Journal of Differential Equations **169**, 614–632 (2001)
doi:10.1006/jdeq.2000.3910, available online at http://www.idealibrary.com on IDEAL®

The Asymptotic Shapes of Periodic Solutions of a Singular Delay Differential System

Yuming Chen and Jianhong Wu

Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada

Received April 9, 1999; revised February 3, 2000

DEDICATED TO PROFESSOR JACK HALE ON THE OCCASION OF HIS 70TH BIRTHDAY

Let $f(\cdot, \lambda)\colon \mathbb{R} \to \mathbb{R}$ be given so that $f(0, \lambda) = 0$ and $f(x, \lambda) = (1+\lambda)\, x + ax^2 + bx^3 + o(x^3)$ as $x \to 0$. We characterize those small values of $\varepsilon > 0$ and $\lambda \in \mathbb{R}$ for which there are periodic solutions of periods approximately $\frac{2}{k}$ with $k \in \mathbb{N}$ of the following system arising from a network of neurons

$$\begin{cases} \varepsilon \dot{x}(t) = -x(t) + f(y(t-1), \lambda), \\ \varepsilon \dot{y}(t) = -y(t) + f(x(t-1), \lambda). \end{cases}$$

The periodic solutions are synchronized if k is even and phase-locked if k is odd. We show that, as $\varepsilon \to 0$, these periodic solutions approach square waves if $a = 0$ and $b < 0$, and pulses if $a = 0$ and $b > 0$ or if $a \neq 0$. © 2001 Academic Press

1. INTRODUCTION

For $\varepsilon > 0$ and $f \in C^m(\mathbb{R} \times \mathbb{R})$, $m \geqslant 3$, the following system of delay differential equations

$$\begin{cases} \varepsilon \dot{x}(t) = -x(t) + f(y(t-1), \lambda), \\ \varepsilon \dot{y}(t) = -y(t) + f(x(t-1), \lambda) \end{cases} \tag{1.1}$$

describes the dynamics of a network of two identical amplifiers (or neurons) with delayed outputs. See, for example, Hopfield [10], Marcus and Westervelt [12], and Wu [13].

For a given λ, if $\frac{\partial}{\partial \xi} f(\xi, \lambda) > 0$ for $\xi \in \mathbb{R}$, then system (1.1) models the delayed excitatory interaction of two identical neurons. We have recently obtained some results about the global dynamics of system (1.1) under some minor technical hypotheses [3–5]. It is shown that system (1.1) has at least two periodic orbits when ε is less than a certain value, one is synchronized and has the minimal period between 1 and 2 and the other one is phase-locked and has the minimal period larger than 2. Here a solution

0022-0396/01 $35.00

(x, y) of (1.1) is *synchronized* if $x \equiv y$ in their domains of definition, and a *phase-locked* T-periodic solution of (1.1) is one satisfying $x(t) = y(t - \frac{T}{2})$ for all $t \in \mathbb{R}$. The purpose of this paper is to study the limiting properties of these periodic solutions of (1.1) as $\varepsilon \to 0$.

More specifically, we assume that

$$f(x, \lambda) = (1 + \lambda)\, x + ax^2 + bx^3 + o(x^3) \qquad \text{as} \quad x \to 0. \tag{1.2}$$

When $a \neq 0$, $f(\cdot, \lambda)$ has only one nontrivial fixed point $c_{0\lambda}$ in a small neighborhood of 0. When $a = 0$, we observe that if $\lambda b < 0$ then $f(\cdot, \lambda)$ has two distinct nonzero fixed points $c_{1\lambda}$ and $c_{2\lambda}$ in a small neighborhood of 0; if $\lambda b > 0$ then 0 is the only fixed point of f in a small neighborhood of 0. Furthermore, $c_{0\lambda}, c_{1\lambda}, c_{2\lambda} \to 0$ as $\lambda \to 0$. One of our objectives here is to understand how these fixed points of the map $f(\cdot, \lambda)$ is reflected into the bifurcation from the origin of periodic solutions whose periods are approximately $\frac{2}{k}$ with some $k \in \mathbb{N}$.

Our work is inspired by that of Chow *et al.* [2] and Hale and Huang [9] for a scalar delay differential equation with negative feedback. In fact, our presentation here is parallel to those in [2, 9]: we show that the aforementioned periodic solutions are determined from the periodic solutions of special perturbed planar Hamiltonian systems that are obtained by an application of the normal form theory for retarded functional differential equations with parameters developed by Faria and Magalhães [7]. We will show that the normal forms on the associated center manifold are exactly the same (up to the third order term) as those for the negative feedback equations if we assume the second order term of $f(x, \lambda)$ vanishes, *i.e.*, $a = 0$, though interpretation for the limiting properties of the resulted periodic solutions for the original system (1.1) as $\varepsilon \to 0$ are different from that of the negative feedback analogue. In the case where $a \neq 0$, however, we will have a different normal form and different limiting profiles of periodic solutions as $\varepsilon \to 0$.

Roughly speaking, our results are as follows: for any $k \in \mathbb{N}$, there exist a neighborhood U_k of $(0, 0)$ in the (λ, ε) plane and a sectorial region S_k in U_k such that, if $(\lambda, \varepsilon) \in U_k$, then there is a periodic solution $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ of (1.1) with period $\frac{2}{k}(1 + \varepsilon) + O(|\varepsilon|\,(|\lambda| + |\varepsilon|))$ as $(\lambda, \varepsilon) \to (0, 0)$ if and only if $(\lambda, \varepsilon) \in S_k$. Moreover, this orbit is unique and the solution $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ is synchronized if k is even and phase-locked if k is odd. When $a = 0$, as $\varepsilon \to 0$, the solution $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ approaches square wave if $b < 0$ or pulse if $b > 0$. When $a \neq 0$, the solution $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ approaches pulse as $\varepsilon \to 0$.

