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Convergence of approximants for demicontinuous pseudo-contractive maps in Hilbert spaces ☆

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1. Introduction

Some fixed point principles for maps can be obtained by studying the convergence of suitable approximants. This approach not only establishes the existence of fixed points but also provides approximations and numeral schemes.

Let K be a closed convex set in a Hilbert space H and $T: K \to H$ a map. There are two natural approximants defined by $x_t = (1-t)Tx_t + tx_0$ and $x_t = (1-t)rTx_t + tx_0$, where $r: H \to K$ is the metric projection. It is known that the above two approximants always exist if T is a nonexpansive self-map or a weakly inward nonexpansive map. We remark that when T is a self-map, the two approximants coincide. Another approximant $x_t = r[(1-t)Tx_t + tx_0]$ is also used in the literature, see [10,15]. As will be shown (see Remark 2.5 below), when T is weakly inward, the approximant is exactly the same as $x_t = (1-t)Tx_t + tx_0$.

A classical result obtained by Browder [1] is that if T is a nonexpansive self-map defined on a bounded closed convex set of a Hilbert space, then the approximant $x_t = (1-t)Tx_t + tx_0$ converges to a fixed point of T. Recently, this result was generalized to nonexpansive nonself-maps defined on a bounded or unbounded closed convex sets. Singh and Watson [14] showed that the result holds if $T(K)$ is bounded and $T(\partial K) \subset K$.

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Xu and Yin [15] proved that the two approximants converge to a fixed point of T if T is a weakly inward nonexpansive map such that $\{x_t\}$ is bounded (see Theorems 2 and 3 in [15]). In Remark 2.4 below, we remark that this boundedness holds if and only if T satisfies the well-known Leray–Schauder condition on K.

In this paper, we study the convergence of approximants for demicontinuous weakly inward pseudo-contractive maps in Hilbert spaces. For a demicontinuous weakly inward pseudo-contractive map defined on a closed convex set, we first prove that the existence of the approximant $x_t = (1-t)Tx_t + tx_0$ by using a recent result obtained by Lan and Webb [6] and then show that the approximant converges to a fixed point of T. This provides not only generalizations of the aforementioned results of [1,14,15] but also some new convergence results when T is defined only on $\bar{D}_K$ and satisfies the Leray–Schauder condition on ∂D_K. We emphasize that these convergence results are new even for nonexpansive maps.

As applications of our general results, we derive new theorems on convergence of the approximant $x_t = (1-t)rTx_t + tx_0$ for generalized inward nonexpansive maps, extending corresponding known results for weakly inward nonexpansive maps obtained in [13,15].

We apply our results to the integral equation of the form

$$x(t) = \int_G k(t,s)f(s,x(s))\,\mathrm{d}s \quad \text{a.e. on } G,$$

where $G \subset \mathbb{R}^n$ is measurable. Such equations with $G = (0,\infty)$ were studied by using the theory of monotone operators and a sequence was obtained which weakly converges to a solution (see, for example, Example 11.2 in [3]). In contrast, we are able to provide sequences which are strongly convergent.

2. Convergence of approximants

Let H be a Hilbert space. Recall that a map $T : D \subset H \to H$ is said to be a k-dissipative map with $k \in \mathbb{R}$ if $(Tx - Ty, x - y) \leqslant k\|x - y\|^2$ for $x, y \in D$. When $k = 1$, T is called a pseudo-contractive map. It is obvious that a nonexpansive map (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$ for $x, y \in D$) is pseudo-contractive. T is said to be demicontinuous if $\{x_n\} \subset D$ and $x_n \to x \in D$ together imply $Ax_n \rightharpoonup Ax$, where $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively.

We start with the following new property of pseudo-contractive maps.

Theorem 2.1. *Let $T : K \to H$ be a demicontinuous pseudo-contractive map. Then the following results hold*:

(P$_1$) *If $\{x_n\} \subset K$ is bounded and $\{t_n\} \subset (0,1)$ with $t_n \to 0$ satisfies $x_n = (1-t_n)Tx_n + t_nx_0$, then $\{x_n\}$ converges to a fixed point of T.*

(P$_2$) *If $\{x_t\} \subset K$ is bounded and satisfies $x_t = (1-t)Tx_t + tx_0$ for $t \in (0,\delta)$. Then $\{x_t\}$ converges to a fixed point of T as $t \to 0$.*

