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Subharmonic solutions of systems of difference equations with periodic perturbations Part I: Existence

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Abstract

We consider the existence of subharmonic solutions of systems of difference equations with periodic perturbations. The theory of coincidence degree, coupled with some detailed a priori estimates, is applied to show that the perturbed system admits subharmonic solutions near a hyperbolic periodic orbit of the unperturbed system if the perturbation is sufficiently small.

1. Introduction

In the last two decades, the existence of subharmonic solutions has been extensively studied for several important classes of dynamical systems such as periodic

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reaction–diffusion equations [4], forced wave equations [16], nonlinear second-order differential equations [10] and nonautonomous Hamiltonian systems [19], to name a few.

Most existing studies deal with the following periodic perturbed systems:

$$\dot{x} = F(x) + \epsilon G(x, t), \tag{1.1}$$

where $x \in \mathbf{R}^N$, $G : R^N \times \mathbf{R} \to \mathbf{R}^N$ is T-periodic with respect to the t-argument, F is a vector field in $\mathbf{R}^N$ such that (1.1) with $\epsilon = 0$ has an orbit homoclinic at a hyperbolic saddle point. See [3,8,9,13,14,20,22,23,25] and references therein.

While subharmonic and superharmonic periodic solutions were studied in [11] for some difference equations, results in the spirit of the aforementioned work on (1.1) were only recently obtained by Agarwal and Zhang [2] for the following scalar difference equation:

$$x(n+1) = f(x(n)) + \epsilon g\big(n, x(n), x(n+1)\big). \tag{1.2}$$

Under the following assumptions:

(a) the map f has an attracting cycle $\gamma = \{w_1, w_2, \ldots, w_p\}$ and its multiplicator satisfies $|\mu_\gamma| < 1$,
(b) both f and g are C^1-smooth,
(c) g is m-periodic in n, where $m \in \mathbf{Z}_+$, the set of all positive integers,

they proved that there is a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the difference equation (1.2) has $[m, p]$ asymptotically stable piecewise continuous $[m, p]$-periodic solutions, where $[m, p]$ denotes the least common multiple of m and p. Other related problems were studied in [7,14,18,21].

In this and a subsequent paper, we consider subharmonic solutions for the discrete perturbed system

$$x(n+1) = f(x(n)) + \mu g(n, x(n), \mu), \tag{1.3}$$

where μ_0 is a positive constant, $\mu \in \mathbf{R}$ with $0 \leqslant |\mu| \leqslant \mu_0$, $x(n) \in \mathbf{R}^N$. Moreover, we assume:

(H_1) The unperturbed system $x(n+1) = f(x(n))$ possesses a k-periodic orbit $\{x_n\}_{n=-\infty}^{\infty}$ for a given positive integer k. $f(x)$ is continuously differential in a neighborhood of $\{x_0, \ldots, x_{k-1}\}$ and the orbit $\{x_n\}_{n \in \mathbf{Z}}$ is hyperbolic [15].
(H_2) The mapping $g : \mathbf{Z} \times \mathbf{R}^N \times [-\mu_0, \mu_0] \to \mathbf{R}^N$ is continuous and w-periodic in n, and there is some $i_* \in \{0, \ldots, k-1\}$ so that for any μ with $|\mu| \leqslant \mu_0$, the minimum period of $g(n, x_{i_*}, \mu)$ with respect to n is ω.

We refer to [1,5,12,17] for a good introduction to the general qualitative theory and the existence of periodic solutions of the autonomous difference equation (1.3) with $\mu = 0$. A solution $x(n)$ of the discrete system (1.3) is said to be subharmonic if it is $m\omega$-periodic for some positive integer m.

Let $\omega_k = (k, \omega)$ denote the greatest common factor of k and ω, and $k_\omega = [k, \omega]$. In this paper, we prove that under (H_1) and (H_2), there are a constant μ_* and a continuous function $r:[-\mu_*, \mu_*] \to (0, \infty)$ satisfying $\lim_{\mu\to 0} r(\mu) = 0$ such that for any μ with $0 < |\mu| \leqslant \mu_*$ and every given integers i and j with $i \in \{0, \ldots, k-1\}$ and $j \in \{0, \ldots, \omega - 1\}$, (1.3) has a k_ω-periodic solution $x^*(n, i, j, \mu)$ satisfying

$$\left|x^*(n, i, j, \mu) - x_{i+n}\right| < r(\mu), \quad \text{for all } n \in \mathbf{Z}.$$

Our approach to the existence problem here is based on a general existence result developed in [24] and the related continuation theorem (see Gaines and Mawhin [6]) as well as some a priori estimates.

The rest of this paper is organized as follows. In Section 2, a general existence result for the existence of periodic solutions of difference equations (from [24]) is introduced. In Section 3, our main existence result for subharmonic solutions of (1.3) is proved by applying the general result of [24] and some a priori estimates.

In a subsequent paper, under the hypothesis that the perturbation is locally Lipschitz continuous, we prove that (1.3) has k distinct k_ω-periodic solutions and ω_k distinct k_ω-period orbits near the periodic orbit of the unperturbed system. We also prove that the k_ω-periodic solutions are stable (unstable) provided that the k-period orbit of the unperturbed system is stable (unstable).

2. Existence of periodic solutions

Let $\mathbf{R}$ and $\mathbf{Z}$ denote the sets of all real numbers and integers, respectively. We fix two integers $\omega^* \geqslant 1$ and $N \geqslant 1$. Define

$$l_N = \left\{x = \{x(n)\}\colon\ x(n) \in \mathbf{R}^N,\ n \in \mathbf{Z}\right\}.$$

For a sequence of mappings $\{G_n\colon\ n \in \mathbf{Z}\}$ with $G_n : l_N \to \mathbf{R}^N$, we use $G = \{G_n\}$ to denote the mapping $G : l_N \to l_N$ defined by

$$G(x) = \{G_n(x)\} \quad \text{for } x \in l_N.$$

For $a = (a_1, \ldots, a_N) \in \mathbf{R}^N$, define $|a| = \max_{1\leqslant j\leqslant q} |a_j|$. Let $l^{\omega^*} \subseteq l_N$ denote the subspace of all ω^*-periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e.,

$$\|x\| = \max_{0\leqslant n\leqslant \omega^*-1} |x(n)| \quad \text{for } x = \left\{x(n)\colon\ n \in \mathbf{Z}\right\} \in l^{\omega^*}.$$

It is easy to see that l^{ω^*} is a finite-dimensional Banach space.

