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Traveling Wave Solutions for Planar Lattice Differential Systems with Applications to Neural Networks

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We obtain some existence results for traveling wave fronts and slowly oscillatory spatially periodic traveling waves of planar lattice differential systems with delay. Our approach is via Schauder's fixed-point theorem for the existence of traveling wave fronts and via S^1-degree and equivarant bifurcation theory for the existence of periodic traveling waves. As examples, the obtained abstract results will be applied to a model arising from neural networks and explicit conditions for traveling wave fronts and global continuation of periodic waves will be obtained. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Recently, lattice differential equations have found a considerable amount of interest and attracted many researchers. The reasons seem to be two-fold. Practically, lattice differential equations have been proposed as models in various contexts. Theoretically, many lattice differential equations can be viewed as the discretization of reaction–diffusion equations along a lattice, but have exhibited much more complicated and colorful dynamics. We refer to the excellent survey papers of Chow and Mallet-Paret [2] for the detailed account of the theory and applications of lattice differential equations.

Among the various important aspects of lattice differential equations is the phenomenon of traveling wave solutions. There have been a few research papers working on the existence of traveling wavefront solutions and periodic traveling waves of systems of lattice differential equations (cf. [2, 6, 7, 23–25]), and it has been shown that the behavior of a system of lattice differential equations can be remarkably different from its continuous version. For example, it is well known that if a reaction–diffusion equation has traveling wave front solutions for some value of the diffusion coefficient d, it does so for all values of $d > 0$, but the corresponding discrete version does not have such a property (cf. [7]).

Evidently, time delay should be and has been taken into consideration in many realistic models and some essential dynamic differences caused by time delay have been observed. A few papers have also been devoted to the existence of traveling wave solutions of lattice differential systems with delay (cf. [19, 21, 26, 27]).

On the other hand, many authors have discussed the existence of traveling plane wave solutions of some reaction–diffusion systems (cf. [8, 17]). However, to the best of our knowledge, little has been done for the existence of traveling plane wave solutions of planar lattice differential equations, many of which can be viewed as the discretization of reaction–diffusion equations defined in a plane (cf. [11, 16, 18]).

The present paper is motivated by the recent work due to Wu and Zou [22] and Zou [26]. Our main goal is to establish some existence results for traveling plane wave solutions of planar lattice differential equations with delay. The remainder of this paper is organized as follows. In Section 2, we discuss the existence of traveling wave fronts of delayed systems of planar lattice differential equations by using some fixed point theorems. In Section 3, we consider periodic traveling wave solutions of delayed systems of planar lattice differential equations from the Hopf bifurcation point of view. Newly established Hopf bifurcation theory for mixed functional differential equations by Erbe *et al.* [3] is applied to the corresponding wave equation to obtain the global existence of slowly oscillatory spatially periodic wave solutions. In particular, the obtained results will be applied to a model arising from neural networks and explicit conditions for traveling wave fronts and periodic traveling plane waves will be obtained.

2. TRAVELING WAVE FRONTS

In this section, we will investigate the existence of traveling wave fronts of the infinitely coupled system of delay differential equations as follows:

$$\frac{d}{dt}u_{m,n}(t) = f((u_{m,n})_t) + D[g(u_{m-1,n}(t)) + g(u_{m+1,n}(t)) + g(u_{m,n-1}(t)) + g(u_{m,n+1}(t)) - 4g(u_{m,n}(t))], \tag{2.1}$$

where $m, n \in \mathbb{Z}$, $t \in \mathbb{R}$, $u_{m,n}(t) \in \mathbb{R}^N$, $D = \operatorname{diag}(d_1, d_2, \ldots, d_N)$ with $d_i \geqslant 0$, $i = 1, \ldots, N$, $g : \mathbb{R}^N \to \mathbb{R}^N$ and $f : C([-\tau, 0], \mathbb{R}^N) \to \mathbb{R}^N$ are continuous, and for any continuous mapping $u : [-\tau, A] \to \mathbb{R}^N$ with $A > 0$ and any $t \in [0, A]$, $u_t \in C([-\tau, 0], \mathbb{R}^N)$ is defined by $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$.

When $\tau = 0$ and $g(x) = x$, $x \in \mathbb{R}^N$, (2.1) becomes

$$\frac{d}{dt}u_{m,n}(t) = f(u_{m,n}(t)) + D[u_{m-1,n}(t) + u_{m+1,n}(t) + u_{m,n-1}(t) + u_{m,n+1}(t) - 4u_{m,n}(t)],$$

which can be viewed as the spatial discretization of the system of reaction–diffusion equations defined in a plane

$$\frac{\partial}{\partial t}u(x, t) = f(u(x, t)) + D\Delta u(x, t).$$

In this section, we will use the usual notations for the standard ordering in $\mathbb{R}^N$. That is, for $u = (u_1, \ldots, u_N)^T$ and $v = (v_1, \ldots, v_N)^T$, we denote $u \leqslant v$ if $u_i \leqslant v_i$, $i = 1, \ldots, N$, and $u < v$ if $u \leqslant v$ but $u \neq v$. In particular, we will denote $u \ll v$ if $u \leqslant v$ but $u_i \neq v_i$, $i = 1, \ldots, N$. If $u \leqslant v$, we also denote $(u, v] = \{w \in \mathbb{R}^N \colon u < w \leqslant v\}$, $[u, v) = \{w \in \mathbb{R}^N \colon u \leqslant w < v\}$. A mapping $G : \mathbb{R}^N \to \mathbb{R}^N$ is said to be nondecreasing if $G(u) \leqslant G(v)$ holds for every u and v with $u \leqslant v$. For any $N \times N$ matrix B, we denote by $\|B\|$ the matrix norm induced by the Euclidean norm $|\cdot|$ in $\mathbb{R}^N$.

A *traveling plane wave solution* of (2.1) is a solution of the form $u_{m,n}(t) = x(t - mc_1 - nc_2)$, where c_1, c_2 are nonnegative constants. Substituting $u_{m,n}(t) = x(t - mc_1 - nc_2)$ into (2.1), we find that (2.1) has a traveling plane wave solution if and only if the following wave equation:

$$\frac{d}{dt}x(t) = f(x_t) + D[g(x(t + c_1)) + g(x(t - c_1)) + g(x(t + c_2)) + g(x(t - c_2)) - 4g(x(t))] \tag{2.2}$$

has a solution. A solution of (2.2) is called a *profile function* of the traveling plane wave solution of (2.1). If a profile function x is nondecreasing and satisfies the following asymptotic boundary condition:

$$\lim_{t\to-\infty} x(t) = x_-, \qquad \lim_{t\to\infty} x(t) = x_+, \tag{2.3}$$

then the corresponding traveling plane wave solution is called a *traveling wavefront*. Obviously, the existence of traveling wavefronts is equivalent to the existence of solutions of the asymptotic boundary value problem (2.2)–(2.3).

Without loss of generality, we assume $x_- = 0$ and $x_+ = K > 0$. Then the asymptotic boundary condition (2.3) takes the form

$$\lim_{t\to-\infty} x(t) = 0, \qquad \lim_{t\to\infty} x(t) = K. \tag{2.4}$$

We will look for monotone solutions of (2.2) and (2.4). For convenience of statement, we now make the following assumptions.

(HF1) $f(\hat{0}) = f(\hat{K}) = 0$, where $\hat{x}$ denotes the constant mapping $x : [-\tau, 0] \to \mathbb{R}^N$ with the value $x \in \mathbb{R}^N$;

(HF2) $g : [0, K] \to \mathbb{R}^N$ is continuously differentiable, monotonically nondecreasing, $g(0) = 0$ and $v := \|Dg(0)\| \geqslant \|Dg(x)\|$ for all $x \in [0, K]$;

(HF3) There exists a matrix $\beta = \mathrm{diag}(\beta_1, \ldots, \beta_N)$ with $\beta_i \geqslant 0$ such that

$$f(\varphi) - f(\psi) + \beta[\varphi(0) - \psi(0)] \geqslant 4D[g(\varphi(0)) - g(\psi(0))]$$

for any $\varphi, \psi \in C([-\tau, 0], \mathbb{R}^N)$ with $0 \leqslant \psi(s) \leqslant \varphi(s) \leqslant K,\ s \in [-\tau, 0]$.

We also denote by $\|\cdot\|$ the supremum norm in $C([-\tau, 0], \mathbb{R}^N)$. We need the following continuity assumption:

(HF4) There exists a constant $L > 0$ such that

$$|f(\varphi) - f(\psi)| \leqslant L\|\varphi - \psi\|$$

for $\varphi, \psi \in C([-\tau, 0], \mathbb{R}^N)$ with $0 \leqslant \varphi(s),\ \psi(s) \leqslant K,\ s \in [-\tau, 0]$.

