An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 45 (2003) 935–942
www.elsevier.nl/locate/camwa

Stable Periodic Orbits in Nonlinear Discrete-Time Neural Networks with Delayed Feedback

ZHAN ZHOU[†]
Department of Applied Mathematics, Hunan University
Changsha, Hunan 410082, P.R. China

JIANHONG WU[‡]
Department of Mathematics and Statistics, York University
Toronto, Ontario, M3J 1P3, Canada

Abstract—We consider a nonlinear discrete-time system

$$\begin{aligned} x(n+1) &= \beta x(n) + g(y(n-k)), \\ y(n+1) &= \beta y(n) - g(x(n-k)), \end{aligned} \qquad n \in N,$$

arising as a discrete-time network of two neurons with McCulloch-Pitts nonlinearity, where $\beta \in (0,1)$, k is a positive integer, and g is a signal transmission function with a threshold σ. We obtain a stable $4(k+1)$-periodic orbit in some regions of the parameters (β, σ), and we describe asymptotic behaviors of the system in other regions of (β, σ).

Keywords—Delay, Discrete neural networks, Periodic orbits.

1. INTRODUCTION

Let Z denote the set of all integers. For any $a, b \in Z$ with $a \le b$, define $N(a) = \{a, a+1, \dots\}$, and $N(a,b) = \{a, a+1, \dots, b\}$. Also, let $N = N(0)$. In this paper, we consider the following nonlinear discrete-time system:

$$\begin{aligned} x(n+1) &= \beta x(n) + g(y(n-k)), \\ y(n+1) &= \beta y(n) - g(x(n-k)), \end{aligned} \qquad n \in N, \tag{1.1}$$

where $\beta \in (0,1)$, $k \in N$, and $g : R \to R$ is given by

$$g(x) = \begin{cases} -\rho, & \text{if } x > \sigma, \\ \rho, & \text{if } x \le \sigma, \end{cases} \tag{1.2}$$

for two parameters $\rho > 0$ and $\sigma \in R$.

[†]This work was carried out while visiting York University. Research partially supported by Natural Science Foundation of Hunan Province of China (Grant No. 02JJY2011).
[‡]Research partially supported by Natural Sciences and Engineering Research Council of Canada, and by the Network of Centers of Excellence "Mathematics for Information Technology and Complex Systems".

PII: S0898-1221(03)00066-X

System (1.1) can be regarded as the discrete analog of the following artifical neural network of two neurons with delayed feedback:

$$\begin{aligned}\frac{dx}{dt} &= -\mu x(t) + g(y(t-\tau)),\\ \frac{dy}{dt} &= -\mu y(t) - g(x(t-\tau)),\end{aligned} \tag{1.3}$$

where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are replaced by the forward difference $x(n+1)-x(n)$ and $y(n+1)-y(n)$, respectively. System (1.3) has found interesting applications in, for example, image processing of moving objects [1,2], and has been recently investigated (see [3] and references therein). In the discrete version (1.1), $\beta \in (0,1)$ is the internal decay rate, g is the McCulloch-Pitts signal function with the threshold σ and the synaptic weight $\rho > 0$, and k is the signal transmission delay. The different signs in front of g represent the "frustrated" nature of the network, and describes the excitatory feedback from neuron y to x and the inhibitory feedback from neuron x to y. The case of mutually excitatory feedback was considered in [4]. For other discrete neural networks, we refer to [5,6].

By a *solution* of (1.1), we mean a sequence $\{(x(n), y(n))\}$ of points in R^2 that is defined for all $n \in N(-k)$ and satisfies (1.1) for $n \in N$. Let X denote the set of mappings from $N(-k,0)$ to R^2. Clearly, for any $\Phi = (\phi,\psi) \in X$, (1.1) has a unique solution $(x^\Phi(n), y^\Phi(n))$ satisfying the initial conditions

$$x^\Phi(i) = \phi(i), \quad y^\Phi(i) = \psi(i), \qquad \text{for } i \in N(-k,0). \tag{1.4}$$

