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SPECIAL SYMMETRIC PERIODIC SOLUTIONS OF DELAYED MONOTONE FEEDBACK SYSTEMS

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ABSTRACT. A delay differential system with monotone feedback is considered. For the positive feedback nonlinearity, it is shown that the existence of a special symmetric 4-periodic solution is equivalent to the existence of a fixed point for a mapping defined on a cone in a Banach space. The coexistence of multiple (including infinitely many) special symmetric 4-periodic solutions is established. Sufficient conditions for the uniqueness of special symmetric 4-periodic solutions are also given. A related singularly perturbed system is considered, where it is shown that the limiting profile of a special symmetric 4-periodic solution is pulse-like of unbounded amplitude and that the product of each component of the solution with the singular parameter approaches a sawtooth wave with the slope determined by the limit of the nonlinearity at infinity. Similar conclusions can be drawn for a negative feedback nonlinearity by the consideration of involved symmetry and, in particular, special symmetric 2-periodic solutions are considered. It is observed that special symmetric 4-periodic solutions of the positive feedback system have line segment-like waves as limiting profiles in the plane, while special symmetric 2-periodic solutions of the negative feedback system approach a diamond-like wave.

1. Introduction. In this paper we consider the following system of delay differential equations

$$(1)\qquad \begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t-1)), \\ \dot{x}_2(t) = f_2(x_2(t), x_1(t-1)), \end{cases}$$

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where $f_i : \mathbf{R}^2 \to \mathbf{R}$, $i = 1, 2$, is locally Lipschitz continuous and satisfies the symmetry condition

$$f_i(x, y) = f_i(-x, y) = -f_i(x, -y) \quad \text{for } (x, y) \in \mathbf{R}^2 \tag{2}$$

and the positive feedback condition

$$y f_i(x, y) > 0 \quad \text{for } (x, y) \in \mathbf{R}^2 \text{ with } y \neq 0. \tag{3}$$

If f_i is C^1-smooth and $(\partial f_i(0,0)/\partial y) > 0$ for $i = 1$ and 2, then, without loss of generality, we can assume $(\partial f_1(0,0)/\partial y) = (\partial f_2(0,0)/\partial y) = a$. This can be achieved in the general case if we rescale x_2 by $\sqrt{(\partial f_1(0,0)/\partial y)/(\partial f_2(0,0)/\partial y)} x_2$.

Linearizing system (1) around the origin gives

$$\begin{cases} \dot{X}_1(t) = a X_2(t-1), \\ \dot{X}_2(t) = a X_1(t-1). \end{cases} \tag{4}$$

The characteristic equation associated with (4) is given by

$$0 = \begin{vmatrix} \lambda & -ae^{-\lambda} \\ -ae^{-\lambda} & \lambda \end{vmatrix} = (\lambda + ae^{-\lambda})(\lambda - ae^{-\lambda}). \tag{5}$$

Equation (5) has purely imaginary solutions only when $a = ((2k-1)/2)\pi$ for some $k \in \mathbf{N}$, the set of all positive integers, and for $a = ((2k-1)/2)\pi$ the only purely imaginary solutions of (5) are $\pm((2k-1)/2)\pi i$. Moreover, for such $a = ((2k-1)/2)\pi$, system (4) has a particular periodic solution

$$\begin{cases} X_1(t) = \sin\left(\dfrac{2k-1}{2}\right)\pi t, \\ X_2(t) = (-1)^{k-1} \sin\left(\dfrac{2k-1}{2}\right)\pi t, \end{cases}$$

with the minimal period $4/(2k-1)$. Clearly this periodic solution satisfies

$$\begin{cases} X_1(t) = -X_1\left(t + \dfrac{2}{2k-1}\right) & \text{for } t \in \mathbf{R}, \\ X_1(t) > 0 & \text{on } \left(0, \dfrac{2}{2k-1}\right), \\ X_2(t) = -X_2\left(t + \dfrac{2}{2k-1}\right) & \text{for } t \in \mathbf{R}. \end{cases} \tag{6}$$

It is therefore natural to study periodic solutions (x_1, x_2) of the nonlinear system (1) with the following symmetry properties:

$$(7)\qquad \begin{cases} x_1(t) = -x_1\Big(t + \dfrac{2}{2k-1}\Big) & \text{for } t \in \mathbf{R}, \\ x_1(t) > 0 & \text{on } \Big(0, \dfrac{2}{2k-1}\Big), \\ x_2(t) = -x_2\Big(t + \dfrac{2}{2k-1}\Big) & \text{for } t \in \mathbf{R}. \end{cases}$$

Obviously, $4/(2k-1)$ is the minimal period of (x_1, x_2). We call such a periodic solution a *special symmetric* $4/(2k-1)$*-periodic solution*.

Let (x_1, x_2) be a given special symmetric $4/(2k-1)$-periodic solution of system (1). Then

$$\begin{cases} x_1(t-1) = (-1)^{k-1} x_1\Big(t - \dfrac{1}{2k-1}\Big), \\ x_2(t-1) = (-1)^{k-1} x_2\Big(t - \dfrac{1}{2k-1}\Big). \end{cases}$$

Hence,

$$\begin{cases} \dot{x}_1(t) = (-1)^{k-1} f_1\Big(x_1(t), x_2\Big(t - \dfrac{1}{2k-1}\Big)\Big), \\ \dot{x}_2(t) = (-1)^{k-1} f_2\Big(x_2(t), x_1\Big(t - \dfrac{1}{2k-1}\Big)\Big). \end{cases}$$

Renaming t by $t/(2k-1)$, x_1 and x_2 by x_1 and $(-1)^{k-1}x_2$, respectively, we get

$$(8)\qquad \begin{cases} \dot{x}_1(t) = \dfrac{1}{2k-1} f_1(x_1(t), x_2(t-1)), \\ \dot{x}_2(t) = \dfrac{1}{2k-1} f_2(x_2(t), x_1(t-1)). \end{cases}$$

Thus (x_1, x_2) is a special symmetric 4-periodic solution of system (8). Note that (8) coincides with system (1) with f_1 and f_2 being replaced by $(1/(2k-1))f_1$ and $(1/(2k-1))f_2$, respectively. So in what follows we will only consider special symmetric 4-periodic solutions of system (1).

We remark that if (3) is replaced by

$$f_i(x, 0) = 0 \quad \text{and} \quad \frac{\partial f_i(x, y)}{\partial y} > 0 \quad \text{for } (x, y) \in \mathbf{R}^2,\ i = 1, 2,$$

then we can show that any periodic solution (x_1, x_2) of system (1) with the minimal period 4 must be a translation of a special symmetric 4-periodic solution. In fact, $(v_1(t), v_2(t)) = (x_1(2t), x_2(2t-1))$ satisfies

$$\begin{cases} \dot{v}_1(t) = 2f_1(v_1(t), v_2(t)), \\ \dot{v}_2(t) = 2f_2(v_2(t), v_1(t-1)), \end{cases} \tag{9}$$

which is a monotone cyclic feedback system with delay considered by Mallet-Paret and Sell [**16**]. Thus by Theorem 7.1 and Theorem 7.2 of [**16**], we have $v_i(t+1) = -v_i(t)$ for $t \in \mathbf{R}$ and $i = 1, 2$ and if $v_1(0) = 0$ then $v_1(t) > 0$ (< 0) on $(0, 1)$. Therefore, it is easy to see that (x_1, x_2) is the translation of a special symmetric 4-periodic solution of system (1).

