Journal of Mathematical Analysis and Applications **259,** 188–208 (2001)
doi:10.1006/jmaa.2000.7410, available online at http://www.idealibrary.com on IDEAL®

Slowly Oscillating Periodic Solutions for a Delayed Frustrated Network of Two Neurons[1]

Yuming Chen[2] and Jianhong Wu[3]

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario M3J 1P3, Canada

Submitted by Hal L. Smith

Received March 20, 2000

In this article, we study a delayed frustrated network of two neurons. We obtain a two-dimensional closed disk bordered by a slowly oscillating periodic orbit, and we give a complete description about the dynamics of the flow restricted to this closed disk.

Key Words: cyclic system; delayed frustrated network; discrete Lyapunov functional; eigenvalue; slowly oscillating periodic orbit.

1. INTRODUCTION

The system of delay differential equations

$$
\begin{aligned}
\dot{x}(t) &= -\mu_1 x(t) + F_{11}(x(t-\tau_{11})) + F_{12}(y(t-\tau_{12})) + I_1, \\
\dot{y}(t) &= -\mu_2 y(t) + F_{21}(x(t-\tau_{21})) + F_{22}(y(t-\tau_{22})) + I_2
\end{aligned} \tag{1.1}
$$

arises as a model for a network of two saturating amplifiers (or neurons) with delayed outputs. Such a system without delays was first proposed by Hopfield [9, 10] and later modified by Marcus and Westervelt [14] by incorporating the time delays in order to account for the finite switching

[1] Research supported partially by Natural Sciences and Engineering Research Council of Canada and by the Network of Centers of Excellence: Mathematics for Information Technology and Complex Systems.

[2] Current address: Department of Mathematical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada. Research is supported by the Izaak Walton Killam Memorial Postdoctoral Fellowship. E-mail: ychen@math.ualberta.ca.

[3] To whom correspondence should be addressed. E-mail: wujh@mathstat.yorku.ca.

0022-247X/01 $35.00

speed of the amplifiers. Some progress has been made towards the description of dynamics of the generated semiflows and the existence of periodic solutions has been the central subject in a few papers including an der Heiden [1], Baptistini and Táboas [3], Chen and Wu [5], Gopalsamy and Leung [8], Ruan and Wei [16], and Táboas [18].

Some of the aforementioned work on the existence of periodic solutions used the Nussbaum's method by applying the ejective fixed point principle of Browder to an invariant cone in the state space (an der Heiden [1], Baptistini and Táboas [3], Táboas [18]). Some used the well-known local Hopf bifurcation theory and the obtained results are local (Gopalsamy and Leung [8]), while others used the S^1-equivariant degree theory of Erbe et al. [7] and the obtained results are global (Ruan and Wei [16]). In [5], using the discrete Lyapunov functional theory developed by Mallet-Paret and Sell [12, 13] and following the geometrical analysis used by Krisztin et al. [11], we studied the positive feedback system

$$\begin{aligned} \dot{x}(t) &= -\mu\tau x(t) + \tau f(y(t-1)), \\ \dot{y}(t) &= -\mu\tau y(t) + \tau f(x(t-1)), \end{aligned} \tag{1.2}$$

which describes the dynamics of two identical saturating amplifiers, with either excitatory or inhibitory interaction. In the case of excitatory interaction, we obtained a phase-locked periodic solution for system (1.2) when τ is larger than a certain critical value τ_d.

The purpose of the present work is to study the dynamics for a network of two neurons with negative feedback. More precisely, we consider

$$\begin{aligned} \dot{x}(t) &= -\mu_1 x(t) + F(y(t-\tau_1)) + I_1, \\ \dot{y}(t) &= -\mu_2 y(t) - G(x(t-\tau_2)) + I_2, \end{aligned} \tag{1.3}$$

where μ_1 and μ_2 are positive constants, τ_1 and τ_2 are nonnegative constants with $\tau := \tau_1 + \tau_2 > 0$, I_1 and I_2 are constants, and F and G are bounded C^1-functions with

$$F'(\xi) > 0 \quad \text{and} \quad G'(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}. \tag{1.4}$$

Such a network is referred to as frustrated network by Bélair et al. [2]. To apply the theory of cyclic systems developed in Mallet-Paret and Sell [12, 13], we will, in Section 2, transform system (1.3) into

$$\begin{aligned} \dot{u}(t) &= -\tau\mu_1 u(t) + \tau f(v(t)), \\ \dot{v}(t) &= -\tau\mu_2 v(t) - \tau g(u(t-1)), \end{aligned} \tag{1.5_τ}$$

with the origin being the only equilibrium point and f and g satisfying $f'(\xi) > 0$ and $g'(\xi) > 0$ for all $\xi \in \mathbb{R}$.

Our first step towards the description of the global attractor of the semiflow generated by (1.5_τ) is to show the existence of a slowly oscillating periodic solution for τ larger than a certain $\tau_0 > 0$. We obtain the existence of such a periodic solution following the geometric approach of Krisztin et al. [11] and using some ideas of Walther [19] for the scalar case. Namely, we will consider the global forward extension of a leading two-dimensional unstable manifold tangent to the eigenspace of the generator of the linearization of (1.5_τ) at the zero solution associated with a pair of eigenvalues with the largest positive real parts. To distinguish the oscillation frequencies, measured by the discrete Lyapunov functional, between the solutions in the aforementioned eigenspace and solutions in its complement, we need some detailed information about the eigenvalues of the linearized equation of (1.5_τ) at zero. We will show, in Section 3, that there exists a $\tau_0 > 0$ such that when $\tau > \tau_0$, all eigenvalues are nonreal, are simple, and are given by conjugate complex pairs $\{a_k \pm ib_k\}_{k \in \mathbb{N}_+}$ with $b_0 \in (0, \pi)$ and $b_k \in ((2k-1)\pi, (2k+1)\pi)$ for $k \in \mathbb{N}$. Moreover, $a_0 > 0$ and $a_1 > a_2 > \cdots > a_k > \cdots \to -\infty$ as $k \to \infty$.

The main result is obtained in Section 4, where we show that for $\tau > \tau_0$ there is a unique slowly oscillating periodic orbit in the closure of the global forward extension W of a two-dimensional C^1-submanifold contained in the unstable set of the origin. Besides this, we show that $\overline{W}$ is homeomorphic to the unit disk in $\mathbb{R}^2$ and we describe completely the dynamics of the flow restricted to $\overline{W}$.

2. MODEL DERIVATION

Consider the system of delay differential equations

$$\begin{aligned} \dot{x}(t) &= -\mu_1 x(t) + F(y(t-\tau_1)) + I_1, \\ \dot{y}(t) &= -\mu_2 y(t) - G(x(t-\tau_2)) + I_2, \end{aligned} \tag{2.1}$$

where μ_1 and μ_2 are positive constants, τ_1 and τ_2 are nonnegative constants with $\tau := \tau_1 + \tau_2 > 0$, I_1 and I_2 are constants, and F and G are bounded C^1-functions with

$$F'(\xi) > 0 \quad \text{and} \quad G'(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}. \tag{2.2}$$

THEOREM 2.1. *For any I_1 and I_2, system* (2.1) *has a unique equilibrium.*

Proof. We first prove the existence of equilibrium. Let $C_{12} = \{(x, y) \in \mathbb{R}^2;\ |x| \le \mu_1^{-1}(M + |I_1|),\ |y| \le \mu_2^{-1}(M + |I_2|)\}$, where $M > 0$ is a bound for both F and G. Then C_{12} is a compact subset of $\mathbb{R}^2$. Define F_{12}: $C_{12} \to \mathbb{R}^2$ by

$$F_{12}(x, y) = \left(\mu_1^{-1}(F(y) + I_1),\ \mu_2^{-1}(-G(x) + I_2)\right). \tag{2.3}$$

It is easy to see that F_{12} is continuous and $F_{12}(C_{12}) \subseteq C_{12}$. Therefore, by the Brouwer fixed point theorem, there exists a fixed point (x_0, y_0) of F_{12} in C_{12} and hence system (2.1) has an equilibrium (x_0, y_0).