Our goal is to determine the limiting behavior of $(x^\Phi(n), y^\Phi(n))$ as $n \to \infty$ for any $\Phi \in X$. In this paper, we concentrate on the case where $\phi - \sigma$ and $\psi - \sigma$ have no sign changes on $N(-k,0)$. Namely, we consider those $\Phi \in X_\sigma^{+,+} \cup X_\sigma^{+,-} \cup X_\sigma^{-,+} \cup X_\sigma^{-,-} = X_\sigma$, where

$$X_\sigma^{\pm,\pm} = \left\{\Phi \in X;\ \Phi = (\phi,\psi),\ \phi \in R_\sigma^\pm, \text{ and } \psi \in R_\sigma^\pm\right\},$$

with

$$R_\sigma^+ = \{\phi;\ \phi : N(-k,0) \to R \text{ and } \phi(i) - \sigma > 0 \text{ for } i \in N(-k,0)\}$$

and

$$R_\sigma^- = \{\phi;\ \phi : N(-k,0) \to R \text{ and } \phi(i) - \sigma \le 0 \text{ for } i \in N(-k,0)\}.$$

For a general background on difference equations, we refer to [7–9].

The main results of this paper are as follows.

THEOREM 1.1. *Let $\beta \in (0,1/2]$ and $|\sigma| < \rho(1+\beta^{2k+3}-2\beta)/(1-\beta)(1-\beta^{2k+3})$. Then there exists $\Phi_0 = (\phi_0,\psi_0) \in X_\sigma^{+,+}$ such that the solution $(x^{\Phi_0}(n), y^{\Phi_0}(n))$ of (1.1) with initial value Φ_0 is periodic with the minimal period $4(k+1)$. Moreover, there exists a positive integer m such that for any $\Phi = (\phi,\psi) \in X_\sigma$,*

$$\lim_{n\to\infty}\left(x^\Phi(n+m) - x^{\Phi_0}(n)\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(y^\Phi(n+m) - y^{\Phi_0}(n)\right) = 0. \tag{1.5}$$

THEOREM 1.2. *Let $\Phi = (\phi,\psi) \in X_\sigma$. Then, $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (\rho/(1-\beta), -\rho/(1-\beta))$ if $\sigma > \rho/(1-\beta)$, and $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (-\rho/(1-\beta), \rho/(1-\beta))$ if $\sigma < -\rho/(1-\beta)$.*

THEOREM 1.3.

(i) *Let $\sigma = \rho/(1-\beta)$. Then $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (\rho/(1-\beta), -\rho/(1-\beta))$ for $\Phi \in X_\sigma^{+,+} \cup X_\sigma^{-,+} \cup X_\sigma^{-,-}$, and $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (\rho/(1-\beta), \rho/(1-\beta))$ for $\Phi \in X_\sigma^{+,-}$.*

(ii) *Let $\sigma = -\rho/(1-\beta)$. Then $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (-\rho/(1-\beta), \rho/(1-\beta))$ for $\Phi \in X_\sigma^{+,+} \cup X_\sigma^{+,-} \cup X_\sigma^{-,-}$, and $\lim_{n\to\infty}(x^\phi(n), y^\Phi(n)) = (-\rho/(1-\beta), -\rho/(1-\beta))$ for $\Phi \in X_\sigma^{-,+}$.*

Theorem 1.1 shows that if $|\sigma|$ is sufficiently small, then system (1.1) has a stable periodic solution with minimal period $4(k+1)$. Theorems 1.2 and 1.3 show that if $|\sigma|$ is large enough, then solutions of system (1.1) with initial data in X_σ converge to a single equilibria.

2. PROOFS OF MAIN RESULTS

In the remaining part of this paper, for any $s \in N$ and a sequence $\{x(n)\}_{n=-k}^{\infty}$, we define $x_s : N(-k,0) \to R$ by $x_s(n) = x(n+s)$ for all $n \in N(-k,0)$.

