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# IRREGULAR OUTSTARS FOR LEARNING/ RECALLING DOMINATED SPATIAL PATTERNS *†

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**Abstract**

We consider an outstar, allowing different spiking frequencies, different transmitter production rates and different thresholds. We show that different thresholds may lead to biased pattern learning of the network described by a system of delay differential equations.

**Keywords** outstar, pattern learning, Pavlovian conditioning, delay differential equation

**1991 MR Subject Classifications** 34K25, 68T05, 92B20

## 1 Introduction

In this paper, we study the long-time behavior of the following system

$$\begin{cases} \frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = -a_0(t)x_0(t) + I_0(t), \\ \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + b_i(t)f_i(x_0(t-\tau_i) - \Gamma_i)z_i + I_i(t), \\ \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_i(t)z_i(t) + d_i(t)g_i(x_0(t-\tau_i^*) - \Gamma_i^*)x_i, \end{cases} \tag{1.1}$$

where $i = 1, \cdots, n$, $\tau_i$, $\tau_i^*$, $\Gamma_i$ and $\Gamma_i^*$ are non-negative constants, $a_0$, $a_i$, $b_i$, $c_i$, $d_i$, $f_i$, $g_i$, $I_0$, $I_i : R \to [0, \infty)$ are continuous functions and $I_0$ and $I_i$ are bounded.

System (1.1) models the evolution of a neural network called outstar, which consists of a command neuron $v_0$ and input neurons $(v_1, \cdots, v_n)$. The short-term memory trace of each neuron $v_i$ is denoted by $x_i(t)$, the long-term memory trace from $v_0$ to $v_i$ is denoted by $z_i(t)$, $I_i(t)$ denotes the external input to neuron

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$v_i$, $f_i$ and $g_i$ denote the signal functions, and $\tau_i$ and $\tau_i^*$ denote the time lags of signal transfer from the command neuron to input neurons. $b_i(t)f_i(x_0(t-\tau_i)-\Gamma_i)$ and $d_i(t)g(x_0(t-\tau_i^*)-\Gamma_i^*)$ are called spiking frequencies and transmitter production rates. The special case where all $(a_i, b_i, c_i, d_i, f_i, g_i, \tau_i, \tau_i^*, \Gamma_i, \Gamma_i^*)$ are independent of $i$ (thus only depending on the source (the command) neuron) was discussed in a series of papers by Grossberg [1-6] in association with pattern learning and Pavlovian conditioning. It was shown by Grossberg that such a network is well suited to perfect (unbiased) pattern learning (See also Harvey [7] and Wu [8]).

In this paper, we are going to consider the more general case admitting different spiking frequencies, different transmitter production rates and different thresholds from the command neuron to input neurons. Several results (Theorem 3.2, Corollary 3.1 and Theorem 3.3) are established to given explicit formulae for $\lim\limits_{t\to\infty}\frac{x_i(t)}{x_j(t)}$ and $\lim\limits_{t\to\infty}\frac{z_i(t)}{z_j(t)}$ in terms of various parameters involved. In particular, we show that even if all parameters are held constant and identical from one input neuron to another, different thresholds can lead to biased pattern learning (Corollary 3.1). This may enable the network to pick up only the dominated spattern and treat other smaller patterns as noise and suppress them altogther after sufficiently many times of training.

## 2 Technical Lemmas

In this section, we establish several technical lemmas which will be needed in the proofs of our main results.

The first lemma, called Gronwall's Inequality, is perhaps well-known, but we cite it here for ease of reference.

**Lemma 2.1** *Suppose that $t_0 \in R$ and that $\varphi, \psi, \omega, \xi : [t_0, \infty) \to R$ are continuous functions with $\omega(t), \xi(t) \geq 0$ for all $t \geq t_0$, If*

$$\varphi(t) \leq \psi(t) + \xi(t)\int_{t_0}^{t} \omega(s)\varphi(s)\mathrm{d}s \quad \text{for } t \geq t_0, \tag{2.1}$$

*then*

$$\varphi(t) \leq \psi(t) + \xi(t)\int_{t_0}^{t} \psi(s)\omega(s)e^{\int_s^t \xi(u)\omega(u)\mathrm{d}u}\mathrm{d}s \quad \text{for } t \geq t_0. \tag{2.2}$$

**Lemma 2.2** *Let $t_0 \in R$, $P, Q, F, G, I : [t_0, \infty) \to R$ be continuous with $I$ being bounded and*

$$P(t), Q(t), F(t), G(t) \to 0 \quad \text{as} \quad t \to \infty.$$

*Assume that $\alpha, \beta, \delta$ and $\gamma$ are non-negative constants such that the real parts of all eigenvalues of* $A = \begin{pmatrix} -\alpha & \delta \\ \gamma & -\beta \end{pmatrix}$ *are negative. Then every solution of*

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + \delta z(t) + P(t)x(t) + F(t)z(t) + I(t), \\ \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \gamma x(t) - \beta z(t) + G(t)x(t) + Q(t)z(t) \end{cases} \tag{2.3}$$

*is bounded on* $[t_0, \infty)$.