Our focus in this paper is on the qualitative properties of special symmetric 4-periodic solutions of system (1): the existence, uniqueness and limiting profiles. Our work relies heavily on the work of Mallet-Paret and Sell [**16**] for cyclic systems of delay differential equations. Our work is also inspired by that of Ivanov [**9**], where the *special symmetric periodic solutions* (SSPSs) of the scalar negative feedback delay differential equation

$$\dot{x}(t) = -f(x(t), x(t-1)), \tag{10}$$

with f satisfying (2) and (3) considered. When $f(x, y)$ is independent of x, equation (10) is a special case studied by Nussbaum [**18**]. In [**18**], Nussbaum obtained some general results on the existence of special symmetric period solutions of

$$\begin{aligned} x'(t) = &-\sum_{j=1}^{n} [f_j(x(t-\gamma_j)) + f_j(x(t-q+\gamma_j))] \\ &-\sum_{k=1}^{m} [g_k(x(t-\delta_k)) - g_k(x(t-2q+\delta_k))] \end{aligned}$$

under conditions on $\lim_{x\to 0} f_j(x)x^{-1}$, $\lim_{x\to\infty} f_j(x)x^{-1}$, $\lim_{x\to 0} g_k(x)x^{-1}$ and $\lim_{x\to\infty} g_k(x)x^{-1}$, where for $1 \le j \le n$ and $1 \le k \le m$, f_j and g_k are continuous and odd and satisfy $xf_j(x) \ge 0$ and $xg_k(x) \ge 0$ for all $x \in \mathbf{R}$. Recently, we [**3**] generalized the approach of Ivanov [**9**] and studied 4/3-periodic solutions for delayed positive feedback systems. An important difference between our work [**3**] and that of Ivanov is that we did not use the standard phase space to construct the cone mapping. As will be shown later, if $f_1 = f_2$, then a special symmetric 4-periodic solution (x_1, x_2) is phase-locked, i.e., $x_2(t) = x_1(t+2)$ for $t \in \mathbf{R}$, Corollary 2.4. Thus, $x_2(t-1) = -x_1(t-1)$ and hence x_1 satisfies

$$\dot{x}(t) = -f_1(x(t), x(t-1)),$$

which is equation (10) with $f = f_1$. However, due to the lack of symmetry (i.e., $f_1 \neq f_2$ in general), we may not have the phase-locked property of the special symmetric 4-periodic solution (x_1, x_2) (i.e., $x_2(t) = x_1(t+2)$ for $t \in \mathbf{R}$ does not always hold). Thus we cannot relate the existence of special symmetric 4-periodic solutions of system (1) to that of SSPSs of equation (10). So the main contribution of this paper is to extend the method developed in [**3, 9**] for scalar equations to the planar case.

Equation (10) arises in the modeling of various physical, biological and ecological phenomena and has been intensively studied. Existence [**10, 11**], uniqueness/nonuniqueness [**19**], stability [**6**] and limiting profiles [**14**] of slowly oscillating periodic solutions and/or SSPS of equation (10) have been obtained. All the results mentioned above and almost all of the existing results are obtained under the condition of negative feedback, except the recent work of Krisztin, Walther and Wu [**13**] and Krisztin and Walther [**12**]. There are a few results on periodic solutions of planar differential delay equations (for example, see [**1, 7, 17, 20**]), but again under the condition of negative feedback. Recently we obtained in [**2, 4, 5**] some results about the global attractor of the positive feedback system

$$\begin{cases} \dot{x}(t) = -\mu x(t) + f(y(t-\tau)), \\ \dot{y}(t) = -\mu y(t) + f(x(t-\tau)), \end{cases} \tag{11}$$

with μ and τ being positive constants, $f(0) = 0$ and $f'(\xi) > 0$ for $\xi \in \mathbf{R}$. Our work here applies to system (11) with $\mu = 0$.

The remaining part of the paper is organized as follows. In Section 2 we show that the existence of a special symmetric 4-periodic solution of system (1) is equivalent to the existence of a fixed point of a mapping constructed on a cone in the Banach space $C_2 := C([0,1]; \mathbf{R}^2)$. This approach enables us to provide, in Section 4, a straightforward example of system (1) which possesses any number (including infinite many) of special symmetric 4-periodic solutions. In Section 3 we obtain sufficient conditions for the uniqueness of special symmetric 4-periodic solutions of system (1). In Section 5 we describe the limiting profiles of the special symmetric 4-periodic solutions of a singularly perturbed system and we obtain pulses of unbounded amplitudes, each component of whose multiplications with the singular parameter tends to a sawtooth wave. Finally, in Section 6 we relate the special symmetric 2-periodic solutions of a negative feedback system to the special symmetric 4-periodic solutions of system (1) with f_1 and f_2 satisfying (2) and (3) under the symmetry condition.

2. Existence. In this section we show the existence of special symmetric 4-periodic solutions of system (1) with f_i $(i = 1, 2)$ being locally Lipschitz continuous and satisfying (2) and (3). First we describe some properties of special symmetric 4-periodic solutions.

Let (x_1, x_2) be a special symmetric 4-periodic solution of system (1). Let $v_1(t) = x_1(2t)$ and $v_2(t) = x_2(2t - 1)$. Then $(v_1(t), v_2(t))$ satisfies the system (9). Let V^+ be the discrete Lyapunov function introduced by Mallet-Paret and Sell [**15, 16**]. By the monotonicity of V^+ and the periodicity of (v_1, v_2), $V^+(((v_1)_t, v_2(t))) = 2l$ for $t \in \mathbf{R}$ and for some $l \in \mathbf{N}$. Thus, all zeros of v_1 and v_2, and hence all zeros of x_1 and x_2, are simple. Since (x_1, x_2) is a special symmetric 4-periodic solution of system (1), we have

$$(12) \qquad \begin{cases} x_i(t+2) = -x_i(t) & \text{for } t \in \mathbf{R} \text{ and } i = 1, 2, \\ x_1(0) = 0, \ x_1(t) > 0 & \text{on } (0, 2). \end{cases}$$

Let $u_1(t) = x_1(t)$ and $u_2(t) = x_2(t-1)$. Using (2) and (12) we get from system (1) the following system of ordinary differential equations

$$(13) \qquad \begin{cases} \dot{u}_1(t) = f_1(u_1(t), u_2(t)), \\ \dot{u}_2(t) = -f_2(u_2(t), u_1(t)). \end{cases}$$

Note that $\dot{u}_1(0) = \dot{x}_1(0)$ must be positive since 0 is a simple zero of x_1, and hence $u_2(0) > 0$.

Proposition 2.1. (i) *The phase portrait of system* (13) *is symmetric with respect to coordinates* $u_1 = 0$ *and* $u_2 = 0$.

(ii) *If* $(u_1, u_2) : \mathbf{R} \to \mathbf{R}^2$ *is a periodic solution of system* (13) *with initial value* $(0, u)$ *and* $u > 0$*, then* u_1 *is odd and* u_2 *is even,* $u_2(\omega/4) = 0$ *and* $u_2(t) > 0$ *on* $(-(\omega/4), (\omega/4))$ *where* ω *is the minimal period of* (u_1, u_2).

Proof. Let $V : \mathbf{R}^2 \to \mathbf{R}^2$ be any vector field which generates a flow $\Phi : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ and S an invertible 2×2 matrix. It is well known that, if V is antisymmetric with respect to S, i.e., $SV(u_1, u_2) = -V(S(u_1, u_2))$ for $(u_1, u_2) \in \mathbf{R}^2$, then $\Phi(t, S(u_1, u_2)) = S\Phi(-t, (u_1, u_2))$ for $(t, (u_1, u_2)) \in \mathbf{R} \times \mathbf{R}^2$.

Now it is easy to check that, because of (2), $V(u_1, u_2) = (f_1(u_1, u_2), -f_2(u_2, u_1))$ is antisymmetric with respect to the following invertible matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the phase portrait of system (13) is symmetric with respect to coordinates $u_1 = 0$ and $u_2 = 0$. □

(ii) follows directly from (i).

As an immediate consequence, we get

Corollary 2.2. *Let* $(x_1, x_2) : \mathbf{R} \to \mathbf{R}^2$ *be a special symmetric 4-periodic solution of system* (1)*. Then both* x_1 *and* x_2 *are odd,* $x_2(0) = 0$ *and* $x_2(t) < 0$ *on* $(0, 2)$.

Corollary 2.3. *If* (u_1, u_2) *is a periodic solution of system* (13) *with the minimal period* 4 *and* $(u_1(0), u_2(0)) = (0, u)$ *with* $u > 0$*, then* (x_1, x_2) *with* $x_1(t) = u_1(t)$ *and* $x_2(t) = u_2(t+1)$ *is a special symmetric 4-periodic solution of system* (1).