Assuming $n_0 \in N$, we first note that the discrete-time equation

$$x(n) = \beta x(n-1) - \rho, \qquad n \in N(n_0+1), \tag{2.1}$$

with initial condition $x(n_0) = a$ is given by

$$x(n) = \left(a + \frac{\rho}{1-\beta}\right)\beta^{n-n_0} - \frac{\rho}{1-\beta}, \qquad n \in N(n_0+1); \tag{2.2}$$

and that the solution of the discrete-time equation

$$x(n) = \beta x(n-1) + \rho, \qquad n \in N(n_0+1), \tag{2.3}$$

with initial condition $x(n_0) = a$ is given by

$$x(n) = \left(a - \frac{\rho}{1-\beta}\right)\beta^{n-n_0} + \frac{\rho}{1-\beta}, \qquad n \in N(n_0+1). \tag{2.4}$$

For the sake of convenience, in the sequel, $x(n)$ denotes $x^{\Phi}(n)$, and $y(n)$ denotes $y^{\Phi}(n)$ when Φ is given.

Proof of Theorem 1.1. We only consider the case where $\Phi = (\phi, \psi) \in X_{\sigma}^{+,+}$ and $\sigma \geq 0$. The other cases are similar.

Since $(x_0, y_0) = (\phi, \psi) \in X_{\sigma}^{+,+}$, we have, for $n \in N(1, k+1)$,

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) + \rho. \end{aligned} \tag{2.5}$$

By (2.2) and (2.4), we get

$$\begin{aligned} x(n) &= \left(\phi(0) + \frac{\rho}{1-\beta}\right)\beta^{n} - \frac{\rho}{1-\beta}, \\ y(n) &= \left(\psi(0) - \frac{\rho}{1-\beta}\right)\beta^{n} + \frac{\rho}{1-\beta}, \end{aligned} \tag{2.6}$$

for $n \in N(1, k+1)$. Let k_0^* be the least integer in N such that $x(k_0^*) \leq \sigma$. That is,

$$x\left(k_0^*\right) \leq \sigma \quad \text{and} \quad x(n) > \sigma, \qquad \text{for } n \in N\left(0, k_0^* - 1\right).$$

Then, (2.5) and (2.6) hold for $n \in N(1, k_0^* + k)$. Let $k_1 = k_0^* + k$. Then $(x_{k_1}, y_{k_1}) \in X_{\sigma}^{-,+}$.

For $n \in N(k_1+1, k_1+k+1)$, we have

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \tag{2.7}$$

which implies that

$$\begin{aligned} x(n) &= \left(x(k_1) + \frac{\rho}{1-\beta}\right)\beta^{n-k_1} - \frac{\rho}{1-\beta}, \\ y(n) &= \left(y(k_1) + \frac{\rho}{1-\beta}\right)\beta^{n-k_1} - \frac{\rho}{1-\beta}. \end{aligned} \tag{2.8}$$

Let k_1^* be the least integer in $N(k_1+1)$ such that $y(k_1^*) \le \sigma$. That is,

$$y\left(k_1^*\right) \le \sigma \quad \text{and} \quad y(n) > \sigma, \qquad \text{for } n \in N\left(k_1, k_1^* - 1\right).$$

Then, (2.7) and (2.8) hold for $n \in N(k_1+1, k_1^*+k)$. Let $k_2 = k_1^* + k$. Then $(x_{k_2}, y_{k_2}) \in X_\sigma^{-,-}$.

For $n \in N(k_2+1, k_2+k+1)$, we have

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \tag{2.9}$$

which implies that

$$\begin{aligned} x(n) &= \left(x(k_2) - \frac{\rho}{1-\beta}\right)\beta^{n-k_2} + \frac{\rho}{1-\beta}, \\ y(n) &= \left(y(k_2) + \frac{\rho}{1-\beta}\right)\beta^{n-k_2} - \frac{\rho}{1-\beta}. \end{aligned} \tag{2.10}$$

Let k_2^* be the least integer in $N(k_2+1)$ such that $x(k_2^*) > \sigma$. That is,

$$x\left(k_2^*\right) > \sigma \quad \text{and} \quad x(n) \le \sigma, \qquad \text{for } n \in N\left(k_2, k_2^* - 1\right).$$

Clearly, (2.9) and (2.10) hold for $n \in N(k_2+1, k_2^*+k)$. Let $k_3 = k_2^*+k$. Then $(x_{k_3}, y_{k_3}) \in X_\sigma^{+,-}$.

