

SMOOTHNESS OF CENTER MANIFOLDS FOR MAPS AND FORMAL ADJOINTS FOR SEMILINEAR FDES IN GENERAL BANACH SPACES*

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Abstract. We develop a formal adjoint theory for retarded linear functional differential equations in Banach spaces and establish the existence and smoothness of center manifolds for nonlinearly perturbed equations. The hypotheses imposed here are significantly weaker than those that usually appear in the literature referring to semigroups for abstract functional differential equations, and the smoothness of the center manifolds for nonlinear perturbed equations is derived from our general results on the smoothness of center manifolds for maps in infinite-dimensional Banach spaces.

Key words. functional differential equations in Banach spaces, formal adjoint equations, resolvent compact operators, perturbations, center manifolds, smoothness

AMS subject classifications. 34K30, 34K06, 34K19, 34K17

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1. Introduction. We consider the autonomous linear functional differential equations (FDEs) of retarded type,

$$(1.1) \quad \dot{u}(t) = A_T u(t) + L(u_t), \quad u(t) \in X,$$

and the nonlinearly perturbed systems

$$(1.2) \quad \dot{u}(t) = A_T u(t) + L(u_t) + F(u_t),$$

where X is a Banach space, $r > 0$, $C := C([-r, 0]; X)$ is the Banach space of continuous mappings from $[-r, 0]$ to X with the sup norm, $u_t \in C$ is defined by $u_t(\theta) = u(t + \theta)$ for $t \in \mathbb{R}$ and $\theta \in [-r, 0]$, $L : C \rightarrow X$ is a bounded linear operator, $A_T : D(A_T) \subset X \rightarrow X$ is the infinitesimal generator of a compact C_0 -semigroup of linear operators on X , and F is a sufficiently smooth nonlinear map with $F(0) = 0, DF(0) = 0$.

In the last two decades, there has been an increasing interest in retarded FDEs in Banach spaces. Typically, these equations depend on both spatial and temporal variables, with the time-dependence involving discrete or distributed delays. Such equations arise from a variety of situations in population dynamics and take the abstract form (1.1) or (1.2), where a diffusion term $d\Delta v(t, x)$ with $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ defines $A_T u(t) = d\Delta v(t, x)$ for $u(t)(x) := v(t, x), x \in \mathbb{R}^n$. See Wu [21] for more details.

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The purpose of the present work is to establish two necessary technical tools—a formal adjoint theory for equations of type (1.1) and the existence and smoothness of center manifolds for nonlinearly perturbed equation (1.2)—in order to develop a normal form theory on invariant manifolds of (1.2).

Several extensions of the formal adjoint and invariant manifold theory for FDEs in \mathbb{R}^n (see Hale [8]) to infinite-dimensional Banach spaces have been developed in different frameworks. Related to our present work is the paper of Travis and Webb [18], where the authors initiated a formal adjoint theory for linear equations of the form (1.1); other related work includes Arino and Sanchez [1], Huang [9], Nakagiri [13], Schumacher [15], Shin and Naito [16], Wu [21], and Yamamoto and Nakagiri [22], to mention a few. We should particularly remark that a quite complete theory has also been developed for FDEs in Banach spaces of type (1.1) and (1.2) regarding duality, formal adjoint theory, and invariant manifolds (cf., e.g., Memory [12], Lin, So, and Wu [11], Wu [21], and Faria [5]) under some quite severe constraints. In fact, assume that the eigenvectors of A_T form a basis for X in the following sense: if $\mu_k, k \in \mathbb{N}$, are the eigenvalues of A_T with associated eigenvectors $\beta_k, k \in \mathbb{N}$, then every $x \in X$ is written in a unique way as $x = \sum_{k \in \mathbb{N}} x_k$, where $x_k \in \text{span}\{\beta_k\}, k \in \mathbb{N}$, with $A_T x = \sum_{k \in \mathbb{N}} \mu_k x_k$. Assume also that $L(\varphi \beta_k) \in \text{span}\{\beta_k\}$ for all $\varphi \in C([-r, 0]; \mathbb{R})$ and all eigenvectors β_k . Then it is possible to decompose the characteristic equation of the abstract FDE into a sequence of characteristic equations in \mathbb{R} . This decomposition yields a decomposition of (1.1) into a sequence of scalar FDEs, to which the standard formal adjoint theory for FDEs in \mathbb{R}^n of Hale [8] can be applied (see [11], [12], [21], and other references therein). A slightly weaker hypothesis was considered in [5], as follows. In addition to the assumption that the eigenvectors of A_T form a basis for X , suppose now that the set of eigenvalues of A_T can be written as $\{\mu_k^{i_k} : k \in \mathbb{N}, i_k = 1, \dots, p_k\}$; for each $k \in \mathbb{N}$, let B_k be the generalized eigenspace for A_T associated with the block of eigenvalues $\{\mu_k^{i_k} : i_k = 1, \dots, p_k\}$, and assume that $L(\mathcal{B}_k) \subset B_k$, where $\mathcal{B}_k = \{\varphi \in C : \varphi(\theta) \in B_k \text{ for } \theta \in [-r, 0]\}$. This means that the eigenvalues of A_T can be organized by blocks in such a way that L does not mix the modes of the generalized eigenspaces associated with the eigenvalues in each block. Under these conditions, (1.1) is decomposed into a sequence of FDEs in finite-dimensional spaces (whose dimensions are now equal to the dimensions of the generalized eigenspaces B_k associated with each block $\{\mu_k^{i_k} : i_k = 1, \dots, p_k\}$), and again one can apply the adjoint theory for FDEs in \mathbb{R}^n . However, these hypotheses impose severe restrictions on the applicability of the approach to a wide range of problems arising from population dynamics. For instance, even if A_T is an n -dimensional elliptic operator with $n > 1$, it is unknown whether the eigenfunctions of A_T form a basis of X . Moreover, the above assumption that the linear operator L does not mix the modes of the eigenfunction spaces of the operator A_T is not realistic, for this almost implies that the operator L is a scalar multiplication.

Our goal is to develop a complete formal adjoint theory and center manifold theory without the aforementioned restrictions. The main sources of inspiration for our work on adjoint theory presented here are the work of Travis and Webb [18] for (1.1) and the work of Arino and Sanchez [1] for equations of the form $\dot{u}(t) = L(u_t)$, with $L : C \rightarrow X$ being a bounded linear operator. More specifically, Travis and Webb [18] set the basis for an adjoint theory by introducing an adequate bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$, which serves as the formal duality between C and its dual C^* , as well as an adequate definition of formal adjoint equation for (1.1). However, their theory was not completed in the following sense: in order to set a suitable framework to construct normal forms for

perturbed FDE (1.2), a formal adjoint theory should eventually provide an analytic formula for the decomposition of the phase space C by a nonempty finite set Λ of characteristic values for (1.1). Here, we present results that enable us to decompose C by Λ as the direct sum $C = P \oplus Q$, where P is the generalized eigenspace associated with Λ and $Q = \{\varphi \in C : \langle \langle \psi, \varphi \rangle \rangle = 0 \text{ for all } \psi \in P^*\}$, where P^* is the generalized eigenspace associated with Λ for the formal adjoint equation.

Since we deal with infinite-dimensional Banach spaces X , rather than finite-dimensional ones, our main difficulty is to use the formal duality to relate the generalized eigenspaces of the infinitesimal generator for the semigroup induced by the solutions of (1.1) with the generalized eigenspaces of its formal adjoint. Without having to impose further hypotheses on X or on the operators A_T and L , we succeeded in expressing the kernel and range for these generalized eigenspaces in terms of the kernel and range for some auxiliary operators. (This is a generalization of the operators introduced by Hale [8] for the case $X = \mathbb{R}^n$.) It turns out that these auxiliary operators are crucial for deriving the decomposition $C = P \oplus Q$ by a nonempty finite set Λ of characteristic eigenvalues because, as we shall prove, they have compact resolvents and closed ranges.

For the sake of exposition, we include some definitions and results from [18]. But we should emphasize that some results about duality in [18] were proven under stronger hypotheses than the ones assumed in this paper. Namely, in the present work the Banach space X is not required to be reflexive; also in [18, Propositions 4.14 and 4.15], some conditions on the characteristic operator were imposed in order to derive some results, such as that the point spectra for the infinitesimal generator of the semigroup defined by the mild solutions of (1.1) and for its formal adjoint coincide. Our techniques and results on formal adjoints are different from those in [1] for equations of type $\dot{u}(t) = L(u_t)$ (i.e., where A_T is absent). In [1] the authors considered only elements in Λ that are not in the essential spectrum, so that their auxiliary operators are Fredholm operators, while in the present paper we prove that the corresponding auxiliary operators have compact resolvents and closed ranges (two key points in establishing a Fredholm alternative result) from which the decomposition $C = P \oplus Q$ is deduced. Also, potential applications of the results in the present paper are much different from those of [1]. For instance, as we have already mentioned, (1.1) includes reaction-diffusion equations with delays as special cases.

As mentioned above, our second goal is to obtain the existence and smoothness of the center manifold. We notice that center manifolds are of particular interest in applications since the qualitative behavior of the solutions of a nonlinear equation in a neighborhood of an equilibrium can be described by the flow on these manifolds. See, for example, Carr [3]. See also Vanderbauwhede and van Gils [20], Vanderbauwhede and Iooss [19], and Diekmann et al. [4] for the theory of center manifolds for FDEs in \mathbb{R}^n . As already observed in the aforementioned papers, the phase space for FDE (1.2) is a Banach space which does not admit a smooth cut-off function, and thus it is a very challenging task to obtain the smoothness of center manifolds. Such a difficult issue was addressed for FDEs in \mathbb{R}^n by Vanderbauwhede and van Gils [20], and the details are presented by Diekmann et al. [4]. In the recent work of Krisztin, Walther, and Wu [10], the existence and C^1 -smoothness of various invariant manifolds for C^1 -maps in general Banach spaces were established. Here we utilize some of the ideas in [10] and prove general C^k -smoothness for C^k -maps, with k being an arbitrary positive integer, and we apply this general smoothness result for maps to obtain the existence and C^k -smoothness of center manifolds for the semiflow generated by (1.2). Such a general

smoothness result is necessary for the normal form theory to be developed later, as the normal forms usually involve Taylor series expansions of various nonlinear maps involved in the center manifold reduction.

Although our final goal is to use formal adjoints and center manifolds as basic tools to develop a normal form theory for equations in the form (1.2), we note that the results presented here are important by themselves, and a decomposition of the phase space for linear equations and center manifolds for semilinear equations could be applied in different frameworks of qualitative theory for FDEs.

The paper is organized as follows. In section 2, some definitions and results are recalled, most of them from [18]. Sections 3 and 4 address a complete formal adjoint theory for FDEs (1.1): the auxiliary operators are introduced in section 3, and we derive some important properties of their spectra and resolvents; in section 4, a Fredholm alternative result is presented, and the phase space C is decomposed by a nonempty finite set Λ of characteristic eigenvalues of (1.1) by using the formal adjoint equation. Section 5 develops general results for the smoothness of center-stable and center-unstable manifolds for maps in Banach spaces, and section 6 applies these results to obtain the existence and regularity of center manifolds for perturbed FDE (1.2) at the zero equilibrium.

Because of space limitations, other important properties of the center manifold, such as the local invariance and attractivity, will be studied in a separate paper.

We now list notation that will be used throughout the paper. For a given Banach space X and for a linear operator A from its domain in X to X , we shall use $D(A)$, $R(A)$, and $N(A)$ to denote the domain, range, and kernel of A , respectively. The spectrum, point spectrum, and resolvent of A are considered as subsets of \mathbb{C} and are denoted by $\sigma(A)$, $\sigma_P(A)$, and $\rho(A)$, respectively. If $\lambda \in \sigma_P(A)$, then $\mathcal{M}_\lambda(A)$ is the generalized eigenspace associated with λ .

2. Preliminary results and definitions. Consider

$$(2.1) \quad \dot{u}(t) = A_T u(t) + L(u_t), \quad t \geq 0, \quad u(t) \in X,$$

where X is a Banach space over the field \mathbb{C} , $r > 0$, $C := C([-r, 0]; X)$ is the Banach space of continuous mappings from $[-r, 0]$ to X with the sup norm, $L : C \rightarrow X$ is a bounded linear operator, and $A_T : D(A_T) \subset X \rightarrow X$ is linear. As usual, $u_t \in C$ denotes the shifted restriction of u to $[t-r, t]$, i.e., $u_t(\theta) = u(t+\theta)$ for $-r \leq \theta \leq 0$. We require the following assumptions:

(H1) A_T generates a C_0 -semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on X , with $\|T(t)\| \leq M e^{\omega t}$ ($t \geq 0$) for some $M \geq 1$, $\omega \in \mathbb{R}$.

(H2) $T(t)$ is a compact operator for each $t > 0$.

For $u \in C([-r, \infty); X)$, u is said to be a *mild solution* of (2.1) with initial condition $\varphi \in C$ if it satisfies

$$(2.2) \quad \begin{cases} u(t) = T(t)\varphi(0) + \int_0^t T(t-s)L(u_s)ds, & t \geq 0, \\ u_0 = \varphi. \end{cases}$$

(See, e.g., [23, p. 75] for the definition of integral used here.) It is known that the initial value problem (2.2) has a unique solution denoted by $u(\varphi)(t)$, $t \in [-r, \infty)$. Moreover, for the operators $U(t)$, $t \geq 0$, given by

$$(2.3) \quad U(t) : C \rightarrow C, \quad U(t)\varphi = u_t(\varphi),$$

from Propositions 2.4, 3.1, and 3.2 in Travis and Webb [18], we have the following proposition.

PROPOSITION 2.1. Assume (H1). Then $\{U(t)\}_{t \geq 0}$ is a C_0 -semigroup of bounded linear operators on C . Its infinitesimal generator $A_U : C \rightarrow C$ is given by

$$(2.4) \quad \begin{aligned} A_U \varphi &= \dot{\varphi}, \\ D(A_U) &= \{\varphi \in C : \dot{\varphi} \in C, \varphi(0) \in D(A_T), \dot{\varphi}(0) = A_T \varphi(0) + L(\varphi)\}. \end{aligned}$$

Moreover, if (H2) holds, then $U(t)$ is a compact operator for each $t > r$.

Since $\{U(t)\}_{t \geq 0}$ is eventually compact (i.e., there exists $t_0 > 0$ such that $U(t)$ is a compact operator for every $t > t_0$), from Greiner [7, p. 209] the next result follows.

PROPOSITION 2.2. Assume (H1), (H2) and let A_U be defined by (2.4). Then we have the following:

- (i) $\sigma(A_U) = \sigma_P(A_U)$ and every $\lambda \in \sigma(A_U)$ is a pole of finite order of the resolvent $R(\lambda; A_U) = (\lambda I - A_U)^{-1}$;
- (ii) for each $\lambda \in \sigma(A_U)$, the generalized eigenspace $\mathcal{M}_\lambda(A_U)$ is finite-dimensional;
- (iii) for each $\alpha \in \mathbb{R}$, the set $\{\lambda \in \sigma(A_U) : \operatorname{Re} \lambda \geq \alpha\}$ is finite.

From the general theory of C_0 -semigroups and compact operators, we also conclude the following.

PROPOSITION 2.3. Assume (H1), (H2) and let $\lambda \in \mathbb{C}$. If $\lambda \in \sigma(A_U)$, then the ascent and descent of $A_U - \lambda I$ are both equal to m , where m is the order of λ as a pole of the resolvent $R(\lambda; A_U)$. Furthermore,

$$(2.5) \quad C = N[(A_U - \lambda I)^m] \oplus R[(A_U - \lambda I)^m],$$

where $N[(A_U - \lambda I)^m] = \mathcal{M}_\lambda(A_U)$ and $R[(A_U - \lambda I)^m]$ is a closed subspace of C .