We let $\Omega \subset l^{\omega^*}$ be an open bounded subset and $G : \Omega \to l^{\omega^*}$ a continuous mapping which maps every bounded subset of Ω into bounded set of l^{ω^*}. For $r > 0$, we set

$$\Omega_r = \left\{x = \{x(n)\} \in l^{\omega^*}\colon\ \|x\| < r\right\}.$$

It is clear that Ω_r is an open, bounded, convex and symmetric neighborhood of $0 \in l^\omega$.

Now, we consider the existence of ω^*-periodic solutions of the difference equation

$$\Delta x(n) = G_n(x), \tag{2.1}$$

where $x = \{x(n)\} \in l_N$, $\Delta x(n) = x(n+1) - x(n)$ and $G_n : l_N \to \mathbf{R}^N$ for $n \in \mathbf{Z}$.

The following existence theorem for ω-periodic solutions of Eq. (2.1) will be needed in next section. This is Theorem 2.4 of [24].

Theorem 2.1. *Assume that there exists $r > 0$ such that*

(i) $G = \{G_n\} : \overline{\Omega}_r \to l^{\omega^*}$ *is a continuous mapping whose image is a bounded set of* l^{ω^*}.
(ii) *There is no point $x \in \partial\Omega_r$ such that*

$$\Delta x(n) = \frac{1}{1+\lambda} G_n(x) - \frac{\lambda}{1+\lambda} G_n(-x)$$

for some $\lambda \in [0, 1]$.

Then (2.1) *has an ω^*-periodic solution $x_{\omega^*} = \{x_{\omega^*}(n)\}$ satisfying $\|x_{\omega^*}\| < r$.*

3. Existence of subharmonic solutions

In this section, we prove the existence theorem of k_ω-periodic solutions of (1.3), after a series technical lemmas.

We denote by $L(\mathbf{R}^N)$ the space of all linear bounded operators from $\mathbf{R}^N$ to $\mathbf{R}^N$. From now on, we write V_r for $\{y \in \mathbf{R}^N\colon |y| < r\}$ and $V_r(x) = \{y \in \mathbf{R}^N\colon |y - x| < r\}$ for a given $x \in \mathbf{R}^N$. Let

$$A_i = Df^k(x_i), \quad 0 \leqslant i \leqslant k-1, \tag{3.1}$$

$$\alpha = \max_{0 \leqslant i \leqslant k-1} \|Df(x_i)\|, \tag{3.2}$$

where $\|\cdot\|$ is the norm of $L(\mathbf{R}^N)$ corresponding to a given Euclidean metric $|\cdot|$ of $\mathbf{R}^N$.

By (H_1) and [15], we infer that A_i is hyperbolic, i.e., there exist constants $K \geqslant 1$ and $0 < \theta < 1$ such that for each $0 \leqslant i \leqslant k-1$,

(1) $\mathbf{R}^N = \mathbf{R}_i^s \oplus \mathbf{R}_i^u$, $A_i[\mathbf{R}_i^s] \subseteq \mathbf{R}_i^s$, $A_i[\mathbf{R}_i^u] \subseteq \mathbf{R}_i^u$, $\mathbf{R}_i^s$ and $\mathbf{R}_i^u$ are closed subspace.
(2)

$$\left|A_i^m y^s\right| \leqslant K\theta^m |y^s|, \quad \text{for all } y^s \in \mathbf{R}_i^s,\ m \geqslant 1, \tag{3.3}$$

$$\left|A_i^{-m} y^u\right| \leqslant K\theta^m |y^u|, \quad \text{for all } y^u \in \mathbf{R}_i^u,\ m \geqslant 1. \tag{3.4}$$

From (H_1) and (H_2), we can choose a constant $r_0 > 0$ sufficiently small so that

$$r_0 < |x_i - x_j|, \quad \text{for } i \not\equiv j \pmod{k}, \tag{3.5}$$

and

$$g(n, y, \mu) \neq g(m, y, \mu), \tag{3.6}$$

with arbitrary $0 < |\mu| \leqslant \mu_0$, $m \not\equiv n \pmod{\omega}$ and $y \in \overline{V}_{r_0}(x_{i_*})$.

For $n \in \mathbf{Z}$ and $y \in \overline{V}_{r_0}$, we have

$$f(x_n + y) = f(x_n) + Df(x_n)y + L_n(0, y)y, \tag{3.7}$$

where $L_n(x, y)$ denotes the map $L_n : \overline{V}_{(1/2)r_0} \times \overline{V}_{r_0} \to L(\mathbf{R}^N)$ defined by

$$L_n(x, y) = \int_0^1 \big[Df(x_n + x + ty) - Df(x_n)\big]\, dt.$$

$L_n(x, y)$ is k-periodic in n. Then the mapping $\gamma : [0, (1/2)r_0] \to R$ defined by

$$\gamma(r) = \sup_{0 \leqslant \rho \leqslant r} \sup_{\substack{(x,y) \in \overline{V}_r \times \overline{V}_{2r} \\ 0 \leqslant i \leqslant k-1}} \|L_i(x, y)\| \tag{3.8}$$

is monotonically nondecreasing in r, $\lim_{r \to 0^+} \gamma(r) = 0$, and

$$\|L_n(0, y)\| \leqslant \gamma(r), \quad \text{for } |y| \leqslant 2r \leqslant r_0.$$

Define projections $P_1 : \mathbf{R}^N \to \mathbf{R}^s$ and $P_2 : \mathbf{R}^N \to \mathbf{R}^u$ by

$$P_1(y) = y^s, \qquad P_2(y) = y^u,$$

where $y = y^s + y^u$, $y^s \in \mathbf{R}^s$, $y^u \in \mathbf{R}^u$. Let

$$\|P\| = \max\big\{\|P_1\|, \|P_2\|, 1\big\}. \tag{3.9}$$

Then for any $y \in \mathbf{R}^N$

$$\max\big\{|y^s|, |y^u|\big\} \leqslant \|P\||y|. \tag{3.10}$$

Choose $0 < r_1 \leqslant \min\{(1/2)r_0, \sum_{l=0}^{k-1}(\alpha + 1)^l\}$ such that

$$\gamma(r_1) < \frac{1 - \theta}{4\|P\|K} \left[\sum_{l=0}^{k-1} (\alpha + 1)^l\right]^{-1}, \tag{3.11}$$

where α, K, θ are defined in (3.2), (3.3) and (3.4), respectively.

