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A Fixed-Point Theorem and Applications to Problems on Sets with Convex Sections and to Nash Equilibria

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Abstract—A new fixed-point theorem for a family of maps defined on product spaces is obtained. The new result requires the functions involved to satisfy the local intersection properties. Previous results required the functions to have the open lower sections which are more restrictive conditions. New properties of multivalued maps are provided and applied to prove the new fixed-point theorem. Applications to problems on sets with convex sections and to the existence of Nash equilibria for a family of continuous functions are given.

Keywords—Fixed-point theorem, Sets with convex sections, Nash equilibrium.

1. INTRODUCTION

Fixed-point problems for a family of multivalued maps defined in product spaces have proved to be useful for the study of problems on sets with convex sections, the existence of Nash equilibria in game theory and minimax inequalities.

Lan and Webb [1] recently obtained fixed-point theorems for a family of multivalued maps defined on product spaces and applied them to problems on sets with convex sections and to some inequalities for a family of functions. A key restriction is that the maps involved are required to have open lower sections, that is, the inverse image of every point is open.

It is known that various results (for example, results in [1–3]) involving maps with open lower sections can be generalized to a larger class of maps possessing the local intersection properties. Examples include the fixed-point theorems of Browder type [4, Theorem 7.2, p. 33], continuous selection theorems [5, Theorem 1] and fixed-point theorems for a family of maps defined on product spaces (different from those in [1]) [6]. It should be also mentioned that the maps having the local intersection properties were also used in [7–11].

In this paper, we generalize Lan and Webb's results to the class of maps which have the local intersection properties. First, we establish some new properties of multivalued maps. In particular, we prove that if a map T has the local intersection property, then $(\bar{T}^*)^*$ has the open

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lower section (see Theorem 2.1); and if also $T(y) \neq \emptyset$, then $(\bar{T}^*)^* \neq \emptyset$ (see Theorem 2.3). These properties enable us to change problems on maps which have the local intersection properties into problems on maps which have open lower sections. Next, we generalize Lan and Webb's results to the class of maps which have local intersection properties. Our proofs are simpler and our conditions imposed on the maps are weaker. Finally, we apply our results to problems on sets with convex sections and to inequalities for a family of functions. We also present an application to the existence of Nash equilibria for a family of continuous functions defined on product spaces, extending many well-known results.

2. PROPERTIES OF MULTIVALUED MAPS

In this section, we establish some new properties of multivalued maps. These new properties will play important roles in the subsequent sections.

We assume that X is a nonempty set and Y a topological space. We denote by 2^Y the family of all subsets of Y, and by $\bar{B}$ and B^0 the closure and interior of a subset B of Y, respectively. Let $G : X \to 2^Y$ be a map. We define $G^c, \bar{G}, G^0 : X \to 2^Y$ by $G^c(x) = \{y \in Y : y \notin G(x)\}$, $\bar{G}(x) = \overline{G(x)}$, and $G^0(x) = (G(x))^0$, respectively. We also define maps $G^{-1}, G^* : Y \to 2^X$ by $G^{-1}(y) = \{x \in X : y \in G(x)\}$ and $G^*(y) = \{x \in X : y \notin G(x)\}$, respectively.

We start with the following known result (see [12, Lemma 3.2]).

LEMMA 2.1. *Let $S, T : X \to Y$ be two maps. Then the following properties hold.*

(h_1) *For each $x \in X$, $S(x) \subset T(x)$ if and only if $T^*(y) \subset S^*(y)$ for each $y \in Y$.*
(h_2) *$y \notin T(x)$ if and only if $x \in T^*(y)$.*
(h_3) *For each $x \in X$, $(T^*)^*(x) = T(x)$.*
(h_4) *For each $x \in X$, $T(x) \neq \emptyset$ if and only if $\bigcap_{y\in Y} T^*(y) = \emptyset$.*
(h_5) *For each $y \in Y$, $(T^c)^*(y) = T^{-1}(y)$.*
(h_6) *For each $y \in Y$, $(T^{-1})^c(y) = T^*(y)$.*

Now, we can state some new properties of multivalued maps.

