i=1}^{3} e^{\tau(t-1)}|x_i(t-1)-q_i(t-1)| \\ &\leqslant 3(-a+2b)\tau L_\beta e^{\tau(1-t)}\gamma^{-1}||\Phi - Q_0||_\tau.\end{aligned}$$

Therefore, we get

$$\begin{aligned}|x_i(t)-q_i(t)| &\leqslant e^{-\tau t}|x_i(0)-q_i(0)| \\ &\quad + 3(-a+2b)\tau L_\beta \gamma^{-1}||\Phi - Q_0||_\tau \int_0^t e^{-\tau(t-s)}e^{\tau(1-s)}\,ds \\ &\leqslant e^{-\tau t}|x_i(0)-q_i(0)| + 3e^{-\tau t}(-a+2b)\tau e^{\tau} L_\beta \gamma^{-1}||\Phi - Q_0||_\tau. \qquad (5.3)\end{aligned}$$

In particular, if $T+s \in [0, 1-\alpha_1^Q-\eta]$ then

$$\begin{aligned}&e^{\tau s}|x_i(T+s)-q_i(T+s)| \\ &\quad = e^{-\tau T}e^{\tau(T+s)}|x_i(T+s)-q_i(T+s)| \\ &\quad \leqslant e^{-\tau T}|\phi_i(0)-q_i(0)| + 3e^{-\tau T}(-a+2b)\tau e^{\tau} L_\beta \gamma^{-1}||\Phi - Q_0||_\tau \\ &\quad \leqslant e^{-\tau T}||\Phi - Q_0||_\tau + 3(-a+2b)\tau e^{\tau(1-T)}\gamma^{-1}L_\beta ||\Phi - Q_0||_\tau. \qquad (5.4)\end{aligned}$$

Therefore, if $s \in [-T, -\alpha+\alpha_0]$, then by Lemma 4.1

$$0 \leqslant s+T \leqslant -\alpha+\alpha_0+T \leqslant 2\alpha_0 \leqslant 1-\alpha-\alpha_0-\eta \leqslant 1-\alpha_1^Q-\eta$$

provided

$$\alpha_0 \leqslant \frac{1-\alpha}{4}, \quad \eta \leqslant \frac{\alpha}{4}. \qquad (5.5)$$

Hence,

$$\begin{aligned}&\gamma e^{\tau s}|x_i(T+s)-q_i(T+s)| \\ &\quad \leqslant \gamma e^{-\tau T}||\Phi - Q_0||_\tau + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}||\Phi - Q_0||_\tau. \qquad (5.6)\end{aligned}$$

Furthermore, using (5.4), we get for $s \in [-\alpha+\alpha_0, -T+1-\alpha_1^Q-\eta]$, that

$$e^{\tau s}|x_i(T+s)-q_i(T+s)| \leqslant [e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}\gamma^{-1}]||\Phi - Q_0||_\tau. \qquad (5.7)$$

Now, we look at the interval $[1-\alpha_1^Q-\eta, 1-\alpha_2^Q+\eta]$ where we have

$$-\alpha-\alpha_0 \leqslant -\alpha_1^Q-\eta \leqslant t-1 \leqslant -\alpha_2^Q+\eta \leqslant -\alpha+\alpha_0$$

provided

$$\eta \leqslant \min\{-\alpha+\alpha_0+\alpha_2^Q, \alpha+\alpha_0-\alpha_1^Q\}. \tag{5.8}$$

Therefore,

$$\dot{x}_i-\dot{q}_i=-\tau(x_i-q_i)+R_{2,i}(t),$$

where for $i=1,3$,

$$\begin{aligned}|R_{2,i}| &\leqslant (-a+b)\tau L_\beta[|x_1(t-1)-q_1(t-1)|+|x_3(t-1)-q_3(t-1)|]\\ &\quad + b\tau L_\infty|x_2(t-1)-q_2(t-1)|\\ &\leqslant 2(-a+b)\tau L_\beta e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau+b\tau L_\infty e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau\\ &=K_i e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau\end{aligned}$$