Now, we show that if $(\tilde{x}_0, \tilde{y}_0)$ is an equilibrium point of system (2.1), then $\tilde{x}_0 = x_0$ and $\tilde{y}_0 = y_0$. In fact, from

$$-\mu_1 x_0 + F(y_0) + I_1 = -\mu_1 \tilde{x}_0 + F(\tilde{y}_0) + I_1$$

and

$$-\mu_2 y_0 - G(x_0) + I_2 = -\mu_2 \tilde{y}_0 - G(\tilde{x}_0) + I_2,$$

we get

$$\mu_1(x_0 - \tilde{x}_0) = F(y_0) - F(\tilde{y}_0) \tag{2.4}$$

and

$$\mu_2(y_0 - \tilde{y}_0) = -(G(x_0) - G(\tilde{x}_0)). \tag{2.5}$$

It follows from (2.2), (2.4), and (2.5) that

$$(x_0 - \tilde{x}_0)(y_0 - \tilde{y}_0) = 0. \tag{2.6}$$

Thus, the combination of (2.2), (2.4), and (2.6) gives $\tilde{x}_0 = x_0$ and $\tilde{y}_0 = y_0$. This completes the proof.

Let (x_0, y_0) be the unique equilibrium obtained in Theorem 2.1 and introduce the following change of variables:

$$\begin{aligned} u(t) &= x(\tau t + \tau_1) - x_0, \\ v(t) &= y(\tau t) - y_0. \end{aligned} \tag{2.7}$$

Then (u, v) satisfies

$$\begin{aligned} \dot{u}(t) &= -\tau\mu_1 u(t) + \tau f(v(t)), \\ \dot{v}(t) &= -\tau\mu_2 v(t) - \tau g(u(t-1)), \end{aligned} \tag{2.8}$$

where

$$f(\xi) = F(\xi + y_0) - F(y_0)$$

and

$$g(\xi) = G(\xi + x_0) - G(x_0)$$

for all $\xi \in \mathbb{R}$. It is easy to see that the origin is the only equilibrium point of (2.8_τ). Moreover, f and g are bounded C^1-functions and satisfy

$$\begin{gathered} f(0) = 0 = g(0), \\ f'(\xi) > 0 \quad \text{and} \quad g'(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}. \end{gathered} \tag{H1}$$

With (H1) holding, system (2.8_τ) is a cyclic system of negative feedback in the sense of Mallet-Paret and Sell [12, 13]. Hence the discrete Lyapunov functional theory developed by Mallet-Paret and Sell can be applied to system (2.8_τ).

Let $\mathbb{K} = [-1,0] \cup \{1\}$,

$$C(\mathbb{K}) = \{\varphi\colon \mathbb{K} \to \mathbb{R};\ \varphi \text{ is continuous}\}.$$

Then $C(\mathbb{K})$ is a Banach space with the supernorm, which we choose as the phase space for (2.8_τ). Throughout this article, we will always tacitly use the identification

$$C(\mathbb{K}) = C([-1,0];\ \mathbb{R}) \times \mathbb{R}$$

and write an element $\psi \in C(\mathbb{K})$ as $(\psi|_{[-1,0]}, \psi(1)) \in C([-1,0];\mathbb{R}) \times \mathbb{R}$. We also use the identification

$$C^1(\mathbb{K}) = C^1([-1,0];\mathbb{R}) \times \mathbb{R}$$

and the C^1-norm on $C^1(\mathbb{K})$ is defined as

$$\|\psi\|_1 = \max\left\{ \sup_{\theta\in[-1,0]} |\psi(\theta)|,\ \sup_{\theta\in[-1,0]} |\dot{\psi}(\theta)|, |\psi(1)|\right\}.$$

Following Smith [17] and Mallet-Paret and Sell [12], for each $\varphi \in C(\mathbb{K})$ there exists a unique pair of continuous maps $u\colon [-1,\infty) \to \mathbb{R}$ and $v\colon [0,\infty) \to \mathbb{R}$ such that $(u,v)\colon (0,\infty) \to \mathbb{R}^2$ is continuously differentiable and satisfies (2.8_τ) for $t > 0$, $u|_{[-1,0]} = \varphi|_{[-1,0]}$, and $v(0) = \varphi(1)$. Let $z^\varphi = (u^\varphi, v^\varphi)$ denote the above unique pair and define $z_t^\varphi = (u_t^\varphi, v^\varphi(t)) \in C(\mathbb{K})$ for $t \geq 0$, where $u_t^\varphi(\theta) = u^\varphi(t+\theta)$ for $\theta \in [-1,0]$. Then the map $\Phi\colon \mathbb{R}^+ \times C(\mathbb{K}) \ni (t,\varphi) \mapsto z_t^\varphi \in C(\mathbb{K})$ is a continuous semiflow, with only one stationary point 0.

Due to the monotonicity condition (H1), one can easily show that, for any given $t \geq 0$, the map $\Phi(t,\cdot)\colon C(\mathbb{K}) \to C(\mathbb{K})$ is injective. In particular, for each $\varphi \in C(\mathbb{K})$ there exists at most one $z = (u,v)\colon \mathbb{R} \to \mathbb{R}^2$ which satisfies (2.8_τ) and such that $z_0 = (u_0, v(0)) = \varphi$. If such a solution exists, it will be also denoted by $z^\varphi\colon \mathbb{R} \to \mathbb{R}^2$.

3. THE EIGENVALUES OF THE LINEARIZED EQUATION

Before studying the periodic solutions of (2.8_τ), we need some information about the eigenvalues of the infinitesimal generator of the C_0-semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$.

The location of these eigenvalues determines the stability of the origin. The linearization of (2.8_τ) at the origin is given by

$$\begin{aligned} \dot{X}(t) &= -\tau\mu_1 X(t) + \tau f'(0)Y(t), \\ \dot{Y}(t) &= -\tau\mu_2 Y(t) - \tau g'(0)X(t-1). \end{aligned} \tag{3.1}$$

The corresponding characteristic equation is

$$\lambda^2 + \tau(\mu_1 + \mu_2)\lambda + \tau^2\mu_1\mu_2 + \tau^2 f'(0)g'(0)e^{-\lambda} = 0. \tag{3.2_τ}$$

The case of $\mu_1 = \mu_2$ was studied by Baptistini and Táboas [3] and Táboas [18]. For the general case, though the analysis is more complicated, we have parallel results to those in the aforementioned two papers. General treatments of equations like (3.2_τ) are given by Bellman and Cooke [4] and Pontryagin [15]. Here, we want to show that solutions of (3.2_τ) are distributed in a union of a sequence of strips S_k (defined later); the real parts of these solutions are monotonically decreasing in $k \in \mathbb{N}$ and unbounded. Results of this nature are expected in Mallet-Paret and Sell [12, Theorem 3.2 and Corollary 3.3], and our discussions in this section seem to confirm their expectation.

For the sake of convenience, let $p = \mu_1 + \mu_2$, $q = \mu_1\mu_2$ and $r = f'(0)g'(0)$. Then (3.2_τ) can be rewritten as

$$(\lambda^2 + \tau p\lambda + \tau^2 q)e^{\lambda} + \tau^2 r = 0. \tag{3.3_τ}$$

Following Theorem 13.9 of Bellman and Cooke [4], we need information about the solution of the following equation:

$$\alpha^2 - \tau^2 q = \tau p\alpha \cot\alpha. \tag{3.4_τ}$$

It is easy to see that for each $\tau \in (0,\infty)$ Eq. (3.4_τ) has a unique solution in $(0,\pi)$, denoted by $\alpha(\tau)$. Then we have a differentiable function α: $(0,\infty) \to (0,\pi)$.

LEMMA 3.1. *The function α is strictly increasing on* $(0,\infty)$.

Proof. Differentiating (3.4_τ) with respect to τ and rearranging the terms, we have

$$\frac{d\alpha}{d\tau} = \frac{2\tau q + p\alpha\cot\alpha}{2\alpha - \tau p\cot\alpha + \tau p\alpha\frac{1}{\sin^2\alpha}}.$$

Using $1/\sin^2\alpha = 1 + \cot^2\alpha$ and (3.4_τ), we get

$$\frac{d\alpha}{d\tau} = \frac{p\alpha(\tau^2 q + \alpha^2)}{\tau p(\tau^2 q + \alpha^2) + \tau^2 p^2\alpha^2 + (\alpha^2 - \tau^2 q)^2} > 0, \tag{3.5}$$

which implies that α is strictly increasing on $(0,\infty)$. This completes the proof.

LEMMA 3.2. *The function* h: $(0,\infty) \to (0,\infty)$ *given by* $h(\tau) = \frac{\tau r \sin\alpha(\tau)}{p\alpha(\tau)}$ *is strictly increasing on* $(0,\infty)$.