PROPOSITION 2.1. *Let $F : Y \to 2^X$ be a map. Then the following assertions hold:*

(1) *$(\bar{F}^*)^c(x) = (F^{-1})^0(x)$ for each $x \in X$;*
(2) *$(\bar{F}^*)^*(y) = ((F^{-1})^0)^{-1}(y)$ for each $y \in Y$.*

PROOF. Let $x \in X$. Since $(F^{-1})^0(x) \subset F^{-1}(x)$, we obtain

$$F^*(x) = \left(F^{-1}\right)^c(x) \subset \left(\left(F^{-1}\right)^0\right)^c(x).$$

Noting that $((F^{-1})^0)^c(x)$ is closed, we have $\bar{F}^*(x) \subset ((F^{-1})^0)^c(x)$. This implies $(F^{-1})^0(x) = [((F^{-1})^0)^c]^c(x) \subset (\bar{F}^*)^c(x)$. On the other hand, since $F^*(x) \subset \bar{F}^*(x)$, we have $(\bar{F}^*)^c(x) \subset (F^*)^c(x) = F^{-1}(x)$. Noting that $(\bar{F}^*)^c(x)$ is open, we have $(\bar{F}^*)^c(x) \subset (F^{-1})^0(x)$. Hence, (1) holds. Moreover, Assertion (2) follows from Assertion (1). ∎

In Proposition 2.1, if we let $F = T^*$, we obtain the following.

PROPOSITION 2.2. *Let $T : X \to 2^Y$ be a map. Then the following assertions hold:*

(i) *$(\bar{T})^c(x) = (T^c)^0(x)$ for each $x \in X$;*
(ii) *$(\bar{T})^*(y) = ((T^c)^0)^{-1}(y)$ for each $y \in Y$.*

Recall that a map $F : Y \to 2^X$ is said to have an open lower section if $F^{-1}(x)$ is open in Y for each $x \in X$. Such maps were first employed in Browder's fixed-point theorem [2] and used in other fixed-point theorems, for example, in [1]. It is well known that some known results involving maps which have open lower sections can be generalized to maps which have the local intersection properties (see, for example, [4–6]).

DEFINITION 2.1. *A map $F : Y \to 2^X$ is said to have the local intersection property if there exists an open neighborhood $N(y)$ of y such that $\bigcap_{z \in N(y)} F(z) \neq \emptyset$ whenever $F(y) \neq \emptyset$.*

The following result provides necessary and sufficient conditions for a map to have the local intersection property. The proof is straightforward, and thus, is omitted.

LEMMA 2.2. *Let $F : Y \to 2^X$ be a map. Then the following assertions are equivalent.*

(1) *F has the local intersection property.*
(2) *If $y \in F^{-1}(x)$, then there exist an open neighborhood $N(y)$ of y and $x_1 \in X$ such that $N(y) \subset F^{-1}(x_1)$.*
(3) $\bigcup_{x \in X} F^{-1}(x) = \bigcup_{x \in X} (F^{-1})^0(x)$.

Condition (2) was used in Theorem 7.2 of [4, p. 33]. By Assertion (3) of Lemma 2.2, we see that if F has an open lower section, then F has the local intersection property. The following example shows that the converse is not true.

EXAMPLE 2.1. Let $E = \mathbb{R}$ and $X = [0, 2]$. We define a map $F : X \to 2^X$ by $F(x) = (0, 2]$ if $x = 0, 2$ and $F(x) = \{1\}$ if $x \in (0, 2)$. Then F has the local intersection property. However, $F^{-1}(y) = \{0, 2\}$ is not open in X for each $y \in (0, 1) \cup (1, 2]$.

The following new result shows that if F has the local intersection property, then $(\bar{F}^*)^*$ has an open lower section. The proof follows from Property (h_5) of Lemma 2.1 and we omit it. The result will play an important role in the following section.

THEOREM 2.1. *Assume that $F : Y \to 2^X$ has the local intersection property. Then the map $(\bar{F}^*)^* : Y \to 2^X$ has an open lower section.*

Now, we study the relations between T and T^*. We need the following concept introduced in [13].

DEFINITION 2.2. *A map $T : X \to 2^Y$ is said to be transfer-closed if $y \in T^c(x)$, there exists $x_1 \in X$ such that $y \notin \bar{T}(x_1)$.*

The following lemma provides necessary and sufficient conditions for a map to be transfer-closed. The proof follows from Assertion (i) of Proposition 2.2 and Lemma 2.1, and thus, is omitted.

LEMMA 2.3. *Let $T : X \to 2^Y$ be a map. Then the following are equivalent.*

(1) *T is transfer-closed.*
(2) $\bigcup_{x \in X} T^c(x) = \bigcup_{x \in X} (\bar{T})^c(x)$.
(3) $\bigcup_{x \in X} T^c(x) = \bigcup_{x \in X} (T^c)^0(x)$.
(4) $\bigcap_{x \in X} T(x) = \bigcap_{x \in X} \bar{T}(x)$.