Proof. (P$_1$) Since $\{x_n\}$ is bounded and $t_n \to 0$, $\{Tx_n\}$ is bounded. Hence, there is a subsequence $\{x_j\}$ of $\{x_n\}$ such that $x_j \rightharpoonup v \in K$ and $x_j - Tx_j \to 0$. This implies

$\lim(x_j - Tx_j, x_j - v) = 0$ and $\lim(x_j - Tx_j, x_j - x) = 0$ for every $x \in K$. Since $I - T$ is demicontinuous and monotone, $I - T$ must be pseudo-monotone (see the remark following Definition 2.1 in [9]). It follows from the definition of a pseudo-monotone map that

$$(v - Tv, v - x) \leqslant \liminf(x_j - Tx_j, x_j - x) = 0 \quad \text{for all } x \in K.$$

This implies $(v - Tv, v - x_j) \leqslant 0$ for each $j \in \mathbb{N}$. Since $x_j = (1 - t_j)Tx_j + t_j x_0$ and T is pseudo-contractive, we have

$$\begin{aligned}\|x_j - v\|^2 &= (1 - t_j)(Tx_j - Tv, x_j - v) + (1 - t_j)(Tv - v, x_j - v)\\ &\quad + t_j(x_0 - v, x_j - v)\\ &\leqslant (1 - t_j)\|x_j - v\|^2 + t_j(x_0 - v, x_j - v).\end{aligned}$$

This implies $\|x_j - v\|^2 \leqslant (x_0 - v, x_j - v)$. This, together with $x_j \rightharpoonup v$, implies $x_j \to v$. Since T is demicontinuous, we have $v = Tv$.

We now prove that $x_n \to v$. The proof is by contradiction. Assume that $x_n \to v$ does not hold. Then there exist $\varepsilon > 0$ and a subsequence $\{x_k\}$ of $\{x_n\}$ such that $\|x_k - v\| \geqslant \varepsilon$. Using the above argument, we obtain $x_k \to u = Tu$ and $u \neq v$. On the other hand, since $x_j = (1 - t_j)Tx_j + t_j x_0$ and $u = Tu$, we have

$$t_j(x_j - x_0, x_j - u) + (1 - t_j)\|x_j - u\|^2 = (1 - t_j)(Tx_j - Tu, x_j - u).$$

As T is pseudo-contractive, it follows that $(x_j - x_0, x_j - u) \leqslant 0$. Since $x_j \to v$, we have $(v - x_0, v - u) \leqslant 0$ for $j \in \mathbb{N}$. Similarly, using $x_k = (1 - t_k)Tx_k + t_k x_0$ and $v = Tv$, we obtain $(u - x_0, u - v) \leqslant 0$. It follows that $\|v - u\|^2 = (v - x_0, v - u) + (x_0 - u, v - u) \leqslant 0$, and hence $v = u$, a contradiction to $\|v - u\| \geqslant \varepsilon$. This proves ($P_1$). Similar arguments can be used to obtain (P_2). □

Remark 2.1. (P_2) of Theorem 2.1 generalizes the 'necessary' part of Theorem 1 in [15], where T is a nonexpansive map. Our method is completely different from that used in [15], which depends heavily on nonexpansiveness. We shall see that the 'sufficiency' part of Theorem 1 in [15] holds when the map involved is a pseudo-contractive map.

Recall that a map $T : D \subset K \to H$ is said to be weakly inward (relative to K) if $Tx \in \bar{I}_K(x)$ for $x \in D$, where $\bar{I}_K(x)$ is the closure of the inward set $I_K(x) := \{x + c(z - x) : z \in K \text{ and } c \geqslant 1\}$ (see [5,7,8] for more details). It is known that a map $T : D \subset K \to H$ is weakly inward if and only if $\lim_{t \to 0^+} d(x + t(Tx - x), K)/t = 0$ for every $x \in D$ (see Lemma 18.1 in [3]).

Let D be a bounded open set in H. We denote by $\bar{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K. Some relations among D_K, $\bar{D}_K$ and ∂D_K can be found in Lemma 2.2 in [6].

The following result is a special case of Theorem 5.1 in [6].

Lemma 2.1. *Let D be a bounded open convex set in H such that $D_K \neq \emptyset$ and $\bar{D}_K \neq K$. Let $T : \bar{D}_K \to H$ be a demicontinuous weakly inward k-dissipative map with $k < 1$. Assume that the following condition holds.*

(LS) There exists $x_0 \in D_K$ *such that* $x \neq (1-t)Tx + tx_0$ *for* $x \in \partial D_K$ *and* $t \in (0,1)$. *Then T has a unique fixed point in* $\bar{D}_K$.