Proof. The first part follows directly from Theorem V.10.1 of Taylor and Lay [17, p. 330]. Now, let $k \in \mathbb{N}$, $t > r$. Since $U(t)$ is compact, $N[(U(t) - \mu I)^k]$ is finite-dimensional for $\mu \in \sigma(U(t))$. On the other hand, from the general theory of C_0 -semigroups,

$$N[(U(t) - \mu I)^k] = \bigoplus_{\lambda \in S_\mu} N[(A_U - \lambda I)^k], \quad \text{where } S_\mu = \{\lambda \in \sigma(A_U) : e^{\lambda t} = \mu\}.$$

Thus, for m the ascent of λ , $N[(A_U - \lambda I)^m] = \mathcal{M}_\lambda(A_U)$ is finite-dimensional and Theorem IV.5.10 of Taylor and Lay [17, p. 217] implies that $R[(A_U - \lambda I)^m]$ is closed. \square

For $\lambda \in \mathbb{C}$, we say that λ is a *characteristic value* for (2.1) if λ satisfies the characteristic equation given by

$$(2.6) \quad \Delta(\lambda)x = 0, \quad x \in D(A_T) \setminus \{0\},$$

where $\Delta(\lambda) : D(A_T) \subset X \rightarrow X$ is defined by

$$(2.7) \quad \Delta(\lambda)x := A_T x + L(e^{\lambda \cdot} x) - \lambda x, \quad x \in D(A_T),$$

and $e^{\lambda \cdot} x \in C$ is given by $(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x$ for $\theta \in [-r, 0]$ and $x \in X$. It is easy to see that $\lambda \in \sigma(A_U)$ if and only if λ is a characteristic value for (2.1), in which case

$$N(A_U - \lambda I) = \{e^{\lambda \cdot} x : x \in N(\Delta(\lambda))\}.$$

Note also that for $\psi \in C$, the equation $\psi = (A_U - \lambda I)\varphi$ has a solution $\varphi \in D(A_U)$ if and only if there is a $b \in D(A_T)$ satisfying the equation

$$(2.8) \quad \Delta(\lambda)b = \psi(0) - L\left(\int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi\right).$$

In this case, the solution φ of $\psi = (A_U - \lambda I)\varphi$ is given by

$$(2.9) \quad \varphi(\theta) = e^{\lambda\theta}b + \int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi, \quad \theta \in [-r, 0].$$

Here and throughout the remainder of this paper, for the sake of simplicity, we abuse notation and write explicitly the value of $\varphi \in C$ at an arbitrary given $\theta \in [-r, 0]$ in the evaluation of $L(\varphi)$. Namely, $L(\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi)$ should be understood as the value of L acting on the mapping $[-r, 0] \ni \theta \mapsto \int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi \in X$.

We now assume that the linear operator L can be expressed in integral form by means of a function of bounded variation:

(H3) There is $\eta : [-r, 0] \rightarrow \mathcal{L}(X, X)$ of bounded variation such that

$$L(\varphi) = \int_{-r}^0 d\eta(\theta)\varphi(\theta), \quad \varphi \in C,$$

where $\mathcal{L}(X, X)$ denotes the Banach space of bounded linear operators from X into X .

Following Travis and Webb [18], we define the formal duality, the formal adjoint operator of L , and the formal adjoint equation of (2.1) below.

Let X^* be the dual of X and $C^* := C([0, r]; X^*)$. The *formal duality* between C^* and C is the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ from $C^* \times C$ to the scalar field, defined by

$$(2.10) \quad \langle\langle \alpha, \varphi \rangle\rangle = \langle \alpha(0), \varphi(0) \rangle - \int_{-r}^0 \int_0^\theta \langle \alpha(\xi - \theta), d\eta(\theta)\varphi(\xi) \rangle d\xi$$

for $\alpha \in C^*$, $\varphi \in C$, where $\langle \cdot, \cdot \rangle$ is the usual duality between X^* and X . For $f \in C([0, r]; \mathbb{R})$ and $u^* \in X^*$, we use fu^* to denote $f \otimes u^*$ in C^* , i.e., $(fu^*)(s) = f(s)u^*$ for $0 \leq s \leq r$. We remark that

$$(2.11) \quad \langle\langle fu^*, \varphi \rangle\rangle = \langle u^*, f(0)\varphi(0) \rangle - \left\langle u^*, L \left(\int_0^\theta f(\xi - \theta)\varphi(\xi)d\xi \right) \right\rangle.$$

To avoid possible confusion, throughout this paper we adopt the following notation: given a densely defined linear operator B in a Banach space, we denote by B^* the (true) adjoint of B , also called the dual of B ; and by *B we denote the formal adjoint of B relative to the formal duality $\langle\langle \cdot, \cdot \rangle\rangle$ defined above, in a sense that will soon be more clearly defined. The *formal adjoint operator* *L of L is given by

$$(2.12) \quad {}^*L : C^* \rightarrow X^*, \quad {}^*L(\alpha) = \int_{-r}^0 d\eta^*(\theta)\alpha(-\theta),$$

where $\eta^*(\theta)$ is the adjoint of $\eta(\theta)$. Since η is of bounded variation, its adjoint operator $\eta^* : [-r, 0] \rightarrow \mathcal{L}(X^*, X^*)$ is also of bounded variation. For (2.1), the *formal adjoint equation* is defined as

$$(2.13) \quad \dot{\alpha}(t) = -A_T^*\alpha(t) - {}^*L(\alpha^t), \quad t \leq 0,$$

where A_T^* is the adjoint of A_T and $\alpha^t \in C^*$ is given by $\alpha^t(s) = \alpha(t+s)$ for $s \in [0, r]$.

Consider the *mild solution* $\alpha^t(\psi)$ for (2.13) with initial condition $\psi \in C^*$, i.e., the solution of the integral equation

$$\begin{cases} \alpha(t) = T^*(-t)\psi(0) + \int_0^t T^*(s-t){}^*L(\alpha^s)ds, & t \leq 0, \\ \alpha^0(\psi) = \psi. \end{cases}$$

As for (2.1), equation (2.13) generates a C_0 -semigroup of linear operators $\{^*U(t)\}_{t \geq 0}$ on C^* defined by $^*U(t)\psi = \alpha^{-t}(\psi)$, whose infinitesimal generator *A_U is given by

$$(2.14) \quad \begin{aligned} &^*A_U \alpha = -\dot{\alpha}, \\ &D(^*A_U) = \{\alpha \in C^* : \dot{\alpha} \in C^*, \alpha(0) \in D(A_T^*), -\dot{\alpha}(0) = A_T^* \alpha(0) + ^*L(\alpha)\} \end{aligned}$$

and has the following properties (see Travis and Webb [18]):

$$(2.15) \quad \langle\langle ^*A_U \alpha, \varphi \rangle\rangle = \langle\langle \alpha, A_U \varphi \rangle\rangle \quad \text{for } \alpha \in D(^*A_U), \varphi \in D(A_U),$$

$$(2.16) \quad \langle\langle \alpha, \varphi \rangle\rangle = 0 \quad \text{for } \alpha \in N(^*A_U - \mu I), \varphi \in N(A_U - \lambda I), \text{ with } \lambda \neq \mu.$$

Note that (2.15) justifies the designation of *A_U as the formal adjoint of A_U , since its behavior relative to the formal duality $\langle\langle \cdot, \cdot \rangle\rangle$ is similar to the behavior of the (true) adjoint of an operator relative to the usual duality between a Banach space and its dual.

3. The point spectrum of *A_U . The classic (formal) adjoint theory for FDEs in \mathbb{R}^n will now be generalized to FDEs in Banach spaces, completing the theory initiated by Travis and Webb [18] and following the ideas of Arino and Sanchez [1], Busenberg and Huang [2], and Huang [9].

Similarly to what is done in section 7.3 of Hale [8] (see also [1]), we introduce some auxiliary operators that allow us to express the null space and range for $(A_U - \lambda I)^m$, $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, in terms of the null space and range of those auxiliary operators. For $\lambda \in \mathbb{C}$, $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, we define the following linear operators:

$$(3.1) \quad L_\lambda^j : X \longrightarrow X, \quad L_\lambda^j(x) = L \left(\frac{\theta^j}{j!} e^{\lambda \theta} x \right),$$

$$(3.2) \quad \mathcal{L}_\lambda^{(m)} : [D(A_T)]^m \longrightarrow X^m, \quad \mathcal{L}_\lambda^{(m)} = \begin{pmatrix} \Delta(\lambda) & L_\lambda^1 - I & L_\lambda^2 & \dots & L_\lambda^{m-1} \\ 0 & \Delta(\lambda) & L_\lambda^1 - I & \dots & L_\lambda^{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta(\lambda) & L_\lambda^1 - I \\ 0 & 0 & \dots & 0 & \Delta(\lambda) \end{pmatrix},$$

$$(3.3) \quad \mathcal{R}_\lambda^{(m)} : C \longrightarrow X^m, \quad \mathcal{R}_\lambda^{(m)}(\psi) = \begin{pmatrix} -L \left(\int_0^\theta e^{\lambda(\theta-\xi)} \frac{(\theta-\xi)^{m-1}}{(m-1)!} \psi(\xi) d\xi \right) \\ \vdots \\ -L \left(\int_0^\theta e^{\lambda(\theta-\xi)} (\theta-\xi) \psi(\xi) d\xi \right) \\ \psi(0) - L \left(\int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi \right) \end{pmatrix}.$$

With the definitions above, it is clear that $\Delta(\lambda) = \mathcal{L}_\lambda^{(1)} = A_T + L_\lambda^0 - \lambda I$. Moreover, from (2.8) and (2.9) it follows that $\psi \in R(A_U - \lambda I)$ if and only if there exists $b \in D(A_T)$ such that $\Delta(\lambda)b = \mathcal{R}_\lambda^{(1)}(\psi)$.

As in [1] and [8], we can carry out direct computations to obtain an explicit characterization of the spaces $N[(A_U - \lambda I)^m]$, $R[(A_U - \lambda I)^m]$, $m \in \mathbb{N}$. So we state the following proposition without a proof.

PROPOSITION 3.1. Assume (H1), (H2) and let $\lambda \in \mathbb{C}, m \in \mathbb{N}$. Then
 (i) $\varphi \in N[(A_U - \lambda I)^m]$ if and only if

$$\varphi(\theta) = \sum_{j=0}^{m-1} \frac{\theta^j}{j!} e^{\lambda\theta} u_j, \quad \theta \in [-r, 0], \quad \text{with} \quad \begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix} \in N(\mathcal{L}_\lambda^{(m)});$$

(ii) $\psi \in R[(A_U - \lambda I)^m]$ if and only if $\mathcal{R}_\lambda^{(m)}(\psi) \in R(\mathcal{L}_\lambda^{(m)})$.
 From the definition of $*L$ in (2.12), one can see that

$$\langle *L(fu^*), u \rangle = \langle u^*, L(\hat{f}u) \rangle$$

for $u^* \in X^*, u \in X, f \in C([0, r]; \mathbb{R})$, where $\hat{f} \in C([-r, 0]; \mathbb{R})$ is given by $\hat{f}(\theta) := f(-\theta)$ for $\theta \in [-r, 0]$. Therefore, the adjoint $(L_\lambda^j)^*$ of L_λ^j ($j \in \mathbb{N}_0, \lambda \in \mathbb{C}$) is given by

$$(3.4) \quad (L_\lambda^j)^* u^* = *L \left(\frac{(-\theta)^j}{j!} e^{-\lambda\theta} u^* \right), \quad u^* \in X^*.$$

Similar to Proposition 3.1, we have an explicit characterization of $N[(^*A_U - \lambda I)^m]$.

PROPOSITION 3.2. Assume (H1)–(H3). For $m \in \mathbb{N}, \lambda \in \mathbb{C}$,

$$\alpha \in N[(^*A_U - \lambda I)^m] \quad \text{if and only if} \quad \alpha(s) = \sum_{j=0}^{m-1} \frac{(-s)^j}{j!} e^{-\lambda s} x_{m-j-1}^*, \quad s \in [0, r],$$

with $(x_0^*, \dots, x_{m-1}^*)^T \in N((\mathcal{L}_\lambda^{(m)})^*)$. In particular, $\alpha \in N(^*A_U - \lambda I)$ if and only if $\alpha(s) = e^{-\lambda s} x^*, s \in [0, r]$, with $x^* \in N(\Delta(\lambda)^*)$.

Proof. We have

$$(\mathcal{L}_\lambda^{(m)})^* = \begin{pmatrix} \Delta(\lambda)^* & 0 & \dots & 0 \\ (L_\lambda^1)^* - I & \Delta(\lambda)^* & \dots & 0 \\ (L_\lambda^2)^* & (L_\lambda^1)^* - I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (L_\lambda^{m-1})^* & \dots & (L_\lambda^1)^* - I & \Delta(\lambda)^* \end{pmatrix},$$

with $(L_\lambda^j)^*$ given by (3.4). Using this and direct computations in the same spirit as in section 7.3 in Hale [8], we can complete the verification of Proposition 3.2. \square

Now, we want to present a Fredholm alternative result relative to the formal adjoint. The following lemmas will establish some properties of the operators $\mathcal{L}_\lambda^{(m)}$ that will play an important role in this setting.

LEMMA 3.3. Assume (H1), (H2) and let $\lambda \in \mathbb{C}$. Then $\lambda \in \rho(A_U)$ if and only if $0 \in \rho(\Delta(\lambda))$.

Proof. For $\lambda \in \mathbb{C}$, it has been shown in section 2 that $\lambda \in \rho(A_U)$ if and only if $N(\Delta(\lambda)) = \{0\}$. On the other hand, $\Delta(\lambda) = A_T + L_\lambda^0 - \lambda I$, where A_T generates a compact C_0 -semigroup of bounded linear operators and $L_\lambda^0 - \lambda I$ is linear and bounded. Hence, $\Delta(\lambda)$ is also the infinitesimal generator of a compact C_0 -semigroup (see Proposition III.1.4 of Pazy [14, p. 79]). From the note in p. 51 of the same book, it follows that $0 \in \rho(\Delta(\lambda))$ if and only if 0 is not an eigenvalue of $\Delta(\lambda)$, or, equivalently, if and only if $N(\Delta(\lambda)) = \{0\}$. \square

LEMMA 3.4. Assume (H1), (H2) and let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$. Then

- (i) if $\mu \in \rho(\Delta(\lambda))$, then $\mu \in \rho(\mathcal{L}_\lambda^{(m)})$ and $(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}$ is a compact operator;
- (ii) $R(\mathcal{L}_\lambda^{(m)})$ is a closed subspace of X^m .

Proof. The proof of (i) is given by induction. For $m = 1$, $\mathcal{L}_\lambda^{(1)} = \Delta(\lambda)$. We have already observed that $\Delta(\lambda)$ is the infinitesimal generator of a compact C_0 -semigroup. Hence, for $\mu \in \rho(\Delta(\lambda))$ the resolvent $[\Delta(\lambda) - \mu I]^{-1}$ is compact (see Theorem II.3.3 of Pazy [14, p. 48]).