**Proof** Let $y(t) = (x(t), z(t))^T$ ($T$ denotes transposition) be a given solution of (2.3) and let $M(t) = \begin{pmatrix} P(t) & F(t) \\ G(t) & Q(t) \end{pmatrix}$ for $t \geq t_0$. Denote by $\|\cdot\|$ the Euclidean norm in $R^2$, and for a $2 \times 2$ real matrix $B$ let $\|B\|$ be the matrix norm so that $\|By\| \leq \|B\|\|y\|$ for all $y \in R^2$. Applying the variantion-of-constants formula to (2.3), we get

$$\begin{aligned} y(t) &= e^{A(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{A(t-s)}(I(s), 0)^T \mathrm{d}s \\ &\quad + \int_{t_0}^{t} e^{A(t-s)}M(s)y(s)\mathrm{d}s, \quad t \geq t_0. \end{aligned}$$

It follows that

$$\begin{aligned} \|y(t)\| &\leq \|e^{A(t-t_0)}\|\|y(t_0)\| + \|\int_{t_0}^{t} e^{A(t-s)}(I(s), 0)^T \mathrm{d}s\| \\ &\quad + \|\int_{t_0}^{t} e^{A(t-s)}M(s)y(s)\mathrm{d}s\|, \quad t \geq t_0. \end{aligned}$$

Since the real parts of all eigenvalues of $A$ are negative, there exist positive constants $K$ and $\lambda$ such that

$$\|e^{At}\| \leq Ke^{-\lambda t} \quad \text{for } t \geq 0.$$

Therefore,

$$\begin{aligned} \|y(t)\| &\leq Ke^{-\lambda(t-t_0)}\|y(t_0)\| + K\int_{t_0}^{t} e^{-\lambda(t-s)}|I(s)|\mathrm{d}s \\ &\quad + K\int_{t_0}^{t} e^{-\lambda(t-s)}\|M(s)\|\|y(s)\|\mathrm{d}s, \quad t \geq t_0. \end{aligned}$$

Let

$$C = \sup_{t \geq t_0}\left[Ke^{-\lambda(t-t_0)}\|y(t_0)\| + K\int_{t_0}^{t} e^{-\lambda(t-s)}|I(s)|\mathrm{d}s\right].$$

Note that $I$ is bounded on $[t_0,\infty)$. Therefore, $C<\infty$ and Lemma 2.1 implies

$$\begin{aligned}\|y(t)\| &\le C+KC\int_{t_0}^{t} e^{-\lambda(t-s)}\|M(s)\|e^{K\int_s^t\|M(u)\|\mathrm{d}u}\mathrm{d}s\\ &= C+KC\int_{t_0}^{t} e^{\int_s^t(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s.\end{aligned}$$

Since $\|M(t)\|\to 0$ as $t\to\infty$, we can find $T\ge t_0$ so that

$$\|M(u)\|<1 \quad \text{and} \quad -\lambda+K\|M(u)\|<-\frac{\lambda}{2} \quad \text{for} \quad u\ge T.$$

Therefore, for $t\ge T$, we have

$$\begin{aligned}\|y(t)\| &\le C+KC\Big[\int_{t_0}^{T} e^{\int_s^t(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s\\ &\quad +\int_T^t e^{\int_s^t(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s\Big]\\ &\le C+KCe^{\int_T^t(-\lambda+K\|M(u)\|)\mathrm{d}u}\int_{t_0}^{T} e^{\int_s^T(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s\\ &\quad +KC\int_T^t e^{-\frac{\lambda}{2}(t-s)}\mathrm{d}s\\ &\le C+KC\int_{t_0}^{T} e^{\int_s^T(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s+\frac{2KC}{\lambda}\left[1-e^{-\frac{\lambda}{2}(t-T)}\right]\\ &\le C+KC\int_{t_0}^{T} e^{\int_s^T(-\lambda+K\|M(u)\|)\mathrm{d}u}\|M(s)\|\mathrm{d}s+\frac{2KC}{\lambda}.\end{aligned}$$

This proves the boundedness of $y(t)$ on $[t_0,\infty)$.

**Lemma 2.3** *Assume that all conditions of Lemma 2.2 are satisfied. Let* $y=(x,z)^T:[t_0,\infty)\to R^2$ *be a solution of (2.3), and let* $Y=(X,Z)^T:[t_0,\infty)\to R^2$ *be the solution of*

$$\begin{cases}\frac{\mathrm{d}X(t)}{\mathrm{d}t}=-\alpha X(t)+\delta Z(t)+I(t),\\ \frac{\mathrm{d}Z(t)}{\mathrm{d}t}=\gamma X(t)-\beta Z(t)\end{cases} \tag{2.4}$$

*with* $Y(t_0)=y(t_0)$. *Then* $\lim_{t\to\infty}[Y(t)-y(t)]=(0,0)^T$.