Note that when k is even, the periodic solution $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ is synchronized and hence $\tilde{x}^{(k)}_{\lambda,\varepsilon}$ satisfies

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1), \lambda). \tag{1.3}$$

Also note that periodic solutions of (1.3) give synchronized periodic solutions of (1.1). Because of the uniqueness, we can deduce similar results for (1.3) from the above results. That is, for any $k \in \mathbb{N}$, there exist a neighborhood V_k of $(0, 0)$ in the (λ, ε) plane and a sectorial region R_k in V_k such that if $(\lambda, \varepsilon) \in V_k$, then there is a periodic solution $x_{\lambda,\varepsilon}^{(k)}$ of (1.3) with period $\frac{1}{k}(1+\varepsilon) + O(|\varepsilon|\,(|\lambda|+|\varepsilon|))$ as $(\lambda, \varepsilon) \to (0, 0)$ if and only if $(\lambda, \varepsilon) \in R_k$. Moreover, as $\varepsilon \to 0$, if $a = 0$ then the solution $x_{\lambda,\varepsilon}^{(k)}$ approaches square wave if $b < 0$ or pulse if $b > 0$; if $a \neq 0$ then the solution $x_{\lambda,\varepsilon}^{(k)}$ approaches pulse as $\varepsilon \to 0$. The results for the case where $a = 0$ are similar to those obtained by Chow *et al.* [2] and Hale and Huang [9] for Eq. (1.3) with f satisfying

$$f(x, \lambda) = -(1+\lambda)\,x + ax^2 + bx^3 + o(x^3) \qquad \text{as} \quad x \to 0. \tag{1.4}$$

See [2, 9] for more details. We observe that in [2, 9], neither a nor b alone but their combination $\beta = a^2 + b$ determines the bifurcation diagrams. However, in our case, if $a \neq 0$ then a itself alone determines the bifurcation diagrams and the bifurcation diagrams are different from the case where $a = 0$ and also from the case considered in [2, 9]. This indicates a difference between excitatory and inhibitory networks of neurons. The precise statements of our main results are given in Section 2. The main results follow from the normal form equations on the center manifold, which are computed in some detail in Section 3 by using the normal form theory developed by Faria and Magalhães [7], and similar arguments to those in [2] and [9].

2. MAIN RESULTS

Before obtaining the planar system on an associated center manifold, let us first do some local analysis for system (1.1). The linear variational system around the equilibrium solution 0 of (1.1) is

$$\begin{cases} \varepsilon \dot{x}(t) = -x(t) + (1+\lambda)\, y(t-1), \\ \varepsilon \dot{y}(t) = -y(t) + (1+\lambda)\, x(t-1). \end{cases}$$

By analyzing the characteristic equation

$$(\varepsilon v + 1 + (1+\lambda)\, e^{-v})(\varepsilon v + 1 - (1+\lambda)\, e^{-v}) = 0,$$

we can see that for given $\lambda > 0$ and $k \in \mathbb{N}$, there exists

$$\varepsilon_k(\lambda) = \frac{\sqrt{\lambda^2 + 2\lambda}}{k\pi - \arccos(1/(1+\lambda))}$$

such that $\varepsilon_k(\lambda)\, v + 1 + (-1)^k (1+\lambda)\, e^{-v} = 0$ has a pair of purely conjugate imaginary solutions $\pm i(k\pi - \arccos \frac{1}{1+\lambda})$. Furthermore, if the complex roots near $\varepsilon = \varepsilon_k(\lambda)$ are denoted by $\mu(\lambda, \varepsilon)$ and $\bar{\mu}(\lambda, \varepsilon)$, then $\partial \mu(\lambda, \varepsilon_k(\lambda))/\partial \varepsilon < 0$. Therefore, there is a Hopf bifurcation of a periodic solution in (1.1) and the period is around $2\pi/k\pi - \arccos(1/(1+\lambda)) \in (\frac{2}{k}, \frac{4}{2k-1})$. It can be shown also that for a given $k \in \mathbb{N}$ there is a unique periodic orbit bifurcating from the origin under the assumption that $a^2 - b \neq 0$ (this follows easily from the calculation of the direction of bifurcation, see, for example, Diekmann *et al.* [6]) and the period of this unique periodic orbit is approximately $2\pi/(k\pi - \arccos(1/(1+\lambda)) \to \frac{2}{k}$ as $\lambda \to 0^+$. The basic problem now is to determine the region near the origin in the parameter space (λ, ε) for the existence of this bifurcating periodic orbit and to determine the limiting properties of this orbit as $\varepsilon \to 0$. We now introduce some scalings in two cases.

Case A. k is even. Let $k = 2l$ for some $l \in \mathbb{N}$. We suppose that (1.1) has a periodic solution (x, y) with period $\frac{1}{l} + r_{2l}\varepsilon$ and let

$$\begin{cases} w_1(t) = x(-\varepsilon r_{2l} l t), \\ w_2(t) = y(-\varepsilon r_{2l} l t). \end{cases} \tag{2.1}$$

Since (x, y) has period $\frac{1}{l} + r_{2l}\varepsilon$, it follows from (1.1) that

$$\begin{cases} \dot{w}_1(t) = r_{2l} l w_1(t) - r_{2l} l f(w_2(t-1), \lambda), \\ \dot{w}_2(t) = r_{2l} l w_2(t) - r_{2l} l f(w_1(t-1), \lambda). \end{cases} \tag{2.2}$$

System (2.2) is now independent of ε. We look for periodic solutions of (2.2) in a neighborhood of the origin. This can be regarded as a two parameter bifurcation problem with (λ, r_{2l}) as parameters.

The next step is to determine the approximate value of the constant r_{2l} in the formula for the period $\frac{1}{l} + r_{2l}\varepsilon$. The appropriate approximate value of r_{2l} is obtained by considering the linear variational system around the zero solution of (2.2) for $\lambda = 0$,

$$\begin{cases} \dot{w}_1(t) = r_{2l} l w_1(t) - r_{2l} l w_2(t-1), \\ \dot{w}_2(t) = r_{2l} l w_2(t) - r_{2l} l w_1(t-1). \end{cases} \tag{2.3}$$

The corresponding characteristic equation of (2.3) is

$$(v - r_{2l} l)^2 - (r_{2l} l e^{-v})^2 = 0. \tag{2.4}$$

Note that $v=0$ is always a zero of (2.4). It is a simple zero if $r_{2l} \neq \frac{1}{l}$ and it is a double zero if $r_{2l} = \frac{1}{l}$. Also note that all other zeros of (2.4) except a unique positive real zero have negative real parts. Since bifurcation from a simple zero does not lead to periodic orbits, it is natural to take $r_{2l} = \frac{1}{l}$.