Let

$$M_r = \max_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant j \leqslant \omega-1}} \left\{ \sup_{0 \leqslant \rho \leqslant r} \sup_{\substack{-\mu_0 \leqslant \mu \leqslant \mu_0 \\ x \in \overline{V}_\rho(x_i)}} |g(j, x, \mu)|, 1 \right\} \quad \text{for } 0 < r \leqslant r_1, \tag{3.12}$$

$$\mu_* = \min\left\{\mu_0, \frac{1-\theta}{4\|P\|KM_{r_1}}\left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]^{-2} r_1\right\}, \tag{3.13}$$

$$r(\mu) = \left[\frac{4\|P\|KM_{r_1}}{1-\theta}\sum_{l=0}^{k-1}(\alpha+1)^l\right]|\mu| \quad \text{for } 0 < |\mu| < \mu_*, \tag{3.14}$$

$$k_\omega = ab\omega_k = a\omega = bk, \tag{3.15}$$

where a and b are relatively prime integers.

Now, we address the existence of k_ω-periodic solutions of (1.3). In the following, "a k_ω-periodic solution of (1.3)" always means the minimum period of such a solution is k_ω.

Our main result can now be stated as follows:

Theorem 3.1. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied. Then, for every given i and j with $i \in \{0,\dots,k-1\}$ and $j \in \{0,\dots,\omega-1\}$ and for any μ with* $0 < |\mu| < \mu_*$, (1.3) *possesses a k_ω-periodic solution $x^*(n,i,j,\mu)$ satisfying*

$$\left|x^*(n+j,i,j,\mu) - x_{i+n}\right| < r(\mu), \tag{3.16}$$

where μ_* and $r(\mu)$ are defined in (3.13) *and* (3.14), *respectively, and $r(\mu) \to 0$ as $\mu \to 0$.*

The remainder of this section is devoted to the proof of Theorem 3.1. We will always fix (i,j,μ) so that $0 \leqslant i \leqslant k-1$, $0 \leqslant j \leqslant \omega-1$ and $0 < |\mu| < \mu_*$, and for the sake of simplicity, we will omit μ from the notation $x^*(n,i,j,\mu)$.

It is obvious that if $x^*(n,i,j)$ is a solution of (1.3) satisfying $x^*(n+k_\omega,i,j) = x^*(n,i,j)$ and (3.16), then with

$$y^*(n,i,j) = x^*(n+j,i,j) - x_{i+n} \quad \text{for } n \in \mathbf{Z}, \tag{3.17}$$

we obtain a k_ω-periodic solution $\{y^*(n,i,j)\}_{n\in\mathbf{Z}}$ of the equation

$$\begin{aligned} y(n+1) = {} & \big[Df(x_{i+n}) + L_{i+n}(0,y(n))\big]y(n) \\ & + \mu g\big(n+j; x_{i+n}+y(n),\mu\big) \end{aligned} \tag{3.18}$$

with $|y^*(n;i,j)| < r(\mu)$ for $n \in \mathbf{Z}$, and vice versa. We divide the long proof of Theorem 3.1 into two steps.

Step 1. We show that Eq. (3.18) has a periodic solution $y^*(n;i,j)$ with

$$\left|y^*(n;i,j)\right| < r(\mu),$$

where k_ω is a period.

Step 2. We prove that the minimum period of $y^*(n;i,j)$ defined in (3.17) is k_ω.

In order to simplify the notations, for $n \in \mathbf{Z}$, $0 \leqslant m \leqslant k-1$ and $v \in \overline{V}_{r(\mu)}$, we define $T_{ij}(n+m,n)v$ as follows:

(i) If $m = 0$, $T_{ij}(n,n)v = v$.

(ii) If $m \geqslant 1$, $T_{ij}(n+m,n)v = y(n+m;n,v)$, where $y(l;n,v)$, $l \geqslant n$, denotes the solution of (3.18) with $y(n;n,v) = u$.

Lemma 3.2. *For $n \in \mathbf{Z}$, we have*

$$T_{ij}(n+m,n)v \in V_{r_1}, \quad \textit{for } v \in V_{r(\mu)} \textit{ and } 1 \leqslant m \leqslant k-1. \tag{3.19}$$

Proof. Let $v_m = T_{ij}(n+m;n)v$, $0 \leqslant m \leqslant k$ and $r_* = [\sum_{l=0}^{k-1}(\alpha+1)^l]^{-1} r_1$. Then $[\sum_{l=0}^{m}(\alpha+1)^l] r_* \leqslant r_1$ for $0 \leqslant m \leqslant k-1$, and (3.13) yields

$$|\mu| M_{r_1} < r_*, \quad r(\mu) < r_* \quad \text{for } 0 < |\mu| < \mu_*. \tag{3.20}$$

To prove that $|v_m| < r_1$ for $0 \leqslant m \leqslant k-1$, it is sufficient to prove that

$$|v_m| < \left[\sum_{l=0}^{m}(\alpha+1)^l\right] r_* \quad \text{for } 0 \leqslant m \leqslant k-1 \tag{3.21}$$

by induction on m. First, it is clear that (3.21) holds for $m = 0$. For the purpose of induction, assume that for $m = p \geqslant 0$, we have already proved

$$|v_p| < \left[\sum_{l=0}^{p}(\alpha+1)^p\right] r_*. \tag{3.22}$$

For $m = p+1$, by (3.2), (3.12), (3.18) and (3.20), we get

$$\begin{aligned} |v_{p+1}| &= \left|\left[Df(x_{i+n+p}) + L_{i+n+p}(0,v_p)\right] v_p + \mu g(n+j+p, x_{i+n+p}, \mu)\right| \\ &< (\alpha+1)|v_p| + \mu_* M_{r_1}. \end{aligned}$$

Therefore, using (3.22) we get

$$|v_{p+1}| < (\alpha+1)\left\{\left[\sum_{l=0}^{p}(\alpha+1)^l\right] r_*\right\} + r_* = \left[\sum_{l=0}^{p+1}(\alpha+1)^l\right] r_*.$$

Thus, we have by the induction principle proved that (3.22) hold for $0 \leqslant m \leqslant k-1$. The proof of Lemma 3.2 is complete. □

The following result describes some elementary properties of $T_{ij}(n+m,n)$ to be used later.

Lemma 3.3. *$T_{ij}(n+m,n)$ satisfies*:

(i) *$T_{ij}(n,n) = I_N$, where $I_N \in L(\mathbf{R}^N)$ is the identity mapping.*
(ii) *$T_{ij}(n+n_1+n_2, n+n_1) T_{ij}(n+n_1,n) = T_{ij}(n+n_1+n_2,n)$, where n_1, n_2 are nonnegative integers.*
(iii) *$T_{ij}(n+k_\omega+n_1, n+k_\omega) = T_{ij}(n+n_1,n)$.*

Proof. The proof is straightforward if we note that the maps L_n and g are k_ω-periodic in n. $\square$

In the following lemma, we express $T_{ij}(n+m,n)v$ with $k \geqslant 2$, $2 \leqslant m \leqslant k$ and $n \in \mathbf{Z}$, in terms of v and $T_{ij}(n+l,n)v$, $0 \leqslant l \leqslant m-1$.