Define the operator $H : C(\mathbb{R}, \mathbb{R}^N) \to C(\mathbb{R}, \mathbb{R}^N)$ by

$$\begin{aligned} H(\varphi)(t) = {} & f(\varphi_t) + \beta\varphi(t) + D[g(\varphi(t - c_1)) + g(\varphi(t + c_1)) \\ & + g(\varphi(t - c_2)) + g(\varphi(t + c_2)) - 4g(\varphi(t))], \\ & \varphi \in C(\mathbb{R}, \mathbb{R}^N). \end{aligned} \tag{2.5}$$

Let

$$C_{[0,K]}(\mathbb{R},\mathbb{R}^N)=\{\varphi\in C(\mathbb{R},\mathbb{R}^N):\ 0\leqslant\varphi(s)\leqslant K, s\in\mathbb{R}\},$$

then we have the following

LEMMA 2.1. *Assume that* (*HF*1) *and* (*HF*2) *hold, then*

(i) $0\leqslant H(\varphi)(t)\leqslant\beta K$, *for* $\varphi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$;

(ii) $H(\varphi)(t)$ *is nondecreasing in* $t\in\mathbb{R}$, *if* $\varphi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *is nondecreasing in* $t\in\mathbb{R}$;

(iii) $H(\psi)(t)\leqslant H(\varphi)(t)$ *for* $t\in\mathbb{R}$, *if* $\varphi,\psi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *are given so that* $\psi(t)\leqslant\varphi(t)$ *for* $t\in\mathbb{R}$.

Lemma 2.1 is easy to be verified and the proof will be omitted. Clearly, with the above notations, (2.2) is equivalent to the following system of ordinary differential equations:

$$x'(t)=-\beta x(t)+H(x)(t),\quad t\in\mathbb{R}. \tag{2.6}$$

Define the operator $F: C_{[0,K]}(\mathbb{R},\mathbb{R}^N)\to C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ by

$$(F\varphi)(t)=e^{-\beta t}\int_{-\infty}^{t}e^{\beta s}H(\varphi)(s)\,ds \tag{2.7}$$

for $\varphi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$.

It is easy to show that $F: C_{[0,K]}(\mathbb{R},\mathbb{R}^N)\to C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ is a well-defined map and a fixed point of F is a solution of (2.6). Furthermore, we have

LEMMA 2.2. *Assume that* (*HF*1) *and* (*HF*2) *hold, then*

(i) $F\varphi(t)$ *is nondecreasing in* $\mathbb{R}$, *if* $\varphi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *is nondecreasing in* $t\in\mathbb{R}$;

(ii) $F\psi(t)\leqslant F\varphi(t)$ *for* $t\in\mathbb{R}$, *if* $\varphi,\psi\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *are given so that* $\psi(t)\leqslant\varphi(t)$ *for* $t\in\mathbb{R}$.

Proof. (ii) follows immediately from Lemma 2.1 (iii). To prove (i), let $t\in\mathbb{R}$ and $s>0$ be given, then for any $i=1,2,\dots,N$, we find

$$\begin{aligned}&(F\varphi)_i(t+s)-(F\varphi)_i(t)\\&\quad=e^{-\beta_i(t+s)}\int_{-\infty}^{t+s}e^{\beta_i\theta}H_i(\varphi)(\theta)\,d\theta-e^{-\beta_i t}\int_{-\infty}^{t}e^{\beta_i\theta}H_i(\varphi)(\theta)\,d\theta\end{aligned}$$

$$= e^{-\beta_i t} \int_{-\infty}^{t} e^{\beta_i \theta} (H_i(\varphi)(\theta + s) - H_i(\varphi)(\theta))\, d\theta$$
$$\geqslant 0.$$

This completes the proof. ∎

Without loss of generality, we assume $\beta_i > 0,\ i = 1, \ldots, N$. Let $\rho > 0$ be such that $\rho < \min_{1 \leqslant i \leqslant N} \beta_i$, and let

$$B_\rho(\mathbb{R}, \mathbb{R}^N) = \left\{ \varphi \in C(\mathbb{R}, \mathbb{R}^N) : \sup_{t \in \mathbb{R}} |\varphi(t)| e^{-\rho|t|} < \infty \right\}, \qquad |\varphi|_\rho = \sup_{t \in \mathbb{R}} |\varphi(t)| e^{-\rho|t|}.$$

Then it is easy to check that $(B_\rho(\mathbb{R}, \mathbb{R}^N), |\cdot|_\rho)$ is a Banach space.

LEMMA 2.3. *Assume that* (*HF*1)–(*HF*4) *hold, then* $F : C_{[0,K]}(\mathbb{R}, \mathbb{R}^N) \to C_{[0,K]}(\mathbb{R}, \mathbb{R}^N)$ *is continuous with respect to the norm* $|\cdot|_\rho$ *in* $B_\rho(\mathbb{R}, \mathbb{R}^N)$.

Proof. Firstly, we claim that $H : C_{[0,K]}(\mathbb{R}, \mathbb{R}^N) \to B_\rho(\mathbb{R}, \mathbb{R}^N)$ *is continuous. In fact*, *for any fixed* $\varepsilon > 0$, *take* $T > 0$ *such that*

$$2L|K| e^{-\rho T} < \varepsilon. \tag{2.8}$$

Let $\delta > 0$ be such that

$$\delta \max\{L e^{\rho(T+\tau)}, ||\beta||, 2v||D||(e^{\rho c_1} + e^{\rho c_2} + 2)\} < \varepsilon. \tag{2.9}$$

Then, if $\varphi, \psi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^N)$ satisfy

$$|\varphi - \psi|_\rho = \sup_{t \in \mathbb{R}} |\varphi(t) - \psi(t)| e^{-\rho|t|} < \delta,$$

from (2.9), we have

$$|\varphi(t) - \psi(t)| \leqslant \delta e^{\rho(T+\tau)} < \varepsilon / L, \qquad \forall t \in [-T - \tau, T].$$

Therefore, for $t \in [-T, T]$, by (2.9), we have

$$|H(\varphi)(t) - H(\psi)(t)| e^{-\rho|t|}$$
$$\leqslant |f(\varphi_t) - f(\psi_t)| + ||\beta||\, |\varphi(t) - \psi(t)| e^{-\rho|t|}$$
$$+\ v||D||\, [|\varphi(t - c_1) - \psi(t - c_1)| + |\varphi(t + c_1) - \psi(t + c_1)|$$

$$
\begin{aligned}
&+ \ |\varphi(t-c_2) - \psi(t-c_2)| + |\varphi(t+c_2) - \psi(t+c_2)| + 4|\varphi(t) - \psi(t)|]e^{-\rho|t|} \\
&\leqslant L\|\varphi_t - \psi_t\| + \|\beta\|\delta + 2v\|D\|(e^{\rho c_1} + e^{\rho c_2} + 2)\delta \\
&\leqslant L\delta e^{\rho(T+\tau)} + \|\beta\|\delta + 2v\|D\|(e^{\rho c_1} + e^{\rho c_2} + 2)\delta \\
&< 3\varepsilon,
\end{aligned}
$$

and for $|t| \geqslant T$, by (2.8) and (2.9), we have

$$
\begin{aligned}
&|H(\varphi)(t) - H(\psi)(t)|e^{-\rho|t|} \\
&\quad \leqslant |f(\varphi_t) - f(\psi_t)|e^{-\rho T} \\
&\quad\quad + \ \|\beta\| \, |\varphi(t) - \psi(t)|e^{-\rho|t|} + 2v\|D\|(e^{\rho c_1} + e^{\rho c_2} + 2)\delta \\
&\quad \leqslant 2L|K|e^{-\rho T} + \|\beta\|\delta + 2v\|D\|(e^{\rho c_1} + e^{\rho c_2} + 2)\delta \\
&\quad < 3\varepsilon.
\end{aligned}
$$

Thus, $|H(\varphi) - H(\psi)|_\rho < 3\varepsilon$. That is, $H : C_{[0,K]}(\mathbb{R}, \mathbb{R}^N) \to B_\rho(\mathbb{R}, \mathbb{R}^N)$ is continuous.

Now, we show that $F : C_{[0,K]}(\mathbb{R}, \mathbb{R}^N) \to C_{[0,K]}(\mathbb{R}, \mathbb{R}^N)$ is continuous.

For $t \geqslant 0$, we find

$$
\begin{aligned}
&|(F\varphi)_i(t) - (F\psi)_i(t)| \\
&\quad \leqslant e^{-\beta_i t} \int_{-\infty}^{t} e^{\beta_i s} |H(\varphi)(s) - H(\psi)(s)| \, ds \\
&\quad \leqslant e^{-\beta_i t} \int_{-\infty}^{t} e^{\beta_i s + \rho|s|} \, ds |H(\varphi) - H(\psi)|_\rho \\
&\quad = e^{-\beta_i t} \left[\int_{0}^{t} e^{(\beta_i + \rho)s} \, ds + \int_{-\infty}^{0} e^{(\beta_i - \rho)s} \, ds \right] |H(\varphi) - H(\psi)|_\rho \\
&\quad = \left[\frac{1}{\beta_i + \rho} e^{\rho t} - \frac{2\rho}{\beta_i^2 - \rho^2} e^{-\beta_i t} \right] |H(\varphi) - H(\psi)|_\rho \\
&\quad \leqslant \left[\frac{1}{\beta_i + \rho} + \frac{2\rho}{\beta_i^2 - \rho^2} \right] e^{\rho t} |H(\varphi) - H(\psi)|_\rho.
\end{aligned}
$$

Hence,

$$
|(F\varphi)_i(t) - (F\psi)_i(t)|e^{-\rho|t|} \leqslant \left[\frac{1}{\beta_i + \rho} + \frac{2\rho}{\beta_i^2 - \rho^2} \right] |H(\varphi) - H(\psi)|_\rho. \tag{2.10}
$$

For $t<0$, we have

$$\begin{aligned}
&|(F\varphi)_i(t) - (F\psi)_i(t)| \\
&\leqslant e^{-\beta_i t}\int_{-\infty}^{t} e^{(\beta_i-\rho)s}\, ds|H(\varphi) - H(\psi)|_\rho \\
&= \frac{1}{\beta_i-\rho}e^{-\rho t}|H(\varphi) - H(\psi)|_\rho
\end{aligned}$$

and hence

$$|(F\varphi)_i(t) - (F\psi)_i(t)|e^{-\rho|t|} \leqslant \frac{1}{\beta_i-\rho}|H(\varphi) - H(\psi)|_\rho. \tag{2.11}$$

Thus, it follows from (2.10) and (2.11) that $F: C_{[0,K]}(\mathbb{R},\mathbb{R}^N) \to C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ is continuous with respect to the norm $|\cdot|_\rho$ in $B_\rho(\mathbb{R},\mathbb{R}^N)$ and the proof is complete. ∎

For the sake of convenience, we introduce the definition of an upper (or a lower) solution of (2.2).