Case B. k is odd. Let $k = 2l-1$ for some $l \in \mathbb{N}$. Suppose that (1.1) has a periodic solution (x, y) with period $\frac{2}{2l-1} + 2r_{2l-1}\varepsilon$. Introducing

$$\begin{cases} w_1(t) = x[-\varepsilon r_{2l-1}(2l-1)\,t], \\ w_2(t) = y[-\varepsilon r_{2l-1}(2l-1)\,t + \varepsilon r_{2l-1}(2l-1) + 1], \end{cases} \tag{2.5}$$

we get

$$\begin{cases} \dot{w}_1(t) = r_{2l-1}(2l-1)\,w_1(t) - r_{2l-1}(2l-1)\,f(w_2(t-1), \lambda), \\ \dot{w}_2(t) = r_{2l-1}(2l-1)\,w_2(t) - r_{2l-1}(2l-1)\,f(w_1(t-1), \lambda). \end{cases} \tag{2.6}$$

Using similar analysis to that for Case A, we should choose $r_{2l-1} = \frac{1}{2l-1}$.

We remark that we need different scalings (2.2) and (2.5) to get the transformed systems (2.3) and (2.6), respectively. In fact, if we assume that (1.1) has a periodic solution (x, y) with period $\frac{2}{k}(1 + r_k\varepsilon)$, then it is natural to introduce

$$\begin{cases} w_1(t) = x(-r_k\varepsilon t), \\ w_2(t) = y(-r_k\varepsilon t). \end{cases}$$

Thus,

$$\begin{cases} \dot{w}_1(t) = r_k w_1(t) - r_k\, f(y(-r_k\varepsilon t - 1), \lambda), \\ \dot{w}_2(t) = r_k w_2(t) - r_k\, f(x(-r_k\varepsilon t - 1), \lambda). \end{cases}$$

Note that

$$\begin{aligned} -r_k\varepsilon t - 1 &= -r_k\varepsilon(t-1) - (r_k\varepsilon + 1) \\ &= -r_k\varepsilon(t-1) - \frac{k}{2}\left(\frac{2}{k}(1 + r_k\varepsilon)\right). \end{aligned}$$

By the periodicities of x and y, it is easy to see that the suggested scaling works only for k being even. For k being odd, there remaining an additional half period. This leads us to introduce the scaling (2.5) for that case.

From the above discussions, for any $l \in \mathbb{N}$, if we let $r_{2l} = \frac{1+h}{l}$ or $r_{2l-1} = \frac{1+h}{2l-1}$, where h is a small parameter, then (2.3) and (2.6) can be rewritten as

$$\begin{cases} \dot{w}_1(t) = (1+h)\, w_1(t) - (1+h)\, f(w_2(t-1), \lambda), \\ \dot{w}_2(t) = (1+h)\, w_2(t) - (1+h)\, f(w_1(t-1), \lambda). \end{cases} \tag{2.7}$$

If we let $w = (w_1, w_2)$, then (2.7) is equivalent to

$$\dot{w} = Lw_t + hLw_t - F_{\lambda, h}(w_t), \tag{2.8}$$

where

$$L\varphi = \varphi(0) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varphi(-1), \tag{2.9}$$

$$F_{\lambda, h}(\varphi) = (1+h) \begin{bmatrix} -\varphi_2(-1) + f(\varphi_2(-1), \lambda) \\ -\varphi_1(-1) + f(\varphi_1(-1), \lambda) \end{bmatrix} \tag{2.10}$$

for $\varphi = \binom{\varphi_1}{\varphi_2} \in C = C([-1, 0]; \mathbb{R}^2)$, the Banach space of all continuous mappings from $[-1, 0]$ to $\mathbb{R}^2$ equipped with the super norm, and $w_t(\theta) = w(t+\theta)$ for $-1 \leqslant \theta \leqslant 0$.

This suggests that we should consider (2.8) as a perturbation of the linear equation

$$\dot{v} = Lv_t. \tag{2.11}$$

Equation (2.11) generates a strongly continuous semigroup $T(t)$ on the phase space C. The infinitesimal generator A of $T(t)$ has domain $\mathfrak{D}(A) = \{\varphi \in C^1 : \dot{\varphi}(0) = L\varphi\}$ and $A\varphi = \dot{\varphi}$, where $C^1 = C^1([-1, 0]; \mathbb{R}^2)$ is the space of all continuously differentiable mappings from $[-1, 0]$ to $\mathbb{R}^2$. The point spectrum of A is given by solutions of the characteristic equation

$$(v-1)^2 - e^{-2v} = 0,$$

which has 0 as a double zero, a unique simple positive real zero and all other zeros have negative real parts. Thus, the small periodic orbits of (2.7) or (2.8) will lie on a two-dimensional center manifold which is tangent to the generalized eigenspace of A associated with the eigenvalue 0.

The generalized eigenspace Λ of A corresponding to the eigenvalue 0 has a basis

$$\Phi(\theta) = \begin{bmatrix} \frac{1}{3} + \theta & -1 \\ \frac{1}{3} + \theta & -1 \end{bmatrix}, \qquad -1 \leqslant \theta \leqslant 0. \tag{2.12}$$

The adjoint linear equation of (2.11) is

$$\dot{u}(t) = -u(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(t+1),$$

with a basis for the generalized eigenspace of the eigenvalue 0 being given by

$$\Psi(s) = \begin{bmatrix} 1 & 1 \\ s & s \end{bmatrix}, \qquad 0 \leqslant s \leqslant 1. \tag{2.13}$$

The associated bilinear form is

$$(\psi, \varphi) = \psi(0)\, \varphi(0) - \int_{-1}^{0} \psi(\xi+1) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varphi(\xi)\, \mathrm{d}\xi. \tag{2.14}$$

With the above choices of basis, we verify easily that $(\Psi, \Phi) = I$, the 2×2 identity. As a consequence, the space C can be decomposed as

$$C = \Lambda \oplus Q,$$

where

$$\begin{aligned} \Lambda &= \{\varphi = \Phi x : x \in \mathbb{R}^2\}, \\ Q &= \{\varphi \in C : (\Psi, \varphi) = 0\}. \end{aligned} \tag{2.15}$$

Each of the closed linear subspaces is invariant under the semigroup $T(t)$. We also note that

$$A\Phi = \Phi B, \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

For details of the above discussion, we refer to Hale [8].

Under the decomposition $w_t = \Phi x(t) + y(t)$, we can decompose (2.8) as

$$\begin{cases} \dot{x} = Bx + \Psi(0)\,[hL(\Phi x + y) - F_{\lambda, h}(\Phi x + y)], \\ \dot{y} = A_{Q^1} y + (I - \Pi)\, X_0\,[hL(\Phi x + y) - F_{\lambda, h}(\Phi x + y)], \end{cases} \tag{2.16}$$

with $x \in \mathbb{R}^2$ and $y \in Q^1 = Q \cap C^1$. Here and throughout this paper, we refer to Faria and Magalhães [7] for explanations of several notations involved. To avoid confusion, we used Π for the projection.