By using Lemma 2.1 and Theorem 4.1 in [9], we obtain the following result.

Theorem 2.2. *Let K be a closed convex set in H. Assume that* $T: K \to H$ *is a demicontinuous weakly inward k-dissipative map with* $k < 1$. *Then T has a unique fixed point in K.*

Proof. When K is bounded, the result follows from Theorem 4.1 in [9]. We assume that K is unbounded. Let $x_0 \in K$ and $r = (1-k)^{-1}\|Tx_0 - x_0\| + \|x_0\|$. We claim that

$$x \neq (1-t)Tx + tx_0 \quad \text{for } x \in K \text{ with } \|x\| > r \text{ and } t \in [0,1], \tag{2.1}$$

In fact, if not, then there exist $t \in [0,1]$ and $x \in K$ with $\|x\| > r$ such that $x = (1-t)Tx + tx_0$. Since

$$\begin{aligned}\|x - x_0\|^2 &= (1-t)(Tx - Tx_0, x - x_0) + (1-t)(Tx_0 - x_0, x - x_0) \\ &\leqslant (1-t)k\|x - x_0\|^2 + (1-t)\|Tx_0 - x_0\|\|x - x_0\|,\end{aligned}$$

we have $(1-k)\|x - x_0\| \leqslant \|Tx_0 - x_0\|$. This implies $\|x\| \leqslant r$, which contradicts $\|x\| > r$. It follows from (2.1) and Lemma 2.1 that T has a unique fixed point in K. □

Remark 2.2. When the space involved is a Hilbert space, Theorem 2.2 generalizes Theorem 7.3, p. 257, in [11], where T is continuous. Note that our method is completely different from that used in [11], which used the theory of semigroup of nonlinear operators.

As a useful corollary of Theorem 2.2, we immediately obtain

Corollary 2.1. *Let K be a closed convex set in H. Assume that* $T: K \to H$ *is a demicontinuous weakly inward pseudo-contractive map and* $x_0 \in K$. *Then for every* $t \in (0,1)$ *there exists a unique* $x_t \in K$ *such that*

$$x_t = (1-t)Tx_t + tx_0. \tag{2.2}$$

Now, we are in a position to give our main results on the convergence of the approximant defined in (2.2) to fixed points of T.

We first consider the case when T is defined on a bounded closed set.

Theorem 2.3. *Let K be a bounded closed convex set in H. Assume that* $T: K \to H$ *is a demicontinuous weakly inward pseudo-contractive map. Then T has a fixed point in K. Moreover, for every* $x_0 \in K$, $\{x_t\}$ *defined in* (2.2) *converges to a fixed point of T.*

Proof. By Corollary 2.1, $\{x_t\}$ defined in (2.2) is well defined. It follows from (P_2) of Theorem 2.1 that $\{x_t\}$ converges to a fixed point of T. □

Remark 2.3. The existence of fixed points of T in Theorem 2.3 was first obtained in (Theorem 4.1, [9]) for a more general map. However, Theorem 2.3 not only shows the existence of fixed points of T but also provides a sequence $\{x_t\}$ which converges strongly to a fixed point of T. Theorem 2.3 generalizes Theorem 1 in [1] where T is a nonexpansive self-map, and Corollary 1 in [15], where T is a weakly inward nonexpansive map.

Now, we consider the case when K is unbounded. We need the following proposition which provides a necessary condition for a pseudo-contractive map defined on a unbounded closed convex set to have at least one fixed point.

Proposition 2.1. *Let K be a unbounded set in H. Assume that $T: K \to H$ is pseudo-contractive and has at least one fixed point in K. Then for every $u \in K, T$ satisfies the following condition $(LS)_u$ on K.*

$(LS)_u$ There exists $r > 0$ such that $x \neq (1-t)Tx + tu$ for $x \in K$ with $\|x\| \geqslant r$ and $t \in (0,1]$.