**Proof** Using the variation-of-constants formula to (2.3) and (2.4) and using the fact that $Y(t_0)=y(t_0)$, we obtain

$$Y(t)-y(t) = \begin{pmatrix}X(t)-x(t)\\ Z(t)-z(t)\end{pmatrix}$$

$$= -\int_{t_0}^{t} e^{A(t-s)} \begin{pmatrix} P(s) & F(s) \\ G(s) & Q(s) \end{pmatrix} \begin{pmatrix} x(s) \\ z(s) \end{pmatrix} \mathrm{d}s$$
$$= -\int_{t_0}^{t} e^{A(t-s)} M(s)y(s)\mathrm{d}s.$$

By Lemma 2.2, we can find a constant $K_1 > 0$ such that $\|y(t)\| \le K_1$ for all $t \ge t_0$. Again note that $\|e^{At}\| \le Ke^{-\lambda t}$ for $t \ge 0$. It follows that

$$\|Y(t) - y(t)\| \le K_1 K \int_{t_0}^{t} e^{-\lambda(t-s)} \|M(s)\| \mathrm{d}s.$$

Let $\varepsilon > 0$ be given. Since $\|M(s)\| \to 0$ as $s \to \infty$, there exists a constant $T_1 \ge t_0$ such that $\|M(s)\| < \frac{\lambda}{2K_1K}\varepsilon$ for $s \ge T_1$. Again note that $e^{-\lambda t} \to 0$ as $t \to \infty$, we can find a constant $T_2 \ge T_1$ such that $K_1Ke^{-\lambda t}\int_{t_0}^{T_1} e^{\lambda s}\|M(s)\|\mathrm{d}s < \frac{1}{2}\varepsilon$. Therefore, for $t \ge T_2$ we have

$$\begin{aligned} \|Y(t) - y(t)\| &\le K_1Ke^{-\lambda t}\int_{t_0}^{T_1} e^{\lambda s}\|M(s)\|\mathrm{d}s + \frac{\lambda\varepsilon}{2}\int_{T_1}^{t} e^{-\lambda(t-s)}\mathrm{d}s \\ &\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon[1 - e^{-\lambda(t-T_1)}] < \varepsilon. \end{aligned}$$

Hence $\|Y(t) - y(t)\| \to 0$ as $t \to \infty$. This completes the proof.

**Lemma 2.4** *Assume that $\alpha \in (0,\infty), t_0 \in R$, $P, I : [t_0,\infty) \to R$ is continuous with $I$ being bounded and that $P(t) \to 0$ as $t \to \infty$. Let $x : [t_0,\infty) \to R$ be a solution of*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + P(t)x(t) + I(t),$$

*and let $X : [t_0,\infty) \to R$ be the solution of*

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\alpha X(t) + I(t)$$

*with $X(t_0)=x(t_0)$. Then $\lim\limits_{t\to\infty}[X(t)-x(t)]=0$. Moreover, if $\lim\limits_{t\to\infty}\int_{t_0}^{t} e^{-\alpha(t-s)}I(s)\mathrm{d}s$ exists, then $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} X(t) = \lim\limits_{t\to\infty}\int_{t_0}^{t} e^{-\alpha(t-s)}I(s)\mathrm{d}s$.*

**Proof** By the variation-of-constants formula, we have

$$x(t) = e^{-\alpha(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-(t-s)}I(s)\mathrm{d}s + \int_{t_0}^{t} e^{-\alpha(t-s)}P(s)x(s)\mathrm{d}s \tag{2.5}$$

and

$$X(t) = e^{-\alpha(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{-(t-s)}I(s)\mathrm{d}s. \tag{2.6}$$

Using (2.5) and a similar argument to that for the proof of Lemma 2.2, one can show that $x$ is bounded on $[t_0, \infty)$. Again by using a similar argument to that for the proof of Lemma 2.3, one can show that $\lim\limits_{t\to\infty}[X(t) - x(t)] = 0$. If $\lim\limits_{t\to\infty}\int_{t_0}^{t} e^{-(t-s)}I(s)\mathrm{d}s$ exists, then it follows from (2.6) that

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} e^{-\alpha(t-t_0)}X(t_0)+\lim_{t\to\infty}\int_{t_0}^{t} e^{-\alpha(t-s)}I(s)\mathrm{d}s = \lim_{t\to\infty}\int_{t_0}^{t} e^{-(t-s)}I(s)\mathrm{d}s.$$

This completes the proof.

## 3 Main Results

In this section, we state and prove our main results. We will require the following assumptions.

(H1) The limits $\alpha_0 \triangleq \lim\limits_{t\to\infty} a_0(t), \alpha_i \triangleq \lim\limits_{t\to\infty} a_i(t), b_i^* \triangleq \lim\limits_{t\to\infty} b_i(t), \beta_i \triangleq \lim\limits_{t\to\infty} c_i(t)$ and $d_i^* \triangleq \lim\limits_{t\to\infty} d_i(t)$ exist for $i = 1, 2, \cdots, n$.

(H2) $\alpha_0 > 0$ and the limit $L_0(-\alpha_0) \triangleq \lim\limits_{t\to\infty}\int_0^t e^{-\alpha_0(t-s)}I_0(s)\mathrm{d}s$ exists.