Proof. Using (3.4_τ) and (3.5), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}h(\tau) &= \frac{p\alpha\left(r\sin\alpha + \tau r\cos\alpha\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right) - \tau pr\sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}}{p^2\alpha^2} \\
&= \frac{rp\alpha\sin\alpha + r(\alpha^2-\tau^2q)\sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} - \tau pr\sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}}{p^2\alpha^2} \\
&= \frac{r\sin\alpha}{p^2\alpha^2}\left[p\alpha + (\alpha^2-\tau^2q-\tau p)\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right] \\
&= \frac{r\sin\alpha}{p\alpha}\left[1 + \frac{(\tau^2q+\alpha^2)(\alpha^2-\tau^2q-\tau p)}{\tau p(\tau^2q+\alpha^2)+\tau^2p^2\alpha^2+(\alpha^2-\tau^2q)^2}\right] \\
&= \frac{r\alpha\left[\tau^2(p^2-2q)+2\alpha^2\right]\sin\alpha}{p\left[\tau p(\tau^2q+\alpha^2)+\tau^2p^2\alpha^2+(\alpha^2-\tau^2q)^2\right]} \\
&= \frac{r\alpha\left[\tau^2(\mu_1^2+\mu_2^2)+2\alpha^2\right]\sin\alpha}{p\left[\tau p(\tau^2q+\alpha^2)+\tau^2p^2\alpha^2+(\alpha^2-\tau^2q)^2\right]} \\
&> 0.
\end{aligned}$$

This completes the proof.

Lemma 3.1 implies that both $\lim_{\tau\to 0^+}\alpha(\tau)$ and $\lim_{\tau\to\infty}\alpha(\tau)$ exist. Moreover, it follows easily from (3.4_τ) that $\lim_{\tau\to 0^+}\alpha(\tau) = 0$ and $\lim_{\tau\to\infty}\alpha(\tau) = \pi$. Note that

$$\tau = \frac{\alpha}{2q\sin\alpha}\left[-p\cos\alpha + \left(p^2\cos^2\alpha + 4q\sin^2\alpha\right)^{1/2}\right]$$

by solving (3.4_τ) with respect to τ. Thus

$$h(\tau) = \frac{r}{2q}\left[-\cos\alpha + \left(\cos^2\alpha + \frac{4q\sin^2\alpha}{p^2}\right)^{1/2}\right].$$

It follows easily that

$$\lim_{\tau\to 0^+} h(\tau) = 0 \tag{3.6}$$

and

$$\lim_{\tau \to \infty} h(\tau) = \frac{r}{q}. \tag{3.7}$$

PROPOSITION 3.3. *Assume* (H1) *and* $\mu_1 \mu_2 \geq f'(0)g'(0)$. *Then the origin is locally asymptotically stable.*

Proof. Since $\mu_1 \mu_2 \geq f'(0)g'(0)$, i.e., $q \geq r$, it follows from (3.7) that

$$\lim_{\tau \to \infty} h(\tau) \leq 1.$$

This, combined with Lemma 3.2, implies that

$$h(\tau) < 1 \qquad \text{for } \tau > 0.$$

It is known that if

$$h(\tau) < 1$$

then all roots of (3.3_τ) and hence of (3.2_τ) lie to the left of the imaginary axis (see, for example, Bellman and Cooke [4]), which means that the origin is (locally) asymptotically stable. This completes the proof.

Now, assume that

$$\mu_1 \mu_2 < f'(0)g'(0). \tag{H2}$$

Again, it follows from (3.7) that

$$\lim_{\tau \to \infty} h(\tau) > 1,$$

which, combined with (3.6) and Lemma 3.2, implies that there exists a unique $\tau_0 \in (0, \infty)$ such that

$$h(\tau) \begin{cases} < 1 & \text{for } \tau \in (0, \tau_0), \\ = 1 & \text{for } \tau = \tau_0, \\ > 1 & \text{for } \tau \in (\tau_0, \infty). \end{cases} \tag{3.8}$$

As discussed in the proof of Proposition 3.3, we know that if $\tau < \tau_0$, then the origin is locally asymptotically stable. Therefore, in order to obtain periodic solutions of (2.8_τ) as a Hopf bifurcation from the origin, it is necessary to assume that

$$\tau > \tau_0. \tag{H3}$$

Note that hypothesis (H3) is equivalent to that (3.2_τ) has at least one root with positive real part.

From now on, we always assume that (H1)–(H3) hold.

Observe that λ is a root of (3.2_τ) if and only if its complex conjugate $\bar{\lambda}$ is. Therefore, we can restrict our study to the upper semi-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \operatorname{Im} \lambda \geq 0\}$.

LEMMA 3.4. *Equation* (3.2_τ) *has no real root.*

Proof. Clearly, (3.2_τ) has no nonnegative real root. We then only need to show that (3.2_τ) has no negative real root. Since $\alpha(\tau) \in (0, \pi)$, we have $0 < \sin \alpha(\tau) \leq \alpha(\tau)$. It follows from (3.8) that $\tau r > p$. We also know that $e^\lambda \geq 1 + \lambda$ for all $\lambda \in \mathbb{R}$. Thus, when $\lambda < 0$, we have

$$\begin{aligned}
&\lambda^2 + \tau p \lambda + \tau^2 q + \tau^2 r e^{-\lambda} \\
&\quad \geq \lambda^2 + \tau p \lambda + \tau^2 q + \tau^2 r (1 - \lambda) \\
&\quad = \lambda^2 + \tau(p - \tau r)\lambda + \tau^2 (q + r) \\
&\quad > 0.
\end{aligned}$$

This implies that (3.2_τ) has no negative real root, completing the proof.

COROLLARY 3.5. *All roots of* (3.2_τ) *are simple.*

Proof. By way of contradiction, we assume that λ_0 is a root of (3.2_τ) with multiplicity larger than 1. Then λ_0 satisfies both

$$\lambda_0^2 + \tau p \lambda_0 + \tau^2 q + \tau^2 r e^{-\lambda_0} = 0 \tag{3.9}$$

and

$$2\lambda_0 + \tau p - \tau^2 r e^{-\lambda_0} = 0. \tag{3.10}$$

Hence, it follows from (3.9) and (3.10) that λ_0 satisfies

$$\lambda_0^2 + (\tau p + 2)\lambda_0 + \tau^2 q + \tau p = 0.$$

Noting $\Delta = (\tau p + 2)^2 - 4\tau^2 q - 4\tau p = \tau^2 p^2 + 4 - 4\tau^2 q = \tau^2(\mu_1 - \mu_2)^2 + 4 > 0$, we know that λ_0 is real, a contradiction to Lemma 3.4. This completes the proof.

LEMMA 3.6. *There is no root of* (3.2_τ) *in line* $a + bi$, *where* $b = (2k + 1)\pi$ *for some* $k \in \mathbb{N}_+ = \mathbb{N} \cup \{0\}$.

Proof. If $\lambda = a + (2k + 1)\pi i$ for some $k \in \mathbb{N}_+$ satisfies (3.2_τ), then, separating the real and imaginary parts, we have

$$\begin{gathered}
a^2 - (2k + 1)^2 \pi^2 + \tau p a + \tau^2 q - \tau^2 r e^{-a} = 0, \\
2a(2k + 1)\pi + \tau p(2k + 1)\pi = 0
\end{gathered}$$

and the system is not compatible. In fact, from the second equation, we have $a = -\frac{\tau p}{2}$. Substituting it into the left hand side of the first equation, we get

$$-(2k+1)^2\pi^2 - \frac{\tau^2 p^2}{4} + \tau^2 q - \tau^2 r e^{\tau p/2}$$
$$= -(2k+1)^2\pi^2 - \frac{\tau^2(\mu_1 - \mu_2)^2}{4} - \tau^2 r e^{\tau p/2} < 0,$$

a contradiction. This completes the proof.

For $k \in \mathbb{N}_+$, define

$$S_k = \begin{cases} \{\lambda \in \mathbb{C};\, 0 < \operatorname{Im} \lambda < \pi\}, & k = 0, \\ \{\lambda \in \mathbb{C};\, (2k-1)\pi < \operatorname{Im} \lambda < (2k+1)\pi\}, & k \in \mathbb{N}. \end{cases}$$

Then Lemma 3.4 combined with Lemma 3.6 asserts that there exists a subset $A \subset \mathbb{N}_+$ such that all roots of (3.2_τ) in $\mathbb{C}_+$ are in the union of strips S_k for $k \in A$.