Proof. By Lemma 2.1 we know that $u_1(t) > 0$ on (0,2). Direct substitution shows that $(-u_1, -u_2)$ is the same trajectory. Therefore, $(u_1(t), u_2(t)) = (-u_1(t+c), -u_2(t+c))$ for $t \in \mathbf{R}$ and some $c \in [0, 4)$. Then $(u_1(t), u_2(t)) = (u_1(t + 2c), u_2(t + 2c))$, which implies $2c = 4m$ or $c = 2m$ for some integer m. Since $c \in [0, 4)$, either $c = 0$ or $c = 2$. If $c = 0$, then $u_1 \equiv 0$, a contradiction. Thus $c = 2$ and hence $u_i(t+2) = -u_i(t)$ for $t \in \mathbf{R}$ and $i = 1, 2$. This, combined with (2) and (13), implies that (x_1, x_2) is a special symmetric 4-periodic solution of system (1). □

Corollary 2.4. *If $f_1 = f_2$ and (u_1, u_2) is a periodic solution of system* (13) *with the minimal period* 4*, then $u_2(t) = u_1(t+1)$ for $t \in \mathbf{R}$. Particularly, if (x_1, x_2) is a special symmetric* 4*-periodic solution of system* (1)*, then $x_2(t) = x_1(t+2)$ for $t \in \mathbf{R}$.*

Proof. As in the proof of Lemma 2.1, V is also antisymmetric with respect to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus the phase portrait of system (13) is symmetric to the diagonal $u_1 = u_2$, and hence also to the diagonal $u_1 + u_2 = 0$, in addition to the coordinates $u_1 = 0$ and $u_2 = 0$. If (u_1, u_2) is a periodic solution of system (13) with the minimal period 4, then direct substitution shows that $(-u_2, u_1)$ is the same trajectory. Therefore, $(u_1(t), u_2(t)) = (-u_2(t+c), u_1(t+c))$ for $t \in \mathbf{R}$ and some $c \in [0, 4)$. From the proof of Corollary 2.3, we know that $u_i(t+2) = -u_i(t)$ for $t \in \mathbf{R}$ and $i = 1, 2$. Hence $u_1(t) = u_1(t+2c+2)$ and $u_2(t) = u_2(t+2c+2)$ for $t \in \mathbf{R}$. This implies that $2c + 2 = 4m$ for some integer m. Since $c \in [0, 4)$, either $c = 1$ or $c = 3$. If $c = 3$, then $u_2(t) = u_1(t-1)$. Thus u_1 satisfies $\dot{x}(t) = f_1(x(t), x(t-1))$. This, combined with $u_1(t) > 0$ on $(0, 2)$ implies that $u_1 > 0$ on $(0, \infty)$, a contradiction. Thus, $c = 1$ and hence $u_2(t) = u_1(t+1)$. This completes the proof. □

Now we are ready to study the existence of special symmetric 4-periodic solutions of system (1). Let $K \subseteq C_2$ be the cone defined by

$$K = \left\{ \varphi = (\varphi_1, \varphi_2) \in C_2; \begin{array}{l} \text{both } \varphi_1 \text{ and } -\varphi_2 \text{ are increasing} \\ \text{on } [0,1] \text{ and } \varphi_1(0) = \varphi_2(0) = 0 \end{array} \right\}.$$

Define $G : K \to C_2$ by

(14)
$$(G\varphi)(t) = \left(-\int_0^t f_1(\varphi_1(s), \varphi_2(1-s))\, ds, -\int_0^t f_2(\varphi_2(s), \varphi_1(1-s))\, ds \right)$$

for $\varphi = (\varphi_1, \varphi_2) \in K$ and $t \in [0, 1]$. Due to the positive feedback condition (3), it is easy to see that K is invariant with respect to G.

The relationship between a fixed point of G and the special symmetric 4-periodic solution of system (1) is given by the following

Theorem 2.5. *System* (1) *has a special symmetric* 4*-periodic solution if and only if the mapping* G *has a nonzero fixed point.*

Before proving Theorem 2.5 we introduce the following special symmetric 4-periodic extension $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) : \mathbf{R} \to \mathbf{R}^2$ for a given $\varphi = (\varphi_1, \varphi_2) \in K$, namely, $\tilde{\varphi}$ is a 4-periodic function and for $i = 1$ and 2,

$$\tilde{\varphi}_i(t) = \begin{cases} \varphi_i(t) & t \in [0,1], \\ \varphi_i(2-t) & t \in [1,2], \\ -\varphi_i(t-2) & t \in [2,3], \\ -\varphi_i(4-t), & t \in [3,4]. \end{cases}$$

Note that the definition of $\tilde{\varphi}$ implies that both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are odd. Also $\tilde{\varphi}_1(t+2) = -\tilde{\varphi}_1(t)$ and $\tilde{\varphi}_2(t+2) = -\tilde{\varphi}_2(t)$ for $t \in \mathbf{R}$.

Theorem 2.5 follows directly from

Lemma 2.6. $0 \neq \varphi \in K$ *is a fixed point of the mapping* G *if and only if* $\tilde{\varphi}$ *is a* 4*-periodic solution of system* (1)*.*

Proof. Necessity. Let $0 \neq \varphi \in K$ be a fixed point of the mapping G. Differentiating (14), we have

$$\begin{cases} \dot{\varphi}_1(t) = -f_1(\varphi_1(t), \varphi_2(1-t)) \\ \dot{\varphi}_2(t) = -f_2(\varphi_2(t), \varphi_1(1-t)), \end{cases}$$

for $t \in [0,1]$. Since $\tilde{\varphi}_i(t-1) = -\tilde{\varphi}_i(1-t)$ for $i = 1, 2$, using (2) we get

$$\begin{cases} \dot{\tilde{\varphi}}_1(t) = f_1(\tilde{\varphi}_1(t), \tilde{\varphi}_2(t-1)) \\ \dot{\tilde{\varphi}}_2(t) = f_2(\tilde{\varphi}_2(t), \tilde{\varphi}_1(t-1)), \end{cases}$$

for $t \in [0,1]$, i.e., $\tilde{\varphi}$ solves system (1) on $[0,1]$. For $t \in [1,2]$, we have

$$\begin{aligned} \dot{\tilde{\varphi}}_1(t) &= -\dot{\varphi}_1(2-t) = -\dot{\tilde{\varphi}}_1(2-t) \\ &= -f_1(\tilde{\varphi}_1(2-t), \tilde{\varphi}_2(1-t)) \\ &= f_1(\tilde{\varphi}_1(t), \tilde{\varphi}_2(t-1)) \end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}_2(t) &= -\dot{\varphi}_2(2-t)\\ &= -f_2(\tilde{\varphi}_2(2-t), \tilde{\varphi}_1(1-t))\\ &= f_2(\tilde{\varphi}_2(t), \tilde{\varphi}_1(t-1))\end{aligned}$$

since both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are odd and $\tilde{\varphi}_i(t-2) = -\tilde{\varphi}_i(t)$ for $t \in \mathbf{R}$ and $i = 1, 2$. Therefore, $\tilde{\varphi}$ solves system (1) on $[1, 2]$. The proof of $\tilde{\varphi}$ solving system (1) on $[2, 4]$ is similar.

Sufficiency. Let $\varphi \in K$ be such that $\tilde{\varphi}$ is a 4-periodic solution of system (1). Then $\varphi \neq 0$. Integrating system (1) for $t \in [0, 1]$ and using $\tilde{\varphi}_i(s-1) = -\tilde{\varphi}_i(1-s)$ for $i = 1$ and 2, we have

$$\begin{cases}\varphi_1(t) = \displaystyle\int_0^t f_1(\varphi_1(s), \tilde{\varphi}_2(s-1))\,ds\\ \qquad\;\; = -\displaystyle\int_0^t f_1(\varphi_1(s), \varphi_2(1-s))\,ds,\\ \varphi_2(t) = \displaystyle\int_0^t f_2(\varphi_2(s), \tilde{\varphi}_1(s-1))\,ds\\ \qquad\;\; = -\displaystyle\int_0^t f_2(\varphi_2(s), \varphi_1(1-s))\,ds,\end{cases}$$

since $\varphi_1(0) = 0$ and $\varphi_2(0) = 0$ by Corollary 2.2. Therefore, $0 \neq \varphi \in K$ is a nonzero fixed point of the mapping G. This completes the proof. $\square$

The following result justifies our extension of $\tilde{\varphi}$ for $\varphi \in K$.

