J. Nonlinear Sci. Vol. 13: pp. 289–310 (2003)
DOI: 10.1007/s00332-003-0524-6

Journal of Nonlinear Science

Travelling Waves and Numerical Approximations in a Reaction Advection Diffusion Equation with Nonlocal Delayed Effects*

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Received April 15, 2002; accepted January 15, 2003
Online publication March 31, 2003
Communicated by J. Bélair

Summary. In this paper, we consider the growth dynamics of a single-species population with two age classes and a fixed maturation period living in a spatial transport field. A Reaction Advection Diffusion Equation (RADE) model with time delay and nonlocal effect is derived if the mature death and diffusion rates are age independent. We discuss the existence of travelling waves for the delay model with three birth functions which appeared in the well-known Nicholson's blowflies equation, and we consider and analyze numerical solutions of the travelling wavefronts from the wave equations for the problems with nonlocal temporally delayed effects. In particular, we report our numerical observations about the change of the monotonicity and the possible occurrence of multihump waves. The stability of the travelling wavefront is numerically considered by computing the full time-dependent partial differential equations with nonlocal delay.

Key words. Travelling wave, reaction advection diffusion equation, structured population, nonlocal time delay, existence, numerical approximation

AMS subject classification. 92D25, 35K55, 65M06, 35N06

1. Introduction

The fast-growing modelling and analyzing of population dynamics with delay in a spatially varying environment have been playing more and more important roles in discov-

* Research partially supported by Natural Sciences and Engineering Research Council of Canada, and by Canada Research Chairs Program.

ering the relation between species and their environment, and in understanding the dynamic processes involved in such areas as the spread and control of diseases and viruses, predator-prey and competition interaction, evolution of pesticide-resistant strains, biological pest control, plant-herbivore systems, and so on.

Single-species models can reflect a telescoping of effects which influence population dynamics. Let $N(t)$ be the population of the species at time t; then the classical logistic model is of the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad t > 0, \tag{1}$$

where $r > 0$ is the birth-rate parameter and $K > 0$ is the carrying capacity of the environment.

In the above model, the birth rate is considered to act instantaneously whereas, in reality, there may be a time delay to take account of the time to reach maturity. Such a delay has been incorporated into the logistic model, leading to the ordinary delay differential equation

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right), \quad t > 0, \tag{2}$$

where $\tau > 0$ is the delay parameter, which models the fact that the regular effect depends on the population at earlier time, $t - \tau$.

Fisher's equation, given below, deals with the diffusion aspect of the spatial effect in the population dynamics and was first suggested by Fisher (1937) as a deterministic version of a stochastic model for the spatial spread of a favoured gene in a population:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{K}\right), \quad t > 0. \tag{3}$$

Incorporating a time delay into the above diffusion model, many investigators in general simply introduce the time delay into the birth term, in the same fashion as the ordinary delay differential equation (ODDE) model above. A typical case is the widely used reaction-diffusion equation with delay and local effect (see Yoshida [21], Memory [9], Yang and So [20], and Feng and Lu [3]), given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru(t,x)\left(1 - \frac{u(t-\tau,x)}{K}\right), \quad t > 0. \tag{4}$$

But in recent years it has been recognized that there are modelling difficulties with this approach, and some alternative approaches have been developed. Britton [1] and Gourley and Britton [5] introduced a weighted spatial-averaging delay term over the whole infinite domain, in which the weighted function is properly derived using probabilistic arguments. Smith [13] developed another approach to derive a scalar delay differential equation from the so-called structured model where the population is divided into immature and mature two age classes with the time delay being the maturation period. Smith's approach was used in Smith and Thieme [14] and in So, Wu, and Zou [15] to derive a system of delay differential equations for matured population distributed in a patchy environment. In So, Wu, and Zou [16], the case of a continuous unbounded spatial domain was considered and a reaction-diffusion equation with temporally delayed and spatially nonlocal effect was derived and analyzed. The existence of a travelling wavefront for the unbounded domain case was studied in [16] following the method developed in Wu and Zou [19].

In this paper, we consider a single-species population with two age classes and a fixed maturation period living in a spatial transport field. We derive a Reaction Advection Diffusion Equation (RADE) model with time delay and nonlocal spatial effect when the mature death and diffusion rates are age independent. Our focus is on the existence and qualitative behaviours of the travelling wavefronts in the case where the birth functions are the ones that appeared in the well-known Nicholson's blowflies equation. We also consider and analyze the numerical computation of travelling wavefronts.

The paper is organized as follows. In Section 2, we derive a RADE model with time delay and nonlocal effect. The existence of travelling wavefronts for three typical birth functions is established theoretically via an iteration method in Section 3, and the numerical computation of these waves is provided in Section 4. Some final remarks are then provided in Section 5.

2. The RADE Model

We consider a single-species population in a spatial transport field. Let $u(t, a, x)$ denote the density of the population of the species at the time $t \geq 0$, the age $a \geq 0$, and the spatial location $x \in \Omega = R$. Let $D(a)$ and $d(a)$ denote the diffusion rate and the death rate, respectively, at age a. Let B be the transport velocity of the field. Then, the population density function $u(t, a, x)$ satisfies

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = \frac{\partial}{\partial x}\left(D(a)\frac{\partial u}{\partial x} + B\, u\right) - d(a)u, \quad t > 0, \quad a > 0, \quad x \in \Omega, \tag{5}$$

and a natural boundary condition

$$|u(t, a, \pm\infty)| < \infty, \quad t \geq 0, \quad a \geq 0. \tag{6}$$

Now, we assume that the population has only two age stages as matured and immatured species, and let $\tau \geq 0$ be the maturation time for the species and $r > 0$ denote the life limit of an individual species. Therefore, $u(t, r, x) = 0$ at any time $t > 0$ and any $x \in \Omega$. We integrate the population density $u(t, a, x)$ with respect to age a from τ to r to obtain the total matured population $w(t, x)$, i.e.,

$$w(t, x) = \int_\tau^r u(t, a, x)\, da, \quad x \in \Omega, \quad t \geq 0.$$

Since only the mature can reproduce, we have

$$u(t, 0, x) = b(w(t, x)), \quad t \geq 0, \quad x \in \Omega, \tag{7}$$

where b: $[0, \infty) \to [0, \infty)$ is the birth function.

If the diffusion and death rates for the matured population are age independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in [\tau, r]$, then integrating (5) leads to

$$\frac{\partial w}{\partial t} = u(t, \tau, x) + \frac{\partial}{\partial x}\left(D_m\frac{\partial w}{\partial x} + B\, w\right) - d_m w. \tag{8}$$

In (8), we need to eliminate $u(t,\tau,x)$ to obtain an equation for $w(t,x)$. This can be achieved as follows. Let us fix $s \geq 0$ and define $V^s(t,x) = u(t, t-s, x)$ for $s \leq t \leq s+\tau$. Then, from (5), it follows, for $s \leq t \leq s+\tau$, that

$$\frac{\partial}{\partial t} V^s(t,x) = \frac{\partial}{\partial x}\left(D(t-s)\frac{\partial V^s(t,x)}{\partial x} + B\ V^s(t,x)\right) - d(t-s)V^s(t,x), \tag{9}$$

with

$$V^s(s,x) = b(w(s,x)), \quad x \in \Omega, \tag{10}$$

and

$$|V^s(t,\pm\infty)| < \infty, \quad t \geq 0. \tag{11}$$

Note that (9) is a linear reaction advection diffusion equation (RADE). From the standard theory of Fourier transforms, we obtain

$$V^s(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{H}(s,\omega) e^{-\int_0^{t-s}(\omega^2 D(a) - i\omega B + d(a))\,da} e^{i\omega x}\, d\omega, \tag{12}$$

where $\mathcal{H}(s,w)$ is determined by

$$\mathcal{H}(s,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(w(s,y)) e^{-i\omega y}\, dy.$$

Let D_{im} and d_{im} denote the diffusion and death rates of the immatured population, respectively, i.e., $D(a) = D_{im}(a)$ and $d(a) = d_{im}(a)$ for $a \in [0,\tau]$. Let

$$\varepsilon = e^{-\int_0^\tau d_I(a)\,da}, \quad \alpha = \int_0^\tau D_{im}(a)\,da. \tag{13}$$

Then, we have

$$\begin{aligned} u(t,\tau,x) &= V^{t-\tau}(t,x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(w(t-\tau,y)) e^{-\int_0^\tau d_{im}(a)\,da} \\ &\quad \times \left(\int_{-\infty}^{+\infty} e^{-(\alpha\omega^2 - iB\tau\omega)} e^{i\omega(x-y)}\, d\omega\right) dy \\ &= \frac{\varepsilon}{\sqrt{4\pi\alpha}}\int_{-\infty}^{+\infty} b(w(t-\tau,y)) e^{\frac{-(x+B\tau-y)^2}{4\alpha}}\, dy. \end{aligned} \tag{14}$$

Let

$$\Phi_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{\frac{-x^2}{4\alpha}}. \tag{15}$$

Then we obtain a RADE with delay

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial}{\partial x}\left(D_m \frac{\partial w}{\partial x} + B\ w\right) - d_m w + F(x, w(t-\tau,\cdot)), \\ & \qquad x \in \Omega, \quad t > 0, \end{aligned} \tag{16}$$

$$w(t,x) = w_0(t,x), \quad x \in \Omega, \quad t \in [-\tau, 0], \tag{17}$$

$$|w(t,\pm\infty)| < \infty, \quad t \geq 0, \tag{18}$$

where

$$F(x, w(t-\tau, \cdot)) = \varepsilon \int_{-\infty}^{+\infty} b(w(t-\tau, y))\Phi_\alpha(x + B\tau - y)\, dy, \tag{19}$$

and w_0 is an initial function which should be specified.