To continue our discussion, we need a result about the continuous dependence of the roots of (3.2_τ) on τ. Since this result will be used repeatedly, we state it for the sake of easy reference.

THEOREM 3.7 [6, Lemma XI.2.8]. *Let Ω be an open set in $\mathbb{C}$. Let ω be an open subset of Ω whose closure $\overline{\omega}$ in $\mathbb{C}$ is compact and contained in Ω. Let $\bar{\tau}$ be such that no root of $(3.2_{\bar{\tau}})$ is on the boundary of ω. Then there exists a neighborhood U of $\bar{\tau}$ in $\mathbb{R}$ such that*

(i) *for any $\tau \in U$, Eq. (3.2_τ) has no zeros on the boundary of ω;*

(ii) *the number of zeros of (3.2_τ) in ω, taking multiplicities into account, is constant for $\tau \in U$.*

For $k \in \mathbb{N}_+$, define

$$b_k = \alpha(\tau_0) + 2k\pi$$

and

$$\tau_k = \frac{\tau_0 b_k}{b_0}.$$

Then it is easy to check that $ib_k \in S_k$ is a root of (3.2_{τ_k}).

LEMMA 3.8. *For any $k \in \mathbb{N}_+$, Eq. (3.2_τ) has at least one root in S_k.*

Proof. Let

$$A_k = \{\tau > \tau_0;\, (3.2_\tau) \text{ has at least one root in } S_k\}.$$

If $k \in \mathbb{N}$, then $\tau_k \in A_k$. If $k = 0$, since $ib_0 \in S_0$ is a root of (3.2_{τ_0}), it follows from Theorem 3.7 that there exists a $\bar{\tau}_0 > \tau_0$ such that $\bar{\tau}_0 \in A_0$. Thus $A_k \neq \emptyset$. It also follows from Theorem 3.7 that A_k is relatively open in (τ_0, ∞). Since (τ_0, ∞) is connected, the proof is complete by showing that A_k is also relatively closed in (τ_0, ∞). Let $\{s_n\}_{n=1}^{\infty} \subseteq A_k$ be a sequence such that $s_n \to s_0$ as $n \to \infty$ for some $s_0 > \tau_0$. For each $n \in \mathbb{N}$, let $\alpha_n + i\beta_n \in S_k$ be a root of (3.2_{s_n}). It follows from (3.2_τ) that $\{\alpha_n + i\beta_n\}_{n=1}^{\infty}$ is bounded. Therefore, there exists a subsequence of $\{\alpha_n + i\beta_n\}$, say itself for the convenience of notation, and $\alpha_0 + i\beta_0$ with $\beta_0 \in [(2k-1)\pi, (2k+1)\pi]$ if $k \in \mathbb{N}$, and $\beta_0 \in [0, \pi]$ if $k = 0$, such that $\alpha_n + i\beta_n \to \alpha_0 + i\beta_0$ as $n \to \infty$. Taking the limit in (3.2_{s_n}), we know that $\alpha_0 + i\beta_0$ is a root of (3.2_{s_0}). It follows from Lemmas 3.4 and 3.6 that $\alpha_0 + i\beta_0 \in S_k$ and hence $s_0 \in A_k$. This completes the proof.

Lemma 3.9. *For any $k \in \mathbb{N}_+$, let $a_k(\tau) + ib_k(\tau) \in S_k$ be a solution of (3.2_τ). Then $a_k(\tau) < 0$ if $\tau \in (\tau_0, \tau_k)$ and $a_k(\tau) > 0$ if $\tau > \tau_k$.*

Proof. We only consider the case where $\tau < \tau_k$ since the other case can be dealt with similarly. It follows from (H3) that $k \in \mathbb{N}$. By way of contradiction, assume that there exist a $\bar{\tau} < \tau_k$ and a solution $a_k(\bar{\tau}) + ib_k(\bar{\tau}) \in S_k$ of $(3.2_{\bar{\tau}})$ such that $\alpha_k(\bar{\tau}) \geq 0$. Let $\alpha_k(\tau) + i\beta_k(\tau) \in S_k$ be the solution curve of (3.2_τ) passing through $a_k(\bar{\tau}) + ib_k(\bar{\tau})$, i.e.; $\alpha_k(\bar{\tau}) + i\beta_k(\bar{\tau}) = a_k(\bar{\tau}) + ib_k(\bar{\tau})$ (this is guaranteed by Lemma 3.6 and Theorem 3.7). Let

$$B_k = \{\tau < \tau_k;\ \alpha_k(\tau) > 0\}.$$

If $\alpha_k(\bar{\tau}) > 0$ then $\bar{\tau} \in B_k$. If $a_k(\bar{\tau}) = 0$, then $b(\bar{\tau}) \neq 0$ by Lemma 3.4. It follows from (3.2_τ) that

$$\frac{d\alpha_k}{d\tau}(\bar{\tau}) = \frac{2[\beta_k(\bar{\tau})]^4 + \bar{\tau}^2[\beta_k(\bar{\tau})]^2[\mu_1^2 + \mu_2^2]}{\bar{\tau}\left[\left(\bar{\tau}p - (\beta_k(\bar{\tau}))^2 + \bar{\tau}^2 q\right)^2 + \left(2\beta_k(\bar{\tau}) + \bar{\tau}p\beta_k(\bar{\tau})\right)^2\right]} > 0.$$

Thus there is a $\tilde{\tau} \in (\bar{\tau}, \tau_k)$ such that $\tilde{\tau} \in B_k$. Therefore, $B_k \neq \emptyset$. Let $\hat{\tau} = \sup B_k$. Then $\hat{\tau} \leq \tau_k$ and $\alpha_k(\hat{\tau}) = 0$ by the definition of $\hat{\tau}$ and Theorem 3.7. Thus, $\hat{\tau} \notin B_k$. However, using the above calculation, we have $(d\alpha_k/d\tau)(\hat{\tau}) > 0$, which means that $\alpha_k(\tau) < 0$ for all $\tau < \hat{\tau}$ and close enough to $\hat{\tau}$, a contradiction to the definition of $\hat{\tau}$. This completes the proof.

Corollary 3.10. *For any $k \in \mathbb{N}_+$, $ib_k \in S_k$ is the unique solution of (3.2_{τ_k}) in S_k.*

Proof. If $\alpha_k + i\beta_k \in S_k$ is a solution of (3.2_{τ_k}), we claim that $\alpha_k = 0$. In fact, if $k \in \mathbb{N}$, the claim follows directly from Lemma 3.9 and Theorem 3.7. For $k = 0$, note that $\alpha_k + i\beta_k$ can be extended to a solution curve of (3.2_τ) around τ_0. Then we can use Lemma 3.9 and Theorem 3.7 to obtain $\alpha_0 = 0$. Thus, the claim is proved. Recall that $ib_k \in S_k$ is a solution of (3.2_{τ_k}). Note that if $ib \in S_k$ is a solution of (3.2_{τ_k}) then it follows from (3.2_{τ_k}) that $b \in (2k\pi, (2k+1)\pi)$. Let $\nu > 0$ be such that $\beta_k = b_k\nu$. Then $\nu \in (2k\pi/b_k, (2k+1)\pi/b_k)$. Again, from (3.2_{τ_k}) we have

$$\cot(b_k\nu) - \frac{b_k^2\nu^2 - \tau_k^2 q}{\tau_k p b_k \nu} = 0. \tag{3.11}$$

Let H: $(2k\pi/b_k, (2k+1)\pi/b_k) \to \mathbb{R}$ be defined by

$$H(\nu) = \cot(b_k\nu) - \frac{b_k^2\nu^2 - \tau_k^2 q}{\tau_k p b_k \nu}.$$

Then $H(1) = 0$ and

$$H'(\nu) = -\frac{b_k}{\sin^2(b_k\nu)} - \frac{\tau_k p b_k^3 \nu^2 + \tau_k^3 p q b_k}{(\tau_k p b_k \nu)^2} < 0.$$

Thus Eq. (3.11) has one unique root $\nu = 1$; i.e., $\beta_k = b_k$. This completes the proof.

It follows from Lemma 3.9 and Corollary 3.10 that Eq. (3.2_τ) has pure imaginary roots only when $\tau = \tau_k$ for some $k \in \mathbb{N}_+$ and the roots are $\pm ib_k$.

Now, we are ready to state the main result of this section.