Definition 2.1. A continuous function $\rho:\mathbb{R}\to\mathbb{R}^n$ is called an *upper solution* of (2.2), *if* ρ *satisfies*

$$\begin{aligned}
\rho'(t) \geqslant f(\rho_t) + D[&g(\rho(t-c_1)) + g(\rho(t+c_1)) \\
&+ g(\rho(t-c_2)) + g(\rho(t+c_2)) - 4g(\rho(t))] \qquad \textit{a.e. } \text{on } \mathbb{R}.
\end{aligned} \tag{2.12}$$

A *lower solution* of (2.2) is defined in a similar way by reversing the inequality in (2.12).

Now, we are in a position to state and show the following existence theorem.

Theorem 2.1. *Assume that* (*HF*1)–(*HF*4) *hold. Suppose that* (2.2) *has an upper solution* $\bar{\rho}\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *and a lower solution* $\underline{\rho}\in C_{[0,K]}(\mathbb{R},\mathbb{R}^N)$ *satisfying*

(I) $\sup_{s\leqslant t}\underline{\rho}(s)\leqslant\bar{\rho}(t)$, *for* $t\in\mathbb{R}$;
(II) $f(\hat{u})\neq 0$, *for* $u\in(0,\inf_{t\in\mathbb{R}}\bar{\rho}(t)]\cup[\sup_{t\in\mathbb{R}}\underline{\rho}(t),K)$.

Then the asymptotic boundary value problem (2.2) *and* (2.4) *has a monotone solution. That is*, (2.1) *has a traveling wavefront solution.*

Proof. Let

$$\Gamma = \left\{ \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^N): \begin{array}{l} \text{(i) } \varphi \text{ is nondecreasing in } \mathbb{R}; \\ \text{(ii) } \underline{\rho}(t) \leqslant \varphi(t) \leqslant \bar{\rho}(t) \quad \forall t \in \mathbb{R}; \\ \text{(iii) } |\varphi(u) - \varphi(v)| \leqslant 2\|\beta\|\,|K|\|u - v| \quad \forall u, v \in \mathbb{R}. \end{array} \right\}.$$

Let $W(t) = (F\bar{\rho})(t) - \bar{\rho}(t),\ t \in \mathbb{R}$. Since $\bar{\rho}(t)$ is an upper solution of (2.2), from (2.5) and (2.12), it follows that

$$\bar{\rho}'(t) \geqslant -\beta\bar{\rho}(t) + H(\bar{\rho})(t). \tag{2.13}$$

On the other hand, (2.7) implies that

$$(F\bar{\rho})'(t) = -\beta(F\bar{\rho})(t) + H(\bar{\rho})(t),$$

which together with (2.13) implies that

$$W'(t) = (F\bar{\rho})'(t) - \bar{\rho}'(t) \leqslant -\beta[(F\bar{\rho})(t) - \bar{\rho}(t)] = -\beta W(t). \tag{2.14}$$

Let $W'(t) + \beta W(t) = r(t)$, then we have

$$W(t) = e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{\beta(t-s)} r(s)\, ds. \tag{2.15}$$

Let $t_0 \to -\infty$ (2.15) yields

$$W(t) = \int_{-\infty}^{t} e^{\beta(t-s)} r(s)\, ds. \tag{2.16}$$

It follows from (2.14) that $r(t) \leqslant 0,\ \forall t \in \mathbb{R}$. Hence, (2.16) implies that $W(t) \leqslant 0$ for all $t \in \mathbb{R}$. This proves that $(F\bar{\rho})(t) \leqslant \bar{\rho}(t),\ t \in \mathbb{R}$.

In a similar way, we can show that $(F\underline{\rho})(t) \geqslant \underline{\rho}(t),\ t \in \mathbb{R}$.

Let $\bar{\varphi}(t) = \sup_{s \leqslant t} \underline{\rho}(s)$, then $\bar{\varphi}(t)$ is nondecreasing in $\mathbb{R}$ and Condition (I) implies that

$$\underline{\rho}(t) \leqslant \bar{\varphi}(t) \leqslant \bar{\rho}(t), \qquad t \in \mathbb{R}.$$

Therefore, Lemma 2.2 implies that

$$\underline{\rho}(t) \leqslant (F\underline{\rho})(t) \leqslant (F\bar{\varphi})(t) \leqslant (F\bar{\rho})(t) \leqslant \bar{\rho}(t), \qquad t \in \mathbb{R}.$$

It is also easy to check that

$$|(F\bar{\varphi})(u) - (F\bar{\varphi})(v)| \leqslant 2\|\beta\|\,|K|\,|u - v|, \qquad u, v \in \mathbb{R}.$$

Consequently, $F\bar{\varphi} \in \Gamma$. That is, Γ is nonempty. It is also easy to verify that Γ is convex and compact in $B_\rho(\mathbb{R}, \mathbb{R}^N)$.

Moreover, by Lemma 2.2 and a similar argument as above, we can show that

$$F(\Gamma) \subset \Gamma.$$

Therefore, the well-known Schauder's fixed-point theorem implies that F has a fixed point φ in Γ. In other words, $\varphi(t)$ is a solution of (2.2).

Finally, we note that

$$0 \leqslant \varphi_- =: \lim_{t\to-\infty} \varphi(t) \leqslant \inf_{t\in\mathbb{R}} \bar{\rho}(t) \tag{2.17}$$

and

$$\sup_{t\in\mathbb{R}} \underline{\rho}(t) \leqslant \varphi_+ =: \lim_{t\to\infty} \varphi(t) \leqslant K. \tag{2.18}$$

Moreover, we can show that

$$f(\hat{\varphi}_-) = 0, \qquad f(\hat{\varphi}_+) = 0.$$

Therefore, it follows from (2.17), (2.18) and Condition (II) that

$$\varphi_- = \lim_{t\to-\infty} \varphi(t) = 0, \qquad \varphi_+ = \lim_{t\to\infty} \varphi(t) = K.$$

Thus, φ is a monotone solution of (2.2) and (2.4). The proof is complete. ∎

Finally, we present an application of our general result obtained in this section to a model arising from neural networks.

Example 2.1. Consider the following system of lattice differential equations

$$\begin{aligned} C\frac{d}{dt}u_n(t) = &-\frac{1}{R}u_n(t) + A_0 g(u_n(t)) + \sum_{j=1}^{N} A_j[g(u_{n+j}(t)) \\ &+ g(u_{n-j}(t))] + I \end{aligned} \tag{2.19}$$

as a model, suggested by Hopfield [4,5], for a network of infinitely many cells located in a linear lattice. Here, it is assumed that each cell is made of a linear capacitor, a nonlinear voltage-controlled current source and a few resistive linear circuit elements, and that cells communicate with each other directly only through its nearest N-neighbors. In this equation, C and R are positive constants denoting the capacitance and resistance of each cell, the transfer (input–output or activation) function $g: \mathbb{R} \to \mathbb{R}$ is a sigmoid (that is, a smooth and nondecreasing function with graph asymptotic to two horizontal lines), $(A_0, A_1, A_2, \dots, A_N)$ are the interactive parameters, and I denotes the input control effect.

Now, we assume that cells are located in a planar lattice and communicate with each other directly only through its adjacent four-neighbors. Then we reach the following system of planar lattice differential equations:

$$C\frac{d}{dt}u_{m,n}(t) = -\frac{1}{R}u_{m,n}(t) + A_0 g(u_{m,n}(t)) + A_1[g(u_{m+1,n}(t)) + g(u_{m-1,n}(t)) + g(u_{m,n+1}(t)) + g(u_{m,n-1}(t))] + I. \tag{2.20}$$

In what follows, we will assume that $I = 0$. This can always be achieved by some translation of coordinates. Note that A_0, A_1 can be either positive or negative, corresponding to the excitatory or inhibitory interaction of cells. (2.20) can be rewritten as

$$\frac{d}{dt}u_{m,n}(t) = -\alpha u_{m,n}(t) + a_0 g(u_{m,n}(t)) + a_1[g(u_{m+1,n}(t)) + g(u_{m-1,n}(t)) + g(u_{m,n+1}(t)) + g(u_{m,n-1}(t))], \tag{2.21}$$

where $\alpha = 1/RC$ and $a_j = A_j/C,\ j = 0, 1$.