**Theorem 3.1** *Suppose that (H1) and (H2) hold. Let* $\delta_i = b_i^* f_i(L_0(-\alpha_0) - \Gamma_i)$, $\gamma_i = d_i^* g_i(L_0(-\alpha_0) - \Gamma_i^*), i = 1, 2, \cdots, n$. *Assume also that*

*(H3)* $\alpha_i\beta_i > \delta_i\gamma_i$ *and* $(\alpha_i - \beta_i)^2 + 4\delta_i\gamma_i \neq 0$ *for* $i = 1, 2, \cdots, n$.

*Let* $y^* = (x_0, x_1, z_1, \cdots, x_n, z_n)^T : [t_0, \infty) \to R^{2n+1}$ *be a solution of (1.1), and let* $Y^* = (X_0, X_1, Z_1, \cdots, X_n, Z_n)^T : [t_0, \infty) \to R^{2n+1}$ *be the solution of*

$$\begin{cases} \frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = -\alpha_0 X_0(t) + I_0(t), \\ \frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = -\alpha_i X_i(t) + \delta_i Z_i(t) + I_i(t), \\ \frac{\mathrm{d}Z_i(t)}{\mathrm{d}t} = -\beta_i Z_i(t) + \gamma_i X_i(t), \quad i = 1, 2, \cdots, n \end{cases} \tag{3.1}$$

*with* $Y^*(t_0) = y^*(t_0)$. *Then* $y^*$ *is bounded on* $[t_0, \infty)$ *and* $\lim\limits_{t\to\infty}[Y^*(t) - y^*(t)] = (0, \cdots, 0)^T \in R^{2n+1}$.

**Proof** Rewrite the first equation of (1.1) as

$$\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = -\alpha_0 x_0(t) + I_0(t) + [\alpha_0 - a_0(t)]x_0(t).$$

Note that $\lim\limits_{t\to\infty}[\alpha_0 - a_0(t)] = 0$. By Lemma 2.4, we have

$$\lim_{t\to\infty} x_0(t) = \lim_{t\to\infty} X_0(t) = \lim_{t\to\infty}\int_{t_0}^{t} e^{-\alpha_0(t-s)}I_0(s)\mathrm{d}s$$

$$= \lim_{t\to\infty}\int_0^t e^{-\alpha_0(t-s)}I_0(s)\mathrm{d}s = L_0(-\alpha_0). \tag{3.2}$$

Let $i \in \{1,2,\cdots,n\}, t_0 \in R$, $y_i = (x_i, z_i)^T : [t_0,\infty) \to R^2$ be a solution of system

$$\begin{cases} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + b_i(t)f_i(x_0(t-\tau_i) - \Gamma)z_i(t) + I_i(t), \\ \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_i(t)z_i(t) + d_i(t)g_i(x_0(t-\tau_i^*) - \Gamma_i^*)x_i(t), \end{cases} \tag{3.3}$$

and let $Y_i = (X_i, Z_i)^T : [t_0,\infty) \to R^2$ be the solution of system

$$\begin{cases} \frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = -\alpha_i X_i(t) + \delta_i Z_i(t) + I_i(t), \\ \frac{\mathrm{d}Z_i(t)}{\mathrm{d}t} = -\beta_i Z_i(t) + \gamma_i X_i(t) \end{cases} \tag{3.4}$$

with $Y_i(t_0) = y_i(t_0)$. Then it suffices to show that $y_i$ is bounded on $[t_0,\infty)$ and $\lim_{t\to\infty}[Y_i(t) - y_i(t)] = (0,0)^T$.

Set

$$A_i = \begin{pmatrix} -\alpha_i & \delta_i \\ \gamma_i & -\beta_i \end{pmatrix}.$$

Then, the eigenvalues $\lambda_{i1}$ and $\lambda_{i2}$ of $A_i$ are given by

$$\lambda_{i1} = \frac{-(\alpha_i+\beta_i) - \sqrt{(\alpha_i-\beta_i)^2 + 4\gamma_i\delta_i}}{2} \tag{3.5}$$

and

$$\lambda_{i2} = \frac{-(\alpha_i+\beta_i) + \sqrt{(\alpha_i-\beta_i)^2 + 4\gamma_i\delta_i}}{2}. \tag{3.6}$$

Moreover, it is straightforward to check that $\lambda_{i1} < \lambda_{i2} < 0$ under the assumption (H3).