THEOREM 3.11. *Equation* (3.2_τ) *has a unique solution, denoted by* $a_k(\tau) + ib_k(\tau)$, *in* S_k *for* $k \in \mathbb{N}_+$. *Moreover*, $a_0 > 0$, $a_{k+1} < a_k$ *for each* $k \in \mathbb{N}_+$ *and* $a_k \to -\infty$ *as* $k \to \infty$.

Proof. The existence and uniqueness of the solution of (3.2_τ) in each S_k follow easily from Lemma 3.4, Corollary 3.5, Lemma 3.6, Theorem 3.7, Lemma 3.8, and Corollary 3.10. From Lemma 3.9, we have $a_0 > 0$. We only need to show that $a_{k+1} < a_k$ for each $k \in \mathbb{N}_+$. If this is true, then $\lim_{k\to\infty} a_k$ exists and it follows from (3.2_τ) that $\lim_{k\to\infty} a_k = -\infty$.

Now, we come to show that $a_{k+1} < a_k$ for each $k \in \mathbb{N}_+$. Let

$$C_k = \{\tau > \tau_0;\ \alpha_{k+1}(\tau) < a_k(\tau)\}.$$

By Lemma 3.9, if $\tau \in (\tau_k, \tau_{k+1})$, then $a_{k+1}(\tau) < 0 < a_k(\tau)$. Thus $C_k \neq \varnothing$. It follows from Theorem 3.7 that C_k is relatively open in (τ_0, ∞). Since

(τ_0, ∞) is connected, $C_k = (\tau_0, \infty)$ if C_k is also relatively closed in (τ_0, ∞). Let $\{s_n\}_{n=1}^{\infty} \subseteq C_k$ be a sequence such that $s_n \to s_0$ as $n \to \infty$ for some $s_0 > \tau_0$. If $s_0 \notin C_k$, then $a_{k+1}(s_0) = a_k(s_0)$. However, it follows from (3.2_{s_0}) that

$$\begin{aligned}
&s_0^4 r^2 e^{-2a_k(s_0)} \\
&\quad = a_k^4(s_0) + b_k^4(s_0) + s_0^2 p^2 a_k^2(s_0) + s_0^4 q^2 + 2s_0 p a_k^3(s_0) + 2s_0 a_k^2(s_0) q \\
&\qquad + s_0^2 b_k^2(s_0)\left(\mu_1^2 + \mu_2^2\right) + 2s_0^3 pq a_k(s_0) \\
&\qquad + 2s_0 p a_k(s_0) b_k^2(s_0) + 2a_k^2(s_0) b_k^2(s_0) \\
&\quad = a_{k+1}^4(s_0) + b_{k+1}^4(s_0) + s_0^2 p^2 a_{k+1}^2(s_0) + s_0^4 q^2 + 2s_0 p a_{k+1}^3(s_0) \\
&\qquad + 2s_0 a_{k+1}^2(s_0) q + s_0^2 b_{k+1}^2(s_0)\left(\mu_1^2 + \mu_2^2\right) + 2s_0^3 pq a_{k+1}(s_0) \\
&\qquad + 2s_0 p a_{k+1}(s_0) b_{k+1}^2(s_0) + 2a_{k+1}^2(s_0) b_{k+1}^2(s_0) \\
&\quad = s_0^4 r^2 e^{-2a_{k+1}(s_0)},
\end{aligned}$$

which gives

$$\begin{aligned}
0 &= \left[b_k^2(s_0) - b_{k+1}^2(s_0)\right] \\
&\qquad \times \left[b_k^2(s_0) + b_{k+1}^2(s_0) + s_0^2\left(\mu_1^2 + \mu_2^2\right) + 2s_0 p a_k(s_0) + 2a_k^2(s_0)\right] \\
&= \left[b_k^2(s_0) - b_{k+1}^2(s_0)\right] \\
&\qquad \times \left[b_k^2(s_0) + b_{k+1}^2(s_0) + \left(s_0\mu_1 + a_k(s_0)\right)^2 + \left(s_0\mu_2 + a_k(s_0)\right)^2\right].
\end{aligned}$$

This is impossible since $a_k(s_0) + ib_k(s_0) \in S_k$ and $a_{k+1}(s_0) - ib_{k+1}(s_0) \in S_{k+1}$. This completes the proof.

4. THE EXISTENCE OF SLOWLY OSCILLATING PERIODIC SOLUTIONS

Recall that the eigenvalues of the generator of the C_0-semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$ coincide with the roots of (3.2_τ). It follows from Theorem 3.11 that the eigenvalues are simple and given by $\{a_k \pm ib_k\}_{k\in\mathbb{N}_+}$, where $a_0 > 0$ and $a_0 > a_1 \cdots > a_k > \cdots \to -\infty$ as $k \to \infty$, $b_0 \in (0, \pi)$, and $b_k \in ((2k-1)\pi, (2k+1)\pi)$ for $k \in \mathbb{N}$.

In what follows, we will let P and Q be the realified generalized eigenspaces of the generator of the semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$ on $C(\mathbb{K})$ associated with the spectral sets $\{a_0 \pm ib_0\}$ and $\{a_k \pm ib_k;\ k \in \mathbb{N}\}$, respectively. Then

$$C(\mathbb{K}) = P \oplus Q.$$

Choose $\beta > 1$ such that $e^{a_1} < \beta < e^{a_0}$. Then we can find convex bounded open neighborhoods N_P and N_Q of 0 in P and Q, respectively, and a C^1-map w: $N_P \to N_Q$ such that $w(0) = 0$, $Dw(0) = 0$, $w(N_P) \subseteq N_Q$, and the graph

$$W_{\text{loc}} = \{\chi + w(\chi);\ \chi \in N_P\}$$

coincides with the set of $\varphi \in N_P + N_Q$ such that there is a sequence $\{\varphi_n\}_{n=-\infty}^{0}$ with $\varphi_{n+1} = \Phi(1, \varphi_n)$ for $n \le -1$, $\varphi_0 = \varphi$, $\varphi_n \beta^{-n} \in N_P + N_Q$ for all $n \le 0$, and $\varphi_n \beta^{-n} \to 0$ as $n \to -\infty$. See Appendix I of Krisztin et al. [11] for details.

Let

$$W = \Phi(\mathbb{R}^+ \times W_{\text{loc}})$$

be the forward extension of W_{loc}. This is inside the unstable set of the origin. Moreover, for each $\varphi \in W$ there exists a unique solution $z^\varphi = (u^\varphi, v^\varphi)$: $\mathbb{R} \to \mathbb{R}^2$ on $\mathbb{R}$ with $z_0^\varphi = \varphi$ and $z_t^\varphi \to 0$ as $t \to -\infty$.

We now derive some important properties of W. For any bounded function m: $\mathbb{R} \to \mathbb{R}$, let $\|m\|_\infty = \sup_{\xi \in \mathbb{R}} |m(\xi)|$. Then it is easy to check that

$$\mathscr{M} = \{\varphi \in C(\mathbb{K});\ |\varphi(\theta)| \le \mu_1^{-1}\|f\|_\infty \quad \text{for } \theta \in [-1, 0] \text{ and } |\varphi(1)| \le \mu_2^{-1}\|g\|_\infty\}$$

is positively invariant with respect to the semiflow Φ.

THEOREM 4.1. (i) *$\overline{W}$ is bounded, compact, and invariant with respect to the semiflow Φ;*

(ii) *the map Φ_W: $\mathbb{R} \times \overline{W} \ni (t, \varphi) \mapsto (u_t^\varphi, v^\varphi(t)) \in \overline{W}$ is a continuous flow;*

(iii) *for each $\varphi \in W$, $z^\varphi = (u^\varphi, v^\varphi)$: $\mathbb{R} \to \mathbb{R}^2$ is C^1-smooth and for each fixed $t \in \mathbb{R}$, the map $\overline{W} \ni \varphi \mapsto z_t^\varphi \in C^1(\mathbb{K})$ is continuous.*

Proof. We only prove the boundedness of $\overline{W}$ since the remaining assertions can be proved in the same way as Theorem 3.3 in Chen and Wu [5]. Let $\varphi \in W$. Then $z_t^\varphi \to 0$ as $t \to -\infty$. Thus there exists $t \le 0$ such that $z_t^\varphi \in \mathscr{M}$. It follows immediately from the positive invariance of $\mathscr{M}$ that $\varphi \in \mathscr{M}$. So $\overline{W} \subseteq \mathscr{M}$ and hence $\overline{W}$ is bounded. This completes the proof.