For $n \in N(k_3+1, k_3+k+1)$, we have

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) + \rho, \end{aligned} \tag{2.11}$$

which implies that

$$\begin{aligned} x(n) &= \left(x(k_3) - \frac{\rho}{1-\beta}\right)\beta^{n-k_3} + \frac{\rho}{1-\beta}, \\ y(n) &= \left(y(k_3) - \frac{\rho}{1-\beta}\right)\beta^{n-k_3} + \frac{\rho}{1-\beta}. \end{aligned} \tag{2.12}$$

Let k_3^* be the least integer in $N(k_3+1)$ such that $y(k_3^*) > \sigma$. That is,

$$y\left(k_3^*\right) > \sigma \quad \text{and} \quad y(n) \le \sigma, \qquad \text{for } n \in N\left(k_3, k_3^* - 1\right).$$

Then, equations (2.11) and (2.12) hold for $n \in N(k_3+1, k_3^*+k)$. Denote k_3^*+k by k_4. Then $(x_{k_4}, y_{k_4}) \in X_\sigma^{+,+}$.

For $n \in N(k_4+1, k_4+k+1)$, we know (2.5) holds, and hence,

$$\begin{aligned} x(n) &= \left(x(k_4) + \frac{\rho}{1-\beta}\right)\beta^{n-k_4} - \frac{\rho}{1-\beta}, \\ y(n) &= \left(y(k_4) - \frac{\rho}{1-\beta}\right)\beta^{n-k_4} + \frac{\rho}{1-\beta}. \end{aligned} \tag{2.13}$$

From (2.12) and (2.10), we obtain

$$\begin{aligned} x(k_4) &= \left(x\left(k_2^*\right) - \frac{\rho}{1-\beta}\right)\beta^{k_4-k_2^*} + \frac{\rho}{1-\beta} \\ &\le \left(\sigma - \frac{\rho}{1-\beta}\right)\beta^{k_4-k_2^*} + \frac{\rho}{1-\beta} < \frac{\rho}{1-\beta}. \end{aligned}$$

This yields

$$x(k_4+1) = \left(x(k_4) + \frac{\rho}{1-\beta}\right)\beta - \frac{\rho}{1-\beta} < \frac{\rho(2\beta-1)}{1-\beta} < 0 \le \sigma.$$

Let $k_5 = k_4+k+1$. Then $(x_{k_5}, y_{k_5}) \in X_\sigma^{-,+}$.

For $n \in N(k_5+1, k_5+k+1)$, (2.7) holds. Similarly, we get $y(k_5+1) < \sigma$. Let $k_6 = k_5+k+1$. Then $(x_{k_6}, y_{k_6}) \in X_\sigma^{-,-}$.

For $n \in N(k_6+1, k_6+k+1)$, (2.9) holds. Moreover,

$$\begin{aligned} x(k_6) &= \left(x(k_4) + \frac{\rho}{1-\beta}\right)\beta^{k_6-k_4} - \frac{\rho}{1-\beta} \\ &> \left(\sigma + \frac{\rho}{1-\beta}\right)\beta^{2(k+1)} - \frac{\rho}{1-\beta}. \end{aligned}$$

Since $\sigma < \rho(1+\beta^{2k+3}-2\beta)/(1-\beta)(1-\beta^{2k+3})$, we get

$$\begin{aligned} x(k_6+1) &= \left(x(k_6) - \frac{\rho}{1-\beta}\right)\beta + \frac{\rho}{1-\beta} \\ &> \left(\left(\sigma + \frac{\rho}{1-\beta}\right)\beta^{2(k+1)} - \frac{2\rho}{1-\beta}\right)\beta + \frac{\rho}{1-\beta} \\ &> \sigma. \end{aligned} \tag{2.14}$$

Let $k_7 = k_6+k+1$. Then $(x_{k_7}, y_{k_7}) \in X_\sigma^{+,-}$.