Put $f(x) = -\alpha x + (a_0 + 4a_1)g(x),\ x \in \mathbb{R}$, then the corresponding wave equation (for $c_1 = c\cos\theta,\ c_2 = c\sin\theta,\ \theta \in [0, \frac{\pi}{2}]$) takes the form

$$\frac{d}{dt}x(t) = f(x(t)) + a_1[g(x(t + c\cos\theta)) + g(x(t - c\cos\theta)) + g(x(t + c\sin\theta)) + g(x(t - c\sin\theta)) - 4g(x(t))]. \tag{2.22}$$

COROLLARY 2.1. *Assume that*

(i) a_0 *and* a_1 *are positive constants*;

(ii) $g \in C^2(\mathbb{R}, \mathbb{R}),\ g(0) = 0,\ \lim_{x\to\pm\infty} g(x) = \pm 1,\ g'(x) > 0$ *and* $xg''(x) < 0$ *for* $x \neq 0$;

(iii) $v(a_0 + 4a_1) > \alpha,\ v =: g'(0)$.

For each $\theta \in [0, \frac{\pi}{2}]$, *let*

$$c^*(\theta) = \sup_{x\in\mathbb{R}} \frac{x}{-\alpha + v[a_0 + a_1(e^{x\cos\theta} + e^{-x\cos\theta} + e^{x\sin\theta} + e^{-x\sin\theta})]}. \tag{2.23}$$

Then for every $c < c^*(\theta)$, (2.21) *has a traveling wavefront* $u_{m,n}(t) = x(t - mc\cos\theta - nc\sin\theta)$ *such that* $\lim_{t\to-\infty} x(t) = 0$ *and* $\lim_{t\to\infty} x(t) = K$,

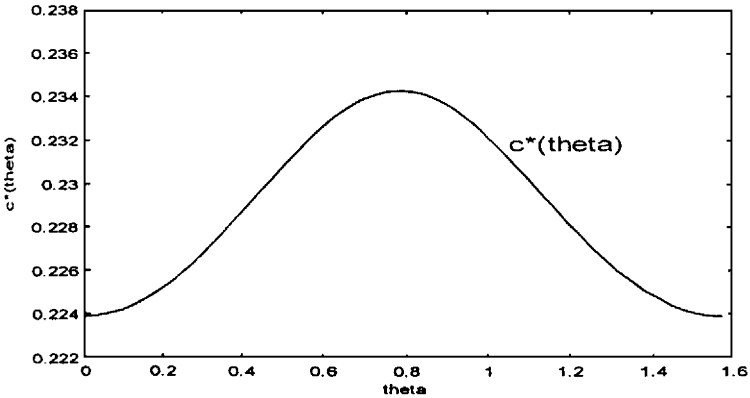

FIG. 1. The graph of the function $c^*(\theta)$ determining the maximal wave velocities in directions $\theta \in [0, \pi/2]$ in the case where $\alpha = v = a_0 = a_1 = 1$.

where K is the unique positive solution of the algebraic equation

$$\alpha K = (a_0 + 4a_1)g(K).$$

We can easily verify that (HF1)–(HF4) hold. Moreover, for any positive number $c < c^*(\theta)$, (2.23) implies that

$$\sup_{\lambda \geqslant 0} \frac{\lambda}{-\alpha + v[a_0 + a_1(e^{\lambda c \cos\theta} + e^{-\lambda c \cos\theta} + e^{\lambda c \sin\theta} + e^{-\lambda c \sin\theta})]} > 1.$$

Therefore, we can find two positive constants λ_1 and λ_2 with $0 < \lambda_1 < \lambda_2$ such that $\Delta(\lambda_1) = \Delta(\lambda_2) = 0$ and $\Delta(\lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$, where $\Delta(\lambda) = \lambda + \alpha - a_0 v - a_1 v[e^{-\lambda c \cos\theta} + e^{\lambda c \cos\theta} + e^{-\lambda c \sin\theta} + e^{\lambda c \sin\theta}]$. Define

$$\bar{\rho}(t) = K \min\{e^{\lambda_1 t}, 1\}$$

and

$$\underline{\rho}(t) = K \max\{0, (1 - Me^{\varepsilon t})e^{\lambda_1 t}\}.$$

It can be verified that $\bar{\rho}(t)$ is an upper solution of (2.22) and for sufficiently small $\varepsilon > 0$ and sufficiently large $M > 0$, $\underline{\rho}(t)$ is a lower solution of (2.22). Clearly, for this pair of upper and lower solutions, Condition (I) and (II) in Theorem 2.1 hold, therefore, for every $c < c^*(\theta)$, (2.21) has a traveling wavefront $u_{m,n}(t) = x(t - mc\cos\theta - nc\sin\theta)$ satisfying $\lim_{t\to-\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = K$.

3. PERIODIC TRAVELING WAVES

In this section, we consider the following planar lattice differential systems:

$$\begin{aligned}\frac{d}{dt}u_{m,n}(t) &= f(u_{m,n}(t), u_{m,n}(t-\tau)) \\ &+ \sum_{j=1}^{N} a_j[g(u_{m-j,n}(t-\tau)) + g(u_{m+j,n}(t-\tau)) - 2g(u_{m,n}(t-\tau))] \\ &+ \sum_{j=1}^{N} b_j[g(u_{m,n-j}(t-\tau)) + g(u_{m,n+j}(t-\tau)) \\ &- 2g(u_{m,n}(t-\tau))], \end{aligned} \tag{3.1}$$

where $m, n \in \mathbb{Z}$, $u_{m,n}(t) \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable. Furthermore, we always assume $f(0,0) = 0$, $g(0) = 0$ and g is nondecreasing in $\mathbb{R}$.

A traveling plane wave $u_{m,n}(t) = x(t - mc_1 - nc_2)$ is said to be *spatially* (p,q)*-periodic* if p, q are positive integers and $u_{m+p,n+q}(t) = u_{m,n}(t)$ for all $t \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. It is easy to verify that $u_{m,n}(t)$ is spatially (p,q)-periodic if and only if $x(t)$ is $pc_1 + qc_2$-periodic.

We now consider the related wave equation

$$\begin{aligned}\frac{d}{dt}x(t) &= f(x(t), x(t-\tau)) \\ &+ \sum_{j=1}^{N} a_j[g(x(t-\tau+jc_1)) + g(x(t-\tau-jc_1)) - 2g(x(t-\tau))] \\ &+ \sum_{j=1}^{N} b_j[g(x(t-\tau+jc_2)) + g(x(t-\tau-jc_2)) - 2g(x(t-\tau))]. \end{aligned} \tag{3.2}$$

Denote $c_1 = c$, $c_2 = rc$, $c > 0$, $r > 0$, then (3.2) takes the form

$$\begin{aligned}\frac{d}{dt}x(t) &= f(x(t), x(t-\tau)) \\ &+ \sum_{j=1}^{N} a_j[g(x(t-\tau+jc)) + g(x(t-\tau-jc)) - 2g(x(t-\tau))] \\ &+ \sum_{j=1}^{N} b_j[g(x(t-\tau+jrc)) + g(x(t-\tau-jrc)) - 2g(x(t-\tau))]. \end{aligned} \tag{3.3}$$

Therefore, finding a (p,q)-periodic traveling plane wave of (3.1) with wave speed (c, rc) is equivalent to finding a $(p+rq)c$-periodic solution of (3.3). In what follows, we will regard the delay τ as a parameter and look for periodic solutions of (3.3) from the Hopf bifurcation point of view.

For the sake of simplicity, we denote $\sigma_r := \frac{2\pi}{p+rq}$. We now normalize the period of $x(t)$ by

$$y(t) = x\left(\frac{(p+rq)c}{2\pi}t\right), \tag{3.4}$$

then $y(t)$ is 2π-periodic if and only if $x(t)$ is $(p+rq)c$-periodic. Substituting (3.4) into (3.3), we get

$$\begin{aligned}\sigma_r\dot{y}(t) = {} & cf(y(t), y(t-\sigma_r\tau/c)) \\ & + c\sum_{j=1}^{N} a_j[g(y(t-\sigma_r\tau/c+\sigma_r j)) + g(y(t-\sigma_r\tau/c-\sigma j)) \\ & - 2g(y(t-\sigma_r\tau/c))] \\ & + c\sum_{j=1}^{N} b_j[g(y(t-\sigma_r\tau/c+\sigma_r jr)) + g(y(t-\sigma_r\tau/c-\sigma_r jr)) \\ & - 2g(y(t-\sigma_r\tau/c))].\end{aligned} \tag{3.5}$$

From now on, we will fix the positive integers p, q and the real $r > 0$. Then, for given constants $c > 0$ and τ and for a given 2π-periodic mapping $y : \mathbb{R} \to \mathbb{R}$, we define

$$\begin{aligned}F(y,\tau,c)(t) = {} & \frac{c}{\sigma_r}f(y(t), y(t-\sigma_r\tau/c)) \\ & + \frac{c}{\sigma_r}\sum_{j=1}^{N} a_j[g(y(t-\sigma_r\tau/c+\sigma_r j)) + g(y(t-\sigma_r\tau/c-\sigma_r j)) \\ & - 2g(y(t-\sigma_r\tau/c))] \\ & + \frac{c}{\sigma_r}\sum_{j=1}^{N} b_j[g(y(t-\sigma_r\tau/c+\sigma_r jr)) + g(y(t-\sigma_r\tau/c-\sigma_r jr)) \\ & - 2g(y(t-\sigma_r\tau/c))].\end{aligned}$$

Then (3.5) can be written as

$$\dot{y}(t) = F(y,\tau,c)(t). \tag{3.6}$$

Restricting to the subspace of all constant mappings, F indices a mapping $\hat{F} : \mathbb{R} \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$ by

$$\hat{F}(x,\tau,c) = \frac{c}{\sigma_r}f(x,x).$$

Recall that a point $(x,\tau,c)\in\mathbb{R}\times\mathbb{R}\times(0,\infty)$ is said to be a *stationary point* if $\hat{F}(x,\tau,c)=0$. In our Hopf bifurcation analysis, we need to verify that $D_x\hat{F}(x,\tau,c)$, the derivative of $\hat{F}$ with respect to the first argument, is an isomorphism at given stationary point $(0,\tau,c)$.