PROPOSITION 1. *Any solution* (w_1, w_2) *of* (2.7) *on the center manifold is synchronized, i.e.,* $w_1 \equiv w_2$.

Proof. Let (w_1, w_2) be a solution of (2.7) on the center manifold. Then it is defined for all $t \in \mathbb{R}$. Write $w_t = \Phi x(t) + y(t)$, where $x(t) \in \mathbb{R}^2$ and $y(t) \in Q^1$ for $t \in \mathbb{R}$. Then (x, y) satisfies (2.16). From the symmetry of (2.7), we know that (w_2, w_1) is also a solution of (2.7) on R. It follows from (2.12)–(2.15) that $((\Phi x)_2, (\Phi x)_1) = \Phi x$ for $x \in \mathbb{R}^2$ and $(Y_2, Y_1) \in Q^1$ if and only if $(Y_1, Y_2) \in Q^1$, here we write $\Phi x = ((\Phi x)_1, (\Phi x)_2)$. If we write $y(t) = (Y_1(t), Y_2(t))$. Then $((w_2)_t, (w_1)_t) = \Phi x(t) + (Y_2(t), Y_1(t))$ for all $t \in \mathbb{R}$. By the uniqueness of solutions of (2.16) we get $Y_1(t) = Y_2(t)$ for all $t \in \mathbb{R}$. Thus, $(w_1)_t = (\Phi x(t))_1 + Y_1(t) = (\Phi x(t))_2 + Y_2(t) = (w_2)_t$ for all $t \in \mathbb{R}$. Hence, $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$, i.e., $w_1 \equiv w_2$. This completes the proof.

Remark 2. Proposition 1, combined with (2.1) and (2.5), implies that if k is even then the periodic solution of (1.1) with period $\frac{2}{k} + r_k \varepsilon$ is synchronized and if k is odd then the periodic solution of (1.1) with period $\frac{2}{k} + 2r_k \varepsilon$ is phase-locked. Also, Proposition 1 implies that to consider the small periodic solutions of (1.1) we only need to consider synchronized solutions of (2.7), i.e., consider

$$\dot{u}(t) = (1+h)\, u(t) - (1+h)\, f(u(t-1), \lambda).$$

We can start from this to calculate the normal form on the center manifold. But this would not simplify our calculation and hence we continue our discussion with system (2.7).

Using the normal form theory for retarded delay differential equations developed by Faria and Magalhães [7], we obtain the normal form of (2.7) or (2.8) on the center manifold, which is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (\frac{4}{3}\lambda + 2h)\, x_1 + 2\lambda\, x_2 - \frac{8}{3}\, a x_1 x_2 - 2a x_2^2 + (2b + \frac{46}{9}\, a^2)\, x_2^3 - \frac{5}{3}\, a\lambda x_2^2 \\ -\frac{4}{3}\, a h x_2^2 + (4b + \frac{428}{45}\, a^2)\, x_1 x_2^2 - \frac{28}{27}\, a\lambda x_1 x_2 - \frac{28}{9}\, a h x_1 x_2 \\ -x_1 \end{pmatrix} \tag{2.17}$$

up through terms of $O((\lambda + h)^2\, |x| + (\lambda + h)\, |x|^3 + |x|^4)$. The details for the calculation of (2.17) are given in the next section. When $a = 0$, system (2.17) reduces to

$$\begin{cases} \dot{x}_1 = (\frac{4}{3}\lambda + 2h)\, x_1 + 2\lambda x_2 + 2b x_2^3 + 4b x_1 x_2^2, \\ \dot{x}_2 = -x_1, \end{cases}$$

which is the same as (1.16) in [9] with $\beta = -b$ and can be transformed into (2.13) of [2] by rescaling x_1 and x_2 if $b < 0$. Thus, we can use Remark 2 and the same arguments in [2, 9] to obtain the existence of periodic orbits and the bifurcation diagrams. When $a \neq 0$, the dynamics of (2.17) is determined by terms up to second order (see Carr [1] and Kopell and Howard [11]). Hence, we only need to consider

$$\begin{cases} \dot{x}_1 = (\frac{4}{3}\lambda + 2h)\, x_1 + 2\lambda x_2 - \frac{8a}{3} x_1 x_2 - 2a x_2^2, \\ \dot{x}_2 = -x_1. \end{cases} \tag{2.18}$$

We can study system (2.18) in a similar fashion as that of Hale and Huang [9]. In fact, if we introduce the scalings

$$\begin{cases} \lambda = -\alpha\mu^2, \\ h = \mu\delta, \\ u_1(t) = -\operatorname{sgn}(a)\, \dfrac{\sqrt{|a|}}{2\mu^2}\, x_2\left(-\dfrac{t}{\sqrt{2}\mu}\right), \\ u_2(t) = -\operatorname{sgn}(a)\, \dfrac{\sqrt{|a|}}{2\sqrt{2}\,\mu^3}\, x_1\left(-\dfrac{t}{\sqrt{2}\,\mu}\right) \end{cases}$$

in (2.18), then we obtain the equivalent system

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = \alpha u_1 + \gamma u_2 - 2\sqrt{|a|}\, u_1^2 - \dfrac{8\sqrt{2\,|a|}}{3}\, \mu u_1 u_2, \end{cases} \tag{2.19}$$

where $\gamma = 2\sqrt{2}\alpha\mu/3 - \sqrt{2}\delta$ and sgn is the sign function. For $\mu = \delta = 0$, we obtain the conservative system

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = \alpha u_1 - 2\sqrt{|a|}\, u_1^2. \end{cases}$$

When $\alpha > 0$, system (2.19) is the same form as system (7.4) in Kopell and Howard [11] and when $\alpha < 0$ we can transform it into the case where $\alpha > 0$ by using the transformations mentioned in Carr [1]. Therefore, we can use the technique in Kopell and Howard [11] to find the periodic solutions. Furthermore, using similar arguments as those in Hale and Huang [9] with the help of the proofs of Lemmas 7.1 and 7.2 in Kopell and Howard [11] (they are so similar and hence are omitted here), we can get the bifurcation diagrams for the case where $a \neq 0$.

In summary, we have obtained the following results.