To characterize $\overline{W}$, we need the discrete Lyapunov functional introduced by Mallet-Paret and Sell [12]. We briefly summarize some of its properties here. For details, we refer to Mallet-Paret and Sell [12, 13].

Let $\varphi \in C(\mathbb{K}) \setminus \{0\}$; define the number of sign changes

$$\mathrm{sc}(\varphi) = \left\{ k \geq 1; \begin{array}{l} \text{there exists } \theta^0 < \theta^1 < \cdots < \theta^k \text{ with } \theta^i \in \mathbb{K} \text{ for} \\ i = 0, 1, \ldots, k \text{ and } \varphi(\theta^{i-1})\varphi(\theta^i) < 0 \text{ for } 1 \leq i \leq k \end{array} \right\}$$

with the convention that

$$\mathrm{sc}(\varphi) = 0 \qquad \text{if } \varphi(\theta) \geq (\leq) 0 \qquad \text{for } \theta \in \mathbb{K}.$$

Set

$$V(\varphi) = \begin{cases} \mathrm{sc}(\varphi) & \text{if } \mathrm{sc}(\varphi) \text{ is odd or infinite}, \\ \mathrm{sc}(\varphi) + 1 & \text{if } \mathrm{sc}(\varphi) \text{ is even}. \end{cases}$$

We know that V is lower semicontinuous on $C(\mathbb{K}) \setminus \{0\}$; that is,

$$V(\varphi) \leq \liminf_{n \to \infty} V(\varphi^n) \qquad \text{if } \varphi^n \to \varphi \in C(\mathbb{K}) \setminus \{0\}.$$

Let

$$R = \left\{ \varphi \in C^1(\mathbb{K}); \begin{array}{l} \varphi(1) = 0 \text{ implies } \varphi(0)\varphi(-1) > 0 \\ \varphi(0) = 0 \text{ implies } \dot{\varphi}(0)\varphi(1) > 0 \\ \varphi(-1) = 0 \text{ implies } \varphi(1)\dot{\varphi}(-1) > 0 \\ \varphi(\theta) = 0 \text{ for some } \theta \in (-1, 0) \text{ implies } \dot{\varphi}(\theta) \neq 0 \end{array} \right\}.$$

Then for each $\varphi \in R$, there exists $\varepsilon > 0$ such that

$$V(\psi) = V(\varphi) \quad \text{for } \psi \in C^1(\mathbb{K}) \text{ with } \|\psi - \varphi\|_1 < \epsilon.$$

Moreover, if I is some interval, $b, c\colon I \to \mathbb{R}$ are some positive continuous functions, $u\colon J_I \to \mathbb{R}$ and $v\colon I \to \mathbb{R}$ are continuous functions with

$$J_I = \{t - 1;\ t \in I\} \cup I$$

such that $u, v\colon I \to \mathbb{R}$ are C^1-smooth and satisfy

$$\begin{aligned} \dot{u}(t) &= -\mu_1 u(t) + b(t)v(t), \\ \dot{v}(t) &= -\mu_2 v(t) - c(t)u(t-1) \end{aligned} \tag{4.1}$$

for $t \in I$, and $z_t = (u_t, v(t)) \in C(\mathbb{K}) \setminus \{0\}$ for $t \in I$, then

$$V(z_t) \leq V(z_s) \quad \text{for } t, s \in I \text{ with } t \geq s. \tag{4.2}$$

(4.3) if $t \in I$ is given so that $t - 4 \in I$ and $V(z_t) = V(z_{t-4}) < \infty$, then $z_t \in R$.

The following results can be established similarly to those in Chen and Wu [5]. Instead of Corollary 4.5 there we need to replace it by the following analog:

Let $(x, y)\colon \mathbb{R} \to \mathbb{R}^2$ *be a nontrivial solution of* (3.1) *on* $(-\infty, 0]$. *Then*

$$(x_0, y(0)) \in P \text{ if and only if } V\big((x_t, y(t))\big) = 1 \text{ for all } t \leq 0.$$

This can be proved similarly to Corollary 4.5 of Chen and Wu [5], and the key to the proof is the information on the eigenvalues we obtained in Section 3. In particular, we need $a_0 > a_1 > \cdots$, $b_0 \in (0, \pi)$, and $b_k \in ((2k-1)\pi, (2k+1)\pi)$ for $k \in \mathbb{N}$ to have different oscillation frequencies for solutions in P and solutions in Q.

THEOREM 4.2. *If* $\varphi, \psi \in \overline{W}$ *and* $\varphi \neq \psi$, *then* $V(\varphi - \psi) = 1$ *and* $\varphi - \psi \in R$.

THEOREM 4.3.

$$W \setminus \{0\} = \left\{ \varphi \in C(\mathbb{K}) \setminus \{0\};\ \begin{array}{l} \text{there exists a solution } z^{\varphi}\colon \mathbb{R} \to \mathbb{R}^2 \\ \text{of } (2.8_\tau) \text{ with } z_0^{\varphi} = \varphi, V(z_t^{\varphi}) = 1 \text{ for} \\ \text{all } t \in \mathbb{R} \text{ and } z_t^{\varphi} \to 0 \text{ as } t \to -\infty \end{array} \right\}.$$

THEOREM 4.4. *Both* $(u^{\varphi})^{-1}(0) \cap (-\infty, 0]$ *and* $(u^{\varphi})^{-1}(0) \cap [0, \infty)$ *are unbounded for* $\varphi \in \overline{W} \setminus \{0\}$.

Proof. It follows from Theorem 4.2 that $\varphi \in R$ and hence all zeros of u^{φ} are simple. Using system (2.8_τ), we see that $(u^{\varphi})^{-1}(0) \cap (-\infty, 0]$ (resp. $(u^{\varphi})^{-1}(0) \cap [0, \infty)$) is unbounded if and only if $(v^{\varphi})^{-1}(0) \cap (-\infty, 0]$ (resp. $(v^{\varphi})^{-1}(0) \cap [0, \infty)$) is.

We now show that $(u^{\varphi})^{-1}(0) \cap (-\infty, 0]$ is unbounded. If not, assume that $(u^{\varphi})^{-1}(0) \cap (-\infty, 0]$ is bounded. Then $(v^{\varphi})^{-1}(0) \cap (-\infty, 0]$ is also bounded. Therefore, there exists $T_0 \leq 0$ such that

Case 1. $u^{\varphi}(t) > 0$ and $v^{\varphi}(t) > 0$ for $t \leq T_0$.

Case 2. $u^{\varphi}(t) > 0$ and $v^{\varphi}(t) < 0$ for $t \leq T_0$.

Case 3. $u^{\varphi}(t) < 0$ and $v^{\varphi}(t) > 0$ for $t \leq T_0$.

Case 4. $u^{\varphi}(t) < 0$ and $v^{\varphi}(t) < 0$ for $t \leq T_0$.

We only discuss Case 1 since the other three cases can be dealt with similarly. In Case 1, it follows from the second equation of (2.8_τ) that $\dot{v}^{\varphi}(t) < 0$ for $t \leq T_0$. Thus, $v^{\varphi}(t) \geq v^{\varphi}(T_0)$ for $t \leq T_0$, a contradiction to $z_t^{\varphi} \to 0$ as $t \to -\infty$. This shows that $(u^{\varphi})^{-1}(0) \cap (-\infty; 0]$ is unbounded.

Next, we prove that $(u^{\varphi})^{-1}(0) \cap [0, \infty)$ is unbounded. Assume, by way of contradiction, $(u^{\varphi})^{-1}(0) \cap [0, \infty)$ is bounded. Then so is $(v^{\varphi})^{-1}(0) \cap [0, \infty)$.

As before, there exists $\overline{T}_0$ such that

Case A. $u^{\varphi}(t) > 0$ and $v^{\varphi}(t) > 0$ for $t \geq \overline{T}_0$.

Case B. $u^{\varphi}(t) > 0$ and $v^{\varphi}(t) < 0$ for $t \geq \overline{T}_0$.

Case C. $u^{\varphi}(t) < 0$ and $v^{\varphi}(t) > 0$ for $t \geq \overline{T}_0$.

Case D. $u^{\varphi}(t) < 0$ and $v^{\varphi}(t) < 0$ for $t \geq \overline{T}_0$.