We now consider $\lambda \in \mathbb{C}$, $\mu \in \rho(\Delta(\lambda))$ and suppose that (i) is true for m . Since

$$\mathcal{L}_\lambda^{(m+1)} - \mu I = \begin{pmatrix} \mathcal{L}_\lambda^{(m)} - \mu I & \begin{pmatrix} L_\lambda^m \\ \vdots \\ L_\lambda^2 \\ L_\lambda^1 - I \end{pmatrix} \\ O & \Delta(\lambda) - \mu I \end{pmatrix},$$

$$(\mathcal{L}_\lambda^{(m+1)} - \mu I)^{-1} = \begin{pmatrix} (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} & -(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} \begin{pmatrix} L_\lambda^m \\ \vdots \\ L_\lambda^2 \\ L_\lambda^1 - I \end{pmatrix} (\Delta(\lambda) - \mu I)^{-1} \\ O & (\Delta(\lambda) - \mu I)^{-1} \end{pmatrix}$$

exists and is bounded. Now, let $(y_n) \subset X^m$, $(z_n) \subset X$ be bounded sequences. The compactness of the operators $(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}$ and $(\Delta(\lambda) - \mu I)^{-1}$ implies that there are subsequences $(y_{n_k}), (z_{n_k})$ such that

$$(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} y_{n_k} \rightarrow w \in X^m, \quad (\Delta(\lambda) - \mu I)^{-1} z_{n_k} \rightarrow x \in X.$$

Then $(\mathcal{L}_\lambda^{(m+1)} - \mu I)^{-1} \begin{pmatrix} y_{n_k} \\ z_{n_k} \end{pmatrix}$ converges, proving that $(\mathcal{L}_\lambda^{(m+1)} - \mu I)^{-1}$ is a compact operator.

To prove (ii), let $(x_n) \subset [D(A_T)]^m$, $\mathcal{L}_\lambda^{(m)} x_n \rightarrow y \in X^m$. For $\mu \in \rho(\Delta(\lambda))$, $\mu \neq 0$,

$$\left[\frac{I}{\mu} + (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} \right] x_n = \frac{1}{\mu} (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} \mathcal{L}_\lambda^{(m)} x_n \rightarrow \frac{1}{\mu} (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} y.$$

The space $R[\frac{I}{\mu} + (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}]$ is closed, because $(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}$ is compact (see Theorem V.7.8 of Taylor and Lay [17, p. 300]). Thus, there exists $x \in X^m$ such that

$$\frac{1}{\mu} (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} y = \left[\frac{I}{\mu} + (\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} \right] x,$$

i.e., $\mathcal{L}_\lambda^{(m)} x = y \in R(\mathcal{L}_\lambda^{(m)})$. \square

The characterization of the point spectrum of $*A_U$ relies on the next lemma.

LEMMA 3.5. Assume (H1)–(H3). Consider $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$. Then

$$\dim N(\mathcal{L}_\lambda^{(m)}) = \dim N((\mathcal{L}_\lambda^{(m)})^*).$$

Proof. We may assume that $\lambda \in \sigma(A_U)$, i.e., $0 \in \sigma(\Delta(\lambda))$ (cf. Lemma 3.3). For $\mu \in \rho(\Delta(\lambda))$, then $\mu \in \rho(\mathcal{L}_\lambda^{(m)})$ by Lemma 3.4, and we conclude that

$$N(\mathcal{L}_\lambda^{(m)}) = N\left((\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} + \frac{I}{\mu}\right),$$

$$N((\mathcal{L}_\lambda^{(m)})^*) = N\left([\mathcal{L}_\lambda^{(m)*} - \mu I]^{-1} + \frac{I}{\mu}\right).$$

Since $\mathcal{L}_\lambda^{(m)}$ is densely defined, we also conclude that $\mu \in \rho((\mathcal{L}_\lambda^{(m)})^*)$ and $[(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}]^* = [(\mathcal{L}_\lambda^{(m)*} - \mu I]^{-1}$ (cf. Lemma I.10.2 of Pazy [14, p. 38]). It remains to be proved that $N((\mathcal{L}_\lambda^{(m)} - \mu I)^{-1} + \frac{I}{\mu})$ and $N([\mathcal{L}_\lambda^{(m)*} - \mu I]^{-1} + \frac{I}{\mu})$ have the same dimension. Since $(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}$ is a compact operator, so is its adjoint $[(\mathcal{L}_\lambda^{(m)} - \mu I)^{-1}]^*$, and the result now follows from Theorem V.7.14 of Taylor and Lay [17, p. 303]. \square

As an immediate and most relevant consequence of this lemma, we can now derive the following result.

PROPOSITION 3.6. *Assume (H1)–(H3). Then*

- (i) $\sigma_P(A_U) = \sigma_P(*A_U)$;
- (ii) $\dim N[(A_U - \lambda I)^m] = \dim N[(*A_U - \lambda I)^m]$, $m \in \mathbb{N}$;
- (iii) *the ascent of $A_U - \lambda I$ and $*A_U - \lambda I$ are equal.*

Proof. Propositions 3.1 and 3.2 and Lemma 3.5 imply (ii), from which (i) and (iii) follow. \square

Remark 3.1. We note that (i) of Proposition 3.6 was proven in Proposition 4.14 of Travis and Webb [18] under the additional hypothesis $N(\Delta(\lambda)) \neq \{0\}$ if and only if $N(\Delta(\lambda)^*) \neq \{0\}$.

Remark 3.2. In the literature dealing with adjoint semigroups for FDEs in Banach spaces (cf., e.g., Nakagiri [13] and Travis and Webb [18, p. 412]), it is often assumed that the Banach space X is reflexive in order to have nice properties for adjoint semigroups. Here, we are able to develop the adjoint theory without imposing such a condition. Of course, if this condition holds, further properties for $*A_U$ and $*U(t)$ are obtained. For example, if the Banach space X is reflexive, then the adjoint A_T^* of A_T is the infinitesimal generator of the adjoint C_0 -semigroup $\{T(t)^*\}_{t \geq 0}$ (cf. Pazy [14, p. 39]). For $t > 0$, $T(t)$ is a compact operator, and hence its adjoint $T(t)^*$ is also compact. Since (H1) and (H2) are fulfilled with $A_T, T(t)$ replaced by $A_T^*, T(t)^*$, respectively, the conclusions of Propositions 2.1, 2.2, and 2.3 hold for $*A_U, *U(t)$ ($t > 0$) instead of $A_U, U(t)$ ($t > 0$). In particular, $\sigma_P(*A_U) = \sigma(*A_U)$.

Remark 3.3. In Arino and Sanchez [1], a formal adjoint theory was established for equations of the form $\dot{u}(t) = L(u_t)$, where $L : C \rightarrow X$ is a bounded linear operator. Since $A_T = 0$, the C_0 -semigroup $\{U(t)\}_{t \geq 0}$ associated with the solutions of this equation is not eventually compact in general. For this reason, in [1] the authors restricted their study to eigenvalues of the infinitesimal generator that are not in the essential spectrum. With this restriction, the corresponding operators $\mathcal{L}_\lambda^{(m)}$ are Fredholm operators, instead of having compact resolvent. However, for our purposes and in view of applications, it is more interesting to consider equations of type (2.1) rather than $\dot{u}(t) = L(u_t)$, and in this situation no restrictions on the eigenvalues have to be assumed.

4. Decomposition of the phase space by using the formal adjoint theory. In this section, we always assume (H1)–(H3). The Fredholm alternative is stated

in the next result.

PROPOSITION 4.1. Consider $\lambda \in \sigma(A_U)$ and $m \in \mathbb{N}$. Then $\psi \in R[(A_U - \lambda I)^m]$ if and only if $\langle \alpha, \psi \rangle = 0$ for all $\alpha \in N[(^*A_U - \lambda I)^m]$. In particular, $\psi \in R(A_U - \lambda I)$ if and only if

$$\langle e^{-\lambda \cdot} u^*, \psi \rangle = 0 \quad \text{for all } u^* \in N(\Delta(\lambda)^*).$$

Proof. Since $R(\mathcal{L}_\lambda^{(m)})$ is closed (Lemma 3.4), we have

$$R(\mathcal{L}_\lambda^{(m)}) = N((\mathcal{L}_\lambda^{(m)})^*)^\perp.$$

Thus, Proposition 3.1 implies that

$$\psi \in R[(A_U - \lambda I)^m] \quad \text{if and only if } \langle Y^*, \mathcal{R}_\lambda^{(m)}(\psi) \rangle = 0$$

for all $Y^* \in N((\mathcal{L}_\lambda^{(m)})^*)$. For $Y^* = (y_0^*, \dots, y_{m-1}^*)^T \in (X^*)^m$, from (2.11) and (3.3) we have

$$\begin{aligned} & \langle Y^*, \mathcal{R}_\lambda^{(m)}(\psi) \rangle \\ &= - \sum_{j=0}^{m-1} \left\langle y_j^*, L \left(\int_0^\theta e^{\lambda(\theta-\xi)} \frac{(\theta-\xi)^{m-j-1}}{(m-j-1)!} \psi(\xi) d\xi \right) \right\rangle + \langle y_{m-1}^*, \psi(0) \rangle \\ &= \sum_{j=0}^{m-1} \left\langle \left\langle e^{-\lambda s} \frac{(-s)^{m-j-1}}{(m-j-1)!} y_j^*, \psi \right\rangle \right\rangle, \end{aligned}$$

and the result follows from Proposition 3.2. \square

We note that the above result was established in Proposition 4.15 of Travis and Webb [18] only for the particular situation $m = 1$ and with the additional hypothesis that $\Delta(\lambda)$ has a closed range. In Proposition 4.1, the most important case is the case m equal to the ascent of $A_U - \lambda I$. For $\lambda \in \sigma(A_U)$, denote by $\mathcal{M}_\lambda(A_U)$ and $\mathcal{M}_\lambda(^*A_U)$ the generalized eigenspaces for A_U and *A_U associated with λ , respectively.

PROPOSITION 4.2. Let $\lambda \in \sigma(A_U)$ and m be the ascent of $A_U - \lambda I$. Then $C = \mathcal{M}_\lambda(A_U) \oplus Q_\lambda$, with $\mathcal{M}_\lambda(A_U) = N[(A_U - \lambda I)^m]$, $\mathcal{M}_\lambda(^*A_U) = N[(^*A_U - \lambda I)^m]$, and

$$(4.1) \quad Q_\lambda = \{\psi \in C : \langle \alpha, \psi \rangle = 0 \quad \text{for all } \alpha \in \mathcal{M}_\lambda(^*A_U)\}.$$

Proof. From Proposition 3.6, m is also the ascent of $^*A_U - \lambda I$. On the other hand, Proposition 4.1 implies that $\psi \in R[(A_U - \lambda I)^m]$ if and only if $\langle \alpha, \psi \rangle = 0$ for all $\alpha \in \mathcal{M}_\lambda(^*A_U)$. Decomposition (2.5) is therefore written as $C = \mathcal{M}_\lambda(A_U) \oplus Q_\lambda$, with $Q_\lambda = R[(A_U - \lambda I)^m]$ defined by (4.1). \square

LEMMA 4.3. For $\lambda, \mu \in \sigma(A_U)$, $\lambda \neq \mu$, and $m, r \in \mathbb{N}$,

$$\langle \alpha, \varphi \rangle = 0 \quad \text{for all } \alpha \in N[(^*A_U - \lambda I)^m] \quad \text{and} \quad \varphi \in N[(A_U - \mu I)^r].$$

Proof. This lemma generalizes formula (2.16) for $m \in \mathbb{N}$. It relies on the identity (2.15) and is easily verified by using arguments as in Lemma 9 of Arino and Sanchez [1], so we omit the details here. \square

Let $\lambda \in \sigma(A_U)$ and choose bases

$$\Phi_\lambda = (\varphi_1, \dots, \varphi_{p_\lambda}), \quad \Psi_\lambda = (\psi_1, \dots, \psi_{p_\lambda})^T$$

of $\mathcal{M}_\lambda(A_U)$ and $\mathcal{M}_\lambda(*A_U)$, respectively, where $p_\lambda = \dim \mathcal{M}_\lambda(A_U) = \dim \mathcal{M}_\lambda(*A_U)$. Define a $p_\lambda \times p_\lambda$ matrix

$$\langle\langle \Psi_\lambda, \Phi_\lambda \rangle\rangle := [\langle\langle \psi_i, \varphi_j \rangle\rangle]_{i,j=1,\dots,p_\lambda}.$$

Suppose that $\langle\langle \Psi, \Phi \rangle\rangle c = 0$ for some constant vector $c = (c_1, \dots, c_{p_\lambda})^T$. Then, $\langle\langle \alpha, c_1 \varphi_1 + \dots + c_{p_\lambda} \varphi_{p_\lambda} \rangle\rangle = 0$ for all $\alpha \in \mathcal{M}_\lambda(*A_U)$, and Proposition 4.2 implies that $c_1 \varphi_1 + \dots + c_{p_\lambda} \varphi_{p_\lambda} \in Q_\lambda \cap \mathcal{M}_\lambda(A_U) = \{0\}$ for Q_λ as in (4.1). This shows that $\langle\langle \Psi_\lambda, \Phi_\lambda \rangle\rangle$ is nonsingular. Therefore, we can always choose bases $\Psi_\lambda, \Phi_\lambda$ such that

$$(4.2) \quad \langle\langle \Psi_\lambda, \Phi_\lambda \rangle\rangle = I_{p_\lambda}, \quad p_\lambda = \dim \mathcal{M}_\lambda(A_U).$$

If the bases are normalized in such a way that (4.2) is fulfilled, then there is a $p_\lambda \times p_\lambda$ constant matrix B_λ , with $\sigma(B_\lambda) = \{\lambda\}$, that satisfies simultaneously

$$(4.3) \quad \dot{\Phi}_\lambda = \Phi_\lambda B_\lambda \quad \text{and} \quad -\dot{\Psi}_\lambda = B_\lambda \Psi_\lambda.$$

Furthermore,

$$(4.4) \quad U(t) = \Phi_\lambda e^{B_\lambda t}, \quad t > 0.$$

We are now in the position to decompose C by a finite set of characteristic eigenvalues of (2.1), using the formal duality $\langle\langle \cdot, \cdot \rangle\rangle$. Consider a nonempty finite set $\Lambda = \{\lambda_1, \dots, \lambda_s\} \subset \sigma(A_U)$ and define $\Phi_\Lambda = (\Phi_{\lambda_1}, \dots, \Phi_{\lambda_s}), \Psi_\Lambda = (\Psi_{\lambda_1}, \dots, \Psi_{\lambda_s})^T$, where $\Phi_{\lambda_j}, \Psi_{\lambda_j}$ are bases of the generalized eigenspaces $\mathcal{M}_{\lambda_j}(A_U), \mathcal{M}_{\lambda_j}(*A_U)$, respectively, such that (4.2) holds ($j = 1, \dots, s$). From Lemma 4.3, it follows that $\langle\langle \Psi_\Lambda, \Phi_\Lambda \rangle\rangle = I_p$, where $p = p_{\lambda_1} + \dots + p_{\lambda_s}$.

PROPOSITION 4.4. *Assume (H1)–(H3), let $\Lambda = \{\lambda_1, \dots, \lambda_s\} \subset \sigma(A_U)$, define*

$$P_\Lambda = \mathcal{M}_{\lambda_1}(A_U) \oplus \dots \oplus \mathcal{M}_{\lambda_s}(A_U), \\ P_\Lambda^* = \mathcal{M}_{\lambda_1}(*A_U) \oplus \dots \oplus \mathcal{M}_{\lambda_s}(*A_U),$$

and consider bases $\Phi_\Lambda, \Psi_\Lambda$ for P_Λ, P_Λ^* such that $\langle\langle \Psi_\Lambda, \Phi_\Lambda \rangle\rangle = I_p$, $p = \dim P_\Lambda$. Then there exists a subspace Q_Λ of C , invariant under A_U and $U(t)$, $t \geq 0$, such that

$$(4.5) \quad C = P_\Lambda \oplus Q_\Lambda$$

with

$$(4.6) \quad Q_\Lambda = \{\varphi \in C : \langle\langle \Psi_\Lambda, \varphi \rangle\rangle = 0\},$$

where $\langle\langle \Psi_\Lambda, \varphi \rangle\rangle := (\langle\langle \Psi_{\lambda_1}, \varphi \rangle\rangle, \dots, \langle\langle \Psi_{\lambda_s}, \varphi \rangle\rangle)^T$. Moreover, $\varphi \in C$ is written according to decomposition (4.6) as $\varphi = \varphi_{P_\Lambda} + \varphi_{Q_\Lambda}$, where $\varphi_{P_\Lambda} = \Phi_\Lambda \langle\langle \Psi_\Lambda, \varphi \rangle\rangle$ and $\varphi_{Q_\Lambda} \in Q_\Lambda$.