THEOREM 3. *Suppose that* $f(x, \lambda)$ *satisfies* (1.2) *with* $a^2 - b \neq 0$. *Then, for any* $k \in \mathbb{N}$, *there is a neighborhood* U_k *of* $(0, 0)$ *in the* (λ, ε) *plane and a sectorial region* S_k *in* U_k *such that, if* $(\lambda, \varepsilon) \in U_k$, *then there is a periodic solution* $(\tilde{x}^{(k)}_{\lambda,\varepsilon}, \tilde{y}^{(k)}_{\lambda,\varepsilon})$ *of* (1.1) *with period* $\frac{2}{k}(1+\varepsilon) + O(|\varepsilon|\,(|\lambda| + |\varepsilon|))$ *as* $(\lambda, \varepsilon) \to (0, 0)$ *if and only if* $(\lambda, \varepsilon) \in S_k$. *Furthermore, this orbit is unique and the solution is synchronized if k is even and phase-locked if k is odd.*

When $a = 0$ *and* $b < 0$, *for small and fixed* $\lambda = \lambda_0 > 0$, *the set* $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ *is an interval* $(0, \varepsilon_k(\lambda_0))$. *At the point* $(\lambda_0, \varepsilon_k(\lambda_0))$, *there is a Hopf bifurcation and the periodic solution approaches a square wave as* $\varepsilon \to 0$; *that is, the periodic solution* $(\tilde{x}^{(k)}_{\lambda_0,\varepsilon}, \tilde{y}^{(k)}_{\lambda_0,\varepsilon})$ *has the property that* $\tilde{x}^{(k)}_{\lambda_0,\varepsilon}(t) \to c_{1\lambda_0}$ (*respectively,* $c_{2\lambda_0}$) *as* $\varepsilon \to 0$ *uniformly on compact subsets of* $(0, \frac{1}{k})$ (*respectively,* $(\frac{1}{k}, \frac{2}{k})$) (*possibly after a translation, same for the other cases*) (*see Fig.* 1). *When* $a = 0$ *and* $b > 0$, *for small and fixed* $\lambda = \lambda_0 > 0$, *the set* $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ *is an interval* $(\varepsilon_k(\lambda_0), \beta_k(\lambda_0))$. *At the point* $(\lambda_0, \varepsilon_k(\lambda_0))$, *there is a Hopf bifurcation. For small and fixed* $\lambda = \lambda_0 < 0$, *the set* $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ *is an interval* $(0, \alpha_k(\lambda_0))$. *As* $\varepsilon \to 0$, *the unique periodic solutions become pulse-like in the following sense: the periodic solution* $(\tilde{x}^{(k)}_{\lambda_0,\varepsilon}, \tilde{y}^{(k)}_{\lambda_0,\varepsilon})$ *has the property that* $\tilde{x}^{(k)}_{\lambda_0,\varepsilon}(t) \to 0$ *as* $\varepsilon \to 0$ *uniformly on compact subsets of* $(0, \frac{1}{k}) \cup (\frac{1}{k}, \frac{2}{k})$. *The magnitude of the pulse exceeds* $\max\{|c_{1\lambda_0}|, |c_{2\lambda_0}|\}$ (*see Fig.* 2). *When* $a \neq 0$, *for small and fixed* $\lambda = \lambda_0 > 0$, *the set* $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ *is an interval* $(\varepsilon_k(\lambda_0), \beta_k(\lambda_0))$. *At the point* $(\lambda_0, \varepsilon_k(\lambda_0))$, *there is a Hopf bifurcation. For small and fixed* $\lambda = \lambda_0 < 0$, *the set* $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ *is an interval* $(0, \alpha_k(\lambda_0))$. *As* $\varepsilon \to 0$, *the unique periodic solutions become pulse-like with the magnitude of the pulse exceeds* $|c_{0\lambda_0}|$ (*see Fig.* 3).

It is interesting to mention that the two nonzero fixed points $c_{1\lambda}$ and $c_{2\lambda}$ play the same important role played by the two period doubling points $d_{1\lambda}$

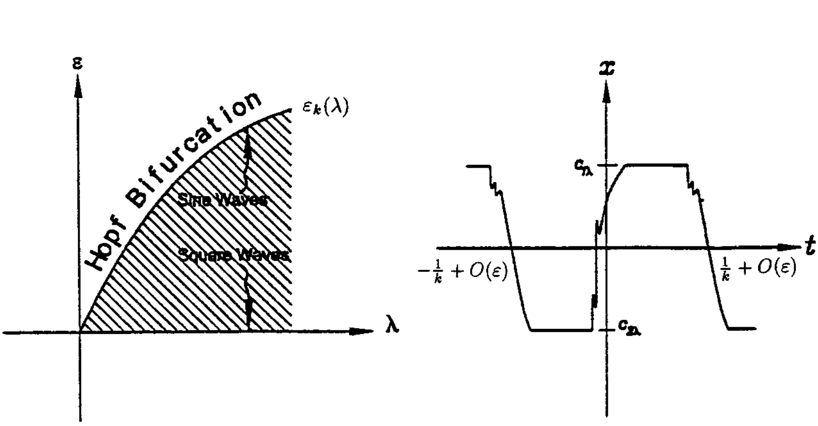

FIG. 1. $a = 0$ and $b < 0$.

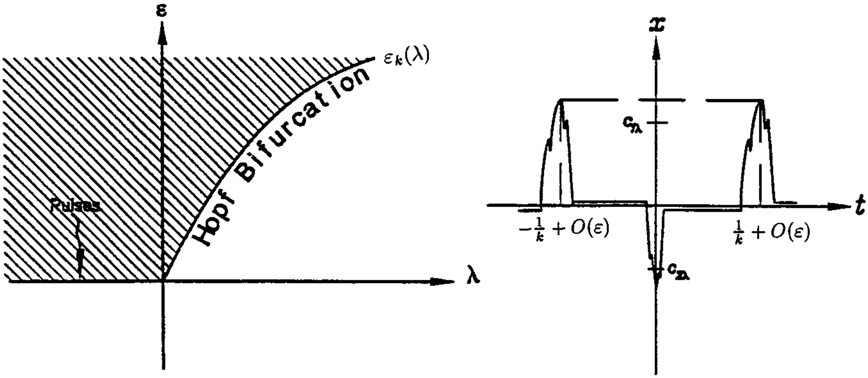

FIG. 2. $a = 0$ and $b > 0$.

and $d_{2\lambda}$ of [2, 9]. Also note that the existence results in Theorem 3 were previously obtained in [3, 5] for some special f.