Again, we only discuss Case A since the other three cases can be dealt with similarly. We claim that, in this case, $(u^{\varphi}(t), v^{\varphi}(t)) \to (0,0)$ as $t \to \infty$. To verify the claim, we note that it follows from (2.8_τ) that $\dot{v}^{\varphi}(t) < 0$ for $t \geq \overline{T}_0 + 1$. Thus, $\lim_{t\to\infty} v^{\varphi}(t) = \bar{v}$ exists. Obviously, $\bar{v} \geq 0$. In fact, we must have $\bar{v} = 0$, for otherwise, from (2.8_τ) it follows that

$$\dot{v}^{\tau}(t) \leq -\tau\mu_2\bar{v} \quad \text{for } t \geq \overline{T}_0 + 1.$$

Thus, $v^{\varphi}(t) \leq v^{\varphi}(\overline{T}_0 + 1) - \tau\mu_2\bar{v}(t - \overline{T}_0 - 1)$ for $t \geq \overline{T}_0 + 1$, a contradiction to $v^{\varphi}(t) > 0$ for $t \geq \overline{T}_0$. It remains to show that $u^{\varphi}(t) \to 0$ as $t \to \infty$. We prove it by showing $\limsup_{t\to\infty} u^{\varphi}(t) = 0$. If not, then there exist $\varepsilon_0 > 0$ and a sequence $\{t_n\}_{n=1}^{\infty} \subseteq [\overline{T}_0 + 1, \infty)$ such that $v^{\varphi}(t_n) \geq \varepsilon_0$ for $n \in \mathbb{N}$ and $t_n \to \infty$ as $n \to \infty$. Since $\overline{W}$ is bounded and invariant, it follows from (2.8_τ) that $\dot{u}^{\varphi}(t)$ is uniformly bounded on $[0,\infty)$. Thus there exists $\eta_0 > 0$ such that $u^{\varphi}(t) \geq \varepsilon_0/2$ for $t \in I_n := [t_n - \eta_0, t_n + \eta_0]$ and for all $n \in \mathbb{N}$. Let us assume, without loss of generality, that $\{I_n\}$ are disjointed from each other; otherwise, we choose an appropriate subsequence of $\{t_n\}$. Again, from (2.8_τ), we have

$$\dot{v}^{\varphi}(t) < -\tau g(u^{\varphi}(t-1)) \quad \text{for } t \geq 0.$$

Thus,

$$v^{\varphi}(t_n + \eta_0 + 1) < v^{\varphi}(t_1 + 1) - 2\tau n\eta_0 g\left(\frac{\varepsilon_0}{2}\right) \to -\infty$$

as $n \to \infty$, a contradiction to $v^{\varphi}(t) > 0$ for $t \geq \overline{T}_0$. Therefore, we have $u^{\varphi}(t) \to 0$ as $t \to \infty$. This proves the claim. The rest of the proof is similar to that of Lemma 5.7 of Chen and Wu [5] and thus is omitted.

Let

$$\Pi\colon C(\mathbb{K}) \ni \varphi \mapsto (\varphi(0), \varphi(1)) \in \mathbb{R}^2.$$

Note that $\varphi, \psi \in \overline{W}$ implies that $V(\varphi - \psi) = 1$ and $\varphi - \psi \in R$. Therefore, $\Pi(\varphi - \psi) \neq 0$. This shows that $\Pi|_{\overline{W}}$ is injective.

Define $\Pi^{-1}\colon \Pi(\overline{W}) \to C(\mathbb{K})$ as the inverse of $\Pi|_{\overline{W}}\colon \overline{W} \to \Pi(\overline{W}) \subseteq \mathbb{R}^2$. Then it can be shown, similar to Lemma 5.8 of Chen and Wu [5], that $\Pi^{-1}\colon \Pi(\overline{W}) \subseteq \mathbb{R}^2 \to \overline{W}$ is Lipschitz continuous.

Let

$$H = \{\varphi \in C(\mathbb{K});\ \varphi(0) = 0\},$$
$$H^+ = \{\varphi \in H;\ \varphi(1) > 0\}.$$

Clearly,

$$\Pi H = \{(x, y);\ x = 0\},$$
$$\Pi H^+ = \{(x, y);\ x = 0,\ y > 0\}.$$

ΠH has a natural ordering $\prec$.

For a given $\chi^0 \in \Pi(\overline{W})$, let $\psi = \Pi^{-1}(\chi^0)$. Then $\psi \in \overline{W}$ and there is a unique solution $z^\psi = (u^\psi, v^\psi)$: $\mathbb{R} \to \mathbb{R}^2$ of (2.8_τ) with $z_t^\psi \in \overline{W}$ for all $t \in \mathbb{R}$. This determines a C^1-curve

$$\mathbb{R} \ni t \mapsto \Pi z_t^\psi = \big(u^\psi(t), v^\psi(t)\big) \in \Pi(\overline{W}) \subseteq \mathbb{R}^2,$$

called the *canonical curve* through χ^0. It is important to note that the images of two canonical curves either coincide or do not intersect.

Observe that if $\chi^0 \in \Pi(\overline{W})$ and $\psi = \Pi^{-1}(\chi^0)$, then

$$\left\langle (1,0), \frac{\mathrm{d}}{\mathrm{d}t}\Pi z_t^\psi \right\rangle = \dot{u}^\psi(t) = \tau f\big(v^\psi(t)\big) > 0$$

at any $t \in \mathbb{R}$ with $\Pi z_t^\psi \in \Pi H^+$,

where $\langle \cdot,\cdot \rangle$ is the usual inner product in $\mathbb{R}^2$. This shows that canonical curves intersect ΠH^+ transversally.

Fix $0 \neq \chi^0 \in \Pi H^+ \cap \Pi(\overline{W})$. Then $\psi = \Pi^{-1}(\chi^0) \in H^+ \cap \overline{W}$. By Theorem 4.4 and the fact that $V(z_t^v) = 1$ for all $t \in \mathbb{R}$, the zeros of u^ψ: $\mathbb{R} \to \mathbb{R}$ are both unbounded from above and below and are all simple. Therefore, there exist the smallest positive and the largest negative zeros $\gamma_+(\psi)$ and $\gamma_-(\psi)$ of u^ψ with $\dot{u}^\psi(\gamma_\pm(\psi)) > 0$. We define the first return map ρ: $\Pi H^+ \cap \Pi(\overline{W}) \to \Pi H^+ \cap \Pi(\overline{W})$ by

$$\rho(\chi^0) = \Pi z^\psi_{\gamma_+(\psi)}, \qquad \psi = \Pi^{-1}(\chi^0).$$

The following two results are analogs of Lemma 5.9 and Theorem 5.10 of Chen and Wu [5] and can be proved similarly.

LEMMA 4.5. (i) *ρ: $\Pi H^+ \cap \Pi(\overline{W}) \to \Pi H^+ \cap \Pi(\overline{W})$ is a homeomorphism. The inverse ρ^{-1} of ρ is given by*

$$\rho^{-1}(\chi^0) = \Pi z^\psi_{\gamma_-(\psi)} \quad \text{for } \chi \in \Pi H^+ \cap \Pi(\overline{W}) \text{ and } \psi = \Pi^{-1}(\chi^0).$$

(ii) *If* $\chi^0, \hat{\chi}^0 \in \Pi H^+ \cap \Pi(\overline{W})$ *and* $\chi^0 \prec \hat{\chi}^0$, *then* $\rho(\chi^0) \prec \rho(\hat{\chi}^0)$.

(iii) *If* $\chi^0 \in \Pi H^+ \cap \Pi(\overline{W})$ *and the sequence* $\{\chi^n\}_{-\infty}^{\infty} \subseteq \Pi H^+ \cap \Pi(\overline{W})$ *satisfies* $\rho(\chi^n) = \chi^{n+1}$ *for all integers* n, *then the limits* $\chi_- = \lim_{n\to-\infty}\chi^n$ *and* $\chi_+ = \lim_{n\to\infty}\chi^n$ *exist with* $\chi_-, \chi_+ \in (\Pi H^+ \cup \{(0,0)\}) \cap \Pi(\overline{W})$, *and* $z^{\Pi^{-1}(\chi_i)}$ *is a nontrivial periodic solution of* (2.8_τ) *provided that* $\chi_i \neq (0,0)$, *where* $i \in \{-,+\}$.