Corollary 2.7. *Let $p = (p_1, p_2) : \mathbf{R} \to \mathbf{R}^2$ be a special symmetric 4-periodic solution of system* (1), *and denote $\varphi := p|_{[0,1]}$. Then $\varphi \in K$ and $\tilde{\varphi} = p$.*

Proof. Since p is a special symmetric 4-periodic solution of system (1), $p_1(0) = p_2(0) = 0$, $p_1 \geq 0$, and $p_2 \leq 0$ on $[0, 1]$, and $\varphi \neq 0$. Note that $p_1(t-1) = -p_1(1-t)$ and $p_2(t-1) = -p_2(1-t)$. Thus

$$\begin{cases}\dot{p}_1(t) = f_1(p_1(t), p_2(t-1)) = -f_1(p_1(t), p_2(1-t)),\\ \dot{p}_2(t) = f_2(p_2(t), p_1(t-1)) = -f_2(p_2(t), p_1(1-t)),\end{cases} \tag{15}$$

which implies that both p_1 and $-p_2$ are increasing on $[0,1]$. Hence $\varphi \in K$. Integrating (15) for $t \in [0,1]$ gives us

$$\begin{cases} \varphi_1(t) = -\displaystyle\int_0^t f_1(\varphi_1(s), \varphi_2(1-s))\, ds, \\ \varphi_2(t) = -\displaystyle\int_0^t f_2(\varphi_2(s), \varphi_1(1-s))\, ds, \end{cases}$$

i.e., $0 \neq \varphi \in K$ is a fixed point of G. Thus $\tilde{\varphi}$ is a 4-periodic solution of system (1) by Lemma 2.6. Observe that if (x_1, x_2) is a special symmetric 4-periodic solution of system (1), then $u_1(t) = x_1(t)$ and $u_2(t) = x_2(t-1)$ satisfy (13). Thus $\tilde{\varphi} = p$ since $\varphi = p|_{[0,1]}$. This completes the proof. □

3. Uniqueness. Recall that if (x_1, x_2) is a special symmetric 4-periodic solution of system (1), then $u_1(t) = x_1(t)$ and $u_2(t) = x_2(t-1)$ satisfy system (13). In what follows, for a periodic solution (u_1, u_2) of system (13) with initial value $(0, u)$ and $u > 0$, we denote by T_u the minimal period.

Definition 3.1. (a) Let $(x_i, y_i) \in \mathbf{R}^2$, $i = 1, 2$. We say that $(x_1, y_1) < (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ but $(x_1, y_1) \neq (x_2, y_2)$.

(b) Let $\phi(x, y)$ be a function defined on $D \subseteq \mathbf{R}^2$. We say that ϕ is *increasing* on D if $(x_i, y_i) \in D$, $i = 1, 2$, and $(x_1, y_1) < (x_2, y_2)$, then $\phi(x_1, y_1) < \phi(x_2, y_2)$. Similarly, we can define *decreasing* ϕ on D.

Theorem 3.2. *Suppose that both $(f_1(x, y)/y)$ and $(f_2(x, y)/y)$ are decreasing (increasing) on $\{(x, y); y > 0\}$. If $0 < u_1 < u_2$, then $T_{u_1} < T_{u_2}$ $(T_{u_1} > T_{u_2})$.*

Proof. We only give the proof for the case where $f_1(x, y)/y$ and $f_2(x, y)/y$ are decreasing.

Let $\big(u_1^{(i)}, u_2^{(i)}\big)$ be the periodic trajectory of system (13) with the initial value $(0, u_i)$, and let θ_i be the angular coordinate of the point $\big(u_1^{(i)}(t), u_2^{(i)}(t)\big)$, $i = 1, 2$. Using system (13), we have

$$\dot{\theta}_i(t) = -\frac{u_1^{(i)} f_2\big(u_2^{(i)}, u_1^{(i)}\big) + u_2^{(i)} f_1\big(u_1^{(i)}, u_2^{(i)}\big)}{\big(u_1^{(i)}\big)^2 + \big(u_2^{(i)}\big)^2} < 0. \tag{16}$$

From this and the monotonicity assumptions we see that

$$\dot{\theta}_1(0) = -\frac{f_1(0, u_1)}{u_1} < -\frac{f_1(0, u_2)}{u_2} = \dot{\theta}_2(0).$$

Therefore, $\theta_1(t) < \theta_2(t)$ for all $t \in (0, \sigma]$ for some $\sigma > 0$. We claim that $\theta_1(t) < \theta_2(t)$ for all $t \in (0, (T_{u_1}/4)]$ which combined with Proposition 2.1 implies that $T_{u_1} < T_{u_2}$. If not, then there must exist a $t_0 \in (0, (T_{u_1}/4)]$ such that $\theta_1(t_0) = \theta_2(t_0)$. Let $t^* \in (0, t_0]$ be the first time such that $0 \leq \theta_1(t^*) = \theta_2(t^*) < \pi/2$. Then $\dot{\theta}_1(t^*) \geq \dot{\theta}_2(t^*)$. If $\big(u_1^{(1)}, u_2^{(1)}\big)$ and $\big(u_1^{(2)}, u_2^{(2)}\big)$ are the corresponding points on the trajectories, then $\big(u_1^{(2)}, u_2^{(2)}\big) = r\big(u_1^{(1)}, u_2^{(1)}\big)$ for some $r > 1$. Equation (16) implies

$$\begin{aligned}\dot{\theta}_2(t^*) &= -\frac{u_1^{(2)} f_2\big(u_2^{(2)}, u_1^{(2)}\big) + u_2^{(2)} f_1\big(u_1^{(2)}, u_2^{(2)}\big)}{\big(u_1^{(2)}\big)^2 + \big(u_2^{(2)}\big)^2} \\ &= -\frac{u^{(1)} f_2\big(ru_2^{(1)}, ru_1^{(1)}\big) + u_2^{(1)} f_1\big(ru_1^{(1)}, ru_2^{(1)}\big)}{r\Big(\big(u_1^{(1)}\big)^2 + \big(u_2^{(1)}\big)^2\Big)}.\end{aligned} \tag{17}$$

Since both $(f_1(x,y)/y)$ and $(f_2(x,y)/y)$ are decreasing on $\{(x,y); y > 0\}$,

$$\frac{f_1\big(ru_1^{(1)}, ru_2^{(1)}\big)}{ru_2^{(1)}} < \frac{f_1\big(u_1^{(1)}, u_2^{(1)}\big)}{u_2^{(1)}}$$

and

$$\frac{f_2\big(ru_2^{(1)}, ru_1^{(1)}\big)}{ru_1^{(1)}} < \frac{f_2\big(u_2^{(1)}, u_1^{(1)}\big)}{u_1^{(1)}}$$

or

$$\frac{u_2^{(1)} f_1\big(ru_1^{(1)}, ru_2^{(1)}\big)}{r} < u_2^{(1)} f_1\big(u_1^{(1)}, u_2^{(1)}\big) \quad \text{for } u_2^{(1)} \neq 0, \tag{18}$$

$$\frac{u_1^{(1)} f_2\big(ru_2^{(1)}, ru_1^{(1)}\big)}{r} < u_1^{(1)} f_2\big(u_2^{(1)}, u_1^{(1)}\big) \quad \text{for } u_1^{(1)} \neq 0. \tag{19}$$

It follows from (16)–(19) that

$$\dot{\theta}_2(t^*) > \dot{\theta}_1(t^*),$$

a contradiction. □

Corollary 3.3. *Under the assumptions of Theorem* 3.2*, if a special symmetric* 4*-periodic solution of system* (1) *exists, then it is unique.*

4. A nonuniqueness example. In this section we consider

$$\begin{cases} \dot{x}_1(t) = f_1(x_2(t-1)) \\ \dot{x}_2(t) = f_2(x_1(t-1)) \end{cases} \tag{20}$$

and assume that both f_1 and f_2 are increasing and continuously differentiable. Recall that we assume without loss of generality $f_1'(0) = f_2'(0)$. Then the cone mapping G becomes

$$(G(\varphi_1, \varphi_2))(t) = \left(-\int_0^t f_1(\varphi_2(1-s))\,ds, -\int_0^t f_2(\varphi_1(1-s))\,ds\right). \tag{21}$$

Definition 4.1. For $\varphi = (\varphi_1, \varphi_2) \in C_2$, we say that φ is *convex* if φ_1 is concave up and φ_2 is concave down on $[0, 1]$.

Proposition 4.2. *If $\psi \in G(K)$, then ψ is convex in* $[0, 1]$.