Let $a=\frac{\partial}{\partial x}f(0,0)$ and $b=\frac{\partial}{\partial y}f(0,0)$, the partial derivatives of $f(x,y)$ with respect to the first and second arguments and evaluating at $(0,0)$, respectively. We assume

(HP1) $a<0,\ a+b\neq 0$ and $xf(x,x)<0$ for $x\neq 0$.

Therefore, for any fixed τ and c, $(0,\tau,c)$ is the only stationary point and $D_x\hat{F}(0,\tau,c)=\frac{c}{\sigma_r}(a+b)\neq 0$, and thus an isomorphism.

The linearization of (3.5) at the stationary point $(0,\tau,c)$ takes the form

$$\begin{aligned}\sigma_r\dot{y}(t) &= cay(t)+cby(t-\sigma_r\tau/c)\\ &\quad + c\sum_{j=1}^{N} a_j v[y(t-\sigma_r\tau/c+\sigma_r j)+y(t-\sigma_r\tau/c-\sigma_r j)\\ &\quad -2y(t-\sigma_r\tau/c)]+c\sum_{j=1}^{N} b_j v[y(t-\sigma_r\tau/c+\sigma_r jr)\\ &\quad + y(t-\sigma_r\tau/c-\sigma_r jr)-2y(t-\sigma_r\tau/c)] \end{aligned}\tag{3.7}$$

here and in what follows, $v:=g'(0)$.

The characteristic values of the stationary point $(0,\tau,c)$ are complex numbers λ satisfying the characteristic equation

$$\begin{aligned}\sigma_r\lambda &= ca+cbe^{-\sigma_r\tau\lambda/c}+c\sum_{j=1}^{N} a_j ve^{-\sigma_r\tau\lambda/c}[e^{\sigma_r j\lambda}+e^{-\sigma_r j\lambda}-2]\\ &\quad + c\sum_{j=1}^{N} b_j ve^{-\sigma_r\tau\lambda/c}[e^{\sigma_r jr\lambda}+e^{-\sigma_r jr\lambda}-2]. \end{aligned}\tag{3.8}$$

A stationary point $(0,\tau,c)$ is a *center* if there exists an integer $k\geqslant 1$ such that ik is a characteristic value. Substituting $\lambda=ik$ into (3.8), we get

$$\begin{aligned}ik\sigma_r &= ca+cbe^{-ik\sigma_r\tau/c}+c\sum_{j=1}^{N} a_j ve^{-ik\sigma_r\tau/c}[e^{ik\sigma_r j}+e^{-ik\sigma_r j}-2]\\ &\quad + c\sum_{j=1}^{N} b_j ve^{-ik\sigma_r\tau/c}[e^{ik\sigma_r jr}+e^{-ik\sigma_r jr}-2]. \end{aligned}\tag{3.9}$$

Since

$$e^{ik\sigma_r j}+e^{-ik\sigma_r j}-2=2(\cos(k\sigma_r j)-1)=-4\sin^2\frac{k\sigma_r j}{2},$$
$$e^{ik\sigma_r jr}+e^{-ik\sigma_r jr}-2=2(\cos(k\sigma_r jr)-1)=-4\sin^2\frac{k\sigma_r jr}{2}$$

it follows from (3.9) that

$$ik\sigma_r=ca+cbe^{-ik\sigma_r\tau/c}-4cve^{-ik\sigma_r\tau/c}\sum_{j=1}^{N}\left[a_j\sin^2\frac{k\sigma_r j}{2}+b_j\sin^2\frac{k\sigma_r jr}{2}\right]. \quad (3.10)$$

Denote

$$\varpi_{r,k}=4v\sum_{j=1}^{N}\left[a_j\sin^2\frac{k\sigma_r j}{2}+b_j\sin^2\frac{k\sigma_r jr}{2}\right] \quad (3.11)$$

and writing (3.10) in terms of its real and imaginary parts, we get

$$(\varpi_{r,k}-b)\cos\frac{k\sigma_r\tau}{c}=a,$$
$$(\varpi_{r,k}-b)\sin\frac{k\sigma_r\tau}{c}=\frac{k\sigma_r}{c}. \quad (3.12)$$

Clearly, from (3.12), we see that ik is a characteristic value only if $\varpi_{r,k}\neq b$. For $k\geqslant 1$ with $\varpi_{r,k}\neq b$, (3.12) can be written as

$$\cos\frac{k\sigma_r\tau}{c}=\frac{a}{\varpi_{r,k}-b},$$
$$\sin\frac{k\sigma_r\tau}{c}=\frac{k\sigma_r}{c(\varpi_{r,k}-b)}$$

or, equivalently,

$$\cot\frac{k\sigma_r\tau}{c}=\frac{ac}{k\sigma_r},$$
$$(\varpi_{r,k}-b)^2=\frac{k^2\sigma_r^2}{c^2}+a^2. \quad (3.13)$$

Therefore, if $(\varpi_{r,k}-b)^2>a^2$, the second equation of (3.13) implies that

$$c_{r,k}=\frac{k\sigma_r}{\sqrt{(\varpi_{r,k}-b)^2-a^2}}.$$

Substituting $c_{r,k}$ into the first equation of (3.13), we can determine the real τ and c. In summary, we have established the following.

LEMMA 3.1. *Assume that* $(\varpi_{r,k} - b)^2 > a^2$ *for some fixed r and k. Let* $\theta_{r,k} \in (\frac{\pi}{2}, \pi)$ *be given so that* $\cot \theta_{r,k} = \frac{ac_{r,k}}{k\sigma_r}$. *For each integer* $\ell \geqslant 0$, *define*

$$\theta_{r,k,\ell} = \theta_{r,k} + \ell\pi,$$

$$c_{r,k} = \frac{k\sigma_r}{\sqrt{(\varpi_{r,k} - b)^2 - a^2}},$$

$$\tau_{r,k,\ell} = \frac{c_{r,k}}{k\sigma_r}\theta_{r,k,\ell} = \frac{c_{r,k}}{k\sigma_r}(\theta_{r,k} + \ell\pi).$$

Then the set of centers of (3.6) *is* $\{(0, \tau_{r,k,\ell}, c_{r,k}); k \geqslant 1 : (\varpi_{r,k}^2 - b)^2 > a^2, \ell \geqslant 0\}$ *and thus is isolated in* $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.

Now, we make the following assumption:
(HP2) $(\varpi_r - b)^2 > a^2$, where

$$\varpi_r := \varpi_{r,1} = 4v \sum_{j=1}^{N} \left(a_j \sin^2 \frac{\sigma_r j}{2} + b_j \sin^2 \frac{r\sigma_r j}{2} \right).$$

Our next step is to evaluate the so-called crossing number of the stationary point $(0, \tau_{r,\ell}, c_r)$, where

$$c_r := \frac{\sigma_r}{\sqrt{(\varpi_r - b)^2 - a^2}},$$

$$\tau_{r,\ell} := \frac{c_r}{\sigma_r}(\theta_r + \ell\pi), \qquad \theta_r := \theta_{r,1}.$$

The crossing number is defined by

$$\gamma(0, \tau_{r,\ell}, c_r) = \deg_B(\Delta, \Omega),$$

where $\deg_B$ is the Brouwer degree and

$$\Delta(\tau, c) = i\frac{\sigma_r}{c} - a - be^{-i\sigma_r\tau/c} + 4ve^{-i\sigma_r\tau/c} \sum_{j=1}^{N} \left(a_j \sin^2 \frac{\sigma_r j}{2} + b_j \sin^2 \frac{r\sigma_r j}{2} \right)$$

$$= i\frac{\sigma_r}{c} - a - be^{-i\sigma_r\tau/c} + \varpi_r e^{-i\sigma_r\tau/c}$$

$$= i\frac{\sigma_r}{c} - a + (\varpi_r - b)e^{-i\sigma_r\tau/c}$$

and $\Omega := (\tau_{r,\ell} - \delta, \tau_{r,\ell} + \delta) \times (c_r - \delta, c_r + \delta)$ for sufficiently small $\delta > 0$.

Define

$$H(\tau, u, c) = \left(u + i\frac{\sigma_r}{c}\right) - a + (\varpi_r - b)e^{-(u+i\sigma_r/c)\tau},$$

where $(u, c) \in D := (0, \varepsilon) \times (c_r - \varepsilon, c_r + \varepsilon)$ for sufficiently small $\varepsilon > 0$. Then we have the following observation:

(i) $H(\tau, 0, c) = \Delta(\tau, c)$,

(ii) $H(\tau, u, c) \neq 0$ if $|\tau - \tau_{r,\ell}| \leqslant \varepsilon$ and $(u, c) \in \partial D \setminus \{(0, c); |c - c_r| < \varepsilon\}$,

(iii) $H(\tau_{r,\ell} \pm \varepsilon, 0, c) \neq 0$ for $|c - c_r| < \varepsilon$.