Proof. Let $v \in K$ with $v = Tv$ and $x \in K$ with $\|x\| \geqslant r$, where $r \geqslant \|u-v\| + \|v\|$. Let $z_t = (1-t)Tx + tu$. Then we have for $t \in (0,1]$,

$$\begin{aligned}(x-z_t, x-u) &= \|x-v\|^2 - (1-t)(Tx-Tv, x-u) - t(u-v, x-u)\\ &\geqslant \|x-v\|^2 - (1-t)\|x-v\|^2 - t\|u-v\|\|x-v\|\\ &= t[\|x-v\|^2 - \|x-v\|\|u-v\|] > 0.\end{aligned}$$

This implies $x \neq (1-t)Tx + tu$. □

Theorem 2.4. *Let K be a unbounded closed convex set in H. Let $T: K \to H$ be a demicontinuous weakly inward pseudo-contractive map. Assume that T satisfies $(LS)_u$ on K for some $u \in K$. Then T has a fixed point in K. Moreover, for every $x_0 \in K$, $\{x_t\}$ defined in (2.2) converges to a fixed point of T as $t \to 0$.*

Proof. Let $t \in (0,1)$. It follows from Corollary 2.1 that there exists $y_t \in K$ such that $y_t = (1-t)Ty_t + tu$. Note that $(LS)_u$ implies the boundedness of $\{y_t\}$. It follows from (P_2) of Theorem 2.1 that $\{y_t\}$ converges to a fixed point of T as $t \to 0$. This, together with Proposition 2.1, implies that T satisfies $(LS)_{x_0}$ on K for every $x_0 \in K$. It follows that $\{x_t\}$ defined in (2.2) converges to a fixed point of T as $t \to 0$. □

Remark 2.4. Let K be a unbounded closed convex set in H and $T: K \to H$ be a demicontinuous weakly inward pseudo-contractive map. Then the following conditions are equivalent.

(i) T satisfies condition $(LS)_{x_0}$ on K.

(ii) $\{x_t\}$ defined in (2.2) is bounded as $t \to 0$.

In fact, by the proof of Theorem 2.4, we see that (i) implies (ii). We now show that (ii) implies (i). In fact, if (i) does not hold, then there exist $\{x_n\} \subset K$ with $\|x_n\| \to \infty$ and $\{t_n\} \subset (0,1]$ with $t_n \to t_0 \in [0,1]$ such that $x_n = (1-t_n)Tx_n + t_n x_0$. By Corollary 2.1, we have $x_n = x_{t_n}$. Since $\{x_t\}$ is bounded as $t \to 0$, we have $t_0 \neq 0$. On the other hand, since

$$\begin{aligned}\|x_n - x_0\|^2 &= (1-t_n)(Tx_n - Tx_0, x_n - x_0) + (1-t_n)(Tx_0 - x_0, x_n - x_0)\\ &\leqslant (1-t_n)\|x_n - x_0\|^2 + \|Tx_0 - x_0\|\|x_n - x_0\|,\end{aligned}$$

we have $t_n\|x_n - x_0\| \leqslant \|Tx_0 - x_0\|$. This implies $t_n \to 0$, a contradiction.

Remark 2.5. Let K be a closed convex set in H and let $T : K \to H$ be a weakly inward map. Let $t \in (0,1)$ and $x_0, x_t \in K$. Define $T_t : K \to H$ defined by $T_t(x) = (1-t)Tx + tx_0$. Since T is weakly inward, T_t is a generalized inward map. It follows from Lemma 3.1 below that $x_t = (1-t)Tx_t + tx_0$ if and only if $x_t = r((1-t)Tx_t + tx_0)$, where $r : H \to K$ is the metric projection.

Remark 2.6. By Proposition 2.1 and Remarks 2.4 and 2.5, we see that Theorem 2.4, together with Theorem 2.3, generalizes Theorem 3 in [15], where T is a weakly inward nonexpansive map and the approximant is $x_t = r((1-t)Tx_t + tx_0)$. Note that if $T(K)$ is bounded, then T satisfies $(LS)_u$ on K; and if $T(\partial K) \subset K$, then T is weakly inward. Hence, Theorem 2.4 generalizes Theorem in [14].

By Proposition 2.1 and Theorem 2.4 we see that when T is a demicontinuous weakly inward pseudo-contractive map, T satisfies $(LS)_u$ on K for some $u \in K$ if and only if T has a fixed point in K. This enables us to obtain the following convergence result which generalizes Corollary 1 in [13] when the space involved is a Hilbert space.