Proof. From

$$\begin{cases} \psi_1(t) = -\displaystyle\int_0^t f_1(\varphi_2(1-s))\,ds \\ \psi_2(t) = -\displaystyle\int_0^t f_2(\varphi_1(1-s))\,ds \end{cases}$$

for some $\varphi = (\varphi_1, \varphi_2) \in K$, we have

$$\begin{cases} \dot{\psi}_1(t) = -f_1(\varphi_2(1-t)) \\ \dot{\psi}_2(t) = -f_2(\varphi_1(1-t)) \end{cases}$$

for $t \in [0, 1]$. Therefore, $\dot{\psi}_1$ decreases and $\dot{\psi}_2$ increases, and hence ψ is convex. □

Proposition 4.2 prompts us to introduce a subset K_0 of the cone K by

$$K_0 = \{\varphi \in K;\ \varphi \text{ is convex in } [0,1]\}.$$

Thus cone K_0 is invariant under G by Proposition 4.2. For $\varphi = (\varphi_1, \varphi_2) \in K$, denote $\|\varphi\| = \max\{\varphi_1(1), -\varphi_2(1)\}$.

Proposition 4.3. *If $f_1'(0) = f_2'(0) > 2$, then the trivial fixed point $\varphi = 0$ is an ejective point of the mapping G, i.e., there exists $\sigma > 0$ such that for every $\phi \in U_\sigma := \{\varphi \in K; \|\varphi\| < \sigma\}$ there exists a positive integer $m = m(\phi)$ such that $G^m\phi \notin U_\sigma$.*

Proof. Let ε be a fixed number in $(0, (f_1'(0)-2)/2)$. Choose $\sigma > 0$ such that $f_1'(x) > 2+\varepsilon$ and $f_2'(x) > 2+\varepsilon$ for $x \in [0,\sigma]$. Let $\phi = (\phi_1, \phi_2) \in U_\sigma$ and $\|\phi| = \kappa$. By Proposition 4.2, we may assume that $\phi_1(t) \geq \kappa t$ and/or $\phi_2(t) \leq -\kappa t$ for $t \in [0,1]$ (if necessary, replace ϕ by $G\phi$). Then either

$$\begin{aligned}(G\phi)_1(1) &= -\int_0^1 f_1(\phi_2(1-s))\,ds \geq \int_0^1 f_1(\kappa(1-s))\,ds \\ &\geq (2+\varepsilon)\int_0^1 \kappa(1-s)\,ds = \left(1+\frac{\varepsilon}{2}\right)\kappa\end{aligned}$$

or

$$\begin{aligned}-(G\phi)_2(1) &= \int_0^1 f_2(\phi_1(1-s))\,ds \geq \int_0^1 f_2(\kappa(1-s))\,ds \\ &\geq (2+\varepsilon)\int_0^1 \kappa(1-s)\,ds = \left(1+\frac{\varepsilon}{2}\right)\kappa.\end{aligned}$$

Thus $\|G\phi\| \geq ((1+\varepsilon)/2)\kappa$ and the claim of the proposition follows. ☐

Proposition 4.4. *Suppose that there exists $u > 0$ such that $\max\{f_1(u), f_2(u)\} \leq u$. Then the set $K_{\leq u} := \{\varphi \in K_0; \|\varphi\| \leq u\}$ is invariant with respect to G.*

Proof. For any $\varphi = (\varphi_1, \varphi_2) \in K_{\leq u}$, we have

$$\|G\varphi\| = \max\left\{ -\int_0^1 f_1(\varphi_2(1-s))\,ds, \int_0^1 f_2(\varphi_1(1-s))\,ds \right\}$$
$$\leq \max\left\{ \int_0^1 f_1(u)\,ds, \int_0^1 f_2(u)\,ds \right\} \leq u. \qquad \square$$

Corollary 4.5. *Assume that both f_1 and f_2 are odd and increasing, $f_1'(0) = f_2'(0) > 2$ and there exists $u > 0$ such that $\max\{f_1(u), f_2(u)\} \leq u$. Then system* (20) *has a special symmetric* 4*-periodic solution.*

Proof. We know that $K_{\leq u}$ is a closed, bounded, convex and infinitely-dimensional set in C_2 and $G : K_{\leq u} \setminus \{0\} \to K_{\leq u}$ is completely continuous. Moreover, by Proposition 4.3, $0 \in K_{\leq u}$ is an ejective point of G. Then G has a fixed point in $K_{\leq u} \setminus \{0\}$, which follows from Theorem 2.1 of Chapter 11 of Hale [**8**]. Now the existence of a special symmetric 4-periodic solution of system (20) follows from Theorem 2.5. This completes the proof. $\square$

Remark 4.6. Let $K_\pm := \{(\varphi_1, \varphi_2) \in C_2; \varphi_1(\theta) \geq 0 \text{ and } \varphi_2(\theta) \leq 0 \text{ for } \theta \in [0,1]\}$. Then $K_\pm$ is a positive cone in C_2 and determines a partial order, denoted by $\preceq$, on C_2. Choose σ as in the proof of Proposition 4.3 and define $\phi = (\phi_1, \phi_2) \in K_0$ by $\phi_1(t) = \sigma t$ and $\phi_2(t) = -\sigma t$ for $t \in [0,1]$. Then it follows from (21) that

$$(G\phi)_1(t) = -\int_0^t f_1(\phi_2(1-s))\,ds \geq -2\int_0^t \phi_2(1-s)\,ds$$
$$= 2\int_0^t \sigma(1-s)\,ds = \sigma t(2-t) \geq \phi_1(t)$$

and

$$(G\phi)_2(t) = -\int_0^t f_2(\phi_1(1-s))\,ds \leq -2\int_0^t \phi_1(1-s)\,ds$$
$$= -2\int_0^t \sigma(1-s)\,ds = -\sigma t(t-s) \leq \phi_2(t)$$

for $t \in [0,1]$. Thus $\phi \preceq G\phi$. Noting that G is order-preserving since both f_1 and f_2 are increasing, we get an increasing sequence $(G^k\phi)_{k=0}^{\infty}$ in $K_{\le u}$ by Proposition 4.4. Therefore, $\lim_{k\to\infty} G^k\phi$ exists and it is a fixed point of G. This gives an iterative scheme to obtain a special symmetric 4-periodic solution numerically.

Proposition 4.7. *Suppose that there exists $v > 0$ such that*

$$\min\left\{\frac{\int_0^v f_1(s)ds}{v^2},\ \frac{\int_0^v f_2(s)\,ds}{v^2}\right\} \ge 1.$$

Then the set $K_{\ge v} := \{\varphi \in K_0; \|\varphi\| \ge v\}$ is invariant with respect to G.

Proof. Let $\phi = (\phi_1, \phi_2) \in K_{\ge v}$. Then we have $\phi_1(t) \ge vt$ (and/or $\phi_2(t) \le -vt$) for $t \in [0,1]$. Thus, from (21),

$$\begin{aligned}\|G\phi\| &= \max\Big\{-\int_0^1 f_1(\phi_2(1-s))\,ds, \int_0^1 f_2(\phi_1(1-s))\,ds\Big\}\\ &\ge \min\Big\{\int_0^1 f_1(v(1-s))\,ds, \int_0^1 f_2(v(1-s))\,ds\Big\}\\ &= \min\left\{\frac{\int_0^v f_1(s)\,ds}{v}, \frac{\int_0^v f_2(s)\,ds}{v}\right\} \ge v. \qquad \square\end{aligned}$$

Corollary 4.8. *If there exist $0 < v < u$ such that*

$$\min\left\{\left(\int_0^v f_1(s)\,ds/v^2\right), \left(\int_0^v f_2(s)\,ds/v^2\right)\right\} \ge 1$$

and $\max\{f_1(u), f_2(u)\} \le u$, *then the set $K_{v,u} := \{\varphi \in K_0; v \le \|\varphi\| \le u\}$ is invariant with respect to G. In particular, the mapping G has a fixed point in $K_{v,u}$.*

Proof. We can use the Schauder fixed point theorem to get the result since $K_{v,u}$ is a bounded, closed, convex set and $G : K_{v,u} \to K_{v,u}$ is completely continuous. $\square$