Therefore, by using Lemma 2.5 of Erbe *et al.* [3], we get

$$\gamma(0, \tau_{r,\ell}, c_r) = \deg_B(H(\tau_{r,\ell} - \varepsilon, \cdot), D) - \deg_B(H(\tau_{r,\ell} + \varepsilon, \cdot), D). \quad (3.14)$$

LEMMA 3.2. *For every integer* $\ell \geqslant 0$, *the crossing number* $\gamma(0, \tau_{r,\ell}, c_r)$ *at* $(0, \tau_{r,\ell}, c_r)$ *is* -1.

Proof. For the sake of simplicity, we let $v = \sigma_r/c$.

Assume that $u = u(\tau)$ and $v = v(\tau)$ are the smooth functions of $\tau \in (\tau_{r,\ell} - \delta, \tau_{r,\ell} + \delta)$ so that

$$u(\tau_{r,\ell}) = 0, \qquad v(\tau_{r,\ell}) = \frac{\sigma_r}{c_r}$$

and

$$u + iv - a + (\varpi_r - b)e^{-(u+iv)\tau} = 0. \quad (3.15)$$

Differentiating both sides of (3.15) with respect to τ and then evaluating at $\tau = \tau_{r,\ell},\ c = c_r$, we find

$$\frac{d}{d\tau}(u + iv) = \frac{(\varpi_r - b)(u + iv)e^{-(u+iv)\tau}}{1 - \tau(\varpi_r - b)e^{-(u+iv)\tau}} = \frac{(u + iv)(a - u - iv)}{1 - \tau(a - u - iv)}$$

$$= \frac{i\frac{\sigma_r}{c_r}(a - i\sigma_r/c_r)}{1 - \tau_{r,\ell}(a - i\sigma_r/c_r)} = \frac{i\sigma_r a/c_r + \sigma_r^2/c_r^2}{1 - \tau_{r,\ell}a + i\sigma_r\tau_{r,\ell}/c_r}.$$

Therefore,

$$u'(\tau_{r,\ell}) = \frac{\sigma_r^2}{c_r^2(1-\tau_{r,\ell}a)^2 + \sigma_r^2\tau_{r,\ell}^2} > 0.$$

Consequently, from (3.14), we get $\gamma(0, \tau_{r,\ell}, c_r) = -1$. The proof is complete. ∎

We can now apply the global Hopf bifurcation theorem in [3] to conclude that the connected component S_r through $(0, \tau_r, c_r)$, $\tau_r := \tau_{r,0} = \frac{c_r\theta_r}{\sigma_r}$, in the closure of the subset

$$\{(y, \tau, c);\ y \text{ is a nonconstant } 2\pi\text{−periodic solution of (3.5) } \tau \in \mathbb{R},\ c \geqslant 0\}$$

of the space $Y \times \mathbb{R}^2$ must be nonempty and unbounded, where Y is the Banach space of 2π-periodic continuous functions equipped with the super-norm. This is equivalent to say that the connected component Σ_r through $(0, \tau_r, c_r)$ in the closure of the subset

$$\{(x, \tau, c);\ x \text{ is a nonconstant } (p+rq)c - \text{ periodic solution of (3.3) } \tau \in \mathbb{R},\ c \geqslant 0\}$$

of the space $X \times \mathbb{R}^2$ must be nonempty and unbounded, X is the Banach space of all bounded continuous functions equipped with the super-norm.

The following results establish a priori bounds for periodic solutions of (3.3).

LEMMA 3.3. *Assume that* $\lim_{x\to\pm\infty} x^{-1}g(x) = \bar{g}$, *and* $\lim_{x\to\pm\infty} x^{-1}f(x, y) = \bar{f}$ *uniformly for* $y \in \mathbb{R}$. *If*

$$\bar{f} < -2\bar{g}\left[\sum_{j=1}^{N}(|a_j| + |b_j|) + \left|\sum_{j=1}^{N}(a_j + b_j)\right|\right] \tag{3.16}$$

holds, then there exists a constant $M > 0$, *independent of* τ *and* c, *such that* $|x(t)| \leqslant M$ *for every given periodic solution* $x(t)$ *of* (3.3).

Proof. Suppose $x(t)$ is a periodic solution of (3.3). Take $t_0 \in \mathbb{R}$ so that $|x(t)| \leqslant |x(t_0)|$ for every $t \in \mathbb{R}$. Without loss of generality, we can assume $x(t_0) \neq 0$. Then there exist two possible cases:

(1) $x(t_0) > 0$.

In this case, we have

$$x(t_0) \geqslant x(t) \geqslant -x(t_0) \qquad \forall t \in \mathbb{R}, \tag{3.17}$$

and

$$\sum_{j=1}^{N} a_j[g(x(t_0-\tau+jc))+g(x(t_0-\tau-jc))-2g(x(t_0)\,\mathrm{sign}(a_j))]$$
$$+\sum_{j=1}^{N} b_j[g(x(t_0-\tau+jrc))+g(x(t_0-\tau-jrc))-2g(x(t_0)\,\mathrm{sign}(b_j))]$$
$$\leqslant 0. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}
0 &= \frac{d}{dt}x(t_0)\\
&= f(x(t_0),x(t_0-\tau))\\
&\quad+\sum_{j=1}^{N} a_j[g(x(t_0-\tau+jc))+g(x(t_0-\tau-jc))-2g(x(t_0-\tau))]\\
&\quad+\sum_{j=1}^{N} b_j[g(x(t_0-\tau+jrc))+g(x(t_0-\tau-jrc))-2g(x(t_0-\tau))]\\
&= f(x(t_0),x(t_0-\tau))-2\sum_{j=1}^{N}(a_j+b_j)g(x(t_0-\tau))\\
&\quad+2\sum_{j=1}^{N} a_j g(x(t_0)\,\mathrm{sign}(a_j))+2\sum_{j=1}^{N} b_j g(x(t_0)\,\mathrm{sign}(b_j))\\
&\quad+\sum_{j=1}^{N} a_j[g(x(t_0-\tau+jc))+g(x(t_0-\tau-jc))-2g(x(t_0)\,\mathrm{sign}(a_j))]\\
&\quad+\sum_{j=1}^{N} b_j[g(x(t_0-\tau+jrc))+g(x(t_0-\tau-jrc))-2g(x(t_0)\,\mathrm{sign}(b_j))]\\
&\leqslant f(x(t_0),x(t_0-\tau))-2\sum_{j=1}^{N}(a_j+b_j)g\left(-x(t_0)\,\mathrm{sign}\left(\sum_{j=1}^{N}(a_j+b_j)\right)\right)\\
&\quad+2\sum_{j=1}^{N} a_j g(x(t_0)\,\mathrm{sign}(a_j))+2\sum_{j=1}^{N} b_j g(x(t_0)\,\mathrm{sign}(b_j)).
\end{aligned}$$

Hence,

$$0 \leqslant \frac{f(x(t_0), x(t_0-\tau))}{x(t_0)} + 2\left|\sum_{j=1}^{N} (a_j+b_j)\right| \frac{g(-x(t_0)\,\mathrm{sign}(\sum_{j=1}^{N} (a_j+b_j)))}{-x(t_0)\,\mathrm{sign}(\sum_{j=1}^{N} (a_j+b_j))}$$

$$+\ 2\sum_{j=1}^{N} |a_j| \frac{g(x(t_0)\,\mathrm{sign}(a_j))}{x(t_0)\,\mathrm{sign}(a_j)} + 2\sum_{j=1}^{N} |b_j|\,\frac{g(x(t_0)\,\mathrm{sign}(b_j))}{x(t_0)\,\mathrm{sign}(b_j)}. \tag{3.19}$$

From (3.16), we observe that there exists a constant $M>0$ such that

$$\frac{f(\pm x, y)}{\pm x} < -2\left[\sum_{j=1}^{N} (|a_j|+|b_j|) + \left|\sum_{j=1}^{N} (a_j+b_j)\right|\right]\frac{g(\pm x)}{\pm x}$$

$$\forall |x| \geqslant M, \qquad y \in \mathbb{R}. \tag{3.20}$$

Therefore, (3.19) and (3.20) imply that $|x(t_0)| \leqslant M$, and hence $|x(t)| \leqslant M$, for all $t \in \mathbb{R}$.

(2) $x(t_0) < 0$.

This case can be proved in a similar way and thus is omitted. This completes the proof. ∎

Now, we are in a position to state and prove the following global Hopf bifurcation theorem.

THEOREM 3.1. *Assume that* (*HP*1) *and Conditions of Lemma* 3.3 *hold. For fixed positive integers p and q, suppose that there exists a constant* $r>0$ *so that* (*HP*2) *holds. Let* $\theta_r \in (\pi/2, \pi)$ *be given so that* $\cot\theta_r = ac_r/\sigma_r$, *and define* $\tau_r = c_r\theta_r/\sigma_r$. *If there exists a real number* $s>0$ *such that* $(p+rq)s \geqslant 4$ *is an even integer and* (3.3) *has no nontrivial* $(p+rq)s\tau$*-periodic solution, then for each* $\tau > \tau_r$ *there exists a constant* $c>0$ *such that* (3.1) *has a spatially* (p,q)*-periodic traveling plane wave* $u_{m,n}(t) = x(t-mc-nrc)$ *and the period of x is between* 2τ *and* $(p+rq)s\tau$.