Remark 4. When considering (1.3) with f satisfying (1.4), Hale and Huang [9] showed that the vector field on the center manifold is odd. Thus the second order terms in the vector field vanish and hence the phenomenon of all periodic solutions approaching pulses cannot happen. More precisely, neither a nor b alone but their combination $\beta = a^2 + b$ determines the bifurcation diagram. But in our model, when k is even, the periodic solutions are synchronized and hence we get the bifurcation diagrams for (1.3) with f satisfying (1.2). However, when $a \neq 0$, a itself

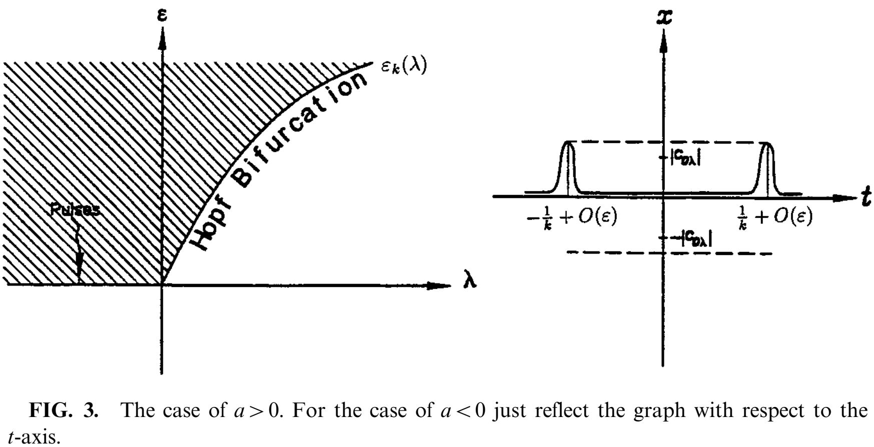

FIG. 3. The case of $a > 0$. For the case of $a < 0$ just reflect the graph with respect to the t-axis.

alone can determine the bifurcation diagram, which is different from that in [2, 9]. This also indicates some differences between excitatory and inhibitory networks of neurons.

3. CALCULATIONS OF THE NORMAL FORM ON CENTER MANIFOLD

In this section, we employ the algorithm and notations of Faria and Magalhães [7] to derive the normal form (2.17) of system (2.8) on the center manifold.

For the convenience of presentation, we let e_1 and e_2 be the standard basis of $\mathbb{R}^2$, i.e., $e_1=(1,0)^T$ and $e_2=(0,1)^T$. We also introduce $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $q=(q_1,q_2,q_3,q_4)\in(\mathbb{N}_0)^4$, let $|q|=\sum_{i=1}^4 q_i$ and $(x,\lambda,h)^q=x_1^{q_1}x_2^{q_2}\lambda^{q_3}h^{q_4}$.

Let $L\varphi=\varphi(0)-\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\varphi(-1)$. We consider the following delay differential equation

$$\dot{w}=Lw_t+hLw_t-F_{\lambda,h}(w_t), \tag{3.1}$$

where $w_t\in C=C([-1,0];\mathbb{R}^2)$ and

$$F_{\lambda,h}(\varphi)=(1+h)\begin{bmatrix}-\varphi_2(-1)+f(\varphi_2(-1),\lambda)\\-\varphi_1(-1)+f(\varphi_1(-1),\lambda)\end{bmatrix},\qquad \varphi=\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}$$

with f satisfying

$$f(x,\lambda)=(1+\lambda)\,x+ax^2+bx^3+o(x^3)\qquad\text{as } x\to 0.$$

We regard (3.1) as a perturbation of the linear equation

$$\dot{v}=Lv_t. \tag{3.2}$$

Equation (3.2) generates a strongly continuous semigroup $T(t)$ on the phase space C. The infinitesimal generator A of T has domain $\mathfrak{D}(A)=\{\varphi\in C^1:\dot{\varphi}(0)=L\varphi\}$ and $A\varphi=\dot{\varphi}$. The point spectrum of A is given by the solution of the characteristic equation

$$(v-1)^2-e^{-2v}=0,$$

which has zero as a double root and no other roots have zero real parts. So we have a two-dimensional center manifold.

We know that the generalized eigenspace Λ of A associated with the eigenvalue 0 has a basis

$$\Phi(\theta)=\begin{bmatrix} \frac{1}{3}+\theta & -1 \\ \frac{1}{3}+\theta & -1 \end{bmatrix}, \qquad -1\leqslant\theta\leqslant 0. \tag{3.3}$$

Let

$$\Psi(s)=\begin{bmatrix} 1 & 1 \\ s & s \end{bmatrix}, \qquad 0\leqslant s\leqslant 1. \tag{3.4}$$

Then under the bilinear form

$$(\psi, \varphi)=\psi(0)\,\varphi(0)-\int_{-1}^{0}\psi(\xi+1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\varphi(\xi)\,\mathrm{d}\xi,$$

it is easy to verify that $(\Psi, \Phi)=I$, the 2×2 identity. Thus, C can be decomposed as

$$C=\Lambda\oplus Q,$$

where

$$\Lambda=\{\varphi=\Phi x : x\in\mathbb{R}^2\},$$

$$Q=\{\varphi\in C : (\Psi, \varphi)=0\}.$$

We also note that

$$A\Phi=\Phi B, \qquad B=\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Under the decomposition $w_t=\Phi x(t)+y(t)$, we can decompose (3.1) as

$$\begin{cases} \dot{x}=Bx+\Psi(0)[hL(\Phi x+y)-F_{\lambda,h}(\Phi x+y)], \\ \dot{y}=A_{Q^1}y+(I-\Pi)\,X_0[hL(\Phi x+y)-F_{\lambda,h}(\Phi x+y)], \end{cases}$$

with $x\in\mathbb{R}^2$ and $y\in Q^1$. We will write the Taylor expression

$$\Psi(0)[hL(\Phi x+y)-F_{\lambda,h}(\Phi x+y)]=\sum_{j\geqslant 2}\frac{1}{j!}\;f_j^1(x, y, \lambda, h), \tag{3.5}$$

where $f_j^1(x, y, \lambda, h)$ are homogeneous polynomials of degree j in (x, y, λ, h) with coefficients in $\mathbb{R}^2$. Then the normal form of (3.1) on the center manifold of the origin at $(\lambda, h) = (0, 0)$ is given by

$$\dot{x} = Bx + \frac{1}{2!} g_2^1(x, 0, \lambda, h) + \frac{1}{3!} g_3^1(x, 0, \lambda, h) + h.o.t., \tag{3.6}$$

where g_2^1 and g_3^1 will be calculated in the following part of this appendix and *h.o.t.* represents the higher order terms.