Now, we state the relations between the transfer-closedness and the local intersection property.

THEOREM 2.2. *The following assertions hold.*

(1) *Let $T : X \to 2^Y$ be a map. Then T is transfer-closed if and only if $T^* : Y \to 2^X$ has the local intersection property.*
(2) *Let $F : Y \to 2^X$ be a map. Then F has the local intersection property if and only if F^* is transfer-closed.*

Let $F : Y \to 2^X$ be a map. In applications, one often requires not only $F(y) \neq \emptyset$ for each $y \in Y$ but also F has the local intersection property. The following result provides necessary and sufficient conditions for maps to satisfy the two conditions.

THEOREM 2.3. *Let $F : Y \to 2^X$ be a map. Then the following assertions are equivalent.*

(i) *$F(y) \neq \emptyset$ for each $y \in Y$ and F has the local intersection property.*
(ii) $\bigcap_{x \in X} \bar{F}^*(x) = \bigcap_{x \in X} F^*(x) = \emptyset$.
(iii) *$(\bar{F}^*)^*(y) \neq \emptyset$ for each $y \in Y$.*
(iv) $Y = \bigcup_{x \in X} (F^{-1})^0(x)$.

PROOF. By Condition (2) of Theorem 2.2 and Assertions (1) and (4) of Lemma 2.3, we see that F has the local intersection property if and only if F^* is transfer-closed if and only if $\bigcap_{x\in X} F^*(x) = \bigcap_{x\in X} \bar{F}^*(x)$. By Property ($h_4$) of Lemma 2.1, $F(y) \neq \emptyset$ for each $y \in Y$ if and only if $\bigcap_{x\in X} F^*(x) = \emptyset$. Hence, Assertions (i) and (ii) are equivalent. By Property (h_4) of Lemma 2.1, Assertions (ii) and (iii) are equivalent. By (1) of Proposition 2.1, we see that Assertions (ii) and (iv) are equivalent. ∎

We refer to Lemma 1 in [7], Lemma 1 in [8], and Lemma 1.1 in [9] for some related results on maps which have the local intersection properties. Item (iv) of Theorem 2.3 was also used in Lemma 1 in [10] and Theorems 3.4 and 3.5 in [11].

Let $F, G : Y \to 2^X$ be two maps such that $F(y) \subset G(y)$ for each $y \in Y$. If one of them has the local intersection property, in general, we do not know whether the other has the local intersection property. The following result gives the sufficient conditions which assure that if F has the local intersection property, then G also has the property. The proof follows from Lemma 2.1 and Theorem 2.3 and is omitted.

THEOREM 2.4. *Let $F, G : Y \to 2^X$ be two maps. Assume that the following conditions hold.*

(a) *$F(y) \subset G(y)$ for each $y \in Y$.*
(b) *$F(y) \neq \emptyset$ for each $y \in Y$.*
(c) *F has the local intersection property.*

Then G has the local intersection property.

We end the section with two results on transfer-closed maps.

THEOREM 2.5. *Let $T : X \to 2^Y$ be a map. Then the following assertions are equivalent.*

(i) *T is transfer-closed and $\bigcap_{x\in X} T(x) = \emptyset$.*
(ii) *$\bigcap_{x\in X} \bar{T}(x) = \bigcap_{x\in X} T(x) = \emptyset$.*
(iii) *$(\bar{T})^*(y) \neq \emptyset$ for each $y \in Y$.*
(iv) *$Y = \bigcup_{x\in X} (T^c)^0(x)$.*

THEOREM 2.6. *Let $T, S : X \to 2^Y$ be two maps. Assume that the following conditions hold.*

(a) *$T(x) \subset S(x)$ for each $y \in Y$.*
(b) *$\bigcap_{x\in X} S(x) = \emptyset$.*
(c) *S is transfer-closed.*

Then T is transfer-closed.

3. FIXED-POINT THEOREMS AND APPLICATIONS

Throughout this section, let I be an index set and for each $i \in I$, let E_i be a Hausdorff topological vector space. Let X_i be a nonempty convex subset in E_i, $X = \prod_{i\in I} X_i$ and $X^i = \prod_{j\neq i,\ j\in I} X_j$. We write $X = X_i \bigotimes X^i$. Then for each $i \in I$ and each $x \in X$, we write $x = (x_i, x^i)$, where $x_i \in X_i$ and $x^i \in X^i$.