Proof. As has been pointed, the connected component Σ_r through $(0, \tau_r, c_r)$ in the closure of the subset

$$\{(x,\tau,c);\ x \text{ is a nonconstant } (p+rq)c - \text{periodic solution of (3.3) } \tau \in \mathbb{R},\ c \geqslant 0\}$$

of the space $X \times \mathbb{R}^2$ must be nonempty and unbounded, here X is the Banach space of all bounded continuous functions equipped with the super-norm. Note that $\sigma_r = \frac{2\pi}{p+rq}$ and $\frac{\sigma_r \tau_r}{c_r} = \theta_r \in (\pi/2, \pi)$ if $a<0$, it follows that

$$\frac{(p+rq)c_r}{\tau_r} = \frac{2\pi c_r}{\sigma_r \tau_r} = \frac{2\pi}{\theta_r} \in (2,4).$$

Therefore, in the neighborhood of $(0, \tau_r, c_r)$, every element $(x, \tau, c) \in \Sigma_r$ must satisfy $\frac{(p+rq)c}{\tau} \in (2,4) \subset (2, (p+rq)s)$. If $(p+rq)s$ is an even integer and (3.3) has no nontrivial $(p+rq)s\tau$-periodic solution, then (3.3) has no nontrivial 2τ-periodic solution. Since Σ_r is connected, Lemma 3.3 implies that the unbounded component Σ_r must satisfy

$$\Sigma_r \subset \left\{(x, \tau, c); \sup_{t\in\mathbb{R}} |x(t)| \leqslant M,\ \frac{(p+rq)c}{\tau} \in (2, (p+rq)s)\right\}.$$

We now claim that Σ_r does not intersect with the hyperplane $\tau = 0$. In fact, if $(x, 0, c) \in \Sigma_r$ for some $x \in X$ and $c \geqslant 0$, then there exists a sequence $(x_n, \tau_n, c_n) \in \Sigma_r$ such that $x_n \to x$ in X, $\tau_n \to 0$ and $c_n \to c$. As $\frac{(p+rq)sc_n}{\tau_n} \in (2, (p+rq)s)$, we must have $c = 0$. Therefore, x must satisfy the ordinary differential equation

$$\frac{d}{dt}x(t) = f(x(t), x(t)).$$

Now the assumption $xf(x,x)<0,\ x\neq 0$ (HP1) implies that $x = 0$. This leads to a contradiction to the obvious fact that $(0,0,0) \notin \Sigma_r$.

Therefore, the projection of Σ_r onto the τ-space is unbounded and is contained in $[0, \infty)$. This shows that for every $\tau > \tau_r$, there exists a $(p+rq)c$-periodic solution of (3.1) with $c_1 = c$, $c_2 = rc$ and $(p+rq)c \in (2\tau, (p+rq)s\tau)$. This completes the proof. ∎

The above result shows that τ_r is the critical value of delay where a branch of spatially (p,q)-periodic plane waves bifurcates from the trivial solution. The *profile* x is of a period larger than 2τ and thus will be called *slowly oscillating plane waves*. See [15,22].

Using a similar argument for $\Sigma_{r,\ell}$ $(\ell \geqslant 1)$, the connected component through $(0, \tau_{r,\ell}, c_r)$ in the closure of the subset

$$\{(x, \tau, c);\ x \text{ is a nonconstant } (p+rq)c - \text{periodic solution of (3.3)},\ \tau \in \mathbb{R},\ c \geqslant 0\}$$

of the space $X \times \mathbb{R}^2$, we get the following coexistence of one slowly oscillating plane wave and multiple rapidly oscillating plane waves of (3.1).

THEOREM 3.2. *Assume that conditions of Theorem* 3.1 *hold. Let* $\theta_r \in (\pi/2, \pi)$ *be given so that* $\cot\theta_r = ac_r/\sigma_r$. *Define*

$$c_r = \frac{\sigma_r}{\sqrt{(\varpi_r - b)^2 - a^2}}, \qquad \tau_r = c_r\theta_r/\sigma_r,$$

$$\tau_{r,\ell} = \frac{c_r}{\sigma_r}\theta_{r,\ell}, \qquad \theta_{r,\ell} = \theta_r + \ell\pi, \qquad \ell \geqslant 0.$$

Then for each fixed integer $\ell \geqslant 0$ *and for each* $\tau > \tau_{r,\ell}$ *there exist constants* $c^{(k)} > 0,\ 0 \leqslant k \leqslant \ell$, *such that* (3.1) *has* $\ell + 1$ *spatially* (p, q)*-periodic traveling plane waves* $u^{(k)}_{m,n}(t) = x^{(k)}(t - mc^{(k)} - nrc^{(k)}),\ k = 0, 1, \dots, \ell$, *and the period* $(p + rq)c^{(k)}$ *of* $x^{(k)}$ *satisfies* $(p + rq)c^{(0)} \in (2\tau, (p + rq)s\tau),\ (p + rq)c^{(k)} \in (\frac{2}{k+1}\tau, \frac{2}{k}\tau),\ 1 \leqslant k \leqslant \ell$, *respectively.*

As an application of the above results, we now consider the following planar lattice differential system:

$$\begin{aligned}\frac{d}{dt}u_{m,n}(t) = &- \alpha u_{m,n}(t) + a_0 g(u_{m,n}(t-\tau)) \\ &+ \sum_{j=1}^{N} a_j[g(u_{m-j,n}(t-\tau)) + g(u_{m+j,n}(t-\tau)) \\ &+ g(u_{m,n-j}(t-\tau)) + g(u_{m,n+j}(t-\tau))],\end{aligned} \tag{3.21}$$

and the associated wave equation

$$\begin{aligned}\frac{d}{dt}x(t) = &- \alpha x(t) + a_0 g(x(t-\tau)) \\ &+ \sum_{j=1}^{N} a_j[g(x(t-\tau+jc)) + g(x(t-\tau-jc)) \\ &+ g(x(t-\tau+jrc)) + g(x(t-\tau-jrc))],\end{aligned} \tag{3.22}$$

where $\alpha > 0,\ g : \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

We have the following existence result for periodic traveling plane waves of (3.21).

THEOREM 3.3. *Assume that*

(i) $g \in C^1(\mathbb{R}, \mathbb{R}),\ g(0) = 0,\ |g(x)| \leqslant M$ *for some* $M > 0$ *and* $v := g'(0) \geqslant g'(x) > 0$;

(ii) $v(a_0 + 4\sum_{j=1}^{N} a_j) < \alpha$;

(iii) *For fixed positive integers p and q, there exists a real number $r>0$ such that* $M_r := (\varpi_r - v(a_0 + 4\sum_{j=1}^N a_j))^2 - \alpha^2 > 0$, *where*

$$\varpi_r = 4v\sum_{j=1}^N a_j\left[\sin^2\frac{\pi j}{p+rq} + \sin^2\frac{r\pi j}{p+rq}\right];$$

(iv) *There exists a real number $s>0$ such that $(p+rq)s \geqslant 4$ is an even integer and $v\gamma_{r,s} < \alpha$, where*

$$\gamma_{r,s} = \max\left\{\Re\left(\sum_{j=1}^{(p+rq)s} b_j e^{i(2\pi/(p+rq)s)(j-1)k}\right);\ k = 0,1,\ldots,(p+rq)s-1\right\},$$

$$b_2 = a_0 + \sum\{a_j;\ 1\leqslant j\leqslant N,\ \ -sj \text{ or } sj \text{ or } -rsj \text{ or } rsj \equiv 0 \ (\mathrm{mod}\,(p+rq)s)\},$$

$$b_i = \sum\{a_j;\ 1\leqslant j\leqslant N,\ \ -sj \text{ or } sj \text{ or } -rsj \text{ or } rsj \equiv i-2 \ (\mathrm{mod}\,(p+rq)s)\},\quad 1\leqslant i\leqslant (p+rq)s,\ \ i\neq 2.$$

Let $\theta_r \in (\pi/2,\pi)$ be given so that $\cot\theta_r = -\alpha/\sqrt{M_r}$, *and define* $\tau_r = \theta_r/\sqrt{M_r}$. *Then for each $\tau > \tau_r$ there exists a constant $c>0$ such that* (3.21) *has a spatially (p,q)-periodic traveling plane wave $u_{m,n}(t) = x(t-mc-nrc)$ and the period of x is between 2τ and $(p+rq)s\tau$.*

Proof. Let $f(x,y) = -\alpha x + (a_0 + 4\sum_{j=1}^N a_j)g(y)$, then (3.22) can be written as

$$\begin{aligned}\frac{d}{dt}x(t) &= f(x(t),x(t-\tau)) \\ &\quad + \sum_{j=1}^N a_j[g(x(t-\tau+jc)) + g(x(t-\tau-jc)) \\ &\quad + g(x(t-\tau+jrc)) + g(x(t-\tau-jrc)) - 4g(x(t-\tau))].\end{aligned} \tag{3.23}$$

Clearly, $a := \frac{\partial}{\partial x}f(0,0) = -\alpha < 0$, $b := \frac{\partial}{\partial y}f(0,0) = v(a_0 + 4\sum_{j=1}^N a_j)$. Therefore, Condition (ii) implies that $a+b\neq 0$ and for $x\neq 0$,

$$\begin{aligned}xf(x,x) &= -\left(\alpha - \left(a_0 + 4\sum_{j=1}^N a_j\right)\frac{g(x)}{x}\right)x^2 \\ &\leqslant \begin{cases} -\alpha x^2 < 0 & \text{if } a_0 + 4\sum_{j=1}^N a_j \leqslant 0, \\ -\left(\alpha - v\left(a_0 + 4\sum_{j=1}^N a_j\right)\right)x^2 < 0 & \text{if } a_0 + 4\sum_{j=1}^N a_j > 0. \end{cases}\end{aligned}$$

Thus, (HP1) holds. Clearly, all conditions of Lemma 3.3 and (HP2) also hold.