5. Center manifolds for maps in general Banach spaces: Smoothness.

We start with the following general results on smooth center-stable manifolds for maps.

THEOREM 5.1. *Let $f : U \rightarrow E$ be a C^1 -map on an open subset U of a Banach space E over \mathbb{R} , with a fixed point p . Let $L = Df(p)$ and assume that E has the following decomposition:*

$$E = E_s \oplus E_c \oplus E_u,$$

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where E_s is a closed subspace, E_c and E_u are finite-dimensional, $L(E_s) \subset E_s$, $L(E_c) \subset E_c$, and $L(E_u) \subset E_u$. We further assume that

$$\sigma_s = \sigma(L|_{E_s} : E_s \rightarrow E_s) \text{ is contained in a compact subset of } \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\begin{aligned} \sigma_c &= \sigma(L|_{E_c} : E_c \rightarrow E_c) \subset S_{\mathbb{C}}^1, \\ \sigma_u &= \sigma(L|_{E_u} : E_u \rightarrow E_u) \subset \{z \in \mathbb{C} : |z| > 1\}. \end{aligned}$$

Let $E_{sc} = E_s \oplus E_c$. Then

(i) there exist open neighborhoods N_{sc} of 0 in E_{sc} , N_u of 0 in E_u , N of p in U , and a C^1 -map $w : N_{sc} \rightarrow E_u$ with $w(0) = 0$, $Dw(0) = 0$, and $w(N_{sc}) \subset N_u$ so that the shifted graph $W = p + \{z + w(z) : z \in N_{sc}\}$ satisfies $f(W \cap N) \subset W$ and $\bigcap_{n=0}^{\infty} f^{-n}(p + N_{sc} + N_u) \subset W$;

(ii) if f is C^k -smooth for an integer $k \geq 2$, then so is w .

Part (i) was proved in [10]. Our argument for the general smoothness in (ii), given below, will be based on the following general C^1 -smoothness result for fixed points of contractions depending on a parameter developed in [10].

LEMMA 5.2. Let Y, Λ be Banach spaces over \mathbb{R} and let an open set $P \subset \Lambda$, a map $h : Y \times P \rightarrow Y$, and a constant $\kappa \in [0, 1)$ be given with $|h(y, p) - h(\tilde{y}, p)| \leq \kappa|y - \tilde{y}|$ for all y, \tilde{y} in Y and all $p \in P$. Consider a convex subset $M \subset Y$ and a map $\Phi : P \rightarrow M$ so that for every $p \in P$, $\Phi(p)$ is the unique fixed point of $h(\cdot, p) : Y \rightarrow Y$. Suppose the following hold:

(i) the restriction $h_0 = h|_{M \times P}$ has a partial derivative $D_2h_0 : M \times P \rightarrow L(\Lambda, Y)$ and the map D_2h_0 is continuous;

(ii) there are a Banach space Y_1 over \mathbb{R} and a continuous injective linear map $j : Y \rightarrow Y_1$ so that the map $k = j \circ h_0$ is continuously differentiable with respect to Y in the sense that there is a continuous map $A : M \times P \rightarrow L(Y, Y_1)$ so that for every $(y, p) \in M \times P$ and every $\epsilon^* > 0$, there exists $\delta > 0$ with $|k(\tilde{y}, p) - k(y, p) - A(y, p)(\tilde{y} - y)| \leq \epsilon^*|\tilde{y} - y|$ for all $\tilde{y} \in M$ with $|\tilde{y} - y| \leq \delta$;

(iii) there exist maps $h^{(1)} : M \times P \rightarrow L(Y, Y)$ and $h_1^{(1)} : M \times P \rightarrow L(Y_1, Y_1)$ such that

$$A(y, p)\hat{y} = jh^{(1)}(y, p)\hat{y} = h_1^{(1)}(y, p)j\hat{y} \quad \text{on } M \times P \times Y$$

and

$$|h^{(1)}(y, p)| \leq \kappa, \quad |h_1^{(1)}(y, p)| \leq \kappa \quad \text{on } M \times P;$$

(iv) the map $(y, p) \in M \times P \rightarrow j \circ h^{(1)}(y, p) \in L(Y, Y_1)$ is continuous.

Then the map $j \circ \Phi : P \rightarrow Y_1$ is C^1 -smooth and

$$D(j \circ \Phi)(p) = h_1^{(1)}(\Phi(p), p) \circ D(j \circ \Phi)(p) + j \circ D_2h_0(\Phi(p), p) \quad \text{for all } p \in P.$$

For a given positive integer k and for given Banach spaces Y_1, \dots, Y_k and Y , let $\mathcal{L}^{(k)}(Y_1 \times \dots \times Y_k, Y)$ be the Banach space of all continuous k -linear maps from $Y_1 \times \dots \times Y_k$ to Y , equipped with the operator norm. If $Y_i = Y_1$ for all $1 \leq i \leq k$, we write $\mathcal{L}^{(k)}(Y_1, Y)$ for $\mathcal{L}^{(k)}(Y_1 \times \dots \times Y_k, Y)$. Also, we will denote the k th derivative of a given map by D^k if it exists.

We now briefly recall some results and associated notation in [10] as a preparation for the proof of Theorem 5.1. Set $b = \inf_{\lambda \in \sigma_u} |\lambda|$, $a = \sup_{\lambda \in \sigma_s} |\lambda|$ and fix $\epsilon > 0$ with

$a + \epsilon < 1 < 1 + \epsilon < (1 + \epsilon)^k < b - \epsilon$. Let P_s, P_c, P_u denote the projections of E onto E_s along $E_c \oplus E_u$, onto E_c along $E_s \oplus E_u$, and onto E_u along $E_c \oplus E_s$, respectively. Whenever convenient, we shall use abbreviations like

$$x_s = P_s x, \quad x_c = P_c x, \quad x_u = P_u x, \quad P_{sc} = P_s + P_c, \quad x_{cu} = x_c + x_u.$$

There exists a norm $|\cdot|$ on E which is equivalent to the originally given one and satisfies

$$\begin{aligned} |x| &= |x_s| + |x_c| + |x_u|, \\ |LP_s x| &\leq (a + \epsilon)|P_s x|, \\ |LP_c x| &\leq (1 + \epsilon)|P_c x|, \\ |LP_u x| &\geq (b - \epsilon)|P_u x| \end{aligned}$$

for all $x \in E$.

Set $V = U - p$. Consider the transformed map $g^* : x \in V \rightarrow f(x + p) - p \in E$ with fixed point 0 and $Dg^*(0) = L$. Define $r^* : V \rightarrow E$ as the nonlinear part of g^* by $r^*(x) = g^*(x) - Lx$, and then extend r^* to a map $r : E \rightarrow E$ by $r(x) = 0$ for all $x \in E \setminus V$. Finally, let $g = L + r$.

To construct small Lipschitz continuous modifications of g which are smooth on strips containing the center-unstable space E_{cu} , we fix a norm $|\cdot|_{cu}$ on E_{cu} which is C^∞ -smooth on $E_{cu} \setminus \{0\}$. The norm $\|\cdot\| : x \in E \rightarrow \max\{|x_s|, |x_{cu}|_{cu}\} \in \mathbb{R}$ is equivalent to $|\cdot|$. For $\delta > 0$, set $E(\delta) = \{x \in E : \|x\| < \delta\}$. Choose a C^∞ -function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ with $\rho([0, \infty)) \subset [0, 1], \rho(t) = 1$ for $0 \leq t \leq 1, \rho(t) = 0$ for $t \geq 2$. For every $\delta > 0$, define $r_\delta : E \rightarrow E$ by

$$r_\delta(x) = \rho\left(\frac{|x_{cu}|_{cu}}{\delta}\right) \rho\left(\frac{|x_s|}{\delta}\right) r(x)$$

and set $g_\delta = L + r_\delta$.

Fix $\delta_0 > 0$ so that $\overline{E(3\delta_0)} \subset V$ and that $r|_{E(3\delta_0)}$ is C^k -smooth and all l th derivatives, $1 \leq l \leq k$, of $r|_{E(3\delta_0)}$ are bounded. Observing that for every $\delta \in (0, \delta_0)$ the restriction $r_\delta|_{\{x \in E : |x_s| < \delta\}}$ is given by $\rho\left(\frac{|x_{cu}|_{cu}}{\delta}\right)r(x)$, it follows that $r_\delta|_{\{x \in E : |x_s| < \delta\}}$ is C^k -smooth and that the restriction of r_δ to $\{x \in E : |x_s| \leq \frac{\delta}{2}\}$ has all l th derivatives bounded, $1 \leq l \leq k$.

It was shown in [10] that there exist $\delta_1 \in (0, \delta_0)$ and a nondecreasing function $\lambda : [0, \delta_1] \rightarrow [0, 1]$ with $\lim_{\delta \rightarrow 0^+} \lambda(\delta) = 0 = \lambda(0)$ so that for each $\delta \in (0, \delta_1]$ and for all x, y in E , $|r_\delta(x)| \leq \delta\lambda(\delta)$ and $|r_\delta(x) - r_\delta(y)| \leq \lambda(\delta)|x - y|$.

For $\eta > 0$, let E_η denote the Banach space of all sequences $\chi = (x_n)_0^\infty \in E^\mathbb{N}$ with

$$\sup_{j \in \mathbb{N}} |x_j| \eta^{-j} < \infty$$

and norm

$$\|\chi\|_\eta = \sup_{j \in \mathbb{N}} |x_j| \eta^{-j}.$$

Consider

$$(5.1) \quad \begin{cases} x_{n+1} = Lx_n + f_n & \text{for } n \geq 0, \\ P_{sc}x_0 = z \end{cases}$$

for given $z \in E_{sc}, \phi = (f_n)_0^\infty \in E_\eta$, and $\eta \in (1 + \epsilon, b - \epsilon)$.

Let $L_{sc} = L|_{E_{sc}} : E_{sc} \rightarrow E_{sc}$. It was shown in [10] that for fixed $z \in E_{sc}$, $1 + \epsilon < \eta < b - \epsilon$, and $\phi \in E_\eta$, if $\chi \in E_\eta$ satisfies (5.1), then

$$x_n = \sum_{j=0}^{n-1} L_{sc}^{n-j-1} P_{sc} f_j - \sum_{j=n}^{\infty} L_u^{n-j-1} P_u f_j + L_{sc}^n z \quad \text{for } n \geq 1$$

and

$$x_0 = z - \sum_{j=0}^{\infty} L_u^{-j-1} P_u f_j.$$

In particular, given $z \in E_{sc}$ and $\phi = (f_j)_0^\infty \in E_\eta$, there is at most one solution of (5.1) in E_η . Let

$$K : \{\chi \in E^\mathbb{N} : \chi \in E_\eta \text{ for some } \eta \in (1 + \epsilon, b - \epsilon)\} \rightarrow E^\mathbb{N}$$

be given by

$$(K\phi)_n = \sum_{j=0}^{n-1} L_{sc}^{n-j-1} P_{sc} f_j - \sum_{j=n}^{\infty} L_u^{n-j-1} P_u f_j \quad \text{for } n \geq 1$$

and

$$(K\phi)_0 = - \sum_{j=0}^{\infty} L_u^{-j-1} P_u f_j.$$

Also, let

$$c(\eta) = \frac{1}{\eta - 1 - \epsilon} + \frac{1}{b - \epsilon - \eta}.$$

Then the linear map $K_\eta : E_\eta \rightarrow E_\eta$ given by $K_\eta \phi = K\phi$ is continuous with $|K_\eta| \leq c(\eta)$. Furthermore, for every $\eta \in (1 + \epsilon, b - \epsilon)$, $z \in E_{sc}$, and $\phi \in E_\eta$, the sequence $\chi = K_\eta \phi + (L_{sc}^n z)_0^\infty \in E_\eta$ solves (5.1).

Consider the substitution operator

$$R_\delta : E^\mathbb{N} \rightarrow E^\mathbb{N} \quad \text{by } R_\delta(\chi) = (r_\delta(x_n))_0^\infty \quad \text{for } \chi = (x_n)_0^\infty \in E^\mathbb{N}.$$

For every $\eta \in (1 + \epsilon, b - \epsilon)$, choose $\delta_\eta \in (0, \delta_1]$ with $\lambda(\delta_\eta)c(\eta) < 1$. Let $\eta \in (1 + \epsilon, b - \epsilon)$ and $\delta \in (0, \delta_\eta)$. It was shown in [10] that $R_\delta(E_\eta) \subset E_\eta$, and the induced map $\gamma_{\delta_\eta} : E_\eta \ni \chi \mapsto R_\delta(\chi) \in E_\eta$ is Lipschitz continuous with a Lipschitz constant $\lambda(\delta)$.

Therefore, for every $z \in E_{sc}$ and $\chi = (x_n)_0^\infty \in E_\eta$ the properties

$$x_{n+1} = g_\delta(x_n) \quad \text{for all } n \geq 0, \quad P_{sc} x_0 = z$$

are equivalent to the fixed point equation $\chi = T_{\delta_\eta}(\chi, z)$, where the map $T_{\delta_\eta} : E_\eta \times E_{sc} \rightarrow E_\eta$ is given by

$$T_{\delta_\eta}(\chi, z) = K_\eta(\gamma_{\delta_\eta}(\chi)) + (L_{sc}^j z)_0^\infty.$$

As

$$|T_{\delta_\eta}(\chi, z) - T_{\delta_\eta}(\chi^*, z)|_\eta \leq c(\eta)\lambda(\delta)|\chi - \chi^*|_\eta$$

for all $\chi, \chi^* \in E_\eta$ and for all $z \in E_{sc}$, there is exactly one fixed point $\chi_{\delta\eta}(z) \in E_\eta$ of the contraction $T_{\delta\eta}(\cdot, z) : E_\eta \rightarrow E_\eta$ for every $z \in E_{sc}$. Moreover, $P_{sc}(\chi_{\delta\eta}(z))_0 = z$. In summary, $\chi \in E_\eta$ is a trajectory of g_δ with $P_{sc}x_0 = z$ if and only if $\chi = \chi_{\delta\eta}(z)$.

It was shown in [10] that the map $\chi_{\delta\eta} : z \in E_{sc} \rightarrow \chi_{\delta\eta}(z) \in E_\eta$ is Lipschitz continuous, and thus $w_{\delta\eta} : z \in E_{sc} \rightarrow P_u(\chi_{\delta\eta}(z))_0 \in E_u$ is Lipschitz continuous. To obtain the differentiability of $w_{\delta\eta}$, [10] proved the following important properties: if $0 < \delta < \delta_\eta$ and $\lambda(\delta) < \frac{(1-a-\epsilon)^2}{2}$, then for every $z \in E_{sc}$ with $|P_s z| < \frac{\delta}{2}$ and for all integers $j \geq 0$,

$$(5.2) \quad |P_s(\chi_{\delta\eta}(z))_j| < \frac{\delta}{2}.$$

We can now give the following proof.

Proof of Theorem 5.1. We divide the long proof into several steps. The first step concerns the proof of the C^1 -smoothness. Except for the last remark, all results in Step 1 belong to [10].