Set

$$\begin{aligned} P_i(t) &= \alpha_i - a_i(t), \qquad Q_i(t) = \beta_i - c_i(t), \\ F_i(t) &= b_i(t)f_i(x_0(t-\tau_i) - \Gamma_i) - \delta_i, \\ G_i(t) &= d_i(t)g_i(x_0(t-\tau_i^*) - \Gamma_i^*) - \gamma_i. \end{aligned}$$

Then, by (3.2) and the given conditions in the theorem, we have

$$P_i(t), Q_i(t), F_i(t), G_i(t) \to 0 \quad \text{as } t \to \infty.$$

Rewrite (3.3) as

$$\begin{cases} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\alpha_i x_i(t) + \delta_i z_i(t) + P_i(t)x_i(t) + F_i(t)z_i(t) + I_i(t), \\ \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = \gamma_i x_i(t) - \beta_i z_i(t) + G_i(t)x_i(t) + Q_i(t)z_i(t). \end{cases} \tag{3.7}$$

Applying Lemmas 2.2 and 2.3 to (3.7), we obtain that $y_i$ is bounded on $[t_0, \infty)$ and $\lim_{t\to\infty}[Y_i(t) - y_i(t)] = (0,0)^T$.

This completes the proof of Theorem 3.1.

**Theorem 3.2** *Suppose that all conditions of Theorem 3.1 are satisfied, and that*

*(H4) The limits* $L_i(\lambda_{i1}) \triangleq \lim_{t\to\infty}\int_0^t e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s$ *and* $L_i(\lambda_{i2}) \triangleq \lim_{t\to\infty}\int_0^t e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s$ *exist for* $i = 1, 2, \cdots, n$, *where* $\lambda_{i1}$ *and* $\lambda_{i2}$ *are given by (3.5) and (3.6). Then every solution of (1.1) is convergent as* $t \to \infty$. *More precisely, we have*

$$\lim_{t\to\infty} x_0(t) = L_0(-\alpha_0), \tag{3.8}$$

$$\lim_{t\to\infty} x_i(t) = \frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}}L_i(\lambda_{i2}) - \frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}}L_i(\lambda_{i1}), \tag{3.9}$$

$$\lim_{t\to\infty} z_i(t) = \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}[L_i(\lambda_{i2}) - L_i(\lambda_{i1})], \quad i = 1, 2, \cdots, n. \tag{3.10}$$

**Proof** The convergence of $x_0(t)$ and the expression (3.8) follow from the proof of Theorem 3.1 and the expression (3.2). We next show the convergence of $x_i(t)$ and $z_i(t)$ as $t \to \infty$ and derive expressions (3.9) and (3.10). In view of Theorem 3.1, it suffices to show that every solution of (3.4) is convergent as $t \to \infty$ and that expressions (3.9) and (3.10) hold for (3.4).

We first consider the case where $\gamma_i \neq 0$. From (3.4), we can obtain

$$\frac{\mathrm{d}^2 Z_i(t)}{\mathrm{d}t^2} + (\alpha_i + \beta_i)\frac{\mathrm{d}Z_i(t)}{\mathrm{d}t} + (\alpha_i\beta_i - \gamma_i\delta_i)Z_i(t) = \gamma_i I_i(t). \tag{3.11}$$

Note that $\lambda_{i1}$ and $\lambda_{i2}$ are the eigenvalues of equation (3.11) and that $\lambda_{i1} < \lambda_{i2} < 0$ under (H3). The variation-of-constants formula leads to

$$\begin{aligned} Z_i(t) &= c_{i1}e^{\lambda_{i1}(t-t_0)} + c_{i2}e^{\lambda_{i2}(t-t_0)} + \int_{t_0}^t \frac{e^{\lambda_{i2}(t-s)} - e^{\lambda_{i1}(t-s)}}{\lambda_{i2}-\lambda_{i1}}\gamma_i I_i(s)\mathrm{d}s \\ &= c_{i1}e^{\lambda_{i1}(t-t_0)} + c_{i2}e^{\lambda_{i2}(t-t_0)} + \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}\left[\int_{t_0}^t e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s \right. \\ &\quad \left. - \int_{t_0}^t e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s\right], \end{aligned} \tag{3.12}$$

where $c_{i1}$ and $c_{i2}$ are constants satisfying the following system

$$\begin{cases} c_{i1} + c_{i2} = Z_i(t_0), \\ \lambda_{i1}c_{i1} + \lambda_{i2}c_{i2} = \left.\frac{\mathrm{d}Z_i(t)}{\mathrm{d}t}\right|_{t=t_0} = -\beta_i Z_i(t_0) + \gamma_i X_i(t_0). \end{cases}$$

That is

$$c_{i1}=\frac{-\gamma_iX_i(t_0)+(\lambda_{i2}+\beta_i)Z_i(t_0)}{\lambda_{i2}-\lambda_{i1}},\quad c_{i2}=\frac{\gamma_iX_i(t_0)-(\lambda_{i1}+\beta_i)Z_i(t_0)}{\lambda_{i2}-\lambda_{i1}}. \tag{3.13}$$

Therefore, it follows from (3.4) and (3.12) that

$$\begin{aligned}X_i(t)&=\frac{1}{\gamma_i}\left[\beta_iZ_i(t)+\frac{\mathrm{d}Z_i(t)}{\mathrm{d}t}\right]\\&=\frac{(\lambda_{i1}+\beta_i)c_{i1}}{\gamma_i}e^{\lambda_{i1}(t-t_0)}+\frac{(\lambda_{i2}+\beta_i)c_{i2}}{\gamma_i}e^{\lambda_{i2}(t-t_0)}\\&\quad+\frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\int_{t_0}^{t}e^{\lambda_{i2}(t-s)}I_i(s)\mathrm{d}s-\frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\int_{t_0}^{t}e^{\lambda_{i1}(t-s)}I_i(s)\mathrm{d}s.\end{aligned} \tag{3.14}$$