Lemma 3.4. *If $k \geqslant 2$, then for $2 \leqslant m \leqslant k$, a given $n \in \mathbf{Z}$ and $v \in V_{r(\mu)}$, we have*

$$T_{ij}(n+m,n)v = \left(\prod_{l=0}^{m-1} \mathbf{Q}_l\right) v + \mu \sum_{l=0}^{m-2} \left(\prod_{q=l+1}^{m-1} \mathbf{Q}_q\right) g_l + \mu g_{m-1}, \tag{3.23}$$

where $\prod_{l=0}^{q} a_l$ denotes the product $a_q a_{q-1} \dots a_1 a_0$, and

$$\begin{aligned} \mathbf{Q}_l &= Df\big(x_{i+n+l} + L_{i+n+l}\big(0, T_{ij}(n+l,n)v\big)\big), \\ g_l &= g\big(j+n+l, x_{i+n+l} + T_{ij}(n+l,n)v, \mu\big). \end{aligned}$$

Proof. Using above notations, we can rewrite (3.23) as

$$v_m = \left(\prod_{l=0}^{m-1} \mathbf{Q}_l\right) v_0 + \mu \sum_{l=0}^{m-2} \left(\prod_{q=l+1}^{m-1} \mathbf{Q}_q\right) g_l + \mu g_{m-1}. \tag{3.24}$$

It follows from (3.18) that

$$v_{l+1} = \mathbf{Q}_l v_l + \mu g_l \quad \text{for } 1 \leqslant l \leqslant k-1. \tag{3.25}$$

We now use (3.25) to prove (3.24) by induction on m. First, for $m = 2$,

$$\begin{aligned} v_2 &= \mathbf{Q}_1 v_1 + \mu g_1 = \mathbf{Q}_1(\mathbf{Q}_0 v_0 + \mu g_0) + \mu g_1 \\ &= \left(\sum_{l=0}^{1} \mathbf{Q}_l\right) v_0 + \mu \sum_{l=0}^{0} \left(\prod_{q=0+1}^{1} \mathbf{Q}_q\right) g_0 + \mu g_1. \end{aligned}$$

Thus, (3.24) holds for $m = 2$.

If $k = 2$, the proof is complete. Assume now $k \geqslant 3$, and assume that for $m = p \leqslant k-1$, we have already proved

$$v_p = \left(\prod_{l=0}^{p-1} \mathbf{Q}_l\right) v_0 + \mu \sum_{l=0}^{p-2} \left(\prod_{q=l+1}^{p-1} \mathbf{Q}_q\right) g_l + \mu g_{p-1}.$$

For $m = p+1$, we have

$$\begin{aligned} v_{p+1} &= \mathbf{Q}_p v_p + \mu g_p \\ &= \left(\prod_{l=0}^{p} \mathbf{Q}_l\right) v_0 + \mu \sum_{l=0}^{p-2} \left(\prod_{q=l+1}^{p} \mathbf{Q}_q\right) g_l + \mu \mathbf{Q}_p g_{p-1} + \mu g_p \\ &= \left(\prod_{l=0}^{(p+1)-1} \mathbf{Q}_l\right) v_0 + \mu \sum_{l=0}^{(p+1)-2} \left(\prod_{q=l+1}^{(p+1)-1} \mathbf{Q}_q\right) g_l + \mu g_{(p+1)-1}. \end{aligned}$$

Therefore we have proved by the induction principle that (3.24) holds. The proof of Lemma 3.4 is complete. □

The expansion formulas established in Lemma 3.4 yield the following consequence immediate consequence:

Lemma 3.5. *If $k \geqslant 2$, then, for $n \in \mathbf{Z}$ and $v \in V_{r(\mu)}$,*

$$T((n+1)k, nk)v = A_i v + F_{ij}(n, v)v + \mu H_{ij}(n, v, \mu), \tag{3.26}$$

where

$$F_{ij}(n, v) = \prod_{l=0}^{k-1}\bigl[Df(x_{i+l}) + L_{i+l}\bigl(0, T_{ij}(nk+l, nk)v\bigr)\bigr] - \prod_{l=0}^{k-1} Df(x_{i+l}), \tag{3.27}$$

$$\begin{aligned} H_{ij}(n, v, \mu) &= \sum_{l=0}^{k-2}\left\{\prod_{q=l+1}^{k-1}\bigl[Df(x_{i+q}) + L_{i+l}\bigl(0, T_{ij}(nk+q, nk)v\bigr)\bigr]\right\} \\ &\quad \times g\bigl(j+nk+l, x_{i+l} + T_{ij}(nk+l, nk)v, \mu\bigr) \\ &\quad + g\bigl(j+nk+k-1, x_{i+k-1} + T_{ij}(nk+k-1, nk)v, \mu\bigr). \end{aligned} \tag{3.28}$$

Both $F_{ij}(n, v)$ and $H_{ij}(n, v, \mu)$ are b-periodic mappings and for $v \in \overline{V}_{r(\mu)}$,

$$\bigl\|F_{ij}(n, v)\bigr\| \leqslant \left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]\gamma(r_1), \tag{3.29}$$

$$\bigl|H_{ij}(n, v, \mu)\bigr| \leqslant \left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]M_{r_1}. \tag{3.30}$$

Proof. Replacing n by nk and m by k in (3.23), respectively, and noting that $\{x_n\}_{n\in\mathbf{Z}}$ is k-periodic, we obtain (3.26)–(3.28). As $\{x_n\}_{n\in\mathbf{Z}}$ and $\{g(n, x, \mu)\}_{n\in\mathbf{Z}}$ both are k_ω-periodic in n and since $k_\omega = bk$, we conclude that $F_{ij}(n, v)$ and $H_{ij}(n, v, \mu)$ are b-periodic in n. It remains to verify the boundedness of $F_{ij}(n, v)$ and $H_{ij}(n, v, \mu)$, as expressed in (3.29) and (3.30).