THEOREM 4.6. (i) *There is a nontrivial periodic solution* $p = (p^1, p^2)\colon \mathbb{R} \to \mathbb{R}^2$ *of* (2.8_τ) *such that*

$$p_0 \in H^+,\ p_t \in \overline{W},\ V(p_t) = V(\dot{p}_t) = 1 \quad \textit{and} \quad p_t, \dot{p}_t \in R \qquad \textit{for all } t \in \mathbb{R},$$

and the minimal period ω *of* p *is larger than* 2 *and is determined by three consecutive zeros of* p^1 *or* p^2;

(ii) $z_t^\varphi \to \mathscr{O} = \{p_t;\ t \in \mathbb{R}\}$ *as* $t \to \infty$ *for all* $\varphi \in W \setminus \{0\}$.

DEFINITION 4.7. A periodic solution (u, v) of (2.8_τ) is said to be *slowly oscillating* if the minimal period is larger than 2 and the distance of any two consecutive zeros of u or v is larger than 1.

THEOREM 4.8. *The periodic solution* $p = (p^1, p^2)$ *of* (2.8_τ) *obtained in Theorem* 4.6 *is slowly oscillating*.

Proof. It follows from $p_t \in R$ for all $t \in \mathbb{R}$ that all zeros of p^1 and p^2 are simple. Since $V(p_t) = 1$ for all $t \in \mathbb{R}$, the distance between any two consecutive zeros of p^1 is larger than 1 and hence the minimal period determined by three consecutive zeros of p^1 is larger than 2. It remains to show that the distance between any two consecutive zeros of p^2 is larger than 1. Assume that there exist two consecutive zeros $t_1 < t_2$ of p^2 such that $t_2 - t_1 \le 1$. Let us assume that $p^2 > 0$ on (t_1, t_2) since the case where $p^2 < 0$ on (t_1, t_2) can be dealt with similarly. Then $\dot{p}^2(t_1) > 0$ and $\dot{p}^2(t_2) < 0$. It follows from (2.8_τ) that $p^1(t_1 - 1) < 0$ and $p^1(t_2 - 1) > 0$. Thus there exists $\bar{t} \in (t_1 - 1, t_2 - 1)$ such that $p^1(\bar{t}) = 0$. Since $0 < t_2 - 1 - \bar{t} < 1$, $p^1 > 0$ on $(\bar{t}, t_2 - 1]$. Note that $p^2 < 0$ on (t_0, t_1), where t_0 is the first zero of p^2 less than t_1. It is easy to see that $t_1 - t_0 > 1$ since the minimal period is larger than 2 and is determined by three consecutive zeros of p^2. Therefore, $p^1(t)p^2(t) < 0$ for $t \in (\bar{t}, t_2 - 1)$. So $\mathrm{sc}(p_t) \ge 2$ for $t \in (\bar{t}, t_2 - 1)$ and $V(p_t) \ge 3$ for $t \in (\bar{t}, t_2 - 1)$, a contradiction to $V(p_t) = 1$ for all $t \in \mathbb{R}$. This completes the proof.

Define $\eta(t) = p_t$ for $t \in [0, \omega]$. Then $[0, \omega] \ni t \mapsto \eta(t) \in C(\mathbb{K})$ and $[0, \omega] \in t \mapsto \Pi(\eta(t)) \in \mathbb{R}^2$ is a smooth closed curve. We have $\mathscr{O} = \{\eta(t);\ 0 \le t \le \omega\}$. Then we have the following results, which can be proved in a similar way to that for Theorem 5.11 of Chen and Wu [5].

THEOREM 4.9. (i) $\Pi(W) = \operatorname{int}(\Pi \circ \eta)$;

(ii) $\operatorname{bd} W = \overline{W} \setminus W = \mathscr{O}$;

(iii) $\Pi(\overline{W} \setminus W) = \Pi \circ (\eta[0, \omega])$.

It follows from Theorem 4.9 that $\mathscr{O}$ is the only nontrivial periodic orbit in $\overline{W}$.

COROLLARY 4.10. *If both f and g are odd, then $p^i(t) = -p^i(t + \frac{\omega}{2})$ for $i = 1, 2$ and $t \in \mathbb{R}$.*

Proof. Since $p = (p^1, p^2)\colon \mathbb{R} \to \mathbb{R}^2$ is a nontrivial periodic solution of (2.8_τ) with $p_t \in \overline{W}$ for $t \in \mathbb{R}$ and both f and g are odd, it is easy to check that $q = (-p^1, -p^2)\colon \mathbb{R} \to \mathbb{R}^2$ is also a nontrivial periodic solution of (2.8_τ) with $q_t \in \overline{W}$ for $t \in \mathbb{R}$. This, combined with the uniqueness of nontrivial periodic orbit in $\overline{W}$, yields

$$-p^1(t) = p^1(t + \alpha),$$
$$-p^2(t) = p^2(t + \alpha)$$

for some $\alpha \in [0, \omega)$. Thus,

$$p^1(t) = p^2(t + 2\alpha),$$
$$p^2(t) = p^1(t + 2\alpha).$$

Therefore, $2\alpha = m\omega$ for some integer m. Note that $0 \le 2\alpha < 2\omega$, so $m = 0$ or $m = 1$. Thus either

$$p^1(t) = -p^1(t),$$
$$p^2(t) = -p^2(t)$$

or

$$p^1(t + \tfrac{\omega}{2}) = -p^1(t),$$
$$p^2(t + \tfrac{\omega}{2}) = -p^2(t).$$

However, $p^1(t) = -p^1(t)$ implies $p^1 \equiv 0$, which is impossible since p is a nontrivial solution of (2.8_τ). This completes the proof.

ACKNOWLEDGMENTS

The authors are very grateful to the referee for her/his careful reading of the original manuscript and for her/his valuable suggestions on improving this article.

REFERENCES

1. U. an der Heiden, Periodic solutions of a nonlinear second order differential equation with delay, *J. Math. Anal. Appl.* **70** (1979), 599–609.
2. J. Bélair, S. A. Campbell, and P. van den Driessche, Frustration, stability and delay-induced oscillations in a neural network model, *SIAM J. Appl. Math.* **46** (1996), 245–255.
3. M. Z. Baptistini and P. Z. Táboas, On the existence and global bifurcation of periodic solutions to planar differential delay equations, *J. Differential Equations* **127** (1996), 391–425.
4. R. Bellman and K. L. Cooke, "Differential-Difference Equations," Academic Press, New York, 1963.
5. Y. Chen and J. Wu, Existence and attraction of a phase-locked oscillation in a delayed network of two neurons, *Integral and Differential Equations* (2001), in press.
6. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, "Delay Equations, Functional-, Complex-, and Nonlinear Analysis," Springer-Verlag, New York, 1995.
7. L. H. Erbe, W. Krawcewicz, K. Gęba, and J. Wu, S^1-degree and global Hopf bifurcation theory of functional differential equations, *J. Differential Equations* **98** (1992), 277–298.
8. K. Gopalsamy and I. Leung, Delay induced periodicity in a neural netlet of excitation and inhibition, *Physica D* **89** (1996), 395–426.
9. J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Nat. Acad. Sci.* **79** (1982), 2554–2558.
10. J. J. Hopfield, Neurons with graded response have collective computational properties like two-stage neurons, *Proc. Nat. Acad. Sci.* **81** (1984), 3088–3092.
11. T. Krisztin, H.-O. Walther, and J. Wu, "Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback," Fields Institute Monographs, Vol. 11, Am. Math. Soc., Providence, RI, 1999.
12. J. Mallet-Paret and G. R. Sell, Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions, *J. Differential Equations* **125** (1996), 385–440.
13. J. Mallet-Paret and G. R. Sell, The Poincaré–Bendixson theorem for monotone cyclic feedback system with delay, *J. Differential Equations* **125** (1996), 441–489.
14. C. M. Marcus and R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39** (1989), 347–356.
15. L. Pontryagin, On the zeros of some elementary transcendental functions, *Amer. Math. Soc. Transl. Ser.* 2 **1** (1955), 95–110.
16. S. Ruan and J. Wei, Periodic solutions of planar systems with two delays, *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), 1017–1032.
17. H. L. Smith, Monotone semiflows generated by functional differential equations, *J. Differential Equations* **66** (1987), 420–442.
18. P. Táboas, Periodic solutions of a planar delay equation, *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 85–101.
19. H.-O. Walther, An invariant manifold of slowly oscillating solutions for $x'(t) = -\mu x(t) + f(x(t-1))$, *J. Reine Angew. Math.* **414** (1991), 67–112.