We next consider the case where $\gamma_i=0$. It follows from the second equation of (3.4) that

$$Z_i(t)=e^{-\beta_i(t-t_0)}Z_i(t_0), \tag{3.15}$$

which coincides with (3.12) as $\gamma_i=0$. Substituting this into the first equation of (3.4), we get

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t}=-\alpha_iX_i(t)+\delta_iZ_i(t_0)e^{-\beta_i(t-t_0)}+I_i(t).$$

Applying the variation-of-constants formula to this equation, and noting that (H3) implies $\alpha_i>0,\beta_i>0$ and $\alpha_i\neq\beta_i$ if $\gamma_i=0$, we obtain

$$\begin{aligned}X_i(t)&=e^{-\alpha_i(t-t_0)}X_i(t_0)+\int_{t_0}^{t}e^{-\alpha_i(t-s)}\left[\delta_iZ_i(t_0)e^{-\beta_i(s-t_0)}+I_i(s)\right]\mathrm{d}s\\&=e^{-\alpha_i(t-t_0)}X_i(t_0)+\frac{\delta_iZ_i(t_0)}{\alpha_i-\beta_i}\left[e^{-\beta_i(t-t_0)}-e^{-\alpha_i(t-t_0)}\right]\\&\quad+\int_{t_0}^{t}e^{-\alpha_i(t-s)}I_i(s)\mathrm{d}s.\end{aligned} \tag{3.16}$$

Note that $\lambda_{i1}<\lambda_{i2}<0$ and that $\lambda_{i1}=\min\{-\alpha_i,-\beta_i\}<0$ and $\lambda_{i2}=\max\{-\alpha_i,-\beta_i\}<0$ if $\gamma_i=0$. From (3.11)-(3.14), it follows that $X_i(t)$ and $Z_i(t)$ are convergent as $t\to\infty$. Moreover,

$$\begin{aligned}\lim_{t\to\infty}x_i(t)&=\lim_{t\to\infty}X_i(t)\\&=\begin{cases}\frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\lim\limits_{t\to\infty}\int_{t_0}^{t}e^{\lambda_{i2}(t-s)}I_i(s)\mathrm{d}s-\frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\lim\limits_{t\to\infty}\int_{t_0}^{t}e^{\lambda_{i1}(t-s)}I_i(s)\mathrm{d}s, & \text{if } \gamma_i\neq0;\\ \lim\limits_{t\to\infty}\int_{t_0}^{t}e^{-\alpha_i(t-s)}I_i(s)\mathrm{d}s, & \text{if } \gamma_i=0\end{cases}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \lim_{t\to\infty}\int_{t_0}^{t} e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s - \frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \lim_{t\to\infty}\int_{t_0}^{t} e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s \\
&= \frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \lim_{t\to\infty}\int_{0}^{t} e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s - \frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \lim_{t\to\infty}\int_{0}^{t} e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s \\
&= \frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}} L_i(\lambda_{i2}) - \frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}} L_i(\lambda_{i1}),
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{t\to\infty} z_i(t) &= \lim_{t\to\infty} Z_i(t) \\
&= \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}\left[\lim_{t\to\infty}\int_{t_0}^{t} e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s - \lim_{t\to\infty}\int_{t_0}^{t} e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s\right] \\
&= \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}\left[\lim_{t\to\infty}\int_{0}^{t} e^{\lambda_{i2}(t-s)} I_i(s)\mathrm{d}s - \lim_{t\to\infty}\int_{0}^{t} e^{\lambda_{i1}(t-s)} I_i(s)\mathrm{d}s\right] \\
&= \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}[L_i(\lambda_{i2}) - L_i(\lambda_{i1})].
\end{aligned}
$$

This completes the proof of Theorem 3.2.

**Corollary 3.1** *Suppose that all conditions of Theorem 3.1 are satisfied and $I_i(t)$ is the constant $I_i$ for $t$ being large and $i = 1, 2, \cdots, n$. Let $I = \sum_{i=1}^{n} I_i > 0$ and $\theta_i = \frac{I_i}{I}$. Then, for each solution $(x_0, x_1, z_1, \cdots, x_n, z_n)^T$ of (1.1), we have*

*(i) $\lim_{t\to\infty} x_0(t) = L_0(-\alpha_0)$. In particular, $\lim_{t\to\infty} x_0(t) = \frac{I_0}{\alpha_0}$ if $I_0(t)$ is the constant $I_0$ for $t$ being large;*