Similarly, for $k_8 = k_7+k+1$, we have $(x_{k_8}, y_{k_8}) \in X_\sigma^{+,+}$. In general, we can get, for $i \in N$,

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) + \rho, \end{aligned} \qquad n \in N(k_4+1+4i(k+1), k_4+(4i+1)(k+1));$$

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \qquad n \in N(k_4+1+(4i+1)(k+1), k_4+(4i+2)(k+1));$$

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \qquad n \in N(k_4+1+(4i+2)(k+1), k_4+(4i+3)(k+1));$$

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) + \rho, \end{aligned} \qquad n \in N(k_4+1+(4i+3)(k+1), k_4+4(i+1)(k+1)).$$

From now on, we denote $1-\beta$ by α. Thus,

$$\begin{aligned} x(n) &= \left(x(k_4) + \frac{\rho}{\alpha}\right)\beta^{n-k_4} + \frac{2\rho\beta^{n-k_4-4i(k+1)}}{\alpha}\left(\frac{1-\beta^{4i(k+1)}}{1+\beta^{2(k+1)}}\right) - \frac{\rho}{\alpha}, \\ y(n) &= \left(y(k_4) - \frac{\rho}{\alpha}\right)\beta^{n-k_4} - \frac{2\rho\beta^{n-k_4-(4i-1)(k+1)}}{\alpha}\left(\frac{1-\beta^{4i(k+1)}}{1+\beta^{2(k+1)}}\right) + \frac{\rho}{\alpha}, \end{aligned}$$

for $n \in N(k_4+1+4i(k+1), k_4+(4i+1)(k+1))$;

$$\begin{aligned} x(n) &= \left(x(k_4) + \frac{\rho}{\alpha}\right)\beta^{n-k_4} + \frac{2\rho\beta^{n-k_4-4i(k+1)}}{\alpha}\left(\frac{1-\beta^{4i(k+1)}}{1+\beta^{2(k+1)}}\right) - \frac{\rho}{\alpha}, \\ y(n) &= \left(y(k_4) - \frac{\rho}{\alpha}\right)\beta^{n-k_4} + \frac{2\rho\beta^{n-k_4-(4i+1)(k+1)}}{\alpha}\left(\frac{1+\beta^{(4i+2)(k+1)}}{1+\beta^{2(k+1)}}\right) - \frac{\rho}{\alpha}, \end{aligned}$$

for $n \in N(k_4+1+(4i+1)(k+1), k_4+(4i+2)(k+1))$;

$$\begin{aligned} x(n) &= \left(x(k_4) + \frac{\rho}{\alpha}\right)\beta^{n-k_4} - \frac{2\rho\beta^{n-k_4-(4i+2)(k+1)}}{\alpha}\left(\frac{1+\beta^{(4i+2)(k+1)}}{1+\beta^{2(k+1)}}\right) + \frac{\rho}{\alpha}, \\ y(n) &= \left(y(k_4) - \frac{\rho}{\alpha}\right)\beta^{n-k_4} + \frac{2\rho\beta^{n-k_4-(4i+1)(k+1)}}{\alpha}\left(\frac{1+\beta^{(4i+2)(k+1)}}{1+\beta^{2(k+1)}}\right) - \frac{\rho}{\alpha}, \end{aligned}$$

for $n \in N(k_4+1+(4i+2)(k+1), k_4+(4i+3)(k+1))$;

$$x(n) = \left(x(k_4)+\frac{\rho}{\alpha}\right)\beta^{n-k_4} - \frac{2\rho\beta^{n-k_4-(4i+2)(k+1)}}{\alpha}\left(\frac{1+\beta^{(4i+2)(k+1)}}{1+\beta^{2(k+1)}}\right)+\frac{\rho}{\alpha},$$
$$y(n) = \left(y(k_4)-\frac{\rho}{\alpha}\right)\beta^{n-k_4} - \frac{2\rho\beta^{n-k_4-(4i+3)(k+1)}}{\alpha}\left(\frac{1-\beta^{4(i+1)(k+1)}}{1+\beta^{2(k+1)}}\right)+\frac{\rho}{\alpha},$$

for $n \in N(k_4+1+(4i+3)(k+1), k_4+4(i+1)(k+1))$.