with

$$K_1=K_3=2(-a+b)\tau L_\beta+b\tau L_\infty \tag{5.9}$$

and

$$\begin{aligned}|R_{2,2}| &\leqslant -a\tau L_\infty|x_2(t-1)-q_2(t-1)|\\ &\quad + b\tau L_\beta[|x_1(t-1)-q_1(t-1)|+|x_3(t-1)-q_3(t-1)|]\\ &\leqslant -aL_\infty\tau e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau+2b\tau L_\beta e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau\\ &=K_2 e^{\tau(1-t)}\gamma^{-1}||\Phi-Q_0||_\tau\end{aligned}$$

with

$$K_2 := -a\tau L_\infty+2b\tau L_\beta. \tag{5.10}$$

Therefore, for $i=1,2,3$, we have

$$\begin{aligned}|x_i(t)-q_i(t)| &\leqslant e^{-\tau(t-1+\alpha_1^Q+\eta)}|x_i(1-\alpha_1^Q-\eta)-q_i(1-\alpha_1^Q-\eta)|\\ &\quad + \int_{1-\alpha_1^Q-\eta}^t e^{-\tau(t-s)}K_i e^{\tau(1-s)}\gamma^{-1}||\Phi-Q_0||_\tau\,ds.\end{aligned}$$

Using (5.3) at $1-\alpha_1^Q-\eta$, we get

$$\begin{aligned}|x_i(t)-q_i(t)| \leqslant{}& e^{-\tau t}|x_i(0)-q_i(0)| \\ &+ 3e^{-\tau t}(-a+2b)\tau e^{\tau} L_\beta \gamma^{-1}||\Phi - Q_0||_\tau \\ &+ K_i e^{\tau} e^{-\tau t}\gamma^{-1}(\alpha_1^Q-\alpha_2^Q+2\eta)||\Phi - Q_0||_\tau. \end{aligned} \tag{5.11}$$

So, if $T+s\in[1-\alpha_1^Q-\eta, 1-\alpha_2^Q+\eta]$ then, by Lemma 5.1 we have

$$\begin{aligned}& e^{\tau s}|x_i(T+s)-q_i(T+s)| \\ &\leqslant e^{-\tau T}|x_i(0)-q_i(0)| + 3(-a+2b)\tau e^{\tau(1-T)}L_\beta\gamma^{-1}||\Phi - Q_0||_\tau \\ &\quad + K_i e^{\tau(1-T)}\gamma^{-1}\left[\frac{2\beta}{\tau(-a+2b)}+\beta O(\beta,\varepsilon)+2\eta\right]||\Phi - Q_0||_\tau \\ &\leqslant A_i||\Phi - Q_0||_\tau, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} A_i ={}& e^{-\tau T} + 3(-a+2b)\tau e^{\tau(1-T)}L_\beta\gamma^{-1} \\ &+ K_i e^{\tau(1-T)}\gamma^{-1}\left[\frac{2\beta}{\tau(-a+2b)}+\beta O(\beta,\varepsilon)+2\eta\right]. \end{aligned} \tag{5.13}$$

Finally, we consider the interval $[1-\alpha_2^Q+\eta, 1-\eta]$ where we have

$$\dot{x}_i - \dot{q}_i = -\tau(x_i - q_i) + R_3(t)$$

with

$$|R_3(t)| \leqslant 3(-a+2b)\tau L_\beta e^{\tau(1-t)}\max\{1,\gamma^{-1}\}||\Phi - Q_0||_\tau.$$

Therefore,

$$\begin{aligned} & |x_i(t)-q_i(t)| \\ &\leqslant e^{-\tau(t-1+\alpha_2^Q-\eta)}|x_i(1-\alpha_2^Q+\eta)-q_i(1-\alpha_2^Q+\eta)| \\ &\quad + 3(-a+2b)\tau L_\beta \int_{1-\alpha_2^Q+\eta}^{t} e^{-\tau(t-s)}e^{\tau(1-s)}ds\,\max\{1,\gamma^{-1}\}||\Phi - Q_0||_\tau. \end{aligned} \tag{5.14}$$

Using (5.11) at $1 - \alpha_2^Q + \eta$, we get

$$\begin{aligned} |x_i(t) - q_i(t)| \leqslant e^{-\tau t}|x_i(0) - q_i(0)| \\ &+ 3e^{-\tau t}(-a+2b)\tau e^{\tau} L_\beta \gamma^{-1}||\Phi - Q_0||_\tau \\ &+ K_i e^{\tau} e^{-\tau t}(\alpha_1^Q - \alpha_2^Q + 2\eta)\gamma^{-1}||\Phi - Q_0||_\tau \\ &+ 3(-a+2b)\tau e^{\tau} L_\beta e^{-\tau t}\max\{1, \gamma^{-1}\}||\Phi - Q_0||_\tau. \end{aligned} \tag{5.15}$$