Step 1. Fix $\eta, \tilde{\eta}, \bar{\eta}$ so that $1 + \epsilon < \eta < \tilde{\eta} \leq \bar{\eta}$ with $\bar{\eta} \in (\eta^k, b - \epsilon)$, and fix $\delta > 0$ so that

$$\delta < \delta_\eta, \quad \lambda(\delta) < \frac{(1-a-\epsilon)^2}{2}, \quad \kappa := \sup_{\tilde{\eta} \in [\eta, \bar{\eta}]} \lambda(\delta)c(\tilde{\eta}) < 1.$$

Let

$$P = \left\{ x \in E_{sc} : |x_s| < \frac{\delta}{2} \right\}.$$

P is an open set in the Banach space $\Lambda = E_{sc}$.

Recall that $r_\delta|_{\{x \in E : |x_s| < \delta\}}$ is C^k -smooth and $\sup\{|D^1 r_\delta(x)| : |x_s| < \delta\} \leq \lambda(\delta)$. It was shown in [10] that for any $\tilde{\eta} \in (\eta, \bar{\eta}]$, the linear map

$$A_{r_\delta}^{(1)}(\chi) : E^\mathbb{N} \ni \hat{\chi} = (\hat{x}_j)_0^\infty \mapsto (D^1 r_\delta(x_j) \hat{x}_j)_0^\infty \in E^\mathbb{N}, \quad \chi = (x_j)_0^\infty, \quad |P_s x_j| < \frac{\delta}{2}, \quad j \in \mathbb{N},$$

induces a continuous map $A_{r_\delta \tilde{\eta} \eta}^{(1)}$ from the convex set

$$M = \left\{ \chi \in E_\eta : |P_s x_j| < \frac{\delta}{2} \text{ for all } j \in \mathbb{N} \right\} \subset E_\eta$$

into $\mathcal{L}(E_\eta, E_{\tilde{\eta}})$.

Let $Y = E_\eta, h = T_{\delta\eta}|_{Y \times P}$. It is important to keep in mind that $\chi_{\delta\eta}(P) \subset M$. Define $\Phi : P \rightarrow M$ by $\Phi(z) = \chi_{\delta\eta}(z)$; we have $h(\Phi(p), p) = \Phi(p)$ for all $p \in P$. The map $h_0 = h|_{M \times P}$ is given by

$$h_0(\chi, z) = T_{\delta\eta}(\chi, z) = K(R_\delta(\chi)) + (L_{sc}^j z)_0^\infty,$$

so for every $(\chi, z) \in M \times P$ the derivative $D_2 h_0(\chi, z)$ exists and is given by

$$D_2 h_0(\chi, z) \tilde{z} = (L_{sc}^j \tilde{z})_0^\infty \in E_\eta.$$

This derivative is constant on $M \times P$ and therefore is continuous.

Set $Y_1 = E_{\tilde{\eta}}$ and define $j_{\tilde{\eta}\eta} : Y \rightarrow Y_1$ by

$$j_{\tilde{\eta}\eta}(\chi) = \chi.$$

Then $j_{\tilde{\eta}\eta}$ is continuous and injective, and the map $k = j_{\tilde{\eta}\eta} \circ h_0$ is given by

$$k(\chi, z) = T_{\delta_{\tilde{\eta}}}(\chi, z) = K_{\tilde{\eta}}(\gamma_{\delta_{\tilde{\eta}}}(\chi)) + (L_{sc}^j z)_0^\infty.$$

It was shown in [10] that the map $A : M \times P \ni (\chi, z) \mapsto K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\eta}(\chi) \in \mathcal{L}(Y, Y_1)$ is continuous, and each $A_{r_\delta}^{(1)}(\chi), \chi \in M$, defines elements

$$A_{r_\delta \eta \eta}^{(1)}(\chi) \in \mathcal{L}(Y, Y) \quad \text{with } |A_{r_\delta \eta \eta}^{(1)}(\chi)| \leq \lambda(\delta)$$

and

$$A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\chi) \in \mathcal{L}(Y_1, Y_1) \quad \text{with } |A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\chi)| \leq \lambda(\delta).$$

Define

$$h^{(1)} : M \times P \rightarrow \mathcal{L}(Y, Y) \quad \text{by } h^{(1)}(\chi, z) = K_\eta \circ A_{r_\delta \eta \eta}^{(1)}(\chi)$$

and

$$h_1^{(1)} : M \times P \rightarrow \mathcal{L}(Y_1, Y_1) \quad \text{by } h_1^{(1)}(\chi, z) = K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\chi).$$

It was shown in [10] that

$$\max\{|h^{(1)}(\chi, z)|, |h_1^{(1)}(\chi, z)|\} \leq \max\{c(\eta), c(\tilde{\eta})\} \lambda(\delta) = \kappa,$$

and all other conditions in Lemma 5.2 are satisfied. Therefore, $j_{\tilde{\eta}\eta} \circ \Phi = j_{\tilde{\eta}\eta} \circ (\chi_{\delta_\eta}|_P)$ is C^1 -smooth and $j_{\tilde{\eta}\eta} \circ \Phi = \chi_{\delta_{\tilde{\eta}}}|_P$. Moreover, $D^1(j_{\tilde{\eta}\eta} \circ \Phi)$ satisfies

$$D^1(j_{\tilde{\eta}\eta} \circ \Phi)(z) = K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\Phi(z)) \circ D^1(j_{\tilde{\eta}\eta} \circ \Phi)(z) + j_{\tilde{\eta}\eta} \circ (L_{sc}^j \cdot)_0^\infty, \quad z \in P.$$

The final remark of this step is essential for the general smoothness to be proved in later steps. Recall that for any $\tilde{\eta} \in [\eta, \bar{\eta}]$, $K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\Phi(z)) \in \mathcal{L}(E_{\tilde{\eta}}, E_{\tilde{\eta}})$ and

$$|K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\Phi(z))|_{\mathcal{L}(E_{\tilde{\eta}}, E_{\tilde{\eta}})} \leq c(\tilde{\eta}) \lambda(\delta) \leq \kappa < 1.$$

Therefore, $K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\Phi(z)) \in \mathcal{L}(E_{\tilde{\eta}}, E_{\tilde{\eta}})$ is a uniform contraction and the map

$$K_{\tilde{\eta}} \circ A_{r_\delta \tilde{\eta}\tilde{\eta}}^{(1)}(\Phi(z))L + j_{\tilde{\eta}\eta} \circ (L_{sc}^j \cdot)_0^\infty, \quad z \in P, L \in \mathcal{L}(\Lambda, E_{\tilde{\eta}}),$$

has a unique fixed point $\Psi_{\tilde{\eta}}^{(1)}(z)$ in $\mathcal{L}(\Lambda, E_{\tilde{\eta}})$. Since $j_{\tilde{\eta}\eta} \circ \Psi_{\tilde{\eta}}^{(1)}(z) \in \mathcal{L}(\Lambda, E_{\tilde{\eta}})$, the uniqueness of a fixed point in $\mathcal{L}(\Lambda, E_{\tilde{\eta}})$ implies

$$\Psi_{\tilde{\eta}}^{(1)}(z) = j_{\tilde{\eta}\eta} \circ \Psi_{\tilde{\eta}}^{(1)}(z).$$

In particular,

$$D^1(j_{\tilde{\eta}\eta} \circ \Phi)(z) = \Psi_{\tilde{\eta}}^{(1)}(z) = j_{\tilde{\eta}\eta} \circ \Psi_{\tilde{\eta}}^{(1)}(z), \quad z \in P.$$

Step 2. We now assume $k \geq 2$. For any given integer l with $1 \leq l \leq k$, consider the operator $A_{r_\delta}^{(l)}$ given by

$$\begin{aligned} A_{r_\delta}^{(l)}(\chi)(\chi^1, \dots, \chi^l) &= (D^l r_\delta(x_j)(x_j^1, \dots, x_j^l))_0^\infty, \\ \chi &= (x_j)_0^\infty, \quad \chi^i = (x_j^i)_0^\infty \in E^{\mathbb{N}}, \quad 1 \leq i \leq l. \end{aligned}$$

Note that $A_{r_\delta}^{(l)}$ with $l = 1$ was introduced in Step 1. The operators $A_{r_\delta}^{(l)}$ with $1 \leq l \leq k$ are the substitution operators of $D^l r_\delta$; they can be regarded as the Nemytskii operators induced by $D^l r_\delta$ in the appropriate spaces.

As $r_\delta|_{\{z \in E; |z_s| \leq \frac{\delta}{2}\}}$ has all l th derivatives bounded, $1 \leq l \leq k$, we can show that

$$A_{r_\delta}^{(l)}(\chi)(E_{\eta^{r_1}} \times \dots \times E_{\eta^{r_l}}) \subset E_{\eta^{r_1 + \dots + r_l}}, \quad \chi \in M, \quad 1 \leq r_i \leq l.$$

We are going to use induction on p with $1 \leq p \leq k$. (Note that for the remainder of this proof, p is not the fixed point of f .) The strategy is to show that the order of the smoothness of $j_{\tilde{\eta}\eta} \circ \Phi : P \rightarrow E_{\tilde{\eta}}$ is increased by at least one as $\tilde{\eta}$ passes η^{p-1} , from (η, η^{p-1}) to (η^{p-1}, η^p) , and to construct higher order derivatives inductively.

Suppose $1 \leq p < k$ and suppose that for all integers q with $1 \leq q \leq p$ and for all $\tilde{\eta} \in [\eta^q, \tilde{\eta}]$, the mapping $j_{\tilde{\eta}\eta} \circ \Phi : P \rightarrow E_{\tilde{\eta}}$ is C^q -smooth with

- (i) $D^q(j_{\tilde{\eta}\eta} \circ \Phi) = j_{\tilde{\eta}\eta} \circ \Psi_{\tilde{\eta}}^{(q)}$;
- (ii) $\Psi_{\tilde{\eta}}^{(q)}(z) \in \mathcal{L}^{(q)}(\Lambda, E_{\eta^q})$ as the unique solution of

$$F = KA_{r_\delta}^{(1)}(\Phi(z))F + H_q(z), \quad F \in \mathcal{L}^{(q)}(\Lambda, E_{\eta^q}), \quad z \in P,$$

with $H_1(z)\tilde{z} = (L_{sc}^j \tilde{z})_0^\infty$, $\tilde{z} \in \Lambda$, and for $q \geq 2$,

$$H_q(z) = \sum_{2 \leq l \leq q, 1 \leq i \leq l, 1 \leq r_i \leq l, r_1 + \dots + r_l = q} KA_{r_\delta}^{(l)}(\Phi(z))(\Psi_{\tilde{\eta}}^{(r_1)}(z), \dots, \Psi_{\tilde{\eta}}^{(r_l)}(z));$$

- (iii) $j_{\tilde{\eta}\eta} \circ \Psi_{\tilde{\eta}}^{(q)} : P \rightarrow \mathcal{L}^{(q)}(\Lambda, E_{\tilde{\eta}})$ being continuous.

We want to show that the above statement is true for $q = p + 1$.

Step 3. Fix $\tilde{\eta} \in [\eta^{p+1}, \tilde{\eta}]$ and let $X = \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})$. For $F \in \mathcal{L}^{(p)}(\Lambda, E_{\eta^p})$ and $z \in P$, let

$$H(F, z) = KA_{r_\delta}^{(1)}(\Phi(z))F + H_p(z).$$

By the induction hypotheses in Step 2 and the estimates in Step 1, for any $\eta^* \in [\eta^p, \tilde{\eta}]$, $F \in \mathcal{L}^{(p)}(\Lambda, E_{\eta^*})$, $z \in P$, we have $H(F, z) \in E_{\eta^*}$ and

$$|H(\tilde{F}, z) - H(F, z)| \leq c(\eta^*)\lambda(\delta)|\tilde{F} - F| \leq \kappa|\tilde{F} - F|, \quad \tilde{F}, F \in \mathcal{L}^{(p)}(\Lambda, E_{\eta^*}).$$

Therefore, $H(\cdot, z)$ has a unique fixed point in $\mathcal{L}^{(p)}(\Lambda, E_{\eta^*})$. Note also that for $\eta^* = \eta^p$ this fixed point is given by $\Psi_{\tilde{\eta}}^{(p)}(z)$. From now on, we restrict $H : X \times P \rightarrow X$ and let $N = \mathcal{L}^{(p)}(\Lambda, E_{\eta^p})$, $H_0 = H|_{N \times P}$.

Step 4. Let $e_j : E^{\mathbb{N}} \rightarrow E$ be given by

$$e_j((z_i)_0^\infty) = z_j, \quad (z_i)_0^\infty \in E^{\mathbb{N}}.$$

Define $\Phi_j = e_j \circ \Phi : P \rightarrow E$ and $\Psi_{\tilde{\eta}j}^{(l)}(z)\tilde{z} = e_j \circ \Psi_{\tilde{\eta}}^{(l)}(z)\tilde{z}$ for $1 \leq l \leq p, z \in P$, and $\tilde{z} \in \Lambda$. We claim that Φ_j is C^1 -smooth and $D\Phi_j(z)\tilde{z} = \Psi_{\tilde{\eta}j}^{(1)}(z)\tilde{z}$. In fact,

$\Phi_j = e_j \circ \Phi = e_j \circ j_{\tilde{\eta}\eta} \Phi$, and thus Φ_j is C^1 -smooth since $j_{\tilde{\eta}\eta} \circ \Phi$ is. Moreover, $D(j_{\tilde{\eta}\eta} \circ \Phi) = j_{\tilde{\eta}\eta} \circ \Psi_{\eta}^{(1)}$, and thus

$$e_j(j_{\tilde{\eta}\eta} \Psi_{\eta}^{(1)}(z)\tilde{z}) = e_j D(j_{\tilde{\eta}\eta} \circ \Phi)(z)\tilde{z}.$$

This shows that $\Psi_{\eta_j}^{(1)}(z)\tilde{z} = D\Phi_j(z)\tilde{z}$.

Step 5. We now prove that for any fixed $F \in \mathcal{L}^{(p)}(\Lambda, E_{\eta^p})$ and $\tilde{\eta} > \eta^{p+1}$, the mapping $P \ni z \mapsto KA_{r_\delta}^{(1)}(\Phi(z))F \in \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})$ has a derivative, which is given by $KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_{\eta}^{(1)}(z)\cdot, F)$, and the map

$$P \times \mathcal{L}^{(p)}(\Lambda, E_{\eta^p}) \ni (z, F) \mapsto KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_{\eta}^{(1)}(z)\cdot, F) \in \mathcal{L}(\Lambda, \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}}))$$

is continuous.

Let

$$|D^l r_\delta|_\infty = \sup \left\{ |D^l r_\delta(z)|; z \in E, |z_s| \leq \frac{\delta}{2} \right\}.$$

Note that for $1 \leq l \leq k$, $|D^l r_\delta|_\infty < \infty$.