We need the following result (see [1, Theorem 2.2]).

LEMMA 3.1. *For each $i \in I$, let $\phi_i : X^i \to 2^{X_i}$ be a map such that the following conditions hold.*

(H_1) *For each $i \in I$ and each $x^i \in X^i$, $\phi_i(x^i) \neq \emptyset$.*
(H_2) *For each $i \in I$ and each $y_i \in X_i$, $\phi_i^{-1}(y_i)$ is open in X^i.*
(H_3) *If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that, for each $x^i \in X^i \setminus D(i)$,*

$$X_i^0 \cap \operatorname{co} \phi_i\left(x^i\right) \neq \emptyset.$$

Then there exists $x \in X$ such that $x_i \in \operatorname{co} \phi_i(x^i)$ for all $i \in I$.

Now, we are in the position to state our new fixed-point theorem.

THEOREM 3.1. *For each $i \in I$, let $\gamma_i, \psi_i : X^i \to 2^{X_i}$ be two maps. Assume that the following conditions hold.*

(S$_1$) *For each $i \in I$ and each $x^i \in X^i$, $\gamma_i(x^i) \subset \psi_i(x^i)$.*

(S$_2$) *For each $i \in I$ and each $x^i \in X^i$, $\gamma_i(x^i) \neq \emptyset$.*

(S$_3$) *For each $i \in I$, γ_i has the local intersection property.*

(S$_4$) *If X^i is not compact, there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that, for each $x^i \in X^i \setminus D(i)$,*

$$X_i^0 \cap \operatorname{co}\left(\bar{\psi}_i^*\right)^*(x^i) \neq \emptyset.$$

Then there exists $x \in X$ such that $x_i \in \operatorname{co}\psi_i(x^i)$ for all $i \in I$.

PROOF. For each $i \in I$, we define a map $\phi_i : X^i \to 2^{X_i}$ by

$$\phi_i(x^i) = \left(\bar{\psi}_i^*\right)^*(x^i).$$

By Conditions (S$_1$)–(S$_3$) and Theorem 2.4, ψ_i has the local intersection property and $\psi_i(x^i) \neq \emptyset$ for each $x^i \in X^i$ and each $i \in I$. It follows from Conditions (i) and (iii) of Theorem 2.3 that $\phi_i(x^i) = (\bar{\psi}_i^*)^*(x^i) \neq \emptyset$ for each $x^i \in X^i$ and Condition (H$_1$) of Lemma 3.1 holds. Since ψ_i has the local intersection property for each $i \in I$, it follows from Theorem 2.1 that $\phi_i^{-1}(y_i)$ is open in X^i for each $y_i \in X_i$. Hence, Condition (H$_2$) of Lemma 3.1 holds. It is clear that Condition (S$_4$) implies Condition (H$_3$) of Lemma 3.1. It follows from Lemma 3.1 that there exists $x \in X$ such that $x_i \in \operatorname{co}\phi_i(x^i) = \operatorname{co}(\bar{\psi}_i^*)^*(x^i)$ for all $i \in I$. Since $(\bar{\psi}_i^*)^*(x^i) \subset \psi_i(x^i)$, we have $x_i \in \operatorname{co}\psi_i(x^i)$ for all $i \in I$. ∎

REMARK 3.1. In general, we have the following inclusion:

$$\left(\bar{\psi}_i^*\right)^*(x^i) \subset \psi_i(x^i), \qquad \text{for each } x^i \in X^i.$$

Moreover, if ψ_i has an open lower section, then

$$\left(\bar{\psi}_i^*\right)^*(x^i) = \psi_i(x^i), \qquad \text{for each } x^i \in X^i. \tag{3.1}$$

Let $\gamma_i = \psi_i = \phi_i$. Then Condition (H$_2$) of Lemma 3.1 implies Condition (S$_3$) and Condition (H$_3$) coincides with Condition (S$_4$) by (3.1). Hence, even when $\gamma_i = \psi_i$, Theorem 3.1 generalizes Lemma 3.1.

Using Theorem 3.1, we obtain the following new result on sets with convex sections.