Corollary 4.9. *Let both* f_1 *and* f_2 *be odd and increasing such that* $\limsup_{x\to\infty} a(x) > 1$ *and* $\liminf_{x\to\infty} b(x) < 1$*, where*

$$a(x) = \min\left\{\frac{\int_0^x f_1(s)\,ds}{x^2}, \frac{\int_0^x f_2(s)\,ds}{x^2}\right\}$$

and

$$b(x) = \max\left\{\frac{f_1(x)}{x}, \frac{f_2(x)}{x}\right\}.$$

Then system (20) *has infinitely many special symmetric* 4*-periodic solutions.*

Proof. The hypotheses imply that there exist two increasing sequences (v_k) and (u_k) such that $v_k < u_k < v_{k+1}$ and $a(v_k) > 1$ and $b(u_k) < 1$. According to Corollary 4.8, each set K_{v_k,u_k} contains a fixed point of G. This fixed point, according to Theorem 2.5, generates a special symmetric 4-periodic solution of system (20). □

5. Limiting profiles. In this section we study the limiting profiles of the special symmetric 4-periodic solutions of the singularly perturbed system

$$\begin{cases} \varepsilon\dot{x}_1(t) = f_1(x_2(t-1)), \\ \varepsilon\dot{x}_2(t) = f_2(x_1(t-1)) \end{cases} \tag{22}$$

with both f_1 and f_2 being odd, increasing and continuously differentiable. Moreover, we assume that $f_1'(0) = f_2'(0) > 0$.

Theorem 5.1. *Assume that both* $(f_1(x)/x)$ *and* $(f_2(x)/x)$ *are decreasing on* $(0,\infty)$ *with* $\lim_{x\to\infty}(f_1(x)/x) = \lim_{x\to\infty}(f_2(x)/x) = 0$. *Let* $(x_1^\varepsilon, x_2^\varepsilon) : \mathbf{R} \to \mathbf{R}^2$ *be the unique special symmetric* 4*-periodic solution of system* (22) *obtained by Corollary* 3.3*. Then*

$$x_1^\varepsilon(t) \to \infty \quad \text{and} \quad x_2^\varepsilon \to \infty \quad \text{uniformly on any subset of } (0,1]$$

$$\text{as } \varepsilon \to 0^+.$$

Proof. For $\varepsilon > 0$, let $f_1^\varepsilon(x) = (f_1(x)/\varepsilon)$ and $f_2^\varepsilon(x) = (f_2(x)/\varepsilon)$. Then system (22) can be written as

$$\begin{cases} \dot{x}_1(t) = f_1^\varepsilon(x_2(t-1)), \\ \dot{x}_2(t) = f_2^\varepsilon(x_1(t-1)). \end{cases} \tag{23}$$

Let G_ε be the cone mapping corresponding to system (23). Then there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ system (23), and hence system (22), has a unique special symmetric 4-periodic solution $(x_1^\varepsilon, x_2^\varepsilon)$ by Corollary 3.3 and Corollary 4.5. Moreover, as in the proof of Proposition 4.3, we can choose a fixed $\gamma \in (0, (1/3))$ and a $\sigma > 0$ such that $f_1'(x) > f_1'(0)((2/3) + \gamma)$ and $f_2'(x) > f_2'(0)((2/3) + \gamma)$ for $x \in [0, \sigma]$. Then, for any $\varepsilon \in (0, \min((f_1'(0)/3), \varepsilon_0))$ and any $\phi \in U_\sigma$ there is a positive integer $m(\varepsilon, \phi)$ such that $G_\varepsilon^{m(\varepsilon,\phi)} \notin U_\sigma$. Thus, we know that $x_1^\varepsilon(1) \geq \sigma$ and/or $x_2^\varepsilon(1) \leq -\sigma$. Let $a_\sigma = (\int_0^\sigma f_1(s)\,ds/\sigma)$, $b_\sigma = (\int_0^\sigma f_2(s)\,ds/\sigma)$ and $c_\sigma = \min\{a_\sigma, b_\sigma\}$. We also assume $\varepsilon < (0, \min((f'(0)/3), \varepsilon_0, c_\sigma))$. In the following we distinguish two cases.

Case 1. $x_1^\varepsilon(1) \geq \sigma$. Then $x_1^\varepsilon(t) \geq \sigma t$ for $t \in [0, 1]$. Thus

$$\begin{aligned} x_2^\varepsilon(1) &= -\frac{1}{\varepsilon}\int_0^1 f_2(x_1^\varepsilon(1-s))\,ds \\ &\leq -\frac{1}{\varepsilon}\int_0^1 f_2(\sigma(1-s))\,ds = -\frac{1}{\varepsilon}b_\sigma. \end{aligned} \tag{24}$$

Noting that $x_2^\varepsilon(t) \leq x_2^\varepsilon(1)t$ for $t \in [0, 1]$, we have

$$\begin{aligned} x_1^\varepsilon(1) &= -\frac{1}{\varepsilon}\int_0^1 f_1(x_2^\varepsilon(1-s))\,ds \\ &\geq -\frac{1}{\varepsilon}\int_0^1 f_1(x_2^\varepsilon(1)(1-s))\,ds \\ &= \frac{1}{\varepsilon}\int_0^1 f_1\left(\frac{1}{\varepsilon}b_\sigma s\right) ds \geq \frac{1}{\varepsilon}\int_0^1 f_1(s)\,ds. \end{aligned} \tag{25}$$

Case 2. $x_2^\varepsilon(1) \leq -\sigma$. Then $x_2^\varepsilon(t) \leq -\sigma t$ for $t \in [0,1]$. Thus

$$\begin{aligned} x_1^\varepsilon(1) &= -\frac{1}{\varepsilon}\int_0^1 f_1(x_2^\varepsilon(1-s))\,ds \\ &\geq -\frac{1}{\varepsilon}\int_0^1 f_1(-\sigma(1-s))\,ds \\ &= \frac{1}{\varepsilon}\int_0^1 f_1(\sigma(1-s))\,ds \\ &= \frac{1}{\varepsilon}\int_0^1 f_1(\sigma s)\,ds = \frac{1}{\varepsilon}a_\sigma. \end{aligned} \tag{26}$$

Noting $x_1^\varepsilon(t) \geq x_1^\varepsilon(t) \geq (1/\varepsilon)a_\sigma t$ for $t \in [0,1]$, we have

$$\begin{aligned} x_2^\varepsilon(1) &= -\frac{1}{\varepsilon}\int_0^1 f_2(x_1^\varepsilon(1-s))\,ds \\ &\leq -\frac{1}{\varepsilon}\int_0^1 f_2\left(\frac{1}{\varepsilon}a_\sigma(1-s)\right)ds \\ &\leq -\frac{1}{\varepsilon}\int_0^1 f_2\left(\frac{1}{\varepsilon}c_\sigma s\right)ds \leq -\frac{1}{\varepsilon}\int_0^1 f_2(s)\,ds. \end{aligned} \tag{27}$$

It follows from (24)–(27) that $x_1^\varepsilon(1) \to \infty$ and $x_2^\varepsilon(1) \to -\infty$ as $\varepsilon \to 0^+$. Noting $x_1^\varepsilon(t) \geq x_1^\varepsilon(1)t$ and $x_2^\varepsilon(t) \leq x_2^\varepsilon(1)t$ for $t \in [0,1]$, the result of the theorem follows immediately. □

Intuitively, one can picture the limiting profiles in Theorem 5.1 as pulses of unbounded amplitudes. Now, for a special class of f_1 and f_2, we show that $x_1^\varepsilon(1) = O(1/\varepsilon)$ and $x_2^\varepsilon(1) = -O(1/\varepsilon)$ as $\varepsilon \to 0^+$ and that, for $i = 1$ and 2, $\varepsilon x_i^\varepsilon(t)$ approaches a sawtooth wave whose slope is determined by $\lim_{x\to\infty} f_i(x)$.