In the RADE, $F(x, w(t-\tau, \cdot))$ represents the nonlocal spatial effect with time delay. In addition to the time delay τ, there are other three important parameters: ε, α, and B. ε reflects the impact of the death rate of the immature and α represents the effect of the dispersal rate of the immature on the growth rate of the matured population, and B is the velocity of the spatial transport field. Therefore, we obtain a general model with nonlocal delay effects, where the effect of the convection during the maturation period is simply the spatial translation by $B\tau$ in the nonlocal delay effect term.

3. Travelling Wavefronts

In population dynamics, a key element or precursor to a developmental process seems to be the appearance of a travelling wave, a special solution which travels without change of shape. In this section, we establish the existence of travelling wavefronts of the problem (16)–(18) with the nonlocal delayed effect (19). We consider, in our analysis, three birth functions which have been widely used in the well-studied Nicholson's blowflies equation for some special parameters (for example, see [6], [9], [11], [12], and [16]). These functions are given by

$$b_1(w) = pwe^{-aw^q}, \quad b_2(w) = \frac{pw}{1 + aw^q}, \tag{20}$$

with constants $p > 0$, $q > 0$, and $a > 0$, and

$$b_3(w) = \begin{cases} pw\left(1 - \dfrac{w^q}{K_c^q}\right), & 0 \le w \le K_c, \\ 0, & w > K_c, \end{cases} \tag{21}$$

with constants $p > 0$, $K_c > 0$, and $q > 0$. For a reaction-diffusion equation model, So, Wu, and Zou [16] analyzed the existence for the special case with the birth function $b(w) = pwe^{-aw}$. In what follows, we will extend the existence results in [16] to the RADE model under nonlocal delayed effects.

In the usual format of a travelling wavefront, we look for solutions $w(t, x) = W(z)$ by setting $z = x + ct$, where $c > 0$ is the wave speed. Then (16) becomes the following second-order functional differential equation:

$$cW'(z) = D_m W''(z) + BW'(z) - d_m W(z) + \varepsilon \mathcal{F}(W)(z), \tag{22}$$

where the operator $\mathcal{F}$: $C(R; R) \to C(R; R)$ is defined, for $W \in C(R, R)$, as

$$\mathcal{F}(W)(z) = \int_{-\infty}^{+\infty} b(W(z - c\tau + y))\Phi_\alpha(y - B\tau)\, dy, \quad z \in R, \tag{23}$$

and $\Phi_\alpha(z)$ is given in (15).

If Problem (22)–(23) has two constant equilibria $0 \le w_- < w_+$, we will seek the leftward-moving wavefront $W(z)$ of the problem. Thus, the boundary conditions are taken as

$$W(-\infty) = w_-, \qquad W(+\infty) = w_+. \tag{24}$$

It is easy to show that

Lemma 1. *If $\frac{\varepsilon p}{d_m} > 1$, Problem (22)–(23) has exactly two constant solutions $0 \le w_i^* < w_i^{**} := M$ for birth function $b_i(\cdot)$, $i = 1, 2, 3$. More precisely, we have*

(i) For $b(w) = b_1(w)$,

$$w_1^* = 0, \quad w_1^{**} = \left(\frac{1}{a} \ln \frac{\varepsilon p}{d_m}\right)^{1/q}. \tag{25}$$

(ii) For $b(w) = b_2(w)$,

$$w_2^* = 0, \quad w_2^{**} = \left(\frac{\varepsilon p - d_m}{a d_m}\right)^{1/q}. \tag{26}$$

(iii) For $b(w) = b_3(w)$,

$$w_3^* = 0, \quad w_3^{**} = K\left(1 - \frac{d_m}{\varepsilon p}\right)^{1/q}. \tag{27}$$

Similar to [16], we have the following list of results:

Lemma 2. *Assume that $1 < \frac{\varepsilon p}{d_m} \le e^{\frac{1}{q}}$ holds for the case $b(w) = b_1(w)$; $1 < \frac{\varepsilon p}{d_m} \le \frac{1+q}{q}$ holds for the case $b(w) = b_3(w)$; either $1 < \frac{\varepsilon p}{d_m} \le \frac{q}{q-1}$ holds if $q > 1$, or $1 < \frac{\varepsilon p}{d_m} < \infty$ holds if $0 < q \le 1$ for the case $b(w) = b_2(w)$. Then, for any nondecreasing function $U \in C(R; R)$ such that $0 \le U(z) \le M$ for $z \in R$, we have*

(a) $\mathcal{F}(U)(z) \ge 0$ for all $z \in R$;
(b) $\mathcal{F}(U)(z)$ is nondecreasing in $z \in R$;
(c) $\mathcal{F}(U)(z) \le \mathcal{F}(V)(z)$ for all $z \in R$, provided $U, V \in C(R; R)$ are such that $0 \le U(z) \le V(z) \le M$ for $z \in R$.

Proof. Consider the case where $b(w) = b_3(w)$. Note that on the interval $[0, \frac{K_c}{(1+q)^{1/q}}]$ the birth function $b_3(w)$ is increasing. Moreover, when $1 < \frac{\varepsilon p}{d_m} \le \frac{1+q}{q}$, we have

$$0 < 1 - \frac{d_m}{\varepsilon p} \le 1 - \frac{q}{1+q} = \frac{1}{1+q}.$$

So,

$$w_3^{**} = K_c\left(1 - \frac{d_m}{\varepsilon p}\right)^{1/q} \le \frac{K_c}{(1+q)^{1/q}}.$$

Therefore, $b_3(w)$ is increasing on the interval $[0, w_3^{**}]$. Following this fact, the conclusion (a)(b)(c) for the case with $b(w) = b_3(w)$ can easily be obtained from the definition of

(23). Similarly, we can verify the lemma for the cases with the birth functions $b_1(w)$ and $b_2(w)$. □

Lemma 3. *For $\lambda \in R$, $c > 0$ and $\tau \geq 0$, define the function*

$$\Delta(\lambda, c) = \varepsilon p e^{\alpha\lambda^2 - \lambda c\tau + \lambda B\tau} - [d_m + c\lambda - B\lambda - D_m\lambda^2]. \tag{28}$$

There exists $c^*(\tau, D_m, d_m, p, \varepsilon, \alpha) > 0$ such that for $c - B > c^*$ the equation $\Delta(\lambda, c) = 0$ has two positive real roots $0 < \lambda_1 < \lambda_2$. Moreover,

$$\Delta(\lambda, c) = \begin{cases} > 0 & \text{for } \lambda < \lambda_1, \\ < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2), \\ > 0 & \text{for } \lambda > \lambda_2. \end{cases} \tag{29}$$

Proof. Consider the nonlinear system

$$\Delta(\lambda, c) = 0, \qquad \frac{\partial}{\partial\lambda}\Delta(\lambda, c) = 0.$$

We can easily prove that this system has a root (c^*, λ^*), which depends on τ, D_m, d_m, p, ε, and α such that $c^* > 0$ and $\lambda^* > 0$ and that for $c - B > c^*$, the equation $\Delta(\lambda, c) = 0$ has two positive real roots $0 < \lambda_1 < \lambda_2$; and $\Delta(\lambda, c) > 0$ for $\lambda < \lambda_1$; $\Delta(\lambda, c) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$; and $\Delta(\lambda, c) > 0$ for $\lambda > \lambda_2$. □

In the following part of this section, we will always assume that the conditions in Lemma 2 for the cases with birth functions $b_1(w)$, $b_2(w)$, and $b_3(w)$ hold. We will focus on the existence of the monotone travelling wavefronts of (22)–(24) by analyzing mainly the cases with birth functions $b_1(w)$, $b_2(w)$, and $b_3(w)$. Following the approach for the existence of travelling waves in the papers Wu and Zou [19] and So, Wu, and Zou [16], we need to construct a pair of upper and lower solutions.