Note

$$\begin{aligned} hL\varphi - F_{\lambda, h}(\varphi) = {} & \left\{ h \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \end{bmatrix} - (\lambda + h)\, h \begin{bmatrix} \varphi_2(-1) \\ \varphi_1(-1) \end{bmatrix} - a \begin{bmatrix} (\varphi_2(-1))^2 \\ (\varphi_1(-1))^2 \end{bmatrix} \right\} \\ & + \left\{ -b \begin{bmatrix} (\varphi_2(-1))^3 \\ (\varphi_1(-1))^3 \end{bmatrix} - ah \begin{bmatrix} (\varphi_2(-1))^2 \\ (\varphi_1(-1))^2 \end{bmatrix} - \lambda h \begin{bmatrix} \varphi_2(-1) \\ \varphi_1(-1) \end{bmatrix} \right\} \\ & + h.o.t., \end{aligned}$$

which, combined with (3.3), (3.4), and (3.5), implies

$$f_2^1(x, 0, \lambda, h) = \begin{pmatrix} 4\,[hx_1 + \frac{2}{3}\lambda x_1 + \lambda x_2 - a(\frac{4}{9}x_1^2 + \frac{4}{3}x_1 x_2 + x_2^2)] \\ 0 \end{pmatrix}. \tag{3.7}$$

These are the second terms in (x, λ, h) of (3.5). Following Faria and Magalhães [7], we have the second terms in (x, λ, h) of the normal form (3.6) on the center manifold as

$$g_2^1(x, 0, \lambda, h) = (I - P_{I,2}^1)\, f_2^1(x, 0, \lambda, h). \tag{3.8}$$

Recall that for $j \geqslant 2$, M_j^1 is the operator defined on $V_j^4(\mathbb{R}^2)$ by

$$M_j^1(p)(x, \lambda, h) = D_x p(x, \lambda, h)\, Bx - Bp(x, \lambda, h),$$

where $V_j^4(\mathbb{R}^2)$ is the linear space of the homogeneous polynomials of degree j in the 4 real variables x_1, x_2, λ and h. Consider the decompositions

$$V_j^4(\mathbb{R}^2) = \mathrm{Im}(M_j^1) \oplus \mathrm{Im}(M_j^1)^c,$$

$$V_j^4(\mathbb{R}^2) = \mathrm{Ker}(M_j^1) \oplus \mathrm{Ker}(M_j^1)^c.$$

The projections associated with the preceding decompositions of $V_j^4(\mathbb{R}^2)$ over $Im(M_j^1)$ and $V_j^4(\mathbb{R}^2)$ over $Ker(M_j^1)^c$ are denoted by $P_{I,j}^1$ and $P_{K,j}^1$, respectively. Denote by $(M_j^1)^{-1}$ the right inverse of M_j^1 with range defined by the spaces complementary to the kernel of M_j^1 with $(M_j^1)^{-1} P_{I,j}^1 M_j^1 = P_{K,j}^1$. Particularly, we have

$$M_2^1(p) = \begin{pmatrix} -x_1 \dfrac{\partial p_1}{\partial x_2} \\ -x_1 \dfrac{\partial p_2}{\partial x_2} + p_1 \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

After computing the images of the basis $\{(x, \lambda, h)^q e_k : k = 1, 2, |q| = 2\}$ of $V_2^4(\mathbb{R}^2)$ under M_2^1, we choose basis for $\mathrm{Im}(M_2^1)$ and $\mathrm{Im}(M_2^1)^c$ as

$$\mathrm{Im}(M_2^1) = \mathrm{span}\left\{ \begin{matrix} x_1^2 e_1, \lambda x_1 e_1 - \lambda x_2 e_2, h x_1 e_1 - h x_2 e_2, 2x_1 x_2 e_1 - x_2^2 e_2, \\ \lambda^2 e_2, h^2 e_2, \lambda h e_2, \lambda x_1 e_2, h x_1 e_2, x_1^2 e_2, x_1 x_2 e_2 \end{matrix} \right\},$$

$$\mathrm{Im}(M_2^1)^c = \mathrm{span}\{\lambda^2 e_1, h^2 e_1, \lambda h e_1, \lambda x_1 e_1, h x_1 e_1, \lambda x_2 e_1, h x_2 e_1, x_1 x_2 e_1, x_2^2 e_1\}.$$

Then

$$\begin{aligned} &(I - P_{I,2}^1)\left(\sum_{|q|=2} (b_q e_1 + c_q e_2)(x, \lambda, h)^q \right) \\ &\quad = (b_{0020} \lambda^2 + b_{0002} h^2 + b_{0011} \lambda h + (b_{1010} + c_{0110})\, \lambda x_1 + b_{0110} \lambda x_2 \\ &\qquad + (b_{1001} + c_{0101})\, h x_1 + \; + b_{0101} h x_2 + (b_{1100} + 2c_{0200})\, x_1 x_2 + b_{0200} x_2^2)\, e_1 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &P_{I,2}^1\left(\sum_{|q|=2} (b_q e_1 + c_q e_2)(x, \lambda, h)^q \right) \\ &\quad = (b_{2000} x_1^2 - c_{0110} \lambda x_1 - c_{0101} h x_1 - 2c_{0200} x_1 x_2)\, e_1 \\ &\qquad + (c_{0020} \lambda^2 + c_{0002} h^2 + c_{0011} \lambda h + c_{1010} \lambda x_1 + c_{1001} h x_1 \\ &\qquad + c_{2000} x_1^2 + c_{1100} x_1 x_2 + c_{0110} \lambda x_2 + c_{0101} h x_2 + c_{0200} x_2^2)\, e_2. \end{aligned}$$

It follows from (3.7)–(3.9) that

$$\frac{1}{2} g_2^1(x, 0, \lambda, h) = \begin{pmatrix} \dfrac{4}{3} \lambda x_1 + 2h x_1 + 2\lambda x_2 - \dfrac{8a}{3} x_1 x_2 - 2a x_2^2 \\ 0 \end{pmatrix}. \tag{3.10}$$