Corollary 5.2. *In addition to the hypotheses in Theorem* 5.1*, we assume that both* f_1 *and* f_2 *are bounded and let* $a_\infty = \lim_{x\to\infty} f_1(x)$ *and* $b_\infty = \lim_{x\to\infty} f_2(x)$*. Then*

$$\left.\begin{aligned} \lim_{\varepsilon\to 0^+} \varepsilon x_1^\varepsilon(t) &= a_\infty t \\ \lim_{\varepsilon\to 0^+} \varepsilon x_2^\varepsilon(t) &= -b_\infty t \end{aligned}\right\} \quad \text{for } t \in [0,1].$$

Proof. Since $(x_1^\varepsilon, x_2^\varepsilon)|_{[0,1]}$ is a fixed point of G_ε (defined in the proof of Theorem 5.1), it follows from (21) that

$$\begin{cases} \varepsilon x_1^\varepsilon(t) = -\displaystyle\int_0^t f_1(x_2^\varepsilon(1-s))\,ds \\ \\ \varepsilon x_2^\varepsilon(t) = -\displaystyle\int_0^t f_2(x_1^\varepsilon(1-s))\,ds \end{cases} \quad \text{for } t \in [0,1].$$

Therefore,

$$\begin{cases} -tf_1(x_2^\varepsilon(1-t)) \le \varepsilon x_1^\varepsilon(t) \le -tf_1(x_2^\varepsilon(1)) \\ \\ -tf_2(x_1^\varepsilon(1)) \le \varepsilon x_2^\varepsilon(t) \le -tf_2(x_1^\varepsilon(1-t)) \end{cases} \quad \text{for } t \in [0,1],$$

which, combined with Theorem 5.1, implies that $\lim_{\varepsilon\to 0^+} \varepsilon x_1^\varepsilon(t) = a_\infty t$ and $\lim_{\varepsilon\to 0^+} \varepsilon x_2^\varepsilon(t) = -b_\infty t$ uniformly on any subset of $[0,1)$. Note that

$$\begin{cases} \varepsilon x_1^\varepsilon(t) \le \varepsilon x_1^\varepsilon(1) \le a_\infty \\ \\ -b_\infty \le \varepsilon x_2^\varepsilon(1) \le \varepsilon x_2^\varepsilon(t) \end{cases} \quad \text{for } t \in [0,1].$$

Hence, $\lim_{\varepsilon\to 0^+} \varepsilon x_1^\varepsilon(1) = a_\infty$ and $\lim_{\varepsilon\to 0^+} \varepsilon x_2^\varepsilon(1) = -b_\infty$. This completes the proof. □

Note that the periodicity of $(x_1^\varepsilon, x_2^\varepsilon)$ implies that each $\varepsilon x_i^\varepsilon(t)$ approaches the sawtooth-like wave $p_i^* : \mathbf{R} \to \mathbf{R}$ with $p_1^*(t) = a_\infty t$ and $p_2^*(t) = -b_\infty t$ for $t \in [-1,1]$ and $p_i^*(t+2) = -p_i^*(t)$ for $t \in \mathbf{R}$. See Figure 1. Furthermore, $(\varepsilon x_1^\varepsilon(t), \varepsilon x_2^\varepsilon(t))$ approaches the line segment $(p_1^*(t), p_2^*(t))$ for $t \in \mathbf{R}$ as $\varepsilon \to 0^+$, see Figure 2.

Example 5.3. We can check that the odd function $g : \mathbf{R} \to \mathbf{R}$ with $g(x) = \alpha \log(x+1)$ for $x \ge 0$ and some positive α satisfies the assumptions of Theorem 5.1 but $\lim_{x\to\infty} g(x) = \infty$.

Example 5.4. It is easy to check that $g(x) = \alpha \tanh(\beta x) = \alpha(e^{\beta x} - e^{-\beta x})/(e^{\beta x} + e^{-\beta x})$ with $\alpha > 0$ and $\beta > 0$ satisfying the assumptions of both Theorem 5.1 and Theorem 5.2.

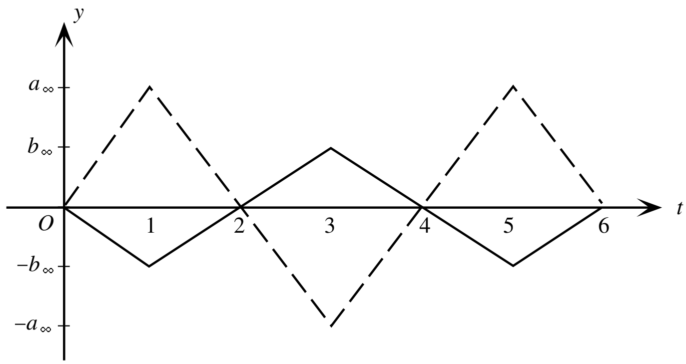

FIGURE 1. Sawtooth waves p_1^* and p_2^* (dashed line for p_1^* and solid line for p_2^*).

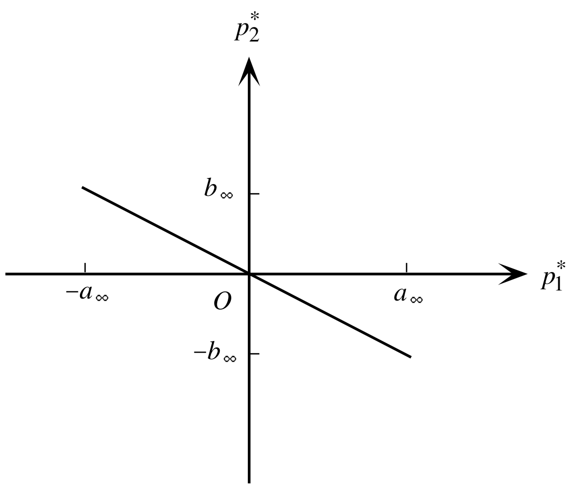

FIGURE 2. Limiting profile of $(\varepsilon x_1^\varepsilon(t), \varepsilon x_2^\varepsilon(t))$ is a line segment in the plane.

6. Implication for negative feedback systems. We now consider the following system of delay differential equations with negative feedback

$$\begin{cases} \dot{x}_1(t) = \quad f_1(x_1(t), x_2(t-1)), \\ \dot{x}_2(t) = -f_2(x_2(t), x_1(t-1)), \end{cases} \tag{28}$$

where $f_i : \mathbf{R}^2 \to \mathbf{R}$, $i = 1, 2$, is locally Lipschitz continuous and satisfies the same symmetry condition (2) and positive feedback condition (3). Linearizing system (28) around the origin gives

$$\begin{cases} \dot{X}_1(t) = \quad aX_2(t-1), \\ \dot{X}_2(t) = -aX_1(t-1). \end{cases} \tag{29}$$

The characteristic equation associated with (29) is given by

$$0 = \begin{vmatrix} \lambda & -ae^{-\lambda} \\ ae^{-\lambda} & \lambda \end{vmatrix} = (\lambda + iae^{-\lambda})(\lambda - iae^{-\lambda}). \tag{30}$$

When and only when $a = k\pi$ for some $k \in \mathbf{N}$, equation (30) has purely imaginary zeros and these zeros are exactly $\pm k\pi i$. Thus system (29) has a particular periodic solution

$$\begin{cases} X_1(t) = \sin(k\pi t) \\ X_2(t) = (-1)^k \cos(k\pi t), \end{cases}$$

with the minimal period $(2/k)$, and it satisfies

$$\begin{cases} X_1(t) = -X_1\left(t + \frac{1}{k}\right) & \text{for } t \in \mathbf{R}, \\ X_1(t) > 0 & \text{on } \left(0, \frac{1}{k}\right), \\ X_2(t) = -X_2\left(t + \frac{1}{k}\right) & \text{for } t \in \mathbf{R}. \end{cases} \tag{31}$$

It is therefore natural to study periodic solutions (x_1, x_2) of system (28) with x_1 and x_2 satisfying

$$\begin{cases} x_1(t) = -x_1\left(t + \frac{1}{k}\right) & \text{for } t \in \mathbf{R}, \\ x_1(t) > 0 & \text{on } \left(0, \frac{1}{k}\right), \\ x_2(t) = -x_2\left(t + \frac{1}{k}\right) & \text{for } t \in \mathbf{R}. \end{cases} \tag{32}$$

Obviously, $(2/k)$ is the minimal period of (x_1, x_2). Using the same technique used in Section 1, we can show that it is sufficient to consider 2-periodic solutions of system (28) satisfying (32). We call such a periodic solution a *special symmetric* 2*-periodic solution.*