Theorem 2.5. *Let K be a unbounded closed convex set in H. Let $T : K \to H$ be a demicontinuous weakly inward pseudo-contractive map. Assume that T has a fixed point in K. Then for every $x_0 \in K$, $\{x_t\}$ defined in (2.2) converges to a fixed point of T as $t \to 0$.*

Finally, we discuss the case when T is defined on $\bar{D}_K$. The following result is new even for nonexpansive maps.

Theorem 2.6. *Let D be a bounded open convex set in H such that $D_K \neq \emptyset$ and $\bar{D}_K \neq K$. Assume that $T : \bar{D}_K \to H$ is a demicontinuous weakly inward pseudo-contractive map and satisfies the following condition*:
(LS) there exists $x_0 \in D_K$ such that $x \neq (1-\lambda)Tx + \lambda x_0$ for $x \in \partial D_K$ and $\lambda \in [0,1)$. Then the following results hold.
(i) *T has a fixed point in D_K.*
(ii) *For every $t \in (0,1)$ there exists $x_t \in D_K$ such that $x_t = (1-t)Tx_t + tx_0$.*
(iii) *$\{x_t\}$ converges to a fixed point of T as $t \to 0$.*

Proof. (i) follows from (iii) which follows from (ii) and (P_2) of Theorem 2.1. We now prove (ii). For each $t \in (0,1)$, we define a map $T_t : \bar{D}_K \to H$ by $T_t(x) = (1-t)Tx + tx_0$. Then $T_t : \bar{D}_K \to H$ is a demicontinuous weakly inward $(1-t)$-dissipative map for

$t \in (0,1)$. Moreover, it is easy to verify that T_t satisfies (LS) on ∂D_K. It follows from Lemma 2.1 that T_t has a fixed point $x_t \in D_K$ for every $t \in (0,1)$. □

Remark 2.7. (i) of Theorem 2.6 generalizes Theorem 13 in [2], where K is a closed ball with center 0 and T is k-dissipative with $k < 1$. The approximant in (iii) of Theorem 2.6 is different from that used in Theorem 13 of [2]. Moreover, (iii) of Theorem 2.6 does not require that T be demicompact. (i) of Theorem 2.6 also generalizes (i) of Theorem 15 in [2], where K is a closed ball with center 0 and $T : H \to H$ is a Lipschitzian pseudo-contractive map. Moreover, (ii) and (iii) of Theorem 2.6 are different from (2) and (3) of Theorem 15 in [2]. Even when $K = H$, (i) of Theorem 2.6 improves Theorem 4.4 in [12] (and Theorem 2 in [4]) in the following ways: (i) T need not be defined on H, (ii) T need not be continuous and (iii) T need not be a k-set contractive or Lipschitzian map. We should point out, however, that the results in [4,12] hold in suitable Banach spaces.

3. Convergence of approximants for generalized inward nonexpansive maps

In this section we apply the results obtained in the above section to generalized inward nonexpansive maps.

Let K be a closed convex set in H. Recall that a map $T : D \subset K \to H$ is called a generalized inward map on D (relative to K) if $d(Tx, K) < \|x - Tx\|$ for $x \in D$ with $Tx \notin K$. It is known that a weakly inward map is generalized inward but the converse is false (see [8] for more details). Let $r : H \to K$ be the metric projection, that is, $\|x - rx\| = d(x, K)$. It is well-known that $r : H \to K$ is nonexpansive (see Proposition 9.2 in [3]).

The following result will be useful, which shows that the fixed point sets of T and rT are same if T is generalized inward (see Lemma 2.12 in [8] for a more general case).

Lemma 3.1. *Assume that $T : D \subset K \to H$ is a generalized inward map on D relative to K. Let $x \in D$. Then x is a fixed point of T if and only if x is a fixed point of rT.*

Using Corollary 2.1, we immediately obtain the following.

Lemma 3.2. *Let $T : K \to H$ be a nonexpansive map and let $r : H \to K$ be the metric projection. Let $x_0 \in K$. Then for every $t \in (0,1)$ there exists $x_t \in K$ such that*

$$x_t = (1-t) rTx_t + t x_0. \tag{3.1}$$

Theorem 3.1. *Let K be a bounded closed convex set in H. Assume that $T : K \to H$ is a generalized inward nonexpansive map. Then T has a fixed point in K. Moreover, for every $x_0 \in K$, $\{x_t\}$ defined in* (3.1) *converges to a fixed point of T.*