On the other hand,

$$Ker(M_2^1) = \text{span}\left\{\begin{matrix} \lambda x_1 e_1 + \lambda x_2 e_2,\ h x_1 e_1 + h x_2 e_2,\ x_1^2 e_1 + x_1 x_2 e_2, \\ \lambda^2 e_2,\ h^2 e_2,\ \lambda h e_2,\ \lambda x_1 e_2,\ h x_1 e_2,\ x_1^2 e_1 \end{matrix}\right\}.$$

We choose $\{\lambda^2 e_1, h^2 e_1, \lambda h e_1, \lambda x_2 e_1, h x_2 e_1, x_2^2 e_1, x_1 x_2 e_1, \lambda x_1 e_1, h x_1 e_1, x_1^2 e_1, x_2^2 e_2\}$ as a basis for $\mathrm{Ker}(M_2^1)^c$. Then

$$\begin{aligned} P_{K,2}^1 &\left(\sum_{|q|=2} (b_q e_1 + c_q e_2)(x, \lambda, h)^q \right) \\ &= (b_{0020} \lambda^2 + b_{0002} h^2 + b_{0011} \lambda h + b_{0110} \lambda x_2 + b_{0101} h x_2 \\ &\quad + b_{0200} x_2^2 + b_{1100} x_1 x_2 + (b_{1010} - c_{0110})\, \lambda x_1 \\ &\quad + (b_{1001} - c_{0101})\, h x_1 + (b_{2000} - c_{1100})\, x_1^2)\, e_1 + c_{0200} x_2^2 e_2. \end{aligned} \tag{3.11}$$

Thus

$$\begin{aligned} U_2(x, \lambda, h) &= M_2^{-1} P_{I,2} f_2(x, 0, \lambda, h) \\ &= \begin{pmatrix} (M_2^1)^{-1}\, P_{I,2}^1 f_2^1(x, 0, \lambda, h) \\ H(x, \lambda, h) \end{pmatrix}, \end{aligned}$$

where $H = \sum_{|q|=2} (H_q^{(1)} e_1 + H_q^{(2)} e_2)(x, \lambda, h)^q \in V_2^4(Q^1)$ is the unique solution of

$$(M_2^2 H)(x_1, x_2, \lambda, h) = f_2^2(x_1, x_2, \lambda, h).$$

Note that $H \in V_2^4(Q^1)$ implies $H_q = H_q^{(1)} e_1 + H_q^{(2)} e_2 \in Q^1$. It is easy to see that H satisfies

$$\begin{aligned} (M_2^2 H)(x_1, x_2, \lambda, h) = 2X_0[\, g(x_1, x_2, \lambda, h)(e_1 + e_2)] \\ - 4\Phi[\, g(x_1, x_2, \lambda, h)\, e_1], \end{aligned} \tag{3.12}$$

where

$$g(x_1, x_2, \lambda, h) = h x_1 + \tfrac{2}{3} \lambda x_1 + \lambda x_2 - a\, (\tfrac{4}{9} x_1^2 + \tfrac{4}{3} x_1 x_2 + x_2^2).$$

Noting

$$P_{I,2}^1 f_2^1(x_1, x_2, 0, \lambda, h) = \begin{pmatrix} -\frac{16}{9} a x_1^2 \\ 0 \end{pmatrix} = P_{I,2}^1 M_2^1 \begin{pmatrix} \frac{16}{9} a x_1 x_2 \\ \frac{8}{9} a x_2^2 \end{pmatrix},$$

it follows from (3.11) that

$$(M_2^1)^{-1}P_{I,2}^1 f_2^1(x_1, x_2, 0, \lambda, h) = (M_2^1)^{-1}P_{I,2}^1 M_2^1 \begin{pmatrix} \frac{16}{9} ax_1x_2 \\ \frac{8}{9} ax_2^2 \end{pmatrix}$$
$$= P_{K,2}^1 \begin{pmatrix} \frac{16}{9} ax_1x_2 \\ \frac{8}{9} ax_2^2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{16}{9} ax_1x_2 \\ \frac{8}{9} ax_2^2 \end{pmatrix}.$$

Before computing $g_3^1(x_1, x_2, 0, \lambda, h)$, we first consider the action of the mapping M_3^1 on $V_3^4(\mathbb{R}^2)$. Take $\{(x, \lambda, h)^q e_k : k = 1, 2, |q| = 3\}$ as a basis for $V_3^4(\mathbb{R}^2)$. After computing the action of M_3^1 on this basis, we can take

$$\left\{\begin{matrix} x_1^2x_2e_1, \lambda x_1^2e_1, hx_1^2e_1, x_1^3e_1, \lambda^3e_2, h^3e_2, x_1^3e_2, \lambda^2he_2, \lambda^2x_1e_2, \lambda h^2e_2, \\ h^2x_1e_2, x_1x_2^2e_2, hx_1^2e_2, \lambda x_1^2e_2, \lambda hx_1e_2, x_1^2x_2e_2, \lambda x_1x_2e_2, hx_1x_2e_2, \\ \lambda^2x_1e_1 - \lambda^2x_2e_2, h^2x_1e_1 - h^2x_2e_2, 3x_1x_2^2e_1 - x_2^3e_2, 2\lambda x_1x_2e_1 - \lambda x_2^2e_2, \\ 2hx_1x_2e_1 - hx_2^2e_2, \lambda hx_1e_1 - \lambda hx_2e_2 \end{matrix}\right\}$$

and

$$\left\{\begin{matrix} \lambda^2x_1e_1, h^2x_1e_1, x_1x_2^2e_1, \lambda x_1x_2e_1, hx_1x_2e_1, \lambda hx_1e_1, \lambda^3e_1, h^3e_1, \\ x_2^3e_1, \lambda^2he_1, \lambda^2x_2e_1, \lambda h^2e_1, h^2x_2e_1, \lambda x_2^2e_1, hx_2^2e_1, \lambda hx_2e_1 \end{matrix}\right\}$$

as basis for $\mathrm{Im}(M_3^1)$ and $\mathrm{Im}(M_3^1)^c$, respectively. Then

$$\begin{aligned}
&(I - P_{I,3}^1)\left(\sum_{|q|=3} (b_qe_1 + c_qe_2)(x, \lambda, h)^q\right) \\
&= (b_{0030}\lambda^3 + b_{0003}h^3 + b_{0300}x_2^3 + b_{0021}\lambda^2h + b_{0120}\lambda^2x_2 \\
&\quad + b_{0012}\lambda h^2 + b_{0102}h^2x_