From (3.19), we get

$$\bigl|T_{ij}(nk+l, nk)v\bigr| < r_1, \quad 0 \leqslant l \leqslant k-1.$$

Thus we have

$$\bigl\|L_{i+l}\bigl(0, T_{ij}(nk+l, nk)v\bigr)\bigr\| \leqslant \gamma(r_1).$$

Using (3.27), we get

$$\begin{aligned}\|F_{ij}(n,v)\| &\leqslant \sum_{l=1}^{k}\binom{l}{k}\gamma^l(r_1)\alpha^{k-l} = [\alpha+\gamma(r_1)]^k-\alpha^k \\ &= \left[\sum_{m=1}^{k}(\alpha+\gamma(r_1))^{k-m}\gamma^{m-1}(r_1)\right]\gamma(r_1) \\ &\leqslant \left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]\gamma(r_1),\end{aligned}$$

where $\binom{l}{k}$ denotes the coefficient of x^l in the expansion of $(1+x)^k$. This gives the desired estimation (3.29).

From (3.12) and arguing as above, we get

$$\left|g\bigl(j+nk+l, x_{i+l}+T_{ij}(nk+l,nk)v,\mu\bigr)\right| \leqslant M_{r_1}.$$

This, together with (3.10), yields (3.30). The proof of Lemma 3.5 is complete. □

Now, we can describe the key observation (to be verified later). If $\{z^*(n)\}_{n\in\mathbf{Z}}$ with $|z^*(n)| < r(\mu)$ is a b-periodic solution of the equation

$$z(n+1) = A_i z(n) + F_{ij}(n,z(n))z(n) + \mu H_{ij}(n,z(n),\mu), \tag{3.31}$$

where $F_{ij}(n,v)$ and $H_{ij}(n,v,\mu)$ are defined as in (3.27) and (3.28), respectively, then $\{y^*(n;i,j)\}$ defined by

$$\begin{aligned}&y^*(nk+m;i,j) = T_{ij}(nk+m,nk)z^*(n;i,j),\\ &\quad n\in\mathbf{Z},\ 0\leqslant m\leqslant k-1,\end{aligned} \tag{3.32}$$

is a k_ω-periodic solution of (3.18) with $|y^*(nk;i,j)| < r(\mu)$.

To prove (3.31) has a b-periodic solution, we will used the argument similar to that in [24]. Namely, we note that (3.31) can be rewritten as

$$\Delta z(n) = G_n(z,\mu,i,j),$$

where $G_n(z,\mu,i,j) = (A_i - I_N)z(n) + F_{ij}(n,z(n))z(n) + \mu H_{ij}(n,z(n),\mu)$. Let

$$G(z,\mu,i,j) = \{G_n(z,\mu,i,j)\}_{n\in\mathbf{Z}}.$$

From (3.11), (3.13), (3.29) and (3.30), we have $|F_{ij}(n,y)| < 1$, $|\mu|\,|H_{ij}(n,y,\mu)| < r_1$ with $n\in\mathbf{Z}$, $|y|\leqslant r(\mu)$ and $0<|\mu|<\mu_*$. These estimates yield that $G=\{G_n\}:\overline{\Omega}_{r(\mu)}\to l^b$ is a continuous mapping whose image is a bounded set of l^b.

In view of Theorem 2.1 in Section 2, to prove that (3.31) has a b-periodic solution $\{z^*(n)\}$, it remains to prove that $G_n(z,\mu,i,j)$ satisfies condition (ii) of Theorem 2.1. This is verified in the following lemma.

Lemma 3.6. *There is no* $z = \{z_n\} \in \partial\Omega_{r(\mu)} \subset l^b$ *and* $\lambda \in [0,1]$ *such that*

$$\Delta z(n) = \frac{1}{1+\lambda} G_n(z, \mu, i, j) - \frac{\lambda}{1+\lambda} G_n(-z, \mu, i, j), \quad n \in \mathbf{Z}.$$

Proof. By way of contradiction, assume there is a constant $0 \leqslant \lambda \leqslant 1$ and a b-periodic sequence $z = \{z(n)\}$ with $\|z\| = r(\mu)$ and

$$\Delta z(n) = \frac{1}{1+\lambda} G_n(z, \mu, i, j) - \frac{\lambda}{1+\lambda} G_n(-z, \mu, i, j),$$

that is,

$$z(n+1) = A_i z(n) + \Phi_{ij}(\lambda, n, z(n), \mu), \tag{3.33}$$

where

$$\Phi_{ij}(\lambda, n, z(n), \mu) = \left[\frac{1}{1+\lambda} F_{ij}(n, z(n)) + \frac{\lambda}{1+\lambda} F_{ij}(n, -z(n))\right] z(n) + \mu\left[\frac{1}{1+\lambda} H_{ij}(n, z(n), \mu) - \frac{\lambda}{1+\lambda} H_{ij}(n, -z(n), \mu)\right].$$

We observe by (3.29) and (3.30) that

$$\left|\Phi_{ij}(\lambda, n, z(n), \mu)\right| \leqslant \left[\sum_{l=0}^{k-1} (\alpha+1)^l\right]\left[\gamma(r_1) r(\mu) + |\mu| M_{r_1}\right]. \tag{3.34}$$

According to the notations in (3.3) and (3.4),

$$z(n) = z_i^s(n) + z_i^u(n), \quad z_i^s(n) \in \mathbf{R}_i^{(s)}, \quad z_i^u(n) \in \mathbf{R}_i^{(u)},$$
$$\Phi_{ij}(\lambda, n, z(n), \mu) = \Phi_{ij}^s(\lambda, n, z(n), \mu) + \Phi_{ij}^u(\lambda, n, z(n), \mu),$$

where $\Phi_{ij}^s(\lambda, n, z(n), \mu) \in \mathbf{R}_i^s$, $\Phi_{ij}^u(\lambda, n, z(n), \mu) \in \mathbf{R}_i^{(u)}$. Let $0 \leqslant n_0 \leqslant b-1$ so that $|z(n_0)| = \|z\| = r(\mu)$. We will lead to a contradiction in two cases.