In particular, if $T + s \in [1 - \alpha_2^Q + \eta, 1 - \eta]$, then

$$\begin{aligned} &e^{\tau s}|x_i(T+s) - q_i(T+s)| \\ &\leqslant e^{-\tau T}|x_i(0) - q_i(0)| \\ &\quad + 3e^{-\tau T}(-a+2b)\tau e^{\tau} L_\beta \gamma^{-1}||\Phi - Q_0||_\tau \\ &\quad + e^{\tau(1-T)} K_i(\alpha_1^Q - \alpha_2^Q + 2\eta)\gamma^{-1}||\Phi - Q_0||_\tau \\ &\quad + 3(-a+2b)\tau e^{\tau(1-T)} L_\beta \max\{1, \gamma^{-1}\}||\Phi - Q_0||_\tau \\ &\leqslant B_i||\Phi - Q_0||_\tau, \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} B_i = e^{-\tau T} &+ 3e^{\tau(1-T)}(-a+2b)\tau L_\beta \gamma^{-1} \\ &+ e^{\tau(1-T)} K_i\left[\frac{2\beta}{\tau(-a+2b)} + \beta O(\beta, \varepsilon) + 2\eta\right]\gamma^{-1} \\ &+ 3(-a+2b)\tau e^{\tau(1-T)} L_\beta \max\{1, \gamma^{-1}\}. \end{aligned} \tag{5.17}$$

Summarizing the above discussions, we obtain

Lemma 5.3. *Let $A = ke^{-\tau} \in (1,2)$ and assume that $\tau > \tau^*$. Let $0 < \alpha_0 \leqslant \min\{\alpha - \frac{1}{2}, \frac{1-\alpha}{4}\}$ and fix $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ as given in Lemma 3.4. Also let $0 < \beta < \beta_{5,0}$, $0 < \varepsilon < \varepsilon^*$ and let $(\delta_2, \delta_3, \alpha_0) = m(z_1, z_2, z_3)$. Then for the chosen $\eta \in \min\{\frac{\alpha}{4}, -\alpha + \alpha_0 + \alpha_2^Q, \alpha + \alpha_0 - \alpha_1^Q\}$ and for every fixed $\Phi \in C_{Q,\omega}$, we have*

$$||X_T^\Phi - Q_T||_\tau \leqslant \max\{B_1, B_2, \gamma e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}\}||\Phi - Q_0||_\tau. \tag{5.18}$$

Proof. This follows from (5.2), (5.6), (5.7),(5.12) and (5.16). □

With the above preparation, we can now state and prove the main stability theorem:

Theorem 5.4. *There exists* $\tau^{**} \geqslant \tau^*$ *so that for every fixed* $\tau > \tau^{**}$ *and every* $\Gamma(\tau) \in (1, e^{\tau(2T-1)})$ *there exist* $L^*_\beta > 0$, $\beta^{**} \in (0, \beta_{5,0}]$ *and* $\varepsilon^{**} \in (0, \varepsilon^*]$ *so that if* $L_\beta \in [0, L^*_\beta)$, $\beta \in (0, \beta^{**})$, $\varepsilon \in (0, \varepsilon^{**})$ *and* $L_\infty \beta \leqslant \Gamma^{-1}(\tau) e^{\tau(2T-1)}$, *then Q is asymptotically stable.*

Proof. Let $\gamma = \Gamma^{-1}(\tau)e^{\tau T}$ in Lemma 5.3. Then as long as τ is sufficiently large then $\gamma^{-1} \leqslant 1$. Hence

$$\begin{aligned} B_i = {} & e^{-\tau T} + 3e^{\tau(1-2T)}(-a+2b)\tau L_\beta \\ & + \Gamma(\tau)e^{\tau(1-2T)} K_i \left[\frac{2\beta}{\tau(-a+2b)} + \beta O(\beta, \varepsilon) + 2\eta \right] \\ & + 3(-a+2b)\tau e^{\tau(1-T)} L_\beta, \end{aligned}$$