*(ii) $\lim_{t\to\infty} x_i(t) = \frac{\beta_i\theta_i I}{\alpha_i\beta_i-\gamma_i\delta_i}$ and $\lim_{t\to\infty} z_i(t) = \frac{\gamma_i\theta_i I}{\alpha_i\beta_i-\gamma_i\delta_i}$ for $i = 1, 2, \cdots, n$. Furthermore, if $\alpha_i = \alpha_j, \beta_i = \beta_j, \delta_i = \delta_j$ and $\gamma_i = \gamma_j$ for $i, j = 1, 2, \cdots, n$, then $\lim_{t\to\infty} \frac{x_i(t)}{x_j(t)} = \lim_{t\to\infty} \frac{z_i(t)}{z_j(t)} = \frac{\theta_i}{\theta_j}$.*

**Proof** Clearly, (H4) is satisfied under the conditions of the corollary. Moreover, after some simple calculations, we have

$$
\begin{aligned}
L_0(-\alpha_0) &= \frac{I_0}{\alpha_0} \ \text{ if } I_0(t) \equiv I_0 \ \text{ for } t \text{ being large and some constant } I_0, \\
L_i(\lambda_{i1}) &= -\frac{\theta_i I}{\lambda_{i1}} \ \text{ and } \ L_i(\lambda_{i2}) = -\frac{\theta_i I}{\lambda_{i2}} \ \text{ for } \ i = 1, 2, \cdots, n.
\end{aligned}
$$

Therefore, from Theorem 3.2, it follows that

$$
\lim_{t\to\infty} x_0(t) = L_0(-\alpha_0) \text{ and } = \frac{I_0}{\alpha_0} \text{ if } I_0(t) \equiv I_0 \text{ for } t \text{ being large and some constant } I_0
$$

$$\lim_{t\to\infty} x_i(t)=\frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\left(-\frac{\theta_i I}{\lambda_{i2}}\right)-\frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\left(-\frac{\theta_i I}{\lambda_{i1}}\right)=\frac{\beta_i\theta_i I}{\lambda_{i1}\lambda_{i2}}=\frac{\beta_i\theta_i I}{\alpha_i\beta_i-\gamma_i\delta_i},$$

$$\lim_{t\to\infty} z_i(t)=\frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}\left[-\frac{\theta_i I}{\lambda_{i2}}-\left(-\frac{\theta_i I}{\lambda_{i1}}\right)\right]=\frac{\gamma_i\theta_i I}{\lambda_{i1}\lambda_{i2}}=\frac{\gamma_i\theta_i I}{\alpha_i\beta_i-\gamma_i\delta_i}.$$

The second conclusion of (ii) follows from the first conclusion of (ii). This completes the proof.

**Remark 3.1** It is interesting to note that even we hold all parameters constant and identical from one input neuron to other neurons, allowing different thresholds, $\Gamma_i$ and $\Gamma_i^*$ lead to biased learning of spatial patterns, since the correspending $\gamma_i$ and $\delta_i$ may be different.

**Theorem 3.3** *Suppose that (H1)-(H3) hold, and that* $I_i : R \to [0,\infty)$ *is* $\omega$*-periodic function for all* $i=1,2,\cdots,n$. *Let* $y_i=(x_0,x_1,z_1,\cdots,x_n,z_n)^T : [t_0,\infty)\to R^{2n+1}$ *be a solution of (1.1). Then* $\lim\limits_{t\to\infty} x_0(t)=L_0(-\alpha_0)$, *and for each* $i\in\{1,2,\cdots,n\}$ *there exists a unique* $\omega$*-periodic solution of* $(X_i^\omega,Z_i^\omega)^T : [t_0,\infty)\to R^2$ *of (3.4), which is given by*

$$X_i^\omega(t)=\frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\int_{-\infty}^t e^{\lambda_{i2}(t-s)}I_i(s)\mathrm{d}s-\frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}}\int_{-\infty}^t e^{\lambda_{i1}(t-s)}I_i(s)\mathrm{d}s \quad (3.17)$$

*and*

$$Z_i^\omega(t)=\frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}}\left[\int_{-\infty}^t e^{\lambda_{i2}(t-s)}I_i(s)\mathrm{d}s-\int_{-\infty}^t e^{\lambda_{i1}(t-s)}I_i(s)\mathrm{d}s\right] \quad (3.18)$$

*such that*

$$\lim_{t\to\infty}[x_i(t)-X_i^\omega(t)]=0 \quad (3.19)$$

*and*

$$\lim_{t\to\infty}[z_i(t)-Z_i^\omega(t)]=0. \quad (3.20)$$

**Proof** Using the same argument as that in the proof of Theorem 3.1, we obtain $\lim\limits_{t\to\infty} x_0(t)=L_0(-\alpha_0)$.