For any $z_i \in \Lambda$ with $1 \leq i \leq p$, let

$$F_j(z_1, \dots, z_p) = e_j(F(z_1, \dots, z_p)).$$

Then for $\tilde{z}, z \in P$ we have

$$\begin{aligned} & \tilde{\eta}^{-j} |D^1 r_\delta(\Phi_j(\tilde{z}))F_j(z_1, \dots, z_p) - D^1 r_\delta(\Phi_j(z))F_j(z_1, \dots, z_p) \\ & \quad - D^2 r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z), F_j(z_1, \dots, z_p))| \\ & \leq \tilde{\eta}^{-j} |D^1 r_\delta(\Phi_j(\tilde{z})) - D^1 r_\delta(\Phi_j(z)) - D^2 r_\delta(\Phi_j(z))\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z)| \eta^{pj} |F| |z_1| \cdots |z_p|. \end{aligned}$$

Therefore, for any $\epsilon > 0$ there exists an integer $J_0 \geq 0$ so that if $j \geq J_0$ and if $|\tilde{z} - z| \leq 1$, then

$$\begin{aligned} & \tilde{\eta}^{-j} |D^1 r_\delta(\Phi_j(\tilde{z}))F_j(z_1, \dots, z_p) - D^1 r_\delta(\Phi_j(z))F_j(z_1, \dots, z_p) \\ & \quad - D^2 r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z), F_j(z_1, \dots, z_p))| \\ & \leq [(\tilde{\eta}\eta^{-p})^{-j} 2|D^1 r_\delta|_\infty |F| + (\tilde{\eta}\eta^{-p})^{-j} |D^2 r_\delta|_\infty \eta^j |\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z)| |F|] |z_1| \cdots |z_p| \\ & \leq \frac{\epsilon}{c(\tilde{\eta}) + 1} |z_1| \cdots |z_p|. \end{aligned}$$

As $r_\delta|_{\{x \in E; |x_s| < \delta\}}$ is C^k -smooth, $k \geq 2$, $\Phi_i : P \rightarrow E$ is C^1 -smooth and $D\Phi_j(z)\tilde{z} = \Psi_{\eta_j}^{(1)}(z)\tilde{z}$ for $z \in P$ and $\tilde{z} \in \Lambda$. For any $\epsilon > 0$, there exists $\delta > 0$ so that when $\tilde{z} \in P$ and $|\tilde{z} - z| < \delta$, then for $0 \leq j \leq J_0$ we have

$$|Dr_\delta(\Phi_j(\tilde{z})) - Dr_\delta(\Phi_j(z)) - D^2 r_\delta(\Phi_j(z))\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z)| < \frac{\tilde{\eta}^j \eta^{-pj}}{|F| + 1} \frac{\epsilon}{c(\tilde{\eta}) + 1},$$

and hence

$$\begin{aligned} & \tilde{\eta}^{-j} |D^1 r_\delta(\Phi_j(\tilde{z}))F_j(z_1, \dots, z_p) - D^1 r_\delta(\Phi_j(z))F_j(z_1, \dots, z_p) \\ & \quad - D^2 r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z), F_j(z_1, \dots, z_p))| \\ & < \tilde{\eta}^{-j} \frac{\tilde{\eta}^j \eta^{-pj}}{|F| + 1} \frac{\epsilon}{c(\tilde{\eta}) + 1} \eta^{pj} |F| |z_1| \cdots |z_p| \\ & \leq \frac{\epsilon}{c(\tilde{\eta}) + 1} |z_1| \cdots |z_p|. \end{aligned}$$

Therefore,

$$\begin{aligned} & |KA_{r_\delta}^{(1)}(\Phi(\tilde{z}))F - KA_{r_\delta}^{(1)}(\Phi(z))F - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)(\tilde{z} - z), F)| \\ & \leq c(\tilde{\eta}) \sup_{z_i \in \Lambda, |z_i| \leq 1, 1 \leq i \leq p, j \geq 0} \tilde{\eta}^{-j} | [D^1 r_\delta(\Phi_j(\tilde{z}))F_j(z_1, \dots, z_p) \\ & \quad - D^1 r_\delta(\Phi_j(z))F_j(z_1, \dots, z_p) - D^2 r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z), F_j(z_1, \dots, z_p))] | \\ & < c(\tilde{\eta}) \frac{\epsilon}{c(\tilde{\eta}) + 1} \leq \epsilon. \end{aligned}$$

This proves the differentiability.

We now prove that the map

$$P \times N \ni (z, F) \mapsto KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), F) \in \mathcal{L}(\Lambda, \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})) = \mathcal{L}^{(p+1)}(\Lambda, E_{\tilde{\eta}})$$

is continuous. Fix $(z, F) \in P \times N$. Then for any $(\tilde{z}, \tilde{F}) \in P \times N$, we have

$$\begin{aligned} & |KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z}), \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), F)| \\ & \leq |KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z}), \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), \tilde{F})| \\ & \quad + |KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), F)| \end{aligned}$$

and

$$\begin{aligned} & |KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z}), \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), \tilde{F})| \\ & = \sup_{z_i \in \Lambda, |z_i| \leq 1, 1 \leq i \leq p+1} |KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})z_{p+1}, \tilde{F}(z_1, \dots, z_p)) \\ & \quad - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)z_{p+1}, \tilde{F}(z_1, \dots, z_p))|_{E_{\tilde{\eta}}} \\ & = \sup_{z_i \in \Lambda, |z_i| \leq 1, 1 \leq i \leq p+1} |K_{\tilde{\eta}}[A_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})z_{p+1}, \tilde{F}(z_1, \dots, z_p)) \\ & \quad - A_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)z_{p+1}, \tilde{F}(z_1, \dots, z_p))]|_{E_{\tilde{\eta}}}. \end{aligned}$$

Moreover,

$$\begin{aligned} & |A_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})z_{p+1}, \tilde{F}(z_1, \dots, z_p)) - A_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)z_{p+1}, \tilde{F}(z_1, \dots, z_p))|_{E_{\tilde{\eta}}} \\ & = \sup_{j \in \mathbb{N}} \tilde{\eta}^{-j} |D^2 r_\delta(\Phi_j(\tilde{z}))(\Psi_{\eta_j}^{(1)}(\tilde{z})z_{p+1}, \tilde{F}_j(z_1, \dots, z_p)) \\ & \quad - D^2 r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)z_{p+1}, \tilde{F}_j(z_1, \dots, z_p))|. \end{aligned}$$

Note that for any $\eta^* \in (\eta, \tilde{\eta}]$, the mapping $j_{\eta^* \eta} \circ \Psi_\eta^{(1)} : P \rightarrow E_{\eta^*}$ is continuous. Fix $\eta^* \in (\eta, \frac{\tilde{\eta}}{\eta^p})$. There exists $\delta_1 > 0$ so that if $\tilde{z} \in P$ and $|\tilde{z} - z| < \delta_1$, then

$$|j_{\eta^* \eta} \circ \Psi_\eta^{(1)}(\tilde{z}) - j_{\eta^* \eta} \circ \Psi_\eta^{(1)}(z)| \leq 1.$$

Therefore, $\eta^{*-j} |\Psi_{\eta_j}^{(1)}(\tilde{z}) - \Psi_{\eta_j}^{(1)}(z)| \leq 1$ for all $j \in \mathbb{N}$. In particular, $|\Psi_{\eta_j}^{(1)}(\tilde{z}) - \Psi_{\eta_j}^{(1)}(z)| \leq \eta^{*j}$ for all $j \in \mathbb{N}$.

Find an integer $J_0 \geq 0$ so that if $j \geq J_0$, then

$$|D^2 r_\delta|_\infty \left(\frac{\tilde{\eta}}{\eta^p} \right)^{-j} [2\eta^j |\Psi_\eta^{(1)}(z)| + (\eta^*)^j] < \frac{\epsilon}{2(c(\eta^*) + 1)(|F| + 1)}.$$

Therefore, for $j \geq J_0$, we have

$$\begin{aligned} & \tilde{\eta}^{-j} |D^2 r_\delta(\Phi_j(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})z_{p+1}, \tilde{F}_j(z_1, \dots, z_p)) - D^2 r_\delta(\Phi_j(z))(\Psi_\eta^{(1)}(z)z_{p+1}, \tilde{F}_j(z_1, \dots, z_p))| \\ & \leq |D^2 r_\delta|_\infty \tilde{\eta}^{-j} [2|\Psi_\eta^{(1)}(z)| + \eta^{*j}] \eta^{pj} |\tilde{F}| |z_1| \cdots |z_p| |z_{p+1}| \\ & \leq |D^2 r_\delta|_\infty \left(\frac{\tilde{\eta}}{\eta^p} \right)^{-j} [2\eta^j |\Psi_\eta^{(1)}(z)| + \eta^{*j}] |\tilde{F}| |z_1| \cdots |z_{p+1}|. \end{aligned}$$

For $0 \leq j \leq J_0$, as $\Phi_j = e_j \Phi$ and $\Psi_{\eta_j}^{(1)} = e_j j \eta^* \eta \Psi_{\eta_j}^{(1)}$ are continuous, we can find $\delta_2 > 0$ so that when $\tilde{z} \in P$ and $|\tilde{z} - z| < \delta_2$, we have

$$\begin{aligned} & \tilde{\eta}^{-j} |D^2 r_\delta(\Phi_j(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})z_{p+1}, \tilde{F}_j(z_1, \dots, z_p)) - D^2 r_\delta(\Phi_j(z))(\Psi_\eta^{(1)}(z)z_{p+1}, \tilde{F}_j(z_1, \dots, z_p))| \\ & < \frac{\epsilon}{2(c(\tilde{\eta}) + 1)(|F| + 1)} |\tilde{F}| |z_1| \cdots |z_{p+1}|. \end{aligned}$$

Therefore, if $|\tilde{F} - F| \leq 1$ and $|\tilde{z} - z| < \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} & |KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z})\cdot, \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)\cdot, \tilde{F})| \\ & \leq c(\tilde{\eta}) \frac{\epsilon}{2(c(\tilde{\eta}) + 1)} < \frac{\epsilon}{2}. \end{aligned}$$

In a similar fashion, we get

$$\begin{aligned} & |KA_{r_\delta}^{(1)}(\Phi(z))(\Psi_\eta^{(1)}(z), \tilde{F}) - KA_{r_\delta}^{(1)}(\Phi(z))(\Psi_\eta^{(1)}(z), F)| \\ & = \sup_{z_i \in \Lambda, |z_i| \leq 1, 1 \leq i \leq p+1} |KA_{r_\delta}^{(2)}(\Psi_\eta^{(1)}(z)z_{p+1}, (\tilde{F} - F)(z_1, \dots, z_p))|_{E_{\tilde{\eta}}} \\ & \leq c(\tilde{\eta}) \sup_{z_i \in \Lambda, |z_i| \leq 1, 1 \leq i \leq p+1, j \geq 0} \tilde{\eta}^{-j} |D^2 r_\delta|_\infty \eta^{-j} |\Psi_\eta^{(1)}(z)| \eta^{-pj} |\tilde{F} - F| |z_1| \cdots |z_{p+1}| \\ & \leq c(\tilde{\eta}) |D^2 r_\delta|_\infty |\Psi_\eta^{(1)}(z)| |\tilde{F} - F|. \end{aligned}$$

Therefore, if $|\tilde{z} - z| < \min\{\delta_1, \delta_2\}$ and if $|\tilde{F} - F| < \min\{1, \frac{\epsilon}{2c(\tilde{\eta})|D^2 r_\delta|_\infty |\Psi_\eta^{(1)}(z)| + 1}\}$, then $|KA_{r_\delta}^{(1)}(\Phi(\tilde{z}))(\cdot, \tilde{F}) - KA_{r_\delta}^{(1)}(\Phi(z))(\cdot, F)| < \epsilon$. This completes the proof of the required continuity.

For the sake of later reference, let us summarize the main idea of the arguments involved in this step. To estimate

$$|KA_{r_\delta}^{(1)}(\Phi(\tilde{z}))F - KA_{r_\delta}^{(1)}(\Phi(z))F - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z)(\tilde{z} - z), F)|$$

in the proof of the differentiability of the mapping $P \ni z \mapsto KA_{r_\delta}^{(1)}(\Phi(z))F \in \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})$, we used the definition of the operator norm for multilinear operators $KA_{r_\delta}^{(1)}(\Phi(z))F$ and the definition of the norm in $E_{\tilde{\eta}}$ and were led to the estimation of the expression

$$\begin{aligned} & \tilde{\eta}^{-j} |[D^1 r_\delta(\Phi_j(\tilde{z}))F_j(z_1, \dots, z_p) - D^1 r_\delta(\Phi_j(z))F_j(z_1, \dots, z_p) \\ & \quad - D^2 r_\delta(\Phi_j(z))(\Psi_\eta^{(1)}(z)(\tilde{z} - z), F_j(z_1, \dots, z_p))| \end{aligned}$$

for each given nonnegative integer j . The above term can be made arbitrarily small if j is sufficiently large, thanks to the choice of $\tilde{\eta} > \eta^{p+1}$ (the essential gradient of the proof). When j is restricted to a finite set, the smallness of the above expression

follows from the continuity of the involved operators and mappings. Similar arguments were used to estimate

$$|KA_{r_\delta}^{(2)}(\Phi(\tilde{z}))(\Psi_\eta^{(1)}(\tilde{z}), \tilde{F}) - KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), F)|$$

in the proof of the continuity of the map

$$P \times N \ni (z, F) \mapsto KA_{r_\delta}^{(2)}(\Phi(z))(\Psi_\eta^{(1)}(z), F) \in \mathcal{L}(\Lambda, \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})).$$

Step 6. Let $2 \leq l \leq p, 1 \leq r_i < l$ with $r_1 + \dots + r_l = p$. For any integer $j \geq 0$, $z \in \Lambda$, and $\hat{z}_{r_i} \in \Lambda^{r_i}$, let

$$\Psi_{\eta_j}^{(r_i)}(z)\hat{z}_{r_i} = e_j(\Psi_\eta^{(r_i)}(z)\hat{z}_{r_i}).$$

Then for $z, \tilde{z} \in \Lambda$ we have

$$\begin{aligned} & \left| KA_{r_\delta}^{(l)}(\Phi(\tilde{z}))(\Psi_\eta^{(r_1)}(\tilde{z}), \dots, \Psi_\eta^{(r_l)}(\tilde{z})) - KA_{r_\delta}^{(l)}(\Phi(z))(\Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_l)}(z)) \right. \\ & \quad - \sum_{k=1}^l KA_{r_\delta}^{(l)}(\Phi(z))(\Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_{k+1})}(z)(\tilde{z} - z), \dots, \Psi_\eta^{(r_l)}(z)) \\ & \quad \left. - KA_{r_\delta}^{(l+1)}(\Phi(z))(\Psi_\eta^{(1)}(z)(\tilde{z} - z), \Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_l)}(z)) \right| \\ & \leq c(\tilde{\eta}) \sup_{\hat{z}_{r_i} \in \Lambda^{r_i}, |z_{r_i}| \leq 1, 1 \leq i \leq p, j \geq 0} \tilde{\eta}^{-j} \left| D^l r_\delta(\Phi_j(\tilde{z}))(\Psi_{\eta_j}^{(r_1)}(\tilde{z})\hat{z}_{r_1}, \dots, \Psi_{\eta_j}^{(r_l)}(\tilde{z})\hat{z}_{r_l}) \right. \\ & \quad - D^l r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(r_1)}(z)\hat{z}_{r_1}, \dots, \Psi_{\eta_j}^{(r_l)}(z)\hat{z}_{r_l}) \\ & \quad - \sum_{k=1}^l D^l r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(r_1)}(z)\hat{z}_{r_1}, \dots, \Psi_{\eta_j}^{(r_{k+1})}(z)(\tilde{z} - z, \hat{z}_{r_k}), \dots, \Psi_{\eta_j}^{(r_l)}(z)\hat{z}_{r_l}) \\ & \quad \left. - D^{l+1} r_\delta(\Phi_j(z))(\Psi_{\eta_j}^{(1)}(z)(\tilde{z} - z), \Psi_{\eta_j}^{(r_1)}(z)\hat{z}_{r_1}, \dots, \Psi_{\eta_j}^{(r_l)}(z)\hat{z}_{r_l}) \right|. \end{aligned}$$

Now we can use the fact that $|D^l r_\delta|_\infty < \infty$ for $1 \leq l \leq p$, and the induction hypothesis implies that the mapping

$$P \ni z \mapsto \Psi_{\eta_j}^{(r_i)}(z) \in \mathcal{L}^{(r_i)}(\Lambda, E_{\eta^{r_i}})$$

is differentiable, and we apply an argument similar to that for the first part of Step 5 to show that for any $2 \leq l \leq p, 1 \leq r_i < l$ with $r_1 + \dots + r_l = p$, the map $P \ni z \mapsto KA_{r_\delta}^{(l)}(\Phi(z))(\Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_l)}(z)) \in \mathcal{L}^{(p)}(\Lambda, E_{\tilde{\eta}})$ is differentiable and the derivative is given by

$$\begin{aligned} & \sum_{j=1}^l KA_{r_\delta}^{(l)}(\Phi(z))(\Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_{j+1})}(z), \dots, \Psi_\eta^{(r_l)}(z)) \\ & \quad + KA_{r_\delta}^{(l+1)}(\Phi(z))(\Psi_\eta^{(1)}(z), \Psi_\eta^{(r_1)}(z), \dots, \Psi_\eta^{(r_l)}(z)). \end{aligned}$$

The continuity of the above derivative, with respect to $z \in P$, can also be verified by using an argument similar to that for the second part of Step 5 and by noting that the induction hypothesis implies that the mapping

$$P \ni z \mapsto \Psi_{\eta_j}^{(r_i+1)}(z) \in \mathcal{L}^{(r_i+1)}(\Lambda, E_{\eta^{r_i}})$$

is continuous.