Define the profile set for travelling wavefronts as

$$\Gamma = \left\{ W \in C(R; R) \;\middle|\; \begin{array}{ll} (i) & W(z) \text{ is nondecreasing for } z \in R; \\ (ii) & \lim_{z \to -\infty} W(z) = 0, \ \lim_{z \to +\infty} W(z) = M. \end{array} \right\} \tag{30}$$

We start with the case of the second birth function $b_2(w)$. Let $c - B > c^*$ and $0 < \lambda_1 < \lambda_2$ be the root of $\Delta(c, \lambda) = 0$ in Lemma 3. Choose $\delta > 0$ sufficiently small so that $\delta < \lambda_1 < \lambda_1 + \delta < \lambda_2$.

Lemma 4. *For the case with birth function $b(w) = b_2(w)$, let $\bar{\phi}(z) = \min\{0, Me^{\lambda_1 z}\}$ and $\underline{\phi}(z) = \max\{0, M(1 - Ne^{\delta z})e^{\lambda_1 z}\}$, where $N > 1$ is a constant. Then, we have*

(a) $\bar{\phi}$ is an upper solution in the sense that

$$c\bar{\phi}'(z) \geq D_m\bar{\phi}''(z) + B\bar{\phi}'(z) - d_m\bar{\phi}(z) + \varepsilon\mathcal{F}(\bar{\phi})(z), \quad a.e. \text{ in } R, \tag{31}$$

and $\bar{\phi} \in \Gamma$.

(b) For sufficiently large N, $\underline{\phi}$ is a lower solution in the sense that

$$c\underline{\phi}'(z) \le D_m\underline{\phi}''(z) + B\underline{\phi}'(z) - d_m\underline{\phi}(z) + \varepsilon\mathcal{F}(\underline{\phi})(z), \quad a.e. \text{ in } R, \tag{32}$$

and $\underline{\phi} \not\equiv 0$.

(c) $0 \le \underline{\phi}(z) \le \bar{\phi}(z) \le M \quad \text{for all } z \in R.$

Proof. It is clear that to prove $\bar{\phi} \in \Gamma$ is an upper solution of (22)–(24), we only need to show that $\bar{\phi}$ satisfies the inequality (31).

From the definition of $\bar{\phi}$, it is obvious that $0 \le \bar{\phi}(y) \le M$ for all $y \in R$. Noting that $0 < M \le \frac{1}{(aq-a)^{1/q}}$ if $q > 1$; and $M > 0$ if $0 < q \le 1$; and the birth function $b_2(w)$ is increasing on the interval $[0, \frac{1}{(aq-a)^{1/q}}]$ for $q > 1$ and on the interval $[0, \infty)$ for $0 < q \le 1$, we have

$$b_2(\bar{\phi}(y)) \le \frac{pM}{1+aM^q}, \quad \textit{for } y \in R.$$

Let $z \in (0, +\infty)$. Then $\bar{\phi}(z) = M$. So,

$$\begin{aligned} c\bar{\phi}'(z) \; &- \; D_m\bar{\phi}''(z) - B\bar{\phi}'(z) + d_m\bar{\phi}(z) - \varepsilon\mathcal{F}(\bar{\phi})(z) \\ &\ge \; M\left(d_m - \frac{\varepsilon p}{1+aM^q}\right) \\ &= \; \varepsilon pM\left(\frac{d_m}{\varepsilon p} - \frac{1}{1+aM^q}\right) = 0, \end{aligned}$$

since $M = (\frac{\varepsilon p - d_m}{a d_m})^{1/q}$.

Let $z \in (-\infty, 0)$. Then $\bar{\phi}(z) = Me^{\lambda_1 z}$. Noting that $\frac{1}{1+aw^q} \le 1$ for $w \in [0, \infty)$ and $q > 0$, we have

$$\begin{aligned} &c\bar{\phi}'(z) - D_m\bar{\phi}''(z) - B\bar{\phi}'(z) + d_m\bar{\phi}(z) - \varepsilon\mathcal{F}(\bar{\phi})(z) \\ &\quad \ge \; (d_m + c\lambda_1 - B\lambda_1 - D_m\lambda_1^2)Me^{\lambda_1 z} - \varepsilon p\int_{-\infty}^{\infty}\bar{\phi}(z - c\tau - y)\Phi_\alpha(y - B\tau)\,dy \\ &\quad \ge \; (d_m + c\lambda_1 - B\lambda_1 - D_m\lambda_1^2)Me^{\lambda_1 z} - M\varepsilon p\int_{-\infty}^{\infty}e^{\lambda_1(z-y-c\tau)}\Phi_\alpha(y - B\tau)\,dy \\ &\quad = \; (d_m + c\lambda_1 - B\lambda_1 - D_m\lambda_1^2 - \varepsilon pe^{\alpha^2\lambda_1^2 - \lambda_1 c\tau + \lambda_1 B\tau})Me^{\lambda_1 z} \\ &\quad = \; -Me^{\lambda_1 z}\Delta(\lambda_1, c) = 0, \end{aligned}$$

since λ_1 is the root of (28). This finishes the proof of part (a).

Now, we prove part (b). Let $z_1 = \frac{1}{\delta}\ln\frac{1}{N}$. Then $z_1 < 0$ for $N > 1$ and

$$\underline{\phi}(z) = \begin{cases} 0, & z \ge z_1, \\ M(1 - Ne^{\delta z})e^{\lambda_1 z}, & z < z_1. \end{cases}$$

It is obvious that $0 \le \underline{\phi}(z) \le M$ for $z \in R$. Thus, for $z \in (z_1, \infty)$, we have $\underline{\phi}(z) = 0$.

So, we have

$$c\underline{\phi}'(z) - D_m\underline{\phi}''(z) - B\underline{\phi}'(z) + d_m\underline{\phi}(z) - \varepsilon\mathcal{F}(\underline{\phi})(z) \le 0,$$

since $b_2(\underline{\phi}(y)) \ge 0$ for $y \in R$.

Let $z \in (-\infty, z_1)$. Noting that $\frac{1}{1+a\underline{\phi}^q(z)} \ge 1 - a\underline{\phi}^q(z)$ for $z \in R$ and $q > 0$, we have

$$\varepsilon\mathcal{F}(\underline{\phi})(z) \ge \varepsilon p \int_{-\infty}^{\infty} \underline{\phi}(z - c\tau - y)(1 - a\underline{\phi}^q(z - c\tau - y))\Phi_\alpha(y - B\tau)\, dy.$$

Using the fact that for $y \in R$,

$$\begin{aligned} 0 &\le \underline{\phi}(y) \le Me^{\lambda_1 y}(1 + Ne^{\delta y}), \\ \underline{\phi}^{1+q}(y) &\le M^{1+q}e^{\lambda_1 y(1+q)}(1 + Ne^{\delta y})^{1+q}, \end{aligned}$$

and choosing $0 < \delta < q\lambda_1$, we have

$$\begin{aligned} &\varepsilon pa \int_{-\infty}^{\infty} \underline{\phi}^{1+q}(z - c\tau - y)\Phi_\alpha(y - B\tau)\, dy \\ &\le \varepsilon paM^{1+q}e^{(\lambda_1+\delta)z} \int_{-\infty}^{\infty} e^{-(q+1)\lambda_1(y+c\tau)}(1 + Ne^{\epsilon(z-y-c\tau)})^{q+1}\Phi_\alpha(y - B\tau)\, dy \\ &\le \varepsilon paM^{1+q}e^{(\lambda_1+\delta)z} \int_{-\infty}^{\infty} e^{-(q+1)\lambda_1(y+c\tau)}(1 + e^{-\delta(y+c\tau)})^{q+1}\Phi_\alpha(y - B\tau)\, dy \\ &= \varepsilon paM^{1+q}e^{(\lambda_1+\delta)z}N^*, \end{aligned}$$