THEOREM 3.2. *Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ be two families of subsets of X. Assume that the following conditions hold.*

(i) *For each $i \in I$, $A_i \subset B_i$.*

(ii) *For each $i \in I$ and each $x^i \in X^i$, $\{y_i \in X_i : (y_i, x^i) \in A_i\} \neq \emptyset$.*

(iii) *For each $i \in I$ and each $(y_i, x^i) \in A_i$, there exist an open neighborhood $N(x^i)$ of x^i and $y_i' \in X_i$ such that $\{y_i'\} \times N(x^i) \subset A_i$.*

(iv) *If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that for each $x^i \in X^i \setminus D(i)$,*

$$X_i^0 \cap \operatorname{co}\left\{y_i \in X_i : x^i \notin \overline{\{z^i \in X^i : (y_i, z^i) \notin B_i\}}\right\}.$$

Then there exists $x \in X$ such that $x_i \in \operatorname{co}\{y_i \in X_i : (y_i, x^i) \in B_i\}$.

PROOF. For each $i \in I$, we define γ_i, $\phi_i : X^i \to 2^{X_i}$ by

$$\gamma_i(x^i) = \left\{y_i \in X_i : (y_i, x^i) \in A_i\right\} \qquad \text{and} \qquad \phi_i(x^i) = \left\{y_i \in X_i : (y_i, x^i) \in B_i\right\}.$$

Then it is easy to verify that $\{\gamma_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$ satisfy all the conditions of Theorem 3.1. It follows from Theorem 3.1 that there exists $x \in X$ such that $x_i \in \operatorname{co}\{y_i \in X_i : (y_i, x^i) \in B_i\}$ for all $i \in I$. The result follows. ∎

REMARK 3.2. If for each $y_i \in X_i$, the set $\{y^i \in X^i : (y_i, x^i) \in A_i\}$ is open in X^i, then Assertion (iii) of Theorem 3.2 holds. Hence, even when $A_i = B_i$, Theorem 3.2 generalizes Theorem 2.3 in [1].

THEOREM 3.3. *Assume that all the conditions of Theorem 3.2 hold. Let $\{C_i\}_{i\in I}$ be a family of subsets of X such that, the following condition holds.*

($*$) *For each $x \in X$, there exists a nonempty subset $I(x) \subset I$ such that, for each $i \in I(x)$,*

$$\operatorname{co}\{y_i \in X_i : (y_i, x^i) \in B_i\} \subset \{y_i \in X_i : (y_i, x^i) \in C_i\}.$$

Then there exists $x \in X$ such that $\bigcap_{i\in I(x)} C_i \neq \emptyset$.

PROOF. By Theorem 3.2, there exists $x \in X$ such that

$$x_i \in \operatorname{co}\{y_i \in X_i : (y_i, x^i) \in B_i\}, \qquad \text{for all } i \in I.$$

For this x, by Condition ($*$), we have $x_i \in \{y_i \in X_i : (y_i, x^i) \in C_i\}$ for all $i \in I(x)$. This implies $x \in C_i$ for all $i \in I(x)$ and $\bigcap_{i\in I(x)} C_i \neq \emptyset$. ∎

REMARK 3.3. Theorem 3.3 generalizes Theorem 3.2 in [1] and those noted in [1], for example, Theorems 15 and 16 in [16] and Theorem 2 in [17].

Now, we give an analytical formulation of Theorem 3.2. We first recall some concepts. Let Z be a topological space. A function $g : Z \to (-\infty, \infty]$ is said to be lower semicontinuous on Z if the set $\{z \in Z : g(z) > \lambda\}$ is open in Z for each $\lambda \in \mathbb{R}$. If, in addition, Z is convex, $g : Z \to \mathbb{R}$ is said to be quasiconcave if for each $\lambda \in \mathbb{R}$, the set $\{z \in Z : g(z) > \lambda\}$ is convex.

DEFINITION 3.1. *Let X be a set, Y a topological space, and $\lambda \in \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$ is said to be λ-transfer-lower-semicontinuous on Y if $f(x,y) > \lambda$, there exist an open neighborhood $N(y)$ of y and $x_1 \in X$ such that*

$$f(x_1, z) > \lambda, \qquad \text{for each } z \in N(y).$$

f is said to be transfer-lower-semicontinuous on Y if f is λ-transfer-lower-semicontinuous on Y for each $\lambda \in \mathbb{R}$.

It is clear that if for each $x \in X$, $f(x, .)$ is lower-semicontinuous on Y, then f is transfer-lower-semicontinuous on Y.