and in particular,

$$\begin{aligned} B_1 = {} & e^{-\tau T} + 3e^{\tau(1-2T)}(-a+2b)\tau L_\beta + 3(-a+2b)\tau e^{\tau(1-T)} L_\beta \\ & + \Gamma(\tau)e^{\tau(1-2T)} \left[\frac{4(-a+b)}{-a+2b} \beta L_\beta + \frac{2b}{-a+2b} \beta L_\infty \right] \\ & + \Gamma(\tau)e^{\tau(1-2T)} [2(-a+b)\tau L_\beta + b\tau L_\infty][\beta O(\beta, \varepsilon) + 2\eta] \end{aligned}$$

and

$$\begin{aligned} B_2 = {} & e^{-\tau T} + 3e^{\tau(1-2T)}(-a+2b)\tau L_\beta + 3(-a+2b)\tau e^{\tau(1-T)} L_\beta \\ & + \Gamma(\tau)e^{\tau(1-2T)} \left[\frac{-2a}{-a+2b} \beta L_\infty + \frac{4b}{-a+2b} \beta L_\beta \right] \\ & + \Gamma(\tau)e^{\tau(1-2T)} [-a\tau L_\infty + 2b\tau L_\beta][\beta O(\beta, \varepsilon) + 2\eta]. \end{aligned}$$

Note also that

$$\gamma e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)} = \Gamma^{-1}(\tau) + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}.$$

Note that $T \geqslant \alpha - \frac{3+\rho}{4}\alpha_0$ and

$$\begin{aligned} 1 - 2T & \leqslant 1 - 2\left(\alpha - \frac{3+\rho}{4}\alpha_0\right) \\ & = 1 - 2\alpha + \frac{3+\rho}{2}\alpha_0. \end{aligned}$$

Therefore, there exists $m^{**} \in (0, m^*)$ so that if $m \in (0, m^{**})$, then

$$\begin{cases} T \geqslant \frac{\alpha}{2}, \quad 1 - 2T \leqslant \frac{1 - 2\alpha}{2}, \\ \alpha_0 = mz_3 \leqslant \min\left\{\frac{1-\alpha}{4}, \alpha - \frac{1}{2}\right\}. \end{cases}$$

Note also that $2b/(-a+2b) \leqslant 1$ and $(-2a)/(-a+2b) = 2/k \to 0$ as $\tau \to \infty$. Consequently, we can find $\tau^{**} \geqslant \tau^*$ so that for every fixed $\tau > \tau^{**}$ there exist $L^*_\beta > 0$, $\beta^{**} \in (0, \beta_{5,0}]$, and $\varepsilon^{**} \in (0, \varepsilon^*)$ so that if $L_\beta \in [0, L^*_\beta)$, $\beta \in (0, \beta^{**})$, $\varepsilon \in (0, \varepsilon^{**})$ and $\beta L_\infty < \Gamma^{-1}(\tau)e^{\tau(2T-1)}$ then we can chose $\eta > 0$ sufficiently small so that

$$K := \max\{B_1, B_2, \gamma e^{-\tau T} + 3(-a+2b)\tau L_\beta e^{\tau(1-T)}\} < 1. \tag{5.19}$$

Using the similar argument for $||X^\Phi_T - Q_T||_\tau \leqslant K||\Phi - Q_0||_\tau$, we get for $\tilde{T} = T(-F(X_{T(\Phi)}))$ that

$$||X^\Phi_{2P} - Q_{2P}||_\tau = ||X^{X^\Phi_T}_{\tilde{T}} - X^{Q_T}_{\tilde{T}}||_\tau \leqslant K||X^\Phi_T - Q_T||_\tau \leqslant K^2||\Phi - Q_0||_\tau,$$

and, in general, for $n \geqslant 2$, that

$$||X^\Phi_{nP} - Q_{nP}||_\tau \leqslant K^n||\Phi - Q_0||_\tau \to 0$$

as $n \geqslant \infty$. This, together with the continuity of the solution of (1.3) with respect to the initial data on the interval $[0, P]$, implies that $||X^\Phi_t - Q_t||_\tau \to$ as $t \to \infty$, completing the proof. □

Acknowledgments

This work would not be done without the valuable comments and continuous encouragement from Hans-otto Walther. The work was partially supported by the Canada Research Chairs Program, by NCE Center of Mathematics for Information Technology and Complex Systems, and by the Natural Sciences and Engineering Research Council of Canada.

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