where N^* is defined as the integral

$$\begin{aligned} N^* &= N^*(\lambda_1, \delta, \tau, c, \alpha, q, B) \\ &:= \int_{-\infty}^{\infty} e^{-(q+1)\lambda_1(y+c\tau)}(1 + e^{-\delta(y+c\tau)})^{q+1}\Phi_\alpha(y - B\tau)\, dy < \infty, \end{aligned}$$

which is convergent. Let $\psi(y) = M(1 - Ne^{\delta y})e^{\lambda_1 y}$. Then $0 \le \psi(y) = \underline{\phi}(y)$ for $y \le z_1$, and $\psi(y) < 0$ for $y > z_1$. So, we have

$$\begin{aligned} \varepsilon\mathcal{F}(\underline{\phi})(z) &\ge \varepsilon p \int_{-\infty}^{\infty} (\psi(z - c\tau - y) - a\underline{\phi}^{q+1}(z - c\tau - y))\Phi_\alpha(y - B\tau)\, dy \\ &\ge \varepsilon paM(e^{\lambda_1 z}e^{\lambda_1(\alpha\lambda_1 - c\tau + B\tau)} - Ne^{(\lambda_1+\delta)z}e^{(\lambda_1+\delta)(\alpha(\lambda_1+\epsilon) - c\tau + B\tau)}) \\ &\quad - \varepsilon paM^{1+q}e^{(\lambda_1+\delta)z}N^*. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &c\underline{\phi}'(z) - D_m\underline{\phi}''(z) - B\underline{\phi}'(z) + d_m\underline{\phi}(z) - \varepsilon\mathcal{F}(\underline{\phi})(z) \\ &\le -Me^{\lambda_1 z}\triangle(\lambda_1, c) + MNe^{(\lambda_1+\delta)z}\triangle(\lambda_1 + \delta, c) + \varepsilon paM^{1+q}e^{(\lambda_1+\delta)z}N^* \\ &= Me^{(\lambda_1+\delta)z}\triangle(\lambda_1 + \delta, c)\left(N + \frac{\varepsilon paM^q N^*}{\triangle(\lambda_1 + \delta, c)}\right) \le 0, \end{aligned}$$

for sufficiently large $N > 1$, since $\triangle(\lambda_1, c) = 0$ and $\triangle(\lambda_1 + \delta, c) < 0$. Thus, (b) holds. (c) is obvious. This completes the proof. □

We now obtain similar results for $b(w) = b_1(w)$ or $b_3(w)$.

Lemma 5. *For the cases with birth functions $b_1(w)$ and $b_3(w)$, let $\bar{\phi}(z) = \min\{0, Me^{\lambda_1 z}\}$ and $\underline{\phi}(z) = \max\{0, M(1 - Ne^{\delta z})e^{\lambda_1 z}\}$. Then, for sufficiently large $N > 1$ and small $\delta > 0$, $\bar{\phi}$ and $\underline{\phi}$ are the upper and lower solutions of the problem (22)–(24) which satisfy (a)(b)(c) of Lemma 4.*

Proof. Consider the case with birth function $b_3(w)$. From the definition of $\bar{\phi}$, it is obvious that $0 \le \bar{\phi}(y) \le M$ for all $y \in R$. Noting that $0 < M \le \frac{K_c}{(1+q)^{1/q}} < K_c$ in Lemma 2 and the birth function $b_3(w)$ is increasing on the interval $[0, \frac{K_c}{(1+q)^{1/q}}]$, we have

$$b_3(\bar{\phi}(y)) \le pM\left(1 - \frac{M^q}{K_c^q}\right), \quad \text{for } y \in R.$$

Let $z \in (0, +\infty)$. Then $\bar{\phi}(z) = M$. So,

$$\begin{aligned} &c\bar{\phi}'(z) - D_m\bar{\phi}''(z) - B\bar{\phi}'(z) + d_m\bar{\phi}(z) - \varepsilon\mathcal{F}(\bar{\phi})(z) \\ &\ge M\left[d_m - \varepsilon p\left(1 - \frac{M^q}{K_c^q}\right)\right] = 0, \end{aligned}$$

since $M = K_c(1 - \frac{d_m}{\varepsilon p})^{1/q}$.

Let $z \in (-\infty, 0)$. Then $\bar{\phi}(z) = Me^{\lambda_1 z}$. Noting that $(1 - \frac{w^q}{K_c^q}) \le 1$ for $w \in [0, K_c)$, we have

$$\begin{aligned} &c\bar{\phi}'(z) - D_m\bar{\phi}''(z) - B\bar{\phi}'(z) + d_m\bar{\phi}(z) - \varepsilon\mathcal{F}(\bar{\phi})(z) \\ &\ge c\bar{\phi}'(z) - D_m\bar{\phi}''(z) - B\bar{\phi}'(z) + d_m\bar{\phi}(z) - \varepsilon p\int_{-\infty}^{\infty}\bar{\phi}(z - c\tau - y)\Phi_\alpha(y - B\tau)\,dy \\ &= (d_m + c\lambda_1 - B\lambda_1 - D_m\lambda_1^2 - \varepsilon pe^{\alpha^2\lambda_1^2 - \lambda_1 c\tau + \lambda_1 B\tau})Me^{\lambda_1 z} = 0, \end{aligned}$$

since λ_1 is the root of (28).

For proving part (b), let $z_1 = \frac{1}{\delta}\ln\frac{1}{N}$. Then $z_1 < 0$ for $N > 1$, and $0 \le \underline{\phi}(z) \le M < K_c$ for $z \in R$. Thus, let $z \in (z_1, \infty)$, then $\underline{\phi}(z) = 0$. From $b_3(\underline{\phi}(y)) \ge 0$ for $y \in R$, we have

$$c\underline{\phi}'(z) - D_m\underline{\phi}''(z) - B\underline{\phi}'(z) + d_m\underline{\phi}(z) - \varepsilon\mathcal{F}(\underline{\phi})(z) \le 0.$$

Let $z \in (-\infty, z_1)$. Let $\psi(y) = M(1 - Ne^{\delta y})e^{\lambda_1 y}$. Then $0 \le \psi(y) = \underline{\phi}(y) \le K_c$ for $y \le z_1$, and $\psi(y) < 0$ for $y > z_1$. For $q > 1$, noting that the function $1 - \frac{\underline{\phi}^q(z)}{K_c^q} \ge 1 - \frac{\underline{\phi}^s(z)}{K_c^s}$

for $z \in R$, where $s < q$ is the largest odd integer, we have

$$\begin{aligned}
\varepsilon\mathcal{F}(\underline{\phi})(z) &\geq \varepsilon p \int_{-\infty}^{\infty} \underline{\phi}(z - c\tau - y)\left(1 - \frac{\underline{\phi}^s(z - c\tau - y)}{K_c^s}\right)\Phi_\alpha(y - B\tau)\,dy \\
&\geq \varepsilon p \int_{-\infty}^{\infty} \psi(z - c\tau - y)\left(1 - \frac{\psi^s(z - c\tau - y)}{K_c^s}\right)\Phi_\alpha(y - B\tau)\,dy \\
&= \varepsilon p M(e^{\lambda_1 z} e^{\lambda_1(\alpha\lambda_1 - c\tau + B\tau)} - N e^{(\lambda_1+\delta)z} e^{(\lambda_1+\delta)(\alpha(\lambda_1+\delta) - c\tau + B\tau)}) \\
&\quad - \varepsilon p \frac{1}{K_c^s} \int_{-\infty}^{\infty} \psi^{s+1}(z - c\tau - y)\Phi_\alpha(y - B\tau)\,dy.
\end{aligned}$$

Then, we estimate the third term above as follows. Note that $s + 1$ is even integer, $z \leq z_1 < 0$, and choosing $0 < \delta < s\lambda_1$, we have

$$\begin{aligned}
&\frac{\varepsilon p}{K_c^s} \int_{-\infty}^{\infty} \psi^{s+1}(z - c\tau - y)\Phi_\alpha(y - B\tau)\,dy \\
&\quad\leq \frac{\varepsilon p M^{s+1}}{K_c^s} e^{(\lambda_1+\delta)z} \int_{-\infty}^{\infty} e^{-(s+1)\lambda_1(y+c\tau)}(1 + e^{-\delta(y+c\tau)})^{s+1}\Phi_\alpha(y - B\tau)\,dy \\
&\quad\equiv \frac{\varepsilon p M^{s+1}}{K_c^s} e^{(\lambda_1+\delta)z} N^*.
\end{aligned}$$