THEOREM 3.4. *Let $\{f_i\}_{i\in I}, \{g_i\}_{i\in I}, \{h_i\}_{i\in I} : X \to \mathbb{R}$ be three families of functions and let $\{t_i\}_{i\in I}$ be a sequence of real numbers. Assume that the following conditions hold.*

(a) *For each $i \in I$ and $x \in X$, $f_i(x) \le g_i(x) \le h_i(x)$.*
(b) *For each $i \in I$ and each $x^i \in X^i$, there exists $y_i \in X_i$ such that $f_i(y_i, x^i) > t_i$.*
(c) *For each $i \in I$, f_i is t_i-transfer-lower-semicontinuous on X^i.*
(d) *If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that for each $x^i \in X^i \setminus D(i)$,*

$$X_i^0 \cap \operatorname{co}\left\{y_i \in X_i : x^i \notin \overline{\{z^i \in X^i : g_i(y_i, z^i) \le t_i\}}\right\}.$$

(e) *For each $i \in I$ and each $x^i \in X^i$, $h_i(., x^i)$ is quasiconcave on X_i.*

Then there exists $x \in X$ such that $h_i(x) > t_i$ for all $i \in I$.

PROOF. For each $i \in I$, we define $\gamma_i, \phi_i : X^i \to 2^{X_i}$ by

$$\gamma_i(x^i) = \{y_i \in X_i : f_i(y_i, x^i) > t_i\} \qquad \text{and} \qquad \phi_i(x^i) = \{y_i \in X_i : g_i(y_i, x^i) > t_i\}.$$

Then it is easy to verify that $\{\gamma_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$ satisfy all the conditions of Theorem 3.1. It follows from Theorem 3.1 that there exists $x \in X$ such that $x_i \in \operatorname{co}\phi_i(x^i)$ for all $i \in I$. Since $g_i(x) \le h_i(x)$ for $x \in X$ and h_i is quasiconcave on X_i, $h_i(x) > t_i$ for all $i \in I$. ∎

REMARK 3.4. If for each $x_i \in X_i$, $f_i(x_i, .)$ is lower-semicontinuous on X^i, then Condition (c) of Theorem 3.4 holds. Hence, even when $f_i = g_i = h_i$, Theorem 3.4 generalizes Theorem 2.5 in [1] and Theorem 3 in [14]. Some related results involved only one lower semicontinuous function can be found for example, in [18].

Now, we can present an application of Lemma 3.1 to the existence of Nash equilibria for a family of continuous functions.

THEOREM 3.5. *Let X_i be a nonempty compact convex subset of E_i for each $i \in I$. Let $\{f_i\}_{i\in I}$: $X \to \mathbb{R}$ be a family of functions. Assume that the following conditions hold.*

(i) *For each $i \in I$, f_i is continuous on X.*

(ii) *For each $i \in I$ and $x^i \in X^i$, $f_i(., x^i)$ is quasiconcave on X_i.*

Then there exists $x \in X$ such that

$$f_i(x) = \max_{y_i \in X_i} f_i\left(y_i, x^i\right), \qquad \textit{for all } i \in I.$$

PROOF. Let $\varepsilon > 0$. For each $i \in I$, we define $\phi_i : X^i \to 2^{X_i}$ by

$$\phi_i\left(x^i\right) = \left\{ y_i \in X_i : f_i\left(y_i, x^i\right) > \sup_{z_i \in X_i} f_i\left(z_i, x^i\right) - \varepsilon \right\}.$$

Then $\{\phi_i\}_{i\in I}$ satisfies all the conditions of Lemma 3.1. It follows from Lemma 3.1 that there exists $x_\varepsilon = (x_i^\varepsilon, x_\varepsilon^i) \in X$ such that

$$f_i(x_\varepsilon) > \sup_{z_i \in X_i} f_i\left(z_i, x_\varepsilon^i\right) - \varepsilon, \qquad \text{for each } i \in I. \tag{3.2}$$

Let $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Since X is compact and $\{x_{\varepsilon_n}\} \subset X$, we assume that $x_{\varepsilon_n} \to x \in X$. Note that f_i and $g_i(x^i) = \sup_{z_i \in X_i} f_i(z_i, x^i)$ are continuous, it follows from (3.2) that $f_i(x_i, x^i) \geq \sup_{z_i \in X_i} f_i(z_i, x^i)$ for each $i \in I$. The result follows. ∎

REMARK 3.5. Theorem 3.5 generalizes Theorem 4 in [14], where I is finite.

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