Given a special symmetric 2-periodic solution (x_1, x_2) of system (28), define $(u_1, u_2) : \mathbf{R} \to \mathbf{R}^2$ by $u_1(t) = x_1(t/2)$ and $u_2(t) = x_2((t-1)/2)$ for $t \in \mathbf{R}$. Then (u_1, u_2) satisfies (7) with $k = 1$. Moreover, using (2) and (32), we can easily check that (u_1, u_2) also satisfies

$$\begin{cases} \dot{u}_1(t) = \dfrac{1}{2}\, f_1(u_1(t), u_2(t-1)), \\[2ex] \dot{u}_2(t) = \dfrac{1}{2}\, f_2(u_2(t), u_1(t-1)). \end{cases} \tag{33}$$

Note that system (33) coincides with system (1) with f_1 and f_2 being replaced by $(1/2)f_1$ and $(1/2)f_2$, respectively. Hence, (u_1, u_2) is a special symmetric 4-periodic solution of system (33). On the other hand, if (u_1, u_2) is a special symmetric 4-periodic solution of system (33), by defining $(x_1, x_2) : \mathbf{R} \to \mathbf{R}^2$ by $x_1(t) = u_1(2t)$ and $x_2(t) = u_2(2t+1)$ for $t \in \mathbf{R}$, then (x_1, x_2) satisfies (32) with $k = 1$. Moreover, using (2) and (7), we can check that (x_1, x_2) also satisfies system (28). Hence, (x_1, x_2) is a special symmetric 2-periodic solution of system (28). Therefore, we have established a one-to-one correspondence between a special symmetric 2-periodic solution of system (28) and a special symmetric 4-periodic solution of system (33). Thus, similar results to those about special symmetric 4-periodic solutions of system (1) can be obtained for special symmetric 2-periodic solutions of system (28). In particular, we have

Theorem 6.1. *Suppose that both* $(f_1(x, y)/y)$ *and* $(f_2(x, y)/y)$ *are decreasing* (*increasing*) *on* $\{(x, y); y > 0\}$*. If a special symmetric* 2*-periodic solution of system* (28) *exists, then it is unique.*

For the case where both $f_1(x, y)$ and $f_2(x, y)$ are independent of x, we have results about existence and nonuniqueness. More precisely, we consider

$$\begin{cases} \dot{x}_1(t) = f_1(x_2(t-1)), \\ \dot{x}_2(t) = -f_2(x_1(t-1)) \end{cases} \tag{34}$$

and assume that both f_1 and f_2 are odd, increasing and continuously differentiable. Also, recall that we assume $f_1'(0) = f_2'(0)$.

Theorem 6.2. *Assume that both f_1 and f_2 are odd and increasing, $f_1'(0) = f_2'(0) > 4$ and there exists $u > 0$ such that $\max\{f_1(u), f_2(u)\} \leq 2u$. Then system (34) has a special symmetric 2-periodic solution.*

Theorem 6.3. *Let both f_1 and f_2 be odd and increasing such that $\limsup_{x\to\infty} a(x) > 1$ and $\liminf_{x\to\infty} b(x) < 1$, where*

$$a(x) = \min\left\{\frac{\int_0^x f_1(s)\,ds}{2x^2}, \frac{\int_0^x f_2(s)\,ds}{2x^2}\right\}$$

and

$$b(x) = \max\left\{\frac{f_1(x)}{2x}, \frac{f_2(x)}{2x}\right\}.$$

Then system (34) has infinitely many special symmetric 2-periodic solutions.

Now we state the results on limiting profiles. Consider

$$\begin{cases} \varepsilon\dot{x}_1(t) = \quad f_1(x_2(t-1)), \\ \varepsilon\dot{x}_2(t) = -f_2(x_1(t-1)), \end{cases} \tag{35}$$

with both f_1 and f_2 being odd, increasing and continuously differentiable. Moreover, we assume that $f_1'(0) = f_2'(0)$.

Theorem 6.4. *Assume that both $(f_1(x)/x)$ and $(f_2(x)/x)$ are decreasing on $(0, \infty)$ with $\lim_{x\to\infty}(f_1(x)/x) = \lim_{x\to\infty}(f_2(x)/x) = 0$. Let $(x_1^\varepsilon, x_2^\varepsilon) : \mathbf{R} \to \mathbf{R}^2$ be the unique special symmetric 2-periodic solution of system (35) obtained by Theorem 6.1. Then*

$$x_1^\varepsilon(t) \to \infty \quad \text{and} \quad x_2^\varepsilon(t) \to -\infty \quad \text{uniformly on any subset of } (0,1) \text{ as } \varepsilon \to 0^+.$$

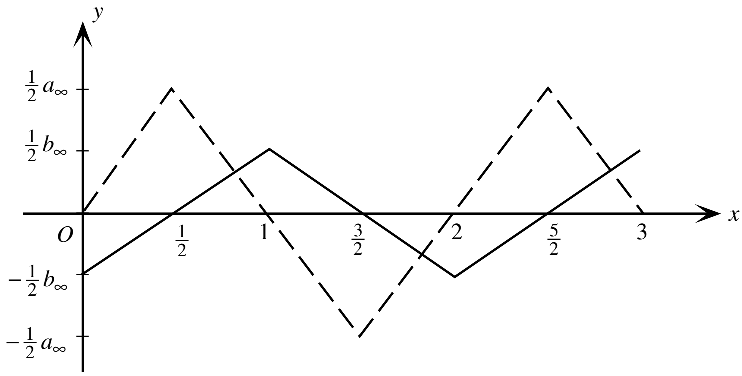

FIGURE 3. Sawtooth waves q_1^* and q_2^* (dashed line for q_1^* and solid line for q_2^*).

Theorem 6.5. *In addition to the hypotheses in Theorem* 6.4*, we assume that both* f_1 *and* f_2 *are bounded and denote* $a_\infty = \lim_{x\to\infty} f_1(x)$ *and* $b_\infty = \lim_{x\to\infty} f_2(x)$*. Then*

$$\left.\begin{aligned} \lim_{\varepsilon\to 0^+} \varepsilon x_1^\varepsilon(t) &= a_\infty t \\ \lim_{\varepsilon\to 0^+} \varepsilon x_2^\varepsilon(t) &= b_\infty(t-(1/2)) \end{aligned}\right. \quad \text{for } t \in \left[0, \frac{1}{2}\right].$$

Note that the periodicity of $(x_1^\varepsilon, x_2^\varepsilon)$ implies that each $\varepsilon x_i^\varepsilon(t)$ approaches the sawtooth-like wave $q_i^* : \mathbf{R} \to \mathbf{R}$ with $q_1^*(t) = a_\infty t$ for $t \in [-(1/2),(1/2)]$ and $q_2^*(t) = b_\infty(t-(1/2))$ for $t \in [0,1]$ and $q_i^*(t+1) = -q_i^*(t)$ for $t \in \mathbf{R}$, see Figure 3. Furthermore, $(\varepsilon x_1^\varepsilon(t), \varepsilon x_2^\varepsilon(t))$ approaches the diamond $(q_1^*(t), q_2^*(t))$ for $t \in \mathbf{R}$ as $\varepsilon \to 0^+$, see Figure 4. Remember that, for the special symmetric 4-periodic solutions $(x_1^\varepsilon(t), x_2^\varepsilon(t))$ of system (22), $(\varepsilon x_1^\varepsilon(t), \varepsilon x_2^\varepsilon(t))$ approaches a line segment. These results indicate a difference between the dynamical behaviors of positive feedback and negative feedback systems.

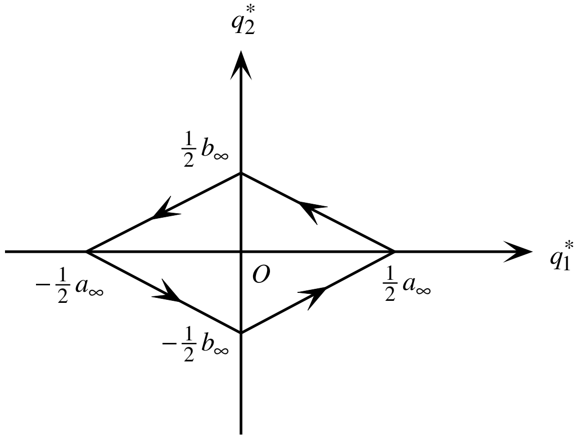

FIGURE 4. Limiting profile of $(\varepsilon x_1^\varepsilon(t), \varepsilon x_2^\varepsilon(t))$ is a diamond in the plane.

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