Therefore, for sufficiently large $N > 1$, noting that $\triangle(\lambda_1, c) = 0$ and $\triangle(\lambda_1 + \delta, c) < 0$, we have that for $q > 1$,

$$\begin{aligned}
&c\underline{\phi}'(z) - D_m\underline{\phi}''(z) - B\underline{\phi}'(z) + d_m\underline{\phi}(z) - \varepsilon\mathcal{F}(\underline{\phi})(z) \\
&\quad\leq -Me^{\lambda_1 z}\triangle(\lambda_1, c) + MNe^{(\lambda_1+\delta)z}\triangle(\lambda_1 + \delta, c) + \frac{\varepsilon p M^{s+1}}{K_c^s} e^{(\lambda_1+\delta)z} N^* \\
&\quad= Me^{(\lambda_1+\delta)z}\triangle(\lambda_1 + \delta, c)\left(N + \frac{\varepsilon p M^s N^*}{K_c^s \triangle(\lambda_1 + \delta, c)}\right) \leq 0.
\end{aligned}$$

Similar to the proof in Lemma 4, we obtain that part (b) is also true for $q \leq 1$. Part (c) is obvious. We finish the proof for the case with $b(w) = b_3(w)$. In the same way, we have the conclusion for the case with $b(w) = b_1(w)$. □

Remark 1. As mentioned in Wu and Zou [19] and So, Wu, and Zou [16], the lower solution $\underline{\phi}$ doesn't need to be in the profile set Γ, which makes the construction of lower solutions a little bit easier. But, it must satisfy $\underline{\phi}(z) \geq 0$, for $z \in R$ and $\underline{\phi} \not\equiv 0$, that will be used in the proof of the following conclusion (see Wu and Zou [19]).

Now, consider the following iteration scheme, for $n = 1, 2, \ldots,$

$$cw_n'(z) = D_m w_n''(z) + Bw_n'(z) - d_m w_n(z) + \varepsilon\mathcal{F}(w_{n-1})(z), \quad z \in R, \tag{33}$$

with the boundary conditions

$$\lim_{z\to-\infty} w_n(z) = 0, \quad \lim_{z\to\infty} w_n(z) = M, \tag{34}$$

where $w_0(z) = \bar{\phi}(z)$ for $z \in R$. Solving (33) and (34), we get a sequence of functions $\{w_n\}_{n=1}^{+\infty}$, given by, for $z \in R$,

$$w_0(z) = \bar{\phi}(z), \tag{35}$$

$$w_n(z) = \frac{\varepsilon}{D_m(\beta_2 - \beta_1)} \left[\int_{-\infty}^{z} e^{\beta_1(z-s)} \mathcal{F}(w_{n-1})(s)\, ds + \int_{z}^{\infty} e^{\beta_2(z-s)} \mathcal{F}(w_{n-1})(s)\, ds \right], \tag{36}$$

where

$$\beta_1 = \frac{(c-B) - \sqrt{(c-B)^2 + 4D_m d_m}}{2D_m}, \qquad \beta_2 = \frac{(c-B) + \sqrt{(c-B)^2 + 4D_m d_m}}{2D_m},$$

and $n = 1, 2, \ldots$.

From Lemma 4 and Lemma 5 above, and using the results of Wu and Zou [19], we can establish the following:

Theorem 1. *Assume that the aforementioned conditions hold for the birth functions* $b_1(w)$ *(resp.* $b_2(w)$ *and* $b_3(w)$*). Then, the sequence of functions* $\{w_n\}_{n=0}^{+\infty}$ *satisfies*

(a) $w_n \in \Gamma$ *for all* $n = 1, 2, \ldots$;
(b) $\underline{\phi}(z) \le w_n(z) \le w_{n-1}(z) \le \bar{\phi}(z)$ *for all* $n = 1, 2, \ldots$ *and* $z \in R$;
(c) Each w_n *is an upper solution of (31);*
(d) The limit $W(z) := \lim_{n\to\infty} w_n(z)$ *is a solution of (22)–(24).*

Therefore, we obtain the following result for the existence of a travelling wavefront $W(z)$ for (22)–(24).

Theorem 2. *Assume that for the case with* $b_1(w)$, $1 < \frac{\varepsilon p}{d_m} \le e^{\frac{1}{q}}$ *holds; for the case* $b_2(w)$, $1 < \frac{\varepsilon p}{d_m} \le \frac{q}{q-1}$ *holds if* $q > 1$ *and* $\frac{\varepsilon p}{d_m} > 1$ *holds if* $0 < q \le 1$*; for the case* $b_3(w)$, $1 < \frac{\varepsilon p}{d_m} \le \frac{1+q}{q}$ *holds. Let* $c^* > 0$ *be the constant as in Lemma 3, then for every* $c - B > c^*$, *(22)–(24) has a monotone travelling wavefront solution* $W(z)$.

Remark 2. The theoretical results inspire our numerical simulation to be reported in the next section. Of particular concern is the existence and monotonicity of travelling wavefronts when the following conditions are violated: the condition $1 < \frac{\varepsilon p}{d_m} \le e^{1/q}$ for the case with $b_1(w)$; the condition $1 < \frac{\varepsilon p}{d_m} \le \frac{q}{q-1}$ if $q > 1$ for the case with $b_2(w)$; and the condition $1 < \frac{\varepsilon p}{d_m} \le \frac{1+q}{q}$ for the case with $b_2(w)$.

Remark 3. When $\alpha \to 0$, that is, when the immatures are immobile, $F(x, w(t-\tau, \cdot))$ in model (19) reduces to

$$F(x, w(t-\tau, \cdot)) = \varepsilon b(w(t-\tau, x + B\tau)). \tag{37}$$

Therefore, we obtain a RADE model with local delayed effect containing a spatial translation $B\tau$. The theoretical and numerical study for the RADE model with local delayed effect and with all three birth functions was reported in [7].

4. Numerical Simulations of Travelling Waves

In this section, we will numerically study and analyze travelling wavefronts of the reaction advection diffusion equations derived in Section 2. For the three general birth functions $b_1(w)$, $b_2(w)$, and $b_3(w)$, we solve the nonlinear travelling wavefront equations (22)–(24) by using finite difference methods coupled with iterative techniques. The numerical results that we report in this section show that biologically realistic travelling waves occur for the RADE models with nonlocal temporally delayed effects for a wide range of parameters.

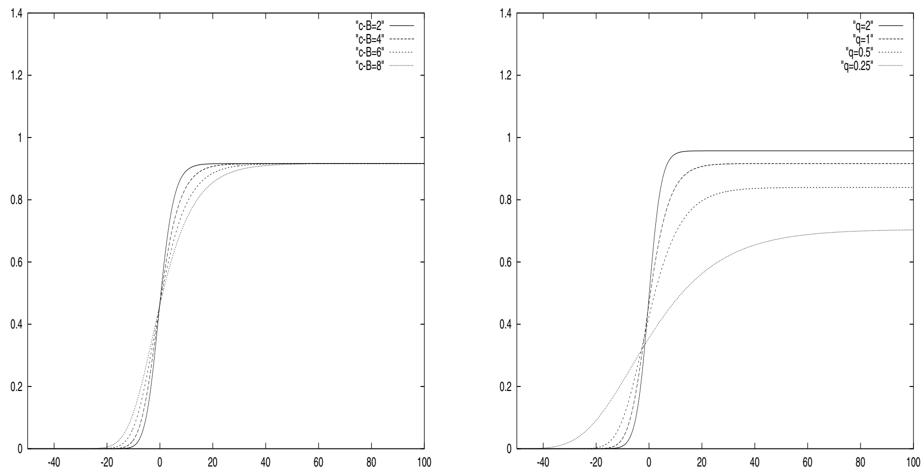
Our numerical experiments show that the travelling wavefronts are, not surprisingly, monotone under the conditions of Theorem 2. However, when these conditions are not met, our simulations show that a one-hump travelling wave occurs when the ratio of the birth rate parameter over the death rate parameter passes a certain value, and then an unsteady multihump (or multilevel) wave may occur when the above ratio is further increased. A proof for the existence of such a multihump wave seems to be very difficult at this stage, though some partial results have recently been obtained in Faria, Huang, and Wu [2] for waves with large speeds.

The nonlinear boundary value problems with nonlocal delay (22)–(24) are solved on the spatial domain $[-L, L]$ subject to either a Dirichlet or a Neumann boundary condition, where $L > 0$ is sufficiently large in comparison with the domain in which the solutions rapidly change shapes. In our illustrative examples below, L is chosen as $L = 1000$, but the figures (Figure 1–Figure 7) only give a small part of the travelling wave solutions in order to present a clear view. Since the wave solutions are invariant with respect to any shift in the z coordinate system, we always translate the numerical wave solutions so that $W(0) = (w_+ - w_-)/2$ in the figures. A finite difference method coupled with iterative techniques for the nonlinear nonlocal delay problems is used in our simulations, and our focus is to examine the roles of varying biological parameters in the change of the existence, shape, and size of the respective waves.