Case A. $|z_i^s(n_0)| > (1/2)|z(n_0)| = r(\mu)$. We have

$$\frac{1}{2} r(\mu) = \frac{1}{2}|z(n_0)| < \left|z_i^s(n_0)\right| \leqslant \|P\| |z_i(n_0)| = \|P\| r(\mu).$$

It follows from (3.33) that

$$z_i^s(n+1) = A_i z_i^s(n) + \Phi_{ij}^s(\lambda, n, z(n), \mu).$$

Furthermore, for any $m \geqslant 1$,

$$z_i^s(n_0 + mb) = A_i^{mb} z_i^s(n_0) + \sum_{l=1}^{mb} A_i^{mb-l} \Phi_{ij}^s\bigl(\lambda, n_0 + l - 1, z(n_0 + l - 1), \mu\bigr).$$

We conclude from (3.3) and (3.34) that

$$\left|z_i^s(n_0+mb)\right| \leqslant \|P\| K\theta^{mb} r(\mu) + \|P\| \left(\sum_{l=1}^{mb} K\theta^{mb-l}\right)\left[\sum_{l=0}^{k-1}(\alpha+1)^l\right] \times \left[\gamma(r_1)r(\mu)+|\mu|M_{r_1}\right].$$

Taking the limit as $m \to \infty$ and using (3.10) and (3.11), we obtain

$$\frac{1}{2}r(\mu) < \left|z_i^s(n_0)\right| \leqslant \frac{1}{4}r(\mu) + \frac{\|P\|K}{1-\theta}\left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]M_{r_1}|\mu|,$$

that is,

$$r(\mu) < \frac{4\|P\|KM_{r_1}}{1-\theta}\left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]|\mu|, \tag{3.35}$$

this contradicts (3.14).

Case B. $|z_i^u(n_0)| \geqslant (1/2)|z(n_0)| = r(\mu)$. Note that (3.33) yields

$$z_i^u(n_0) = A_i^{-mb} z_i^u(n_0+mb) - \sum_{l=1}^{mb} A_i^{-l}\Phi_{ij}^u\big(\lambda, n_0+l-1, z(n_0+l-1), \mu\big).$$

Using this and following the procedure used in Case A, we get

$$r(\mu) \leqslant \frac{4\|P\|K\theta}{1-\theta}\left[\sum_{l=0}^{k-1}(\alpha+1)^l\right]\left[\gamma(r_1)r(\mu)+|\mu|M_{r_1}\right].$$

From this and $0<\theta<1$, we obtain (3.35) which contradicts (3.14). The proof of Lemma 3.6 is complete. □

We can now complete the arguments in Step 1. From now on, the symbol $\{z^*(n)\}_{n\in\mathbf{Z}}$ will denote the b-periodic solution of (3.31) with $|z^*(n)| < r(\mu)$. For the remaining part of the proof of the assertion in Step 1, it suffices to prove

Lemma 3.7. $\{y^*(n;i,j)\}_{n\in\mathbf{Z}}$ *defined in* (3.32) *is a periodic solution of* (3.18) *with*

$$\left|y^*(n;i,j)\right| < r(\mu) \quad \text{for } n \in \mathbf{Z}. \tag{3.36}$$

Proof. We first prove that $\{y^*(n;i,j)\}_{n\in\overline{\mathbf{Z}}}$ is a periodic solution of (3.18). Using (3.32) and the definition of $T_{ij}(n+m,n)$, we get

$$\begin{aligned} y^*(nk+m;i,j) &= y\big(nk+m; nk, z^*(n;i,j)\big) \\ &= y\big(nk+m; nk, y^*(nk;i,j)\big). \end{aligned}$$

As $\{z^*(n;i,j)\}_{n\in\mathbf{Z}}$ is a solution of (3.31), we have

$$y^*\big((n+1)k;i,j\big)=z^*(n+1;i,j)=T_{ij}(nk+k;nk)z^*(nk;i,j)$$
$$=y\big(nk+k;nk,y^*(nk;i,j)\big).$$

So $\{y^*(n;i,j)\}$ satisfies (3.18).

By $k_\omega=bk$ and $z^*(n+b;i,j)=z^*(n;i,j)$ for $n\in\mathbf{Z}$, we obtain

$$y^*(nk+m+k_\omega;i,j)=T_{ij}(nk+bk+m,nk+bk)z^*(n+b;i,j)$$
$$=T_{ij}(nk+k_\omega+m,nk+k\omega)z^*(n;i,j).$$

Lemma 3.3(iii) then yields

$$y^*(nk+m+k_\omega;i,j)=T_{ij}(nk+m,nk)z^*(n;i,j)=y^*(nk+m;i,j),$$

i.e., k_ω is a period of $\{y^*(n;i,j)\}_{n\in\mathbf{Z}}$.

We now show that $\{y^*(n,i,j)\}_{n\in\mathbf{Z}}$ satisfies (3.36). The case where $k=1$ is trivial. So we only consider the case where $k\geqslant 2$. It is sufficient to prove that for each $1\leqslant m\leqslant k-1$ and $n\in\mathbf{Z}$, $|y^*(nk+m;i,j)|<r(\mu)$. If this were false, there would exist $1\leqslant m_0\leqslant k-1$ and $n_0\in\mathbf{Z}$ such that

$$\big|y^*(n_0k+m_0;i,j)\big|=r^0\geqslant r(\mu).$$

Note that

$$\big|y^*(n;i,j)\big|<r_1,\quad n\in\mathbf{Z},\tag{3.37}$$

by virtue of (3.19) and (3.32). Let $0\leqslant i_0\leqslant k-1$ and $0\leqslant j_0\leqslant\omega-1$ such that

$$i+m_0\equiv i_0\ (\mathrm{mod}\,k),\qquad j+m_0\equiv j_0\ (\mathrm{mod}\,\omega).\tag{3.38}$$

Let

$$z^0(n,i_0,j_0)=y^*(nk+m_0;i,j),\quad n\in\mathbf{Z}.\tag{3.39}$$

Our strategy below, intuitively speaking, is to show that $\{z^0(n;i_0,j_0)\}_{n\in\mathbf{Z}}$ is a b-periodic solution of (3.31) with i being replaced by i_0, and j by j_0, respectively. This, as in the proof of Lemma 3.6, will lead to a contradiction. To carry out this strategy, our key step is to prove that

$$T_{ij}(nk+m_0+l,nk+m_0)=T_{i_0j_0}(nk+l,nk)\quad\text{for }1\leqslant l\leqslant k.\tag{3.40}$$

To begin with, we note that it is possible to write $T_{ij}((nk+m_0)+l,nk+m_0)\times z^0(n;i_0,j_0)$ for $1\leqslant l\leqslant k$ and $n\in\mathbf{Z}$ by virtue of (3.37) and

$$z^0(n+1;i_0,j_0)=y^*\big((n+1)k+m_0;i,j\big)$$
$$=T_{ij}\big((nk+m_0)+k,nk+m_0\big)y^*(nk+m_0;i,j)$$
$$=T_{ij}\big((nk+m_0)+k,nk+m_0\big)z^0(n;i_0,j_0).$$

In the following, we show that (3.40) holds by induction. For $l = 1$, (3.18) gives

$$\begin{aligned}
&T_{ij}(nk + m_0 + 1, nk + m_0)v \\
&\quad = \big[Df(x_{i+nk+m_0}) + L_{i+nk+m_0}(0, v)\big]v + \mu g(nk + m_0 + j, v, \mu) \\
&\quad = T_{i_0 j_0}(nk + 1, nk)v.
\end{aligned}$$