We claim that under Condition (iv), (3.22) has no nontrivial $(p+rq)s\tau$-periodic solution, thus, by virtue of Theorem 3.1, for each $\tau > \tau_r$ there exists a constant $c > 0$ such that (3.21) has a spatially (p, q)-periodic traveling plane wave $u_{m,n}(t) = x(t - mc - nrc)$ and the period of x is between 2τ and $(p+rq)s\tau$.

In what follows, we prove that (3.22) has no nonconstant $(p+rq)s\tau$-periodic solution. By way of contradiction, we suppose that $x(t)$ is a nontrivial $(p+rq)s$-periodic solution of (3.22). Define

$$\begin{aligned} &x_i(t) = x(t - i\tau), \qquad 1 \leqslant i \leqslant (p+rq)s; \\ &X(t) = (x_1(t), x_2(t), \ldots, x_{(p+rq)s}(t))^T; \\ &G(X(t)) = (g(x_1(t)), g(x_2(t)), \ldots, g(x_{(p+rq)s}(t)))^T. \end{aligned}$$

Then we get

$$\dot{X}(t) = -\alpha X(t) + BG(X(t)), \tag{3.24}$$

where B is the $(p+rq)s \times (p+rq)s$ circulant matrix

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_{(p+rq)s} \\ b_{(p+rq)s} & b_1 & b_2 & \cdots & b_{(p+rq)s-1} \\ b_{(p+rq)s-1} & b_{(p+rq)s} & b_1 & \cdots & b_{(p+rq)s-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_2 & b_3 & b_4 & \cdots & b_1 \end{pmatrix}.$$

Let $V(X) = \sum_{j=1}^{(p+rq)s} \int_0^{x_j} g(x)\,dx$. Then

$$\dot{V}_{(3.24)}(X(t)) = -\alpha[X(t)]^T G(X(t)) + [G(X(t))]^T BG(X(t)).$$

By using Nussbaum's spectral theorem for circulant matrices [14], we find

$$[G(X(t))]^T BG(X(t)) \leqslant \gamma_{r,s}[G(X(t))]^T G(X(t)).$$

Therefore, we have

$$\begin{aligned} \dot{V}_{(3.24)} &\leqslant -\sum_{j=1}^{(p+rq)s} x_j(t) g(x_j(t)) \left[\alpha - \frac{g(x_j(t))}{x_j(t)} \gamma_{r,s}\right] \\ &\leqslant -[\alpha - v\gamma_{r,s}] \sum_{j=1}^{(p+rq)s} x_j(t) g(x_j(t)). \end{aligned}$$

By virtue of LaSSalle's Invariance Principle [9], we conclude that $X(t)$ is convergent to a constant as $t \to \infty$. This contradicts with the fact that $x(t)$ is a nonconstant periodic solution. The proof of Theorem 3.3 is complete.

Corresponding to Theorem 3.2, we have the coexistence of at least one slowly oscillating and multiple rapidly oscillating plane waves.

THEOREM 3.4. *Assume that all conditions of Theorem* 3.3 *hold. Let* $\theta_r \in (\pi/2, \pi)$ *be given so that* $\cot\theta_r = -\alpha/\sqrt{M_r}$, *and define*

$$\tau_{r,\ell} = \theta_{r,\ell}/\sqrt{M_r}, \qquad \theta_{r,\ell} = \theta_r + \ell\pi, \ \ \ell \geqslant 0.$$

Then for each fixed integer $\ell \geqslant 0$ *and for each* $\tau > \tau_{r,\ell}$ *there exist constants* $c^{(k)} > 0, \ 0 \leqslant k \leqslant \ell$, *such that* (3.21) *has* $\ell + 1$ *spatially* (p, q)*-periodic traveling plane waves* $u^{(k)}_{m,n}(t) = x^{(k)}(t - mc^{(k)} - nrc^{(k)}), \ k = 0, 1, \ldots, \ell$, *and the period* $(p + rq)c^{(k)}$ *of* $x^{(k)}$ *satisfies* $(p + rq)c^{(0)} \in (2\tau, (p + rq)s\tau), \ (p + rq)c^{(k)} \in (\frac{2}{k+1}\tau, \frac{2}{k}\tau), \ 1 \leqslant k \leqslant \ell$, *respectively.*

Example 3.1. We return to the neural network model (2.21). It was observed by Hopfield [4,5], Marcus and Westervelt [12] and Wu and Zou [21] that cells do not communicate and response instantaneously and sustained oscillations can arise from large relative size of the delay (relative to the relaxation time of the system) in the communication and response among cells. This naturally leads to the following infinite system of delay differential equations:

$$\begin{aligned} \frac{d}{dt}u_{m,n}(t) = & -\alpha u_{m,n}(t) + a_0 g(u_{m,n}(t-\tau)) \\ & + a_1[g(u_{m-1,n}(t-\tau)) + g(u_{m+1,n}(t-\tau)) \\ & + g(u_{m,n-1}(t-\tau)) + g(u_{m,n+1}(t-\tau))]. \end{aligned} \tag{3.25}$$

Clearly, its wave equation takes the form

$$\begin{aligned} \frac{d}{dt}x(t) = & -\alpha x(t) + a_0 g(x(t-\tau)) \\ & + a_1[g(x(t-\tau+c)) + g(x(t-\tau-c)) \\ & + g(x(t-\tau+rc)) + g(x(t-\tau-rc))]. \end{aligned} \tag{3.26}$$

In the case where $p = 1, \ q = 2, \ r = 1/2$ and $s = 2$, we see that $b_1 = a_1, \ b_2 = a_0, \ b_3 = a_1, \ b_4 = 2a_1$ and

$$\Re\left(\sum_{j=1}^{4} b_j\right) = 4a_1 + a_0, \qquad \Re\left(\sum_{j=1}^{4} b_j e^{i\frac{\pi}{2}(j-1)}\right) = b_1 - b_3 = 0,$$

$$\Re\left(\sum_{j=1}^{4} b_j e^{i\pi(j-1)}\right) = b_1 - b_2 + b_3 - b_4 = -a_0,$$
$$\Re\left(\sum_{j=1}^{4} b_j e^{i\frac{3\pi}{2}(j-1)}\right) = b_1 - b_3 = 0.$$

Therefore, we have

$$\gamma_{r,s} = \max\left\{\Re\left(\sum_{j=1}^{4} b_j e^{i\frac{\pi}{2}(j-1)k}\right);\ k = 0,1,2,3\right\} = \max\{a_0 + 4a_1, 0, -a_0\}.$$

Moreover, by Theorem 3.3, we can prove the following.

COROLLARY 3.1. *Assume that* $g \in C^2(\mathbb{R},\mathbb{R})$, $g(0) = 0$, $\lim_{x\to\pm\infty} g(x) = \pm 1$, $g'(x) > 0$ *and* $xg''(x) < 0$ *for* $x \neq 0$, *and that*

$$a_0 + 4a_1 < \frac{\alpha}{v},$$
$$|a_0 - 2a_1| > \frac{\alpha}{v},$$
$$a_0 > -\frac{\alpha}{v}$$

that is, (a_0, a_1) *belongs to the shaded region in Fig. 2. Let* $\theta_{1/2} \in (\pi/2, \pi)$ *be given so that* $\cot\theta_{1/2} = -\frac{\alpha}{\sqrt{v^2(2a_1-a_0)^2-\alpha^2}}$, *then for each* $\tau > \tau_{1/2} := \frac{\theta_{1/2}}{\sqrt{v^2(2a_1-a_0)^2-\alpha^2}}$ *there exists a constant* $c > 0$ *such that* (3.25) *has a spatially*

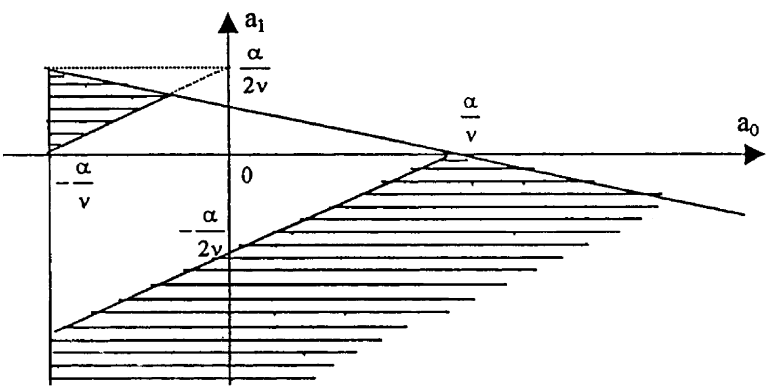

FIG. 2. The region of (a_0, a_1) when (3.25) has a slowly oscillating spatially $(1,2)$ – periodic traveling plane wave.

(1, 2)-periodic traveling plane wave $u_{m,n}(t) = x(t - mc - nc/2)$ *and the period of* x *is between* 2τ *and* 4τ.

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