4.1. Nonlocal Delayed Effects with $b_1(w) = pwe^{-aw^q}$

First, we consider the problem with nonlocal delayed effects and with the birth function $b_1(w) = pwe^{-aw^q}$. This birth function with $q = 1$ has been widely used in the well-studied Nicholson's blowflies equation. It increases monotonically before reaching the peak, then decays almost exponentially to zero. The travelling wavefront equation, derived in Section 3, for $W(z)$ is as follows:

$$\begin{aligned} cW'(z) \;=\; & D_m W''(z) + BW'(z) - d_m W(z) \\ & + \varepsilon p \int_{-\infty}^{+\infty} W(z - c\tau + y) e^{-a(W(z-c\tau+y))^q} \Phi_\alpha(y - B\tau)\, dy, \\ & z \in R, \end{aligned} \tag{38}$$



(a) (b)

Fig. 1. The shapes of travelling wavefronts under nonlocal delayed effects with the birth function $b_1(w) = pwe^{-aw^q}$. The data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $p = 2.5$, $a = 1$, and $\tau = 0.5$. We illustrate the effects of the speed $c - B$ and the parameter q: (a) $q = 1$ and $c - B = 2, 4, 6, 8$; (b) $c - B = 4$, and $q = 0.25, 0.5, 1, 2$.

$$W(-\infty) = 0, \qquad W(+\infty) = \left(\frac{1}{a}\ln\frac{\varepsilon p}{d_m}\right)^{1/q}. \tag{39}$$

Example 1. Consider the case where $1 < \frac{\varepsilon p}{d_m} \le e^{1/q}$. Let the diffusion coefficient $D_m = 1$ and death rate $d_m = 1$ for the mature population, $\alpha = 1$ and $\varepsilon = 1$ for the immature population, and the maturation age (the time delay) $\tau = 0.5$. Let the birth rate parameter $p = 2.5$ and $a = 1$ for the birth function. We then numerically observe travelling wave solutions when (a) Fix $q = 1$, and choose the total speed $c - B$ from the value $c - B = 2$ to $c - B = 8$; (b) Fix the speed $c - B = 4$, and vary the parameter q from $q = 0.25$ to $q = 2$.

The numerical results are shown in Figure 1. It is clear to see that since the condition $1 < \frac{\varepsilon p}{d_m} \le e^{1/q}$ holds, monotone travelling wavefront solutions exist for both cases (a) and (b). This is consistent with the theoretical results obtained in Section 3. Figure 1(a) illustrates the change of shapes of the travelling wavefronts when the total speed $c - B$ varies. In Figure 1(b), the shape of the travelling wave changes significantly as the parameter q varies. The front of the travelling wave is quite smooth when q is small, but it becomes sharper and sharper at zero when q is increased.

Example 2. The birth rate parameter p describes the capability of the reproduction of matured species. In this example, we consider the effect of the large birth rate parameter p on the monotonicity of travelling wave solutions when other parameters are fixed. Let $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, and $a = 1$. Let the time delay $\tau = 1$, the total speed

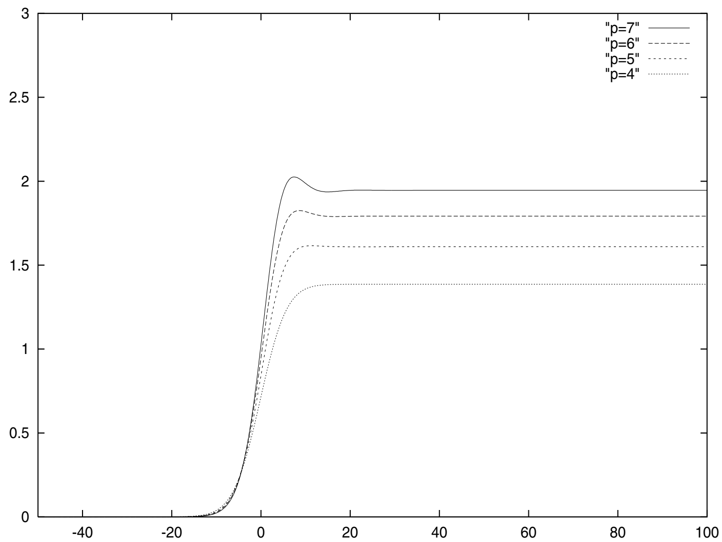


Fig. 2. The shapes of travelling wavefronts with birth function $b_1(w) = pwe^{-aw^q}$, while the birth rate parameter $p = 4, 5, 6, 7$ is varied. Other data are $D_m = 5$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$, $\tau = 1$, and $c - B = 3$. Note that a one-hump wave occurs when $p = 6$ and 7.

$c - B = 4$, and the parameter $q = 1$. We numerically compute the travelling wavefront solutions with different birth rate parameter values $p = 4, 5, 6, 7, 8, 9$.

In this case, it is obvious that the condition $1 < \frac{\varepsilon p}{d_m} \leq e^{1/q}$ required in Section 3 is not satisfied. However, the numerical results in Figure 2 show that the monotone travelling wave solutions still exist as long as the parameter p is moderate (see $p = 4, 5$ in Fig. 2). As the birth rate parameter p increases further, the peak value of the birth function $b_1(w)$ increases beyond the value at the equilibrium $w_1^{**} = (\frac{1}{a} \ln \frac{\varepsilon p}{d_m})^{1/q}$, and our numerical results show that the monotone travelling wave disappears, but a one-hump travelling wave solution occurs (see $p = 6, 7$ in Fig. 2). In Figure 2, we cut the picture at $z = 100$, computed on $[-1000, 1000]$. If we cut at $z = 800$, as in Figure 3, then we can see clearly that if the birth rate parameter p is further increased, an unsteady multihump (or multilevel) solution seems to occur (see $p = 8, 9$ in Fig. 3). Note that when $p = 6, 7$, we can only obtain one-hump travelling waves even when the figure is cut at $z = 800$. The first hump with a fixed shape and size remains stable on the front of the wave, but the second hump (or level) moves and expands in width to the positive z-direction as the iteration number is increased for one group data.

4.2. Nonlocal Delayed Effects with $b_2(w) = pw/(1 + aw^q)$

We now turn to the birth function $b_2(w) = pw/(1+aw^q)$. For this case, we have obtained the condition for the existence of a monotone travelling wave $1 < \frac{\varepsilon p}{d_m} \leq \frac{q}{q-1}$ if $q > 1$ or $\frac{\varepsilon p}{d_m} > 1$ if $0 < q \leq 1$.

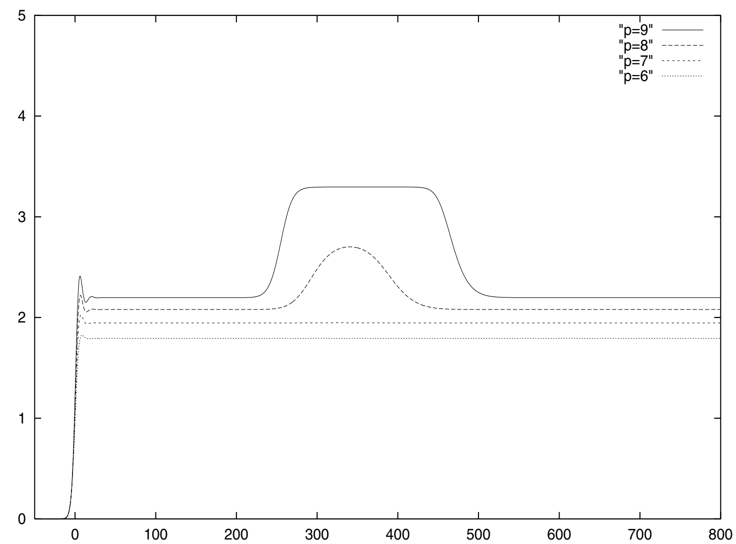


Fig. 3. The shape of travelling wavefronts with birth function $b_1(w) = pwe^{-aw^q}$, while the birth rate parameter $p = 6, 7, 8, 9$ is varied. Other data are $D_m = 5$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$, $\tau = 1$, and $c - B = 3$. Unsteady multihump waves appear when $p = 8$ and $p = 9$.