Let $\Phi_0 = (\phi_0, \psi_0) \in X_\sigma^{+,+}$ with

$$\phi_0(0) = \frac{\rho\left(1-\beta^{2(k+1)}\right)}{\alpha\left(1+\beta^{2(k+1)}\right)}, \qquad \psi_0(0) = \frac{\rho\left(1+\beta^{2(k+1)}-2\beta^{k+1}\right)}{\alpha\left(1+\beta^{2(k+1)}\right)}.$$

Then,

$$\begin{aligned} x^{\phi_0}(n) &= \frac{2\rho\beta^{n-4i(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}-\frac{\rho}{\alpha}, \\ y^{\phi_0}(n) &= -\frac{2\rho\beta^{n-(4i-1)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}+\frac{\rho}{\alpha}, \end{aligned} \qquad n \in N(4i(k+1)+1, (4i+1)(k+1));$$

$$\begin{aligned} x^{\phi_0}(n) &= \frac{2\rho\beta^{n-4i(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}-\frac{\rho}{\alpha}, \\ y^{\phi_0}(n) &= \frac{2\rho\beta^{n-(4i+1)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}-\frac{\rho}{\alpha}, \end{aligned} \qquad n \in N((4i+1)(k+1)+1, (4i+2)(k+1));$$

$$\begin{aligned} x^{\phi_0}(n) &= -\frac{2\rho\beta^{n-(4i+2)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}+\frac{\rho}{\alpha}, \\ y^{\phi_0}(n) &= \frac{2\rho\beta^{n-(4i+1)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}-\frac{\rho}{\alpha}, \end{aligned} \qquad n \in N((4i+2)(k+1)+1, (4i+3)(k+1));$$

$$\begin{aligned} x^{\phi_0}(n) &= -\frac{2\rho\beta^{n-(4i+2)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}+\frac{\rho}{\alpha}, \\ y^{\phi_0}(n) &= -\frac{2\rho\beta^{n-(4i+3)(k+1)}}{\alpha\left(1+\beta^{2(k+1)}\right)}+\frac{\rho}{\alpha}, \end{aligned} \qquad n \in N((4i+3)(k+1)+1, 4(i+1)(k+1)).$$

Clearly, $\{x^{\phi_0}(n), y^{\phi_0}(n)\}$ is periodic with minimal period $4(k+1)$, and as $j \to \infty$,

$$x(k_4+j)-x^{\Phi_0}(j) = \left(x(k_4)-\frac{\rho\left(1-\beta^{2(k+1)}\right)}{\alpha\left(1+\beta^{2(k+1)}\right)}\right)\beta^j \to 0,$$
$$y(k_4+j)-y^{\Phi_0}(j) = \left(y(k_4)-\frac{\rho\left(1+\beta^{2(k+1)}-2\beta^{k+1}\right)}{\alpha\left(1+\beta^{2(k+1)}\right)}\right)\beta^j \to 0.$$

Denote $m = k_4$. This completes the proof.

PROOF OF THEOREM 1.2. We only prove the case where $\sigma > \rho/(1-\beta)$, the case where $\sigma < -\rho/(1-\beta)$ is similar.

In view of (1.1), we have

$$\begin{aligned} x(n)-\beta x(n-1) &\le \rho, \\ y(n)-\beta y(n-1) &\le \rho, \end{aligned} \qquad n \in N.$$

By induction, this implies

$$x(n) \le \left(\phi(0) - \frac{\rho}{1-\beta}\right)\beta^n + \frac{\rho}{1-\beta}, \qquad n \in N. \tag{2.15}$$
$$y(n) \le \left(\psi(0) - \frac{\rho}{1-\beta}\right)\beta^n + \frac{\rho}{1-\beta},$$

Since $\rho/(1-\beta) < \sigma$ and $\beta \in (0,1)$, from (2.15), we conclude that there exists $m_1 \in N(1)$ such that $x(n) < \sigma$, $y(n) < \sigma$ for $n \in N(m_1)$. Thus,

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \qquad n \in N(m_1 + k + 1). \tag{2.16}$$

Therefore,

$$x(n) = \left(x(m_1+k) - \frac{\rho}{1-\beta}\right)\beta^{n-m_1-k} + \frac{\rho}{1-\beta},$$
$$y(n) = \left(y(m_1+k) + \frac{\rho}{1-\beta}\right)\beta^{n-m_1-k} - \frac{\rho}{1-\beta},$$

which implies that $(x(n), y(n)) \to (\rho/(1-\beta), -\rho/(1-\beta))$ as $n \to \infty$.