Proof. Since $rT : K \to K$ is nonexpansive map, it follows from Theorem 2.3 that $x_t \to v \in K$ as $t \to 0$ and $v = rTv$. Since T is generalized inward, it follows from Lemma 3.1 that v is a fixed point of T. □

By a similar argument and using Theorem 2.4, we obtain

Theorem 3.2. *Let K be a unbounded closed convex set in H. Let $T:K\to H$ be a generalized inward nonexpansive map. Assume that rT satisfies $(LS)_u$ on K for some $u\in K$. Then T has a fixed point in K. Moreover, for every $x_0\in K$, $\{x_t\}$ defined in* (3.1) *converges to a fixed point of T.*

Remark 3.1. Theorem 3.2, together with Theorem 3.1, generalizes Theorem 2 in [15], where T is a weakly inward nonexpansive map.

We also have the following result which generalizes Corollary 1 in [13] when the space involved is a Hilbert space.

Theorem 3.3. *Let K be a unbounded closed convex set in H. Let $T:K\to H$ be a generalized inward nonexpansive map. Assume that T has a fixed point in K. Then for every $x_0\in K$, $\{x_t\}$ defined in* (3.1) *converges to a fixed point of T.*

Finally, using Theorem 2.6 we obtain the following new result.

Theorem 3.4. *Let D be a bounded open convex set in H such that $D_K\neq\emptyset$ and $\bar{D}_K\neq K$. Let $T:\bar{D}_K\to H$ be a generalized inward nonexpansive map. Assume that rT satisfies (LS) of Theorem 2.6 on ∂D_K. Then the following assertions hold.*
(i) *T has a fixed point in $\bar{D}_K$.*
(ii) *For each $t\in(0,1)$ there exists $x_t\in D_K$ such that $x_t=(1-t)rTx_t+tx_0$.*
(iii) *$\{x_t\}$ converges to a fixed point of T.*

Remark 3.2. In Theorems 3.1, 3.2 and 3.4, if T is not assumed to be generalized inward, then, by Proposition 2.1 in [9], $\{x_t\}$ defined in (3.1) converges to a nearest point v of T, that is, $\|Tv-v\|=d(Tv,K)$, or, equivalently, to a solution of the variational inequality $(v-Tv,v-x)\leqslant 0$ for all $x\in K$.

4. Applications

In this section, we consider the integral equation of the form

$$x(t)=\int_G k(t,s)f(s,x(s))\,\mathrm{d}s+g(t)\quad\text{a.e. on } G, \tag{4.1}$$

where G is a measurable set in $\mathbb{R}^n$ and $g\in L^2(G)$.

A well-known result on the existence of solutions for such equations with $G=(0,\infty)$ can be found in Example 11.2 of [3], where a sequence which weakly converges to a solution is provided. In this section we shall provide a sequence of approximating solutions which strongly converges to a solution.

We always assume that the following conditions hold.

(C_1) $k:G\times G\to\mathbb{R}$ is such that the linear operator K defined by $(Kx)(t)=\int_G k(t,s)x(s)\,\mathrm{d}s$ maps $L^2(G)$ into $L^2(G)$ and satisfies $(Kx,x)\geqslant 0$ for $x\in L^2(G)$.

(C_2) $f: G \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions on $G \times \mathbb{R}$ and there exists $h \in L^2(G)$ and $c > 0$ such that

$$|f(t,u)| \leqslant h(t) + c|u| \quad \text{for } u \geqslant 0 \text{ and a.e. } t \in G.$$

Theorem 4.1. *Assume that* $0 < \text{meas}(G) < \infty$ *and the following conditions hold.*

(i) *k is symmetric on $G \times G$, that is, $k(t,s) = k(s,t)$ on $G \times G$.*
(ii) $(f(t,u) - f(t,v))(u - v) \leqslant \|K\|^{-1}(u - v)^2$ *for* $u, v \in \mathbb{R}$.
(iii) *There exist $a \in L^2(G)$, $q > 2$ and $b > 0$ such that*

$$f(t,u)u \leqslant a(t)|u| + b|u|^{(q+2)/q} \quad \textit{for } u \in \mathbb{R} \textit{ and a.e. } t \in G.$$

Then the following assertions hold.

(1) *Eq.* (4.1) *has a solution in* $L^2(G)$.
(2) *For every $\lambda \in (0,1)$, there exists $x_\lambda \in L^2(G)$ such that*

$$x_\lambda(t) = (1 - \lambda) \int_G k(t,s) f(s, x_\lambda(s))\, \mathrm{d}s + g(t) \quad \textit{a.e. on } D,$$

(3) $\{x_\lambda\}$ *converges to a solution of Eq.* (4.1) *as* $\lambda \to 0^+$.