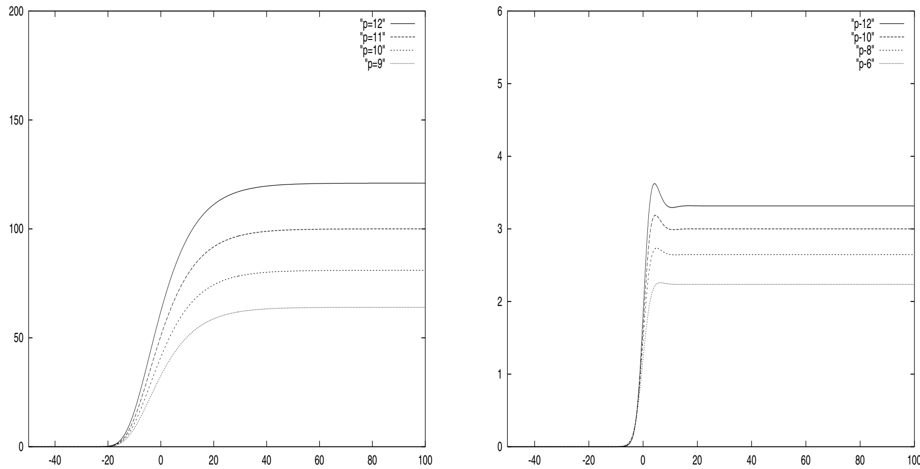
The travelling wavefront equation for this case is

$$\begin{aligned} cW'(z) &= D_m W''(z) + BW'(z) - d_m W(z) \\ &\quad + \varepsilon p \int_{-\infty}^{+\infty} \frac{W(z - c\tau + y)}{1 + a(W(z - c\tau + y))^q} \Phi_\alpha(y - B\tau)\, dy, \quad z \in R, \end{aligned} \tag{40}$$

$$W(-\infty) = 0, \qquad W(+\infty) = \left(\frac{\varepsilon p - d_m}{a d_m}\right)^{1/q}. \tag{41}$$

Example 3. We consider the effect of varying the birth rate parameter p for two different cases $0 < q \leq 1$ and $q > 1$. Let $\alpha = 1$, $\varepsilon = 1$ for the immature population, and $a = 1$ for the birth function. Let the death rate $d_m = 1$ and the diffusion constant $D_m = 1$ for the matured population. Fix the maturation period $\tau = 1$ and the total speed $c - B = 3$. (a) $q = 0.5$ and the birth rate parameter $p = 9, 10, 11, 12$; (b) $q = 2$ and the birth rate parameter $p = 6, 8, 10, 12$.

The numerical travelling wavefronts are given in Figure 4. In case (a), as $q = 0.5$, the ratio of the birth rate parameter over the death rate satisfies $\frac{\varepsilon p}{d_m} > 1$ for all $p > 1$ and fixed constant $d_m = 1$ and $\varepsilon = 1$. Thus, the travelling wavefront solutions are monotone for large values $p = 9, 10, 11, 12$ in Figure 4(a). However, in case (b) with the same set of values $p = 6, 8, 10, 12$, the monotone travelling wave disappears for $q = 2$. It is obvious that for these values, the condition in Theorem 2 is not met. On the other hand,



(a) (b)

Fig. 4. The shapes of travelling wavefronts with birth function $b_2(w) = pw/(1 + aw^q)$. The data are $D_m = 1, \alpha = 1, \varepsilon = 1, a = 1, c - B = 3, \tau = 1$. The figures illustrate the effects of the birth rate parameter p with different values q: (a) $q = 0.5$, and $p = 9$, 10, 11, 12; (b) $q = 2$, and $p = 6$, 8, 10, 12.

a one-hump travelling wave exists for a large range of the birth rate parameter for this birth function as shown in Figure 4(b).

4.3. Nonlocal Delayed Effects with $b_3(w) = pw(1 - \frac{w^q}{K_c^q})$

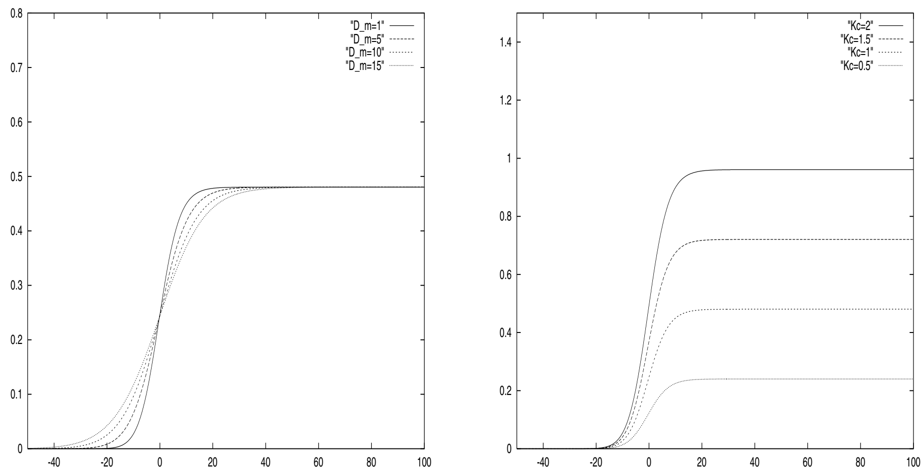
In this subsection, we consider the numerical travelling wavefronts with birth function

$$b_3(w) = \begin{cases} pw\left(1 - \dfrac{w^q}{K_c^q}\right), & 0 \le w \le K_c, \\ 0, & w > K_c, \end{cases} \tag{42}$$

with constants $p > 0$, $K_c > 0$ and $q > 0$.

In this part, we show the effects of the varying diffusion rate D_m of the matured population and the carrying capacity parameter K_c on the travelling wavefronts. Meanwhile, we also consider the effect of varying the birth rate parameter p.

Example 5. Consider the case $1 < \frac{\varepsilon p}{d_m} \le \frac{q+1}{q}$. Let $\alpha = 1, \varepsilon = 1$ for the immature population, and $p = 1.3$ for the birth function. Let the death rate $d_m = 1$. Fix $\tau = 1$ and the speed $c - B = 2$. We observe the effects of the diffusion constant D_m and the carrying capacity parameter K_c on the travelling waves: (a) $K_c = 1$ and vary the diffusion constant D_m from $D_m = 1$ to $D_m = 25$; (b) $D_m = 1$ and $K_c = 0.5$, 1, 1.5, 2.



(a) (b)

Fig. 5. The travelling wavefronts with the birth function $b_3(w) = pw(1-\frac{w^q}{K_c^q})$. The data are $q = 2$, $p = 1.3$, $d_m = 1$, $\tau = 1$, and $\epsilon = 1$, $\alpha = 1$. We illustrate the effect of the diffusion rate D_m and the carrying capacity constant K_c: (a) $K_c = 1$, $D_m = 1, 5, 10, 15$; (b) $D_m = 1$, $K_c = 0.5, 1, 1.5, 2$.

The numerical travelling wavefronts are presented in Figure 5 for this example. In Figure 5(a), the travelling wavefront solutions are monotone, and increasing the diffusion constant D_m increases the smoothness of the travelling wavefronts. The monotone waves exist even for any $K_c > 0$ in Figure 5(b), and the constant K_c affects only the limit value of the travelling wave.

Example 6. We compute with another example to illustrate the effect of the birth rate parameter p on the travelling wave solution. Let $D_m = 5$, $d_m = 1$, $\alpha = 1$, and $\varepsilon = 1$. Let the time delay $\tau = 1$, the total speed $c - B = 2$ and the parameter $q = 2$. We numerically compute the travelling wavefront solutions with different values $p = 1.6, 1.7, 1.8, 1.9, 2, 2.1$.

In these cases, the condition $1 < \frac{\varepsilon p}{d_m} \leq \frac{q+1}{q}$ required in Theorem 2 does not hold. Nevertheless, monotone travelling wave solutions still exist for a proper range of large parameter p, as shown in Figure 6. It should be noted that as the birth rate parameter p continuously increases, the monotone travelling wave disappears, and a one-hump travelling wave solution may occur. However, this one-hump travelling wave solution occurs only for a very small range of large p. Therefore, for this birth function the effect of varying p on the shape of fronts is sensitive. In particular, as can be seen in Figure 7 (see $p = 2.1$), an unsteady multihump (or multilevel) solution seems to occur as parameter p continuously increases. The first stable hump on the front of the wave is very small; however, the second hump (or level) shifts and expands in width to the right as the iteration number increases.