PROOF OF THEOREM 1.3. We prove (i) only; (ii) can be dealt with similarly. First, let $\Phi \in X_\sigma^{+,+}$. In this case,

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) + \rho, \end{aligned} \qquad n \in N(1, k+1).$$

Similar to the proof of Theorem 1.1, we can get $k_1 \in N(k+1)$ such that $(x_{k_1}, y_{k_1}) \in X_\sigma^{-,+}$. And thus,

$$\begin{aligned} x(n) &= \beta x(n-1) - \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \qquad n \in N(k_1+1, k_1+k+1);$$

and that there is $k_2 \in N(k_1+k+1)$ such that $(x_{k_2}, y_{k_2}) \in X_\sigma^{-,-}$. Therefore,

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) - \rho, \end{aligned} \qquad n \in N(k_2+1, k_2+k+1). \tag{2.17}$$

So,

$$\begin{aligned} x(n) &= \left(x(k_2) - \frac{\rho}{1-\beta}\right)\beta^{n-k_2} + \frac{\rho}{1-\beta}, \\ y(n) &= \left(y(k_2) + \frac{\rho}{1-\beta}\right)\beta^{n-k_2} - \frac{\rho}{1-\beta}. \end{aligned} \tag{2.18}$$

By induction, $x(n) \le \rho/(1-\beta) = \sigma$ and $y(n) \le \rho/(1-\beta) = \sigma$, which implies (2.18) for $n \in N(k_2+1)$. Therefore, $\lim_{n\to\infty}(x(n), y(n)) = (\rho/(1-\beta), -\rho/(1-\beta))$.

From the above argument, we see that the conclusion holds when $\Phi \in X_\sigma^{-,+} \cup X_\sigma^{-,-}$.

Now consider $\Phi \in X_\sigma^{+,-}$. Then, for $n \in N(1, k+1)$, we have

$$\begin{aligned} x(n) &= \beta x(n-1) + \rho, \\ y(n) &= \beta y(n-1) + \rho. \end{aligned} \tag{2.19}$$

So,

$$\begin{aligned} x(n) &= \left(\phi(0) - \frac{\rho}{1-\beta}\right)\beta^{n} + \frac{\rho}{1-\beta}, \\ y(n) &= \left(\psi(0) - \frac{\rho}{1-\beta}\right)\beta^{n} + \frac{\rho}{1-\beta}. \end{aligned} \tag{2.20}$$

Thus, $(x_{k+1}, y_{k+1}) \in X_\sigma^{+,-}$, and (2.19) holds for $n \in N(k+2, 2(k+1))$. By induction, we see that (2.19) and (2.20) hold for $n \in N$. Therefore, $\lim_{n\to\infty}(x(n), y(n)) = (\rho/(1-\beta), \rho/(1-\beta))$. This completes the proof of (i).

REFERENCES

1. T. Roska and L.O. Chua, Cellular neural networks with nonlinear and delay-type template elements, *Int. J. Circuit Theory Appl.* **20**, 469–481, (1992).
2. L. Huang and J. Wu, The role of threshold in preventing delay-induced oscillations of frustrated neural networks with McCulloch-Pitts nonlinearity (to appear).
3. J. Wu, *Introduction to Neural Dynamics and Signal Tranmission Delay*, De-Gruyter, Berlin, (2001).
4. Z. Zhou, J.S. Yu and L.H. Huang, Asymptotic behavior of delay difference systems, *Computers Math. Applic.* **42** (3/5), 283–290, (2001).
5. R.D. DeGroat, L.R. Hunt, D.A. Linebarger and M. Verma, Discrete-time nonlinear system stability, *IEEE Trans. Circuits Syst. I* **39**, 834–840, (1992).
6. T. Ushio, Limitation of delay feedback control in nonlinear discrete-time systems, *IEEE Trans. Circuits Syst. I* **43**, 815–816, (1996).
7. R.P. Agarwal, *Difference Equations and Inequalities: Theory, Method and Applications*, Marcel Dekker, New York, (1992).
8. R.P. Agarwal and P.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer, Dordrecht, (1997).
9. V.L.J. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, (1993).