In the proof of Theorem 3.2, we obtained the analytic representations (3.12)-(3.16) of solutions for (3.4). Assume that (3.4) has a periodic solution $(X_i^\omega,Y_i^\omega)^T$, then such a solution must be bounded on $R$. If $\gamma_i\neq 0$, then letting $t_0\to-\infty$ in (3.14) and (3.12) we can get (3.17) and (3.18), respectively. If $\gamma_i=0$, then letting $t_0\to-\infty$ in (3.16) and (3.15) we obtain

$$X_i^\omega(t)=\int_{-\infty}^t e^{-\alpha_i(t-s)}I_i(s)\mathrm{d}s \quad (3.21)$$

and

$$Z_i^\omega(t) = 0. \tag{3.22}$$

But (3.21) and (3.22) coincide with (3.17) and (3.18) respectively if $\gamma_i = 0$. After some simple calculations, it is easy to check that $(X_i^\omega, Z_i^\omega)^T$ given by (3.17) and (3.18) is an $\omega$-periodic solution of (3.4). On the other hand, for an arbitrary solution $(X_i, Z_i)^T$ of (3.4) with the initial value $(X_i(t_0), Z_i(t_0))^T$, from (3.12)-(3.16), (3.17), (3.18), (3.21) and (3.22), it follows that

$$\begin{aligned} & X_i(t) - X_i^\omega(t) \\ = & \begin{cases} \frac{\lambda_{i1}+\beta_i}{\gamma_i} c_{i1} e^{\lambda_{i1}(t-t_0)} + \frac{\lambda_{i2}+\beta_i}{\gamma_i} c_{i2} e^{\lambda_{i2}(t-t_0)} - \frac{\lambda_{i2}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \int_{-\infty}^{t_0} e^{\lambda_{i2}(t-s)} I_i(s) \mathrm{d}s & \\ + \frac{\lambda_{i1}+\beta_i}{\lambda_{i2}-\lambda_{i1}} \int_{-\infty}^{t_0} e^{\lambda_{i1}(t-s)} I_i(s) \mathrm{d}s, & \text{if } \gamma_i \neq 0; \\ e^{-\alpha_i(t-t_0)} X_i(t_0) + \frac{\delta_i Z_i(t_0)}{\alpha_i - \beta_i} \left[ e^{-\beta_i(t-t_0)} - e^{-\alpha_i(t-t_0)} \right] & \\ - \int_{-\infty}^{t_0} e^{-\alpha_i(t-s)} I_i(s) \mathrm{d}s, & \text{if } \gamma_i = 0. \end{cases} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} & Z_i(t) - Z_i^\omega(t) \\ = & \begin{cases} c_{i1} e^{\lambda_{i1}(t-t_0)} + c_{i2} e^{\lambda_{i2}(t-t_0)} - \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}} \int_{-\infty}^{t_0} e^{\lambda_{i2}(t-s)} I_i(s) \mathrm{d}s & \\ + \frac{\gamma_i}{\lambda_{i2}-\lambda_{i1}} \int_{-\infty}^{t_0} e^{\lambda_{i1}(t-s)} I_i(s) \mathrm{d}s, & \text{if } \gamma_i \neq 0; \\ e^{-\beta_i(t-t_0)} Z_i(t_0), & \text{if } \gamma_i = 0, \end{cases} \end{aligned} \tag{3.24}$$

where $c_{i1}$ and $c_{i2}$ are given by (3.13). Since $\lambda_{i1} < \lambda_{i2} < 0, \alpha_i > 0$ and $\beta_i > 0$ for $i \in \{1, 2, \cdots, n\}$ under the given conditions in the theorem, it is clear that (3.23) and (3.24) imply that

$$\lim_{t\to\infty} [X_i(t) - X_i^\omega(t)] = 0 \tag{3.25}$$

and

$$\lim_{t\to\infty} [Z_i(t) - Z_i^\omega(t)] = 0 \tag{3.26}$$

for $i = 1, 2, \cdots, n$. This means that $(X_i^\omega, Z_i^\omega)^T$ attracts every solution of (3.4), and so $(X_i^\omega, Z_i^\omega)^T$ is a unique $\omega$-periodic solution of (3.4).

Let $Y_i = (X_0, X_1, Z_1, \cdots, X_n, Z_n)^T : [t_0, \infty) \to R^{2n+1}$ be a solution of (3.1) with $Y_i(t_0) = y_i(t_0)$. Then $(X_i, Z_i)^T$ is the solution of (3.4) with the initial value $X_i(t_0) = x_i(t_0)$ and $Z_i(t_0) = z_i(t_0)$, and it follows from Theorem 3.1 that

$$\lim_{t\to\infty} [x_i(t) - X_i(t)] = 0 \quad \text{and} \quad \lim_{t\to\infty} [z_i(t) - Z_i(t)] = 0.$$

This, together with (3.25) and (3.26), implies that

$$\lim_{t\to\infty} [x_i(t) - X_i^\omega(t)] = \lim_{t\to\infty} [x_i(t) - X_i(t)] + \lim_{t\to\infty} [X_i(t) - X_i^\omega(t)] = 0$$

and

$$\lim_{t\to\infty}[z_i(t) - Z_i^{\omega}(t)] = \lim_{t\to\infty}[z_i(t) - Z_i(t)] + \lim_{t\to\infty}[Z_i(t) - Z_i^{\omega}(t)] = 0$$

for all $i = 1, 2, \cdots, n$.

This completes the proof of Theorem 3.3.

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