Step 7. Let $\hat{\eta}$ be given so that $\hat{\eta} \in (\tilde{\eta}, \bar{\eta}]$. Define the continuous linear injective map $J : X \rightarrow X_1 = \mathcal{L}^{(p)}(\Lambda, E_{\hat{\eta}})$ by

$$J(L)(z_1, \dots, z_p) = j_{\hat{\eta}\tilde{\eta}}L(z_1, \dots, z_p), \quad z_1, \dots, z_p \in \Lambda, \quad L \in X.$$

Then

$$JH_0(F, z) = j_{\hat{\eta}\tilde{\eta}}KA_{r_\delta}^{(1)}(\Phi(z))F + j_{\hat{\eta}\eta}H_p(z), \quad z \in P, \quad F \in \mathcal{L}^{(p)}(\Lambda, E_{\eta^p}).$$

Let $A : P \rightarrow \mathcal{L}(X, X_1)$ be given by

$$(A(z)F)(z_1, \dots, z_p) = j_{\hat{\eta}\tilde{\eta}}K \circ A_{r_\delta}(\Phi(z))F(z_1, \dots, z_p), \quad z \in P, \quad F \in X, \quad x_1, \dots, x_p \in \Lambda.$$

Again, we can use arguments similar to those in Step 5 (see the remarks at the end of Step 5) to show that A is continuous. Moreover, we have

$$JH_0(\tilde{F}, z) - JH_0(F, z) = A(z)(\tilde{F} - F), \quad z \in P, \quad \tilde{F}, F \in N.$$

Note that for any $\eta^* \geq \eta$, $KA_{r_\delta}^{(1)}(\Phi(z))$ induces a bounded linear map from $\mathcal{L}^{(p)}(\Lambda, E_{\eta^*})$ into itself by

$$Q_{\eta^*}(L)(z_1, \dots, z_p) = K_{\eta^*}A_{r_\delta \eta^* \eta^*}^{(1)}(\Phi(z))L(z_1, \dots, z_p)$$

and

$$|Q_{\eta^*}| \leq c(\eta^*)\lambda(\delta).$$

Define $H^{(1)} : P \rightarrow \mathcal{L}(X, X)$ and $H_1^{(1)} : P \rightarrow \mathcal{L}(X_1, X_1)$ by

$$H^{(1)}(z) = Q_{\hat{\eta}}, \quad H_1^{(1)}(z) = Q_{\hat{\eta}}, \quad z \in P.$$

Clearly, we have for $F \in X$ the following:

$$\begin{aligned} A(z)F &= j_{\hat{\eta}\tilde{\eta}}KA_{r_\delta}^{(1)}(\Phi(z))F \\ &= j_{\hat{\eta}\tilde{\eta}}Q_{\hat{\eta}}F = JH_1^{(1)}(z)F \\ &= Q_{\hat{\eta}}j_{\hat{\eta}\tilde{\eta}}F = H_1^{(1)}(z)JF \end{aligned}$$

and

$$|H^{(1)}(z)| \leq c(\tilde{\eta})\lambda(\delta) \leq \kappa, \quad |H_1^{(1)}(z)| \leq c(\hat{\eta})\lambda(\delta) \leq \kappa.$$

Moreover, the mapping

$$P \ni z \mapsto J \circ H^{(1)}(z) = j_{\hat{\eta}\tilde{\eta}} \circ Q_{\hat{\eta}} = J_{\hat{\eta}\tilde{\eta}}K_{\hat{\eta}}A_{r_\delta \hat{\eta}\tilde{\eta}}^{(1)}(\Phi(z)) = A \in \mathcal{L}(X, X_1)$$

is continuous. Therefore, by Lemma 5.2, the map $j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)} = j_{\hat{\eta}\eta} \circ j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)} : P \rightarrow X_1$ is C^1 -smooth and

$$D(j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)})(z) = K A_{r_{\delta}}^{(1)}(\Phi(z)) D(j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)}) + j_{\hat{\eta}\eta} \circ D_2 H_0(\Psi_{\eta}^{(p)}, z), \quad z \in P.$$

Step 8. We now prove that the mapping $j_{\hat{\eta}\eta} \circ \Phi : P \rightarrow E_{\hat{\eta}}$ is C^{p+1} -smooth. Indeed, as $\hat{\eta} > \eta^{p+1} > \eta^p$, $j_{\hat{\eta}\eta} \circ \Phi : P \rightarrow E_{\hat{\eta}}$ is C^p -smooth and

$$D^p(j_{\hat{\eta}\eta} \circ \Phi) = j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)}.$$

Since $j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)}$ is C^1 -smooth, we conclude that $j_{\hat{\eta}\eta} \circ \Phi$ is C^{p+1} -smooth and $D^{p+1}(j_{\hat{\eta}\eta} \circ \Phi) = D(j_{\hat{\eta}\eta} \circ \Psi_{\eta}^{(p)})$. Let $H_{p+1}(z) = D_2 H_0(\Psi_{\eta}^p(z), z)$ and let $\Psi_{\eta}^{(p+1)}(z)$ be the unique fixed point of the contraction

$$\mathcal{L}^{(p+1)}(\Lambda, E_{\eta^{p+1}}) \ni F \mapsto K_{\eta^{p+1}} A_{r_{\delta} \eta^{p+1} \eta^{p+1}}^{(1)}(\Phi(z)) F + H_{p+1}(z) \in \mathcal{L}^{(p+1)}(\Lambda, E_{\eta^{p+1}});$$

then $D^{p+1}(j_{\hat{\eta}\eta} \circ \Phi) = j_{\hat{\eta}\eta} \Psi_{\eta}^{(p+1)}$. This proves all conclusions in the case of $p + 1$.

Therefore, we have proved that for a fixed $\tilde{\eta} > \eta^k$ the mapping $j_{\tilde{\eta}\eta} \circ \Phi : P \rightarrow E_{\eta^k}$ is C^k -smooth, and hence $\chi_{\delta_{\tilde{\eta}}}|_P = j_{\tilde{\eta}\eta} \Phi$ is C^k smooth. Consequently, $w_{\delta_{\tilde{\eta}}}|_P = P_u \circ e_0 \chi_{\delta_{\tilde{\eta}}}|_P$ is C^k -smooth. \square

Similarly, we have the following center-unstable manifold theorem.

THEOREM 5.3. *Let $f : U \rightarrow E$ be a C^1 -map on an open subset U of a Banach space E over \mathbb{R} , with a fixed point p . Let $L = Df(p)$ and assume that E has the following decomposition:*

$$E = E_s \oplus E_c \oplus E_u,$$

where E_s is a closed subspace, E_c and E_u are finite-dimensional, $L(E_s) \subset E_s$, $L(E_c) \subset E_c$, and $L(E_u) \subset E_u$. We further assume that

$$\sigma_s = \sigma(L|_{E_s} : E_s \rightarrow E_s) \text{ is contained in a compact subset of } \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\begin{aligned} \sigma_c &= \sigma(L|_{E_c} : E_c \rightarrow E_c) \subset S_{\mathbb{C}}^1, \\ \sigma_u &= \sigma(L|_{E_u} : E_u \rightarrow E_u) \subset \{z \in \mathbb{C} : |z| > 1\}. \end{aligned}$$

Let $E_{cu} = E_u \oplus E_c$. Then

(i) *there exist open neighborhoods N_{cu} of 0 in E_{cu} , N_s of 0 in E_s , N of p in U , and a C^1 -map $w : N_{cu} \rightarrow E_s$ with $w(0) = 0$, $Dw(0) = 0$, and $w(N_{cu}) \subset N_s$ so that the shifted graph $W = p + \{z + w(z) : z \in N_{cu}\}$ satisfies $f(W \cap N) \subset W$ and $\{x \in E; \text{ there exists a trajectory } (x_n)_{-\infty}^0 \text{ of } f \text{ in } p + N_{cu} + N_s \text{ with } x_0 = x\} \subset W$;*

(ii) *if f is C^k -smooth for an integer $k \geq 2$, then so is w .*

We can now state the following smoothness theorem for center manifolds in general Banach spaces.

THEOREM 5.4. *Let $f : U \rightarrow E$ be a C^1 -map on an open subset U of a Banach space E over \mathbb{R} , with a fixed point p . Let $L = Df(p)$ and assume that E has the following decomposition:*

$$E = E_s \oplus E_c \oplus E_u,$$

where E_s is a closed subspace, E_c and E_u are finite-dimensional, $L(E_s) \subset E_s$, $L(E_c) \subset E_c$, and $L(E_u) \subset E_u$. We further assume that

$$\sigma_s = \sigma(L|_{E_s} : E_s \rightarrow E_s) \text{ is contained in a compact subset of } \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\begin{aligned} \sigma_c &= \sigma(L|_{E_c} : E_c \rightarrow E_c) \subset S_{\mathbb{C}}^1, \\ \sigma_u &= \sigma(L|_{E_u} : E_u \rightarrow E_u) \subset \{z \in \mathbb{C} : |z| > 1\}. \end{aligned}$$

Let $E_{su} = E_s \oplus E_c$. Then

(i) there exist open neighborhoods N_c of 0 in E_c , N_{su} of 0 in E_{su} , N of p in U , and a C^1 -map $w : N_c \rightarrow E_{su}$ with $w(0) = 0$, $Dw(0) = 0$, and $w(N_c) \subset N_{su}$ so that the shifted graph $W = p + \{z + w(z) : z \in N_c\}$ satisfies $f(W \cap N) \subset W$, and if there exists $(x_n)_{-\infty}^{\infty}$ such that $x_n = f(x_{n-1})$ and $x_n \in p + N_c + N_{su}$ for every integer n , then $x_0 \in W$;

(ii) if f is C^k -smooth for an integer $k \geq 2$, then so is w .

Proof. Without loss of generality, we may assume $p = 0$. By Theorem 5.1, there exist convex open neighborhoods \tilde{N}_{cs} of 0 in $E_c + E_s$, \tilde{N}_u of 0 in E_u , \tilde{N} of 0 in U , and a C^k -map ($k = 1$ in case of (i) and $k \geq 2$ in case of (ii)) $\tilde{w}_{cs} : \tilde{N}_{cs} \rightarrow E_u$ with

$$\begin{aligned} \tilde{w}_{cs}(0) &= 0, & D\tilde{w}_{cs}(0) &= 0; \\ \tilde{w}_{cs}(\tilde{N}_{cs}) &\subset \tilde{N}_u, \end{aligned}$$

and such that the graph

$$\tilde{W}_{cs} = \{z_{cs} + \tilde{w}_{cs}(z_{cs}) : z_{cs} \in \tilde{N}_{cs}\}$$

satisfies

$$f(\tilde{W}_{cs} \cap \tilde{N}) \subset \tilde{W}_{cs}$$

and

$$(5.3) \quad \bigcap_{n=0}^{\infty} f^{-n}(\tilde{N}_{cs} + \tilde{N}_u) \subset \tilde{W}_{cs}.$$

By Theorem 5.3, there exist open neighborhoods \hat{N}_{cu} of 0 in $E_c \oplus E_u$, \hat{N}_s of 0 in E_s , \hat{N} of 0 in U , and a C^k -map ($k = 1$ in case (i) and $k \geq 2$ in case (ii)) $\hat{w}_{cu} : \hat{N}_{cu} \rightarrow E_s$ with

$$\begin{aligned} \hat{w}_{cu}(0) &= 0, & D\hat{w}_{cu}(0) &= 0; \\ \hat{w}_{cu}(\hat{N}_{cu}) &\subset \hat{N}_s, \end{aligned}$$

and the graph

$$\hat{W}_{cu} = \{z_{cu} + \hat{w}_{cu}(z_{cu}) : z_{cu} \in \hat{N}_{cu}\}$$

satisfies

$$f(\hat{W}_{cu} \cap \hat{N}) \subset \hat{W}_{cu}$$

and

$$(5.4) \quad z \in \hat{W}_{cu} \text{ if there exists } \{z_n\}_{n=-\infty}^0 \subset \hat{N}_{cu} + \hat{N}_s \text{ such that } z_{n+1} = f(z_n) \text{ for } n \leq -1 \text{ and that } z_0 = z.$$

Choose open neighborhoods N_c^* of 0 in E_c , N_s^* of 0 in E_s , N_c^* of 0 in E_c , N^* of 0 in E such that

$$\begin{cases} N^* \subset \widehat{N} \cap \widetilde{N}; \\ N_c^* + N_s^* \subset \widetilde{N}_{cs}; \\ N_c^* + N_u^* \subset \widehat{N}_{cu}; \\ z_c \in N_c^*, z_s \in N_s^* \text{ if } z \in f(N^*); \\ z_c \in N_c^*, z_u \in N_u^* \text{ if } z \in f(N^*); \\ \widetilde{w}_{cs}(z_c + z_s) \in N_u^* \text{ if } z_c \in N_c^* \text{ and } z_s \in N_s^*. \end{cases}$$

Define

$$\begin{aligned} W_{cs}^* &= \{z_{cs} + \widetilde{w}_{cs}(z_{cs}) : z_{cs} = z_c + z_s \in N_c^* + N_s^*\}, \\ W_{cu}^* &= \{z_{cu} + \widehat{w}_{cu}(z_{cu}) : z_{cu} = z_c + z_u \in N_c^* + N_u^*\}, \end{aligned}$$

and

$$W^* = W_{cs}^* \cap W_{cu}^*.$$

For $z \in W^*$, we have

$$\begin{aligned} z &= z_c + z_s + \widetilde{w}_{cs}(z_c + z_s) \\ &= z_c + z_u + \widehat{w}_{cu}(z_c + z_u) \end{aligned}$$

with $z_c \in N_c^*$, $z_s \in N_s^*$, and $z_u \in N_u^*$. Therefore,

$$z_s = \widehat{w}_{cu}(z_c + z_u) = \widehat{w}_{cu}(z_c + \widetilde{w}_{cs}(z_c + z_s)).$$

Consider the equation

$$(5.5) \quad z_s = \widehat{w}_{cu}(z_c + \widetilde{w}_{cs}(z_c + z_s)).$$

As both \widehat{w}_{cu} and \widetilde{w}_{cs} are C^k -smooth and $D\widehat{w}_{cu}(0) = 0$, $D\widetilde{w}_{cs}(0) = 0$, the implicit function theorem implies that there are open neighborhoods N_c of 0 in N_c^* and N_s of 0 in N_s^* and a C^k -map $w_s : N_c \rightarrow N_s$ such that for every $z_c \in N_c$ equation (5.5) has the unique solution $z_s = w_s(z_c)$. It is easy to verify that $w_s(0) = 0$ and $Dw_s(0) = 0$.