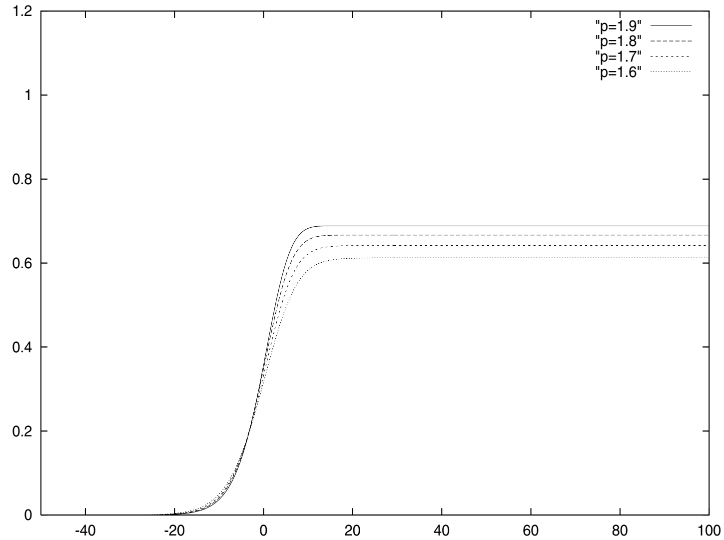


Fig. 6. The shapes of travelling wavefronts with birth function $b_3(w) = pw(1 - \frac{w^q}{K_c^q})$. The data are $q = 2$, $D_m = 5$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 1$, and $c - B = 2$, $\tau = 1$. Vary the birth rate parameter p: $p = 1.6, 1.7, 1.8, 1.9$.

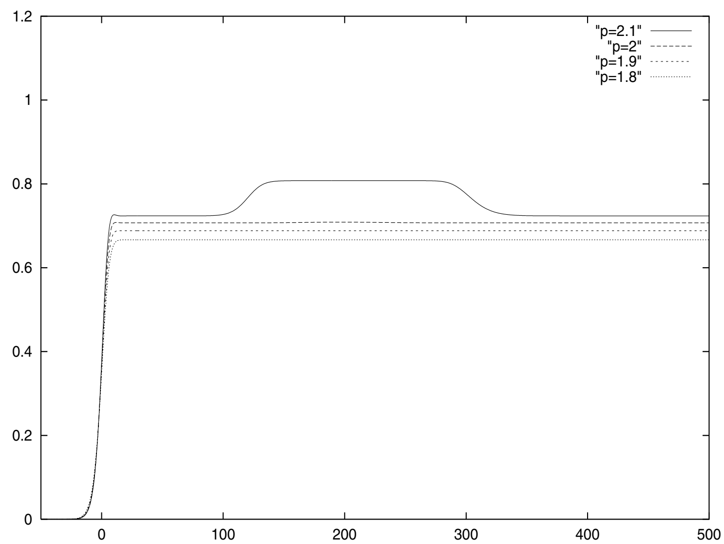


Fig. 7. The shapes of travelling wavefronts with birth function $b_3(w) = pw(1 - \frac{w^q}{K_c^q})$. The data are $q = 2$, $D_m = 5$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 1$, and $c - B = 2$, $\tau = 1$. The effect of the birth rate parameter p ($p = 1.8, 1.9, 2, 2.1$) on the shape of waves is quite sensitive.

4.4. The Stability of the Travelling Wavefronts

Finally, we investigate numerically the stability of travelling waves by computing the time-dependent problem (16)–(19).

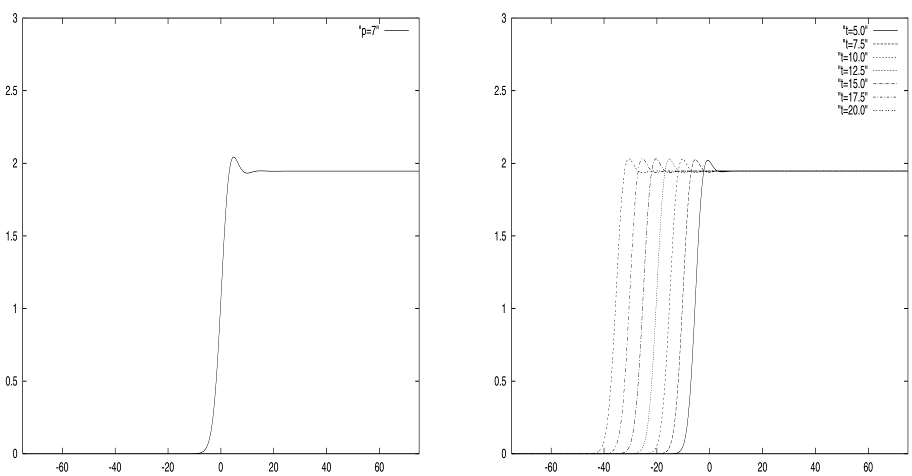
Example 7. The full time-dependent partial differential equations (16)–(19) are numerically considered for the birth function $b_1(w) = pwe^{-aw^q}$. The data are chosen as $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, and $q = 1$. The time delay is $\tau = 1$, the birth rate parameter $p = 7$.

The initial condition is given as

$$w_0(t, x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \quad \text{for } t \in [-\tau, 0]. \tag{43}$$

The full partial differential equations with nonlocal delay are discretised by using the finite different method on a finite spatial domain with either a Dirichlet or a Neumann boundary condition.

The numerical travelling wavefronts from the full time-dependent partial differential equations are shown in Figure 8(b). The solution gets smooth immediately from the discontinuous initial function. Then the shape of the solution promptly converges to a



(a) (b)

Fig. 8. The stability of travelling wavefronts with birth function $b_1(w) = pwe^{-aw^q}$. The data are $D_m = 1, d_m = 1, \alpha = 1, \varepsilon = 1, q = 1, \tau = 1$. The birth rate parameter is $p = 7$. (a) The travelling wavefront obtained from the wave equation (38)–(39) with the total wave speed $c - B = 2$. (b) The leftward-moving travelling wavefronts obtained from the time-dependent equation (16)–(17). The times are $t = 5, 7.5, 10, 12.5, 15, 17.5, 20$. The wave speed of the moving travelling wavefront is about 2.

stable one-hump wave (at $t = 12.5$) for the case with $p = 7$. The stable travelling wave moves in the negative x-direction as the time increases (at $t = 15$, 17.5, and 20). The speed of the moving travelling wave is about 2. Moreover, using the same data and choosing the total wave speed $c - B = 2$, we solve the wave equation (38)–(39) as in Section 4.1. The numerical travelling wave obtained from the wave equation in Figure 8(a) is in excellent agreement with the one obtained from the full partial differential equations in Figure 8(b) (at $t = 12.5$, 15, 17.5, and 20).

5. Conclusion

In this paper, we develop a reaction advection diffusion equation (RADE) model for the growth dynamics of a single-species population living in a spatial transport field. The model is derived from an age-structured population model and contains a time delay and nonlocal effect term, in which the fixed maturation period is considered as the time delay. The model can be used to study the behaviour of the mature population.

Our theoretical and numerical analyses for the travelling wavefront solutions of the model with three widely used birth functions are reported in Section 3 and Section 4. It is shown, both theoretically and numerically, that the travelling waves exist for the RADE model with nonlocal and delayed effects when the ratio of the birth rate parameter p over the death rate parameter d_m is in a certain range. Outside of this range, numerical simulations suggest monotone travelling waves may still exist, and hence our theoretical results can be further improved. Also, outside of this range, numerical simulations suggest possible occurrence of one-hump travelling waves. This was also observed in Gourley [4] for the Fisher equation with nonlocal weighted spatial averaging effect and was partially confirmed for waves and large speeds in the recent work of Faria, Huang, and Wu [2]. Furthermore, we numerically investigate the stability of the travelling wavefronts by computing the full time-dependent partial differential equations. The numerical results show the stability of travelling wavefronts and indicate that the wave shapes obtained from both the wave equation and initial boundary problem are agreeable. Moreover, we numerically observe that the wave solutions may exhibit unsteady multihumps (or multilevels) when a certain parameter is increased moderately. In these cases, the first hump (as far as the shape, size, and location) remains stable on the front of the waves, but the second hump (level) moves and expands in width to the positive z-direction as the iteration number is increased. Detailed theoretical investigation of these one-hump and multihump waves would be a challenging task.

Acknowledgments

The authors would like to thank the anonymous referees whose suggestions improved the readability of this paper.

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