For the purpose of induction, assume that for $l = p \leqslant k - 1$, (3.40) holds true for $l \leqslant p$. Then for $l = p + 1$, Lemma 3.3 yields

$$\begin{aligned}
&T_{ij}(nk + m_0 + p + l, nk + m_0)v \\
&\quad = T_{ij}(nk + m_0 + p + l, nk + m_0 + p)T_{ij}(nk + m_0 + p, nk + m_0)v \\
&\quad = \big[Df(x_{i+nk+m_0+p}) + L_{i+nk+m_0+p}\big(T_{i_0 j_0}(nk + p, nk)v\big)\big] \\
&\qquad\quad \times T_{i_0 j_0}(nk + p, nk)v \\
&\qquad + \mu g\big(nk + m_0 + p + j, T_{i_0 j_0}(nk + p, nk)v, \mu\big) \\
&\quad = \big[Df(x_{i_0+p}) + L_{i_0+p}\big(0, T_{i_0 j_0}(nk + p, nk)v\big)\big]T_{i_0 j_0}(nk + p, nk)v \\
&\qquad + \mu g\big(nk + p + j_0, T_{i_0 j_0}(nk + p, nk)v, \mu\big) \\
&\quad = T_{i_0 j_0}(nk + p + 1, nk + p)T_{i_0 j_0}(nk + p, nk)v \\
&\quad = T_{i_0 j_0}(nk + p + 1, nk)v.
\end{aligned}$$

Thus, by the induction principle, we show that (3.40) holds for $1 \leqslant l \leqslant k$. Specially, we get $T_{i_0 j_0}(nk + m_0 + k, nk + m_0) = T_{i_0 j_0}(nk + k, nk)$. It follows that

$$\begin{aligned}
z^0(n + 1; i_0, j_0) &= T_{ij}\big((nk + m_0) + k, nk + m_0\big)z^0(n; i_0, j_0) \\
&= T_{i_0 j_0}((n + 1)k, nk)z^0(n; i_0, j_0).
\end{aligned}$$

It is obvious that $\{z^0(n; i_0, j_0)\}$ has similar properties as $\{z^*(n; i, j)\}$ as a b-periodic solution of the equation

$$z(n + 1) = A_{i_0}z(n) + F_{i_0 j_0}(n, z(n))z(n) + \mu H_{i_0 j_0}(n, z(n), \mu).$$

We can further proceed as in the proof of Lemma 3.6 and show that $|z^0(n_0; i_0, j_0)| = r^0 \geqslant r(\mu)$ contradicts (3.14). This proves that $\{y^*(n, i, j)\}_{n\in\mathbf{Z}}$ satisfies (3.36), and completes the proof of Lemma 3.7. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. According to (3.17), we have

$$x^*(n + j; i, j) = x_{i+n} + y^*(n; i, j) \quad \text{for } n \in \mathbf{Z}. \tag{3.41}$$

It is easy to see that $x^*(n + k_\omega; i, j) = x^*(n; i, j)$ for $n \in \mathbf{Z}$ and

$$\big|x^*(n + j; i, j) - x_{i+n}\big| < r(\mu).$$

We also obtain from (3.41) and (3.18), for any $n \in \mathbf{Z}$, that

$$\begin{aligned} x^*(n+j+1;i,j) &= x_{i+n+1} + y^*(n+1;i,j) \\ &= f(x_{i+n}) \\ &\quad + \big[Df(x_{i+n}) + L_{i+n}\big(0, y^*(n;i,j)\big)\big] y^*(n;i,j) \\ &\quad + \mu g\big(n+j; x_{i+n} + y^*(n;i,j), \mu\big), \\ y^*(n,i,j,\mu) &= f\big(x_{i+n} + y^*(n;i,j)\big) + \mu g\big(n+j; x^*(n+j;i,j), \mu\big) \\ &= f\big(x^*(n+j;i,j)\big) + \mu g\big(n+j; x^*(n+j;i,j), \mu\big), \end{aligned}$$

that is, $\{x^*(n;i,j)\}_{n\in\mathbf{Z}}$ is a solution of (1.3).

Note that the conclusion stated in Step 1 has been verified above, in Lemma 3.7. We now complete the proof for the conclusion in Step 2. Namely, we prove that k_ω is the minimum period of $x^*(n;i,j)$. It will suffice to prove that if ω_0 makes $x^*(n+\omega_0;i,j) = x^*(n;i,j)$ for $n \in \mathbf{Z}$, then $\omega_0 \equiv 0 \pmod{k}$ and $\omega_0 \equiv 0 \pmod{\omega}$. To do this, let

$$\omega_0 = p^0 k + q^0 = p_0\omega + q_0. \tag{3.42}$$

All we need to prove is $q^0 = q_0 = 0$.

The proof for $q^0 = 0$ is clear if $k = 1$. Assume now $k \geqslant 2$. By way of contradiction, if $1 \leqslant q^0 \leqslant k-1$, then $x_{i+p^0k+q^0} = x_{i+q^0} \neq x_i$. So we have by this and (3.5), (3.14), (3.16) that

$$\begin{aligned} 0 &= \big|x^*(j+p^0k+q^0;i,j) - x^*(j;i,j)\big| \\ &\geqslant |x_{i+p^0k+q^0} - x_i| \\ &\quad - \big[\big|x^*(j+p^0k+q^0;i,j) - x_{i+p^0k+q^0}\big| + \big|x_i - x^*(j;i,j)\big|\big] \\ &> 2r(\mu) - \big[r(\mu) + r(\mu)\big] = 0. \end{aligned}$$

This is a contradiction. Thus, $q^0 = 0$ and we have by (3.42) that

$$\omega_0 = p^0 k = p_0\omega + q_0. \tag{3.43}$$

It remains to prove that $q_0 = 0$. This is trivial if $\omega = 1$. For $\omega \geqslant 2$, we prove $q_0 = 0$, again by way of contradiction. Suppose $1 \leqslant q_0 \leqslant \omega - 1$. Choosing $0 \leqslant n_* \leqslant k-1$ such that $n_* + i \equiv i_* \pmod{k}$. Then (3.16) gives

$$\big|x^*(n_*+j;i,j) - x_{i_*}\big| < r(\mu). \tag{3.44}$$

(3.43) yields

$$x^*(l + p_0\omega + q_0 + n_* + j; i,j) = x^*(l + n_* + j; i,j), \quad l \in \mathbf{Z}.$$