Proof. Let $H = L^2(G)$. We write Eq. (4.1) as $x = KF(x+g)$. We define a map $T: H \to H$ by $Tx = K^{1/2}F(K^{1/2}x + g)$, where $K^{1/2}$ is the square root of K. It is easy to verify that the equation $x = KF(x + g)$ is equivalent to the fixed point equation $y = Ty$. Hence, it suffices to show that T satisfies all the conditions of Theorem 2.4. It is obvious that $T: H \to H$ is continuous. Moreover, we have for $x, y \in H$,

$$\begin{aligned}(Tx - Ty, x - y) &= (F(K^{1/2}x + g) - F(K^{1/2}y + g), K^{1/2}x - K^{1/2}y)\\ &\leqslant \|K\|^{-1}\|K^{1/2}(x - y)\|^2 \leqslant \|x - y\|^2.\end{aligned}$$

This shows that $T: H \to H$ is pseudo-contractive. To show that T satisfies $(LS)_0$ on H, it suffices to prove the set $\{x \in H: x = \lambda Tx, 0 \leqslant \lambda \leqslant 1\}$ is bounded. Assume that $x \in H$ is such that $x = \lambda Tx$ for some $\lambda \in (0,1]$. Let $\alpha = 2q/(2 + q)$ and $\beta > 1$ satisfy $1/\beta + 1/\alpha = 1$. Using (iii), we obtain

$$\|x\| \leqslant \|a\| \|K^{1/2}\| \|x\| + b \text{ meas}(G)(\beta)^{-1}\|K\|^{1/\alpha}\|x\|^{2/\alpha}.$$

This implies $\|x\| \leqslant m$ for a suitable constant m, independent of λ. The result follows. □

The following result allows $\text{meas}(G) = \infty$ and k need not be symmetric, but conditions on f are different from those used in Theorem 4.1.

Theorem 4.2. *Assume that the following conditions hold.*

(h_1) $(f(t,u) - f(t,v))(u - v) \leqslant 0$ *for* $u, v \in \mathbb{R}$.
(h_2) *There exist $\delta \in L^2(G)$ and $\gamma > 0$ such that*

$$f(t,u)u \leqslant \delta(t)|u| - \gamma|u|^2 \quad \textit{for } u \in \mathbb{R} \textit{ and } t \in G.$$

Then the following assertions hold.

(1) *Eq.* (4.1) *has a solution in* $L^2(G)$.
(2) *For every* $\lambda > 0$, *there exists* $x_\lambda \in L^2(G)$ *such that*

$$\lambda x_\lambda(t) + K^* x_\lambda = KF(K^* x_\lambda + g),$$

where K^* *is the adjoint of* K.
(3) $\{x_\lambda\}$ *converges to a solution of Eq.* (4.1) *as* $\lambda \to 0^+$.
(4) $\{K^* x_\lambda\}$ *converges to a solution of Eq.* (4.1) *as* $\lambda \to 0^+$.

Proof. Let $H = L^2(G)$. We write Eq. (4.1) as $x = KF(x+g)$. We define a map $T : H \to H$ by $Ty = y + KF(K^*y + g) - K^*y$. Let $x = K^*y$. Then $y = Ty$ implies $x = KF(x+g)$. It is easy to verify that $T : H \to H$ is a continuous pseudo-contractive map. It follows from Corollary 2.1 that for every $\lambda > 0$, there exists $x_\lambda \in H$ such that $x_\lambda = [1 - \lambda/(1+\lambda)]Tx_\lambda$. This implies that (2) holds. Now, by a similar argument to that used in Example 11.2 in [3], one can show that T has a fixed point in H, that is, (1) holds. This, together with Theorem 2.5, implies that (3) holds. It is obvious that (2) and (3) imply (4). □

Remark 4.1. Theorem 4.2 generalizes Example 11.2 in [3], where $G = (0, \infty)$. Theorem 4.2 not only shows the existence of solutions of Eq. (4.1) but also provides sequences which strongly converge to solutions of Eq. (4.1). Example 11.2 in [3] only provides a sequence which weakly converges to a solution of Eq. (4.1) (see the proof of the Example 11.2).

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