We now define $w_c : N_c \rightarrow E_s \oplus E_u$ by

$$w_c(z_c) = w_s(z_c) + \widetilde{w}_{cs}(z_c + w_s(z_c)), \quad z_c \in N_c.$$

Clearly, w_c is C^k -smooth, $w_c(0) = 0$, $Dw_c(0) = 0$, and

$$w_c(N_c) \subset N_s + N_u$$

with

$$N_u = N_u^*.$$

Let

$$W_c = \{z_c + w_c(z_c) : z_c \in N_c\}.$$

We prove that if there exists $\{z_n\}_{n=-\infty}^{\infty} \subset N_c + N_s + N_u$ such that $z_{n+1} = f(z_n)$ for $n \in \mathbf{Z}$, then $z = z_0 \in W_c$. In fact, (5.3) and (5.4) imply that $z \in \widehat{W}_{cu} \cap \widetilde{W}_{cs}$. As $z_c \in N_c \subset N_c^*$, $z_s \in N_s \subset N_s^*$, and $z_u \in N_u = N_u^*$, we have $z \in W^*$ and

$$z_0 = z_c + z_s + \widetilde{w}_{cs}(z_c + z_s) = z_c + z_u + \widehat{w}_{cu}(z_c + z_u),$$

from which it follows that

$$z_s = \widehat{w}_{cu}(z_c + \widetilde{w}_{cs}(z_c + z_s)), \quad z_s \in N_s, \quad z_c \in N_c.$$

Therefore, we must have $z_s = w_s(z_c)$ and $z_u = \widetilde{w}_{cs}(z_c + w_s(z_c))$. This shows that $z \in W_c$.

Other properties in Theorem 5.4 are straightforward consequences of Theorems 5.1 and 5.3. \square

6. Center manifolds for nonlinear FDEs in Banach spaces. We now start to consider semilinear FDEs

$$(6.1) \quad \dot{u}(t) = A_T u(t) + L(u_t) + F(u_t),$$

where we assume A_T, L are as in the previous sections, and, in particular, that (H1)–(H3) are satisfied. We also assume that $F : V_1 \rightarrow X$ is a C^k -mapping ($k \geq 1$) from a neighborhood V_1 of $0 \in C$ into X with $F(0) = 0$ and $DF(0) = 0$.

Fix $\omega > r$. Using the arguments of Fitzgibbon [6] (see also Theorems 2.1 and 2.2 in Chapter 2 of Wu [21]), we can find an open neighborhood $V_2 \subset V_1$ of 0 in C such that for any $\phi \in V_2$ there exists a unique continuous function $u^\phi : [-r, \omega] \rightarrow X$ such that $u_0^\phi = \phi$ and

$$u^\phi(t) = T(t)\phi(0) + \int_0^t T(t-s)[L(u_s^\phi) + F(u_s^\phi)]ds$$

for $t \in [0, \omega]$. Define $\tilde{f} : V_2 \rightarrow C$ by

$$\tilde{f}(\phi) = u_\omega^\phi \quad \text{for } \phi \in V_2.$$

As $\omega > r$, we can show that \tilde{f} is compact (using the argument in Travis and Webb [18]; see also Theorem 1.8 of Chapter 2 of Wu [21]). The next lemma shows that there exists an open neighborhood $V \subset V_2$ of 0 in C such that $f = \tilde{f}|_V : V \rightarrow C$ is C^k -smooth and

$$Df(0) = U(\omega) : C \rightarrow C.$$

LEMMA 6.1. *There exists an open neighborhood $V \subset V_2$ of 0 in C such that for each $t \in [0, \omega]$, u_t^ϕ is C^k -smooth with respect to $\phi \in V$. Moreover, for each $\psi \in C$, $D_\phi u^\phi(t)\psi$ satisfies the linear variational equation*

$$(6.2) \quad \begin{cases} v(t) = T(t)\psi(0) + \int_0^t T(t-s)[L(v_s) + DF(u_s^\phi)v_s]ds, \\ v_0 = \psi. \end{cases}$$

In particular, $Df(0) = U(\omega)$.

Proof. We are going to apply the same argument as that for Theorem 4.1 in Hale [8] based on [8, Lemma 4.2, p. 46]. Let $\hat{F}(\phi) = L(\phi) + F(\phi)$. Fix $\chi \in V_2$. There exist $M > 0$, $\delta > 0$, and $N > 0$ such that

$$\begin{cases} \|T(t)\| \leq M & \text{for } t \in [0, 1]; \\ \overline{B_\delta(\chi)} \subset V_2 & \text{with } B_\delta(\chi) = \{\psi \in C : \|\psi - \chi\| < \delta\}; \\ |\hat{F}(\psi)| \leq N, \quad |D\hat{F}(\psi)| \leq N & \text{for } \psi \in \overline{B_\delta(\chi)}. \end{cases}$$

Now choose $\epsilon \in (0, 1)$ and $\beta \in (0, 1)$ so that

$$\begin{cases} \beta < \frac{\delta}{2}; \\ \sup_{\theta, \theta' \in [-r, 0], |\theta - \theta'| \leq \epsilon} |\chi(\theta) - \chi(\theta')| < \frac{\delta}{8}; \\ \sup_{t \in [0, \epsilon]} |T(t)\chi(0) - \chi(0)| < \frac{\delta}{8}; \\ \epsilon < \frac{\beta}{MN}. \end{cases}$$

Let

$$K(\epsilon, \beta) = \{y \in C([-r, \epsilon]; X) : y_0 = 0, \|y_t\| \leq \beta \text{ for } t \in [0, \epsilon]\}.$$

Clearly, $K(\epsilon, \beta)$ is a closed subset of the Banach space $C_0([-r, \epsilon]) = \{z \in C([-r, \epsilon]; X) : z(s) = 0 \text{ for } s \in [-r, 0]\}$ equipped with the supremum norm.

For each $\phi \in C$, define $\tilde{\phi} : [-r, \infty) \rightarrow X$ by $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}(t) = T(t)\phi(0)$ for $t \geq 0$. Now, for fixed $\phi \in B_{\frac{\delta}{8(1+M)}}(\chi)$ define $A(\phi)$ on $K(\epsilon, \beta)$ by

$$A(\phi)y(t) = \begin{cases} \int_0^t T(t-s)\hat{F}(y_s + \tilde{\phi}_s)ds, & y \in K(\epsilon, \beta), \quad t \in [0, \epsilon]; \\ 0, & t \in [-r, 0]. \end{cases}$$

Clearly, $A(\phi)y \in C([-r, \epsilon]; X)$. Moreover, since for $s \in [0, \epsilon]$, $\|y_s\| \leq \beta$, and

$$\begin{aligned} \|\tilde{\phi}_s - \chi\| &\leq \|\tilde{\phi}_s - \tilde{\chi}_s\| + \|\tilde{\chi}_s - \chi\| \\ &\leq \|\phi - \chi\| + \sup_{s \in [0, \epsilon]} \|T(s)\| \|\phi(0) - \chi(0)\| \\ &\quad + \sup_{\theta \in [-r, 0], s \in [0, \epsilon], s+\theta \in [-r, 0]} |\chi(\theta + s) - \chi(\theta)| \\ &\quad + \sup_{\theta \in [-r, 0], s \in [0, \epsilon], s+\theta \geq 0} |T(s+\theta)\chi(0) - \chi(0)| \\ &\quad + \sup_{\theta \in [-r, 0], s \in [0, \epsilon], s+\theta \geq 0} |\chi(\theta) - \chi(0)| \\ &\leq (1+M)\|\phi - \chi\| + \frac{\delta}{8} + \frac{\delta}{8} + \frac{\delta}{8} < \frac{\delta}{2}, \end{aligned}$$

we have

$$\|y_s + \tilde{\phi}_s - \chi\| < \beta + \frac{\delta}{2} < \delta,$$

and hence

$$|\hat{F}(y_s + \tilde{\phi}_s)| \leq N \quad \text{for } s \in [0, \epsilon].$$

This implies that

$$|A(\phi)y(t)| \leq MN\epsilon < \beta \quad \text{for } t \in [0, \epsilon].$$

So, $A(\phi)y \in K(\epsilon, \beta)$ and $A(\phi)K(\epsilon, \beta) \subset K(\epsilon, \beta)$.

Moreover, using $\|D\hat{F}(\psi)\| \leq N$ for all $\psi \in \overline{B_\delta(\chi)}$, for $y, \hat{y} \in K(\epsilon, \beta)$ and $t \in [0, \epsilon]$ we have

$$\begin{aligned} & |A(\phi)y(t) - A(\phi)\hat{y}(t)| \\ & \leq \left| \int_0^t T(t-s)[\hat{F}(y_s + \tilde{\phi}_s) - \hat{F}(\hat{y}_s + \tilde{\phi}_s)] ds \right| \\ & \leq MN\epsilon \sup_{s \in [0, t]} \|y_s - \hat{y}_s\| \\ & \leq MN\epsilon \sup_{s \in [-r, \epsilon]} |y(s) - \hat{y}(s)| \\ & \leq \beta \sup_{s \in [-r, \epsilon]} |y(s) - \hat{y}(s)|. \end{aligned}$$

As $\beta < 1$, we conclude that for each $\phi \in \overline{B_{\frac{\delta}{8(1+M)}}(\chi)}$, the mapping $A(\phi) : K(\epsilon, \beta) \rightarrow K(\epsilon, \beta)$ is a contraction. By Lemma 4.2 of Hale [8], for each fixed $\phi \in \overline{B_{\frac{\delta}{8(1+M)}}(\chi)}$, $A(\phi)$ has a unique fixed point $y(\phi) \in K(\epsilon, \beta)$ which is continuous in ϕ .

Note that $\overline{B_{\frac{\delta}{8(1+M)}}(\chi)}$ is the closure of the open set $B_{\frac{\delta}{8(1+M)}}(\chi)$ and $A(\phi)y$ has a continuous k th derivative with respect to $(\phi, y) \in B_{\frac{\delta}{8(1+M)}}(\chi) \times K^0(\epsilon, \beta)$, where

$$K^0(\epsilon, \beta) = \{y \in K(\epsilon, \beta) : \|y_t\| < \beta \text{ for } t \in [0, \epsilon]\}$$

is open in $C_0([-r, \epsilon])$ and $K(\epsilon, \beta) = \overline{K^0(\epsilon, \beta)}$. Therefore, by Lemma 4.2 in Hale [8], $y(\phi)$ is C^k -smooth with respect to $\phi \in \overline{B_{\frac{\delta}{8(1+M)}}(\chi)}$, and hence $u_t^\phi = \tilde{\phi}_t + (y(\phi))_t$ is C^k -smooth in $\phi \in \overline{B_{\frac{\delta}{8(1+M)}}(\chi)}$ for each fixed $t \in [0, \epsilon]$. A standard continuation argument then leads to the C^k -smoothness of $u(\phi)$ with respect to ϕ for $t \in [0, \omega]$. The remaining part of the lemma can be easily verified. \square

Let

$$\begin{aligned} \Sigma_s &= \{\lambda \in \sigma_P(A_U) : \operatorname{Re} \lambda < 0\}, \\ \Sigma_u &= \{\lambda \in \sigma_P(A_U) : \operatorname{Re} \lambda > 0\}, \\ \Sigma_c &= \{\lambda \in \sigma_P(A_U) : \operatorname{Re} \lambda = 0\}, \end{aligned}$$

and assume $\Sigma_c \neq \emptyset$. We know that $\Sigma_c \cup \Sigma_u$ is a finite set.

Let

$$\begin{aligned} C^s &= \bigoplus_{\lambda \in \Sigma_s} \mathcal{M}_\lambda(A_U), \\ C^u &= \bigoplus_{\lambda \in \Sigma_u} \mathcal{M}_\lambda(A_U), \\ C^c &= \bigoplus_{\lambda \in \Sigma_c} \mathcal{M}_\lambda(A_U). \end{aligned}$$

C^s , C^u , and C^c are realified generalized eigenspaces associated with Σ_s , Σ_u , and Σ_c , respectively. Then C^u and C^c are finitely dimensional and

$$C = C^s \oplus C^u \oplus C^c.$$

Recall that C^s , C^u , and C^c are called the *stable*, *unstable*, and *center* subspaces of the C_0 -semigroup $\{U(t)\}_{t \geq 0}$.

We can now state the main result of this section.

THEOREM 6.2. *There exist open neighborhoods N_c of 0 in C^c , N_s of 0 in C^s , N_u of 0 in C^u , and a C^k -map $w_c : N_c \rightarrow C^s \oplus C^u$ such that*

- (i) $w_c(0) = 0, Dw_c(0) = 0, w_c(N_c) \subset N_s + N_u$;
- (ii) *for any $\phi \in V$, if there exists a continuous mapping $u^\phi : \mathbb{R} \rightarrow X$ such that $u_0^\phi = \phi$,*

$$u^\phi(t) = T(t-s)u^\phi(s) + \int_s^t T(t-\theta)[L(u_\theta^\phi) + F(u_\theta^\phi)]d\theta$$

for $t, s \in \mathbb{R}$ with $t \geq s$, and $u_t^\phi \in N_s + N_u + N_c$ for all $t \in \mathbb{R}$, then $u_t^\phi \in W_c$ for $t \in \mathbb{R}$, where

$$W_c = \{\phi_c + w_c(\phi_c) : \phi_c \in N_c\}.$$

Proof. Recall that $f : V \rightarrow C$ is C^k -smooth, $f(0) = 0$, $Df(0) = U(\omega)$, and

$$\begin{cases} C = C^s \oplus C^u \oplus C^c, \\ U(\omega)C^s \subset C^s, U(\omega)C^u \subset C^u, U(\omega)C^c \subset C^c, \\ \sigma(U(\omega)|_{C^s}) \text{ is a compact subset of } \{z \in \mathbb{C} : |z| < 1\}, \\ \sigma(U(\omega)|_{C^c}) \subset S_{\mathbb{C}}^1, \\ \sigma(U(\omega)|_{C^u}) \subset \{z \in \mathbb{C} : |z| > 1\}. \end{cases}$$

See Chapter IV.2 in Diekmann et al. [4].

By Theorem 5.4, there exist open neighborhoods N_c of 0 in C^c , N_s of 0 in C^s , N_u of 0 in C^u , and a C^k -map $w : N_c \rightarrow C^s \oplus C^u$ such that $w_c(0) = 0$, $Dw_c(0) = 0$, and $w_c(N_c) \subset N_s + N_u$. Moreover, for $W_c = \{\phi_c + w_c(\phi_c) : \phi_c \in N_c\}$, if there exists $(\phi^n)_{-\infty}^\infty$ such that $\phi^n = f(\phi^{n-1})$ and $\phi^n \in N_c + N_{su}$ for $n \in \mathbf{Z}$, then $\phi^0 \in W_c$.

Fix $\phi \in V$ such that condition (ii) of this theorem is satisfied. Then for any fixed $t \in \mathbb{R}$, $u_t^\phi \in N_s + N_u + N_c \subset V$, and if we let

$$\phi^n = u_{t+n\omega}^\phi, \quad n \in \mathbf{Z},$$

then $\phi^{n+1} = f(\phi^n)$ for $n \in \mathbf{Z}$ and $\phi^n \in N_c + N_s + N_u$ for all $n \in \mathbf{Z}$. Therefore, the result in the last step implies that $\phi^0 = u_t^\phi \in W_c$. This completes the proof. \square

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