By (1.3) and (3.43), we have

$$\begin{aligned} & x^*(p_0\omega + q_0 + n_* + j + 1; i,j) \\ &\quad = f\big(x^*(p_0\omega + q_0 + n_* + j; i,j)\big) \\ &\qquad + \mu g\big(p_0\omega + q_0 + n_* + j; x(p_0\omega + q_0 + n_* + j; i,j), \mu\big) \\ &\quad = f\big(x^*(n_* + j; i,j)\big) + \mu g\big(q_0 + n_* + j, x^*(n_* + j; i,j), \mu\big) \end{aligned}$$

and

$$x^*(n_* + j + 1; i, j) = f\big(x^*(n_* + j; i, j)\big) + \mu g\big(n_* + j; x^*(n_* + j; i, j), \mu\big).$$

It follows that

$$g\big(n_* + j + q_0; x^*(n_* + j; i, j), \mu\big) = g\big(n_* + j; x^*(n_* + j; i, j), \mu\big).$$

But by (3.44), $x^*(n_* + j; i, j) \in V_{r(\mu)}(x_{i_*}) \subset \overline{V}_{r_1}(x_{i_*})$ with $n_* + j + q_0 \not\equiv n_* + j \pmod{\omega}$. This contradicts (3.6). This prove $q_0 = 0$. The proof of Theorem 3.1 is then complete. □

We conclude this section with a simple consequence of Theorem 3.1. If $g(n, x, \mu) = g(x, \mu)$ in (1.3), we have the following equation:

$$x(n+1) = f(x(n)) + \mu g(x(n), \mu). \tag{3.45}$$

Theorem 3.8. *Suppose that the condition* (H_1) *is satisfied and the mapping* $g : \mathbf{R}^N \times [-\mu_0, \mu_0] \to \mathbf{R}^N$ *is continuous. Then, for any* μ *with* $0 < |\mu| < \mu_*$, (3.45) *has a k-periodic solution* $x^*(n, \mu)$ *satisfying*

$$\big|x^*(n, \mu) - x_n\big| < r(\mu),$$

where μ_* *and* $r(\mu)$ *are defined in* (3.13) *and* (3.14), *respectively.*

References

[1] A.M. Amleh, E.A. Grove, C.M. Kent, G. Ladas, On some difference equations with eventually periodic solutions, J. Math. Anal. Appl. 223 (1998) 196–215.

[2] R.P. Agarwal, W. Zhang, Periodic solutions of difference equations with general periodicity, Comput. Math. Appl. 42 (2001) 719–727.

[3] F. Battelli, M. Fečkan, Subharmonic solutions in singular systems, J. Differential Equations 132 (1996) 21–45.

[4] E.N. Dancer, P. Hess, Stable subharmonic solutions in periodic reaction–diffusion equations, J. Differential Equations 108 (1994) 190–200.

[5] F. Dannan, S. Elaydi, P. Liu, Periodic solutions of difference equations, J. Difference Equations Appl. 6 (2000) 203–232.

[6] R.E. Gaines, J. Mawhin, Coincidence Degree and Nonlinear Differential Equations, in: Lecture Notes in Math., Vol. 568, Springer, 1977.

[7] M.I. Gil', S.S. Cheng, Periodic solutions of a perturbed difference equation, Appl. Anal. 76 (2000) 241–248.

[8] B.D. Greenspan, P.J. Holmes, Homoclinic orbits, subharmonics and global bifurcations in forced oscillations, in: Nonlinear Dynamics and Turbulence, in: Interaction of Mechanics and Mathematics Series, Pitman, Boston, MA, 1983, pp. 172–214.

[9] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, in: Applied Mathematical Sciences, Vol. 42, Springer, New York, 1983.

[10] N. Hirano, Subharmonic solutions for second order differential systems, J. Math. Anal. Appl. 196 (1995) 266–281.
[11] A. Krishnan, Difference equation analysis of nonlinear subharmonic and superharmonic oscillations, J. Sound Vibration 79 (1981) 121–131.
[12] V. Lakshmikanthan, D. Trigiante, Theory of Difference Equations, Numerical Methods and Applications, Academic Press, 1988.
[13] X.B. Lin, Using Mel'nikov's method to solve Šilnikov's problems, Proc. Roy. Soc. Edinburgh Sect. A Math. 116 (1990) 295–325.
[14] G.P. Pelyukh, On the existence and properties of periodic solutions of discrete difference equations, Ukrain. Mat. Zh. 46 (10) (1994) 1387–1387; English translation: Ukrainian Math. J. (June 1995) 1527–1533.
[15] J. Quandt, On the Hartman–Grobman theorem for maps, J. Differential Equations 64 (1986) 154–164.
[16] P.H. Rabinowitz, Subharmonic solutions of a forced wave equation, in: Contributions to Analysis and Geometry (Baltimore, MD, 1980), Johns Hopkins University Press, Baltimore, MD, 1981, pp. 285–292.
[17] A.N. Ronto, M. Ronto, A.M. Samoilenko, S.I. Trofimchuk, On periodic solutions of autonomous difference equations, Georgian Math. J. 8 (2001) 135–164, English summary.
[18] A.N. Sharkovsky, Yu.L. Maistrenko, E.Yu. Romanenko, Difference Equations and Their Applications, Naukova, Kyiv, 1986, in Russian; English translation: Kluwer, Dordrecht, 1993.
[19] L.P. Šhil'nikov, The generation of a periodic motion from a trajectory which is doubly asymptotic to a saddle type equilibrium state, Mat. Sb. Novaya Ser. 77 (119) (1968) 461–472.
[20] E.A. de B. e Silva, Subharmonic solutions for subquadratic Hamiltonian systems, J. Differential Equations 115 (1995) 120–145.
[21] Kh. Turaev, On the existence and uniqueness of periodic solutions of a class of nonlinear difference equations, Uzbek. Math. Zh. 2 (1994) 52–54, in Russian; English, Uzbek summaries.
[22] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, 1991.
[23] X.S. Yang, On existence of subharmonic orbits in perturbed Hamiltonian systems, Bull. Polish Acad. Sci. Math. 48 (2000) 221–225.
[24] R.Y. Zhang, Z.C. Wang, Y. Chen, J. Wu, Periodic solutions of a single species discrete population model with periodic harvest/stock, Comput. Math. Appl. 39 (2000) 77–90.
[25] D. Zhu, M. Han, Invariant tori and subharmonic bifurcations from periodic manifolds, Acta Math. Sinica New Ser. 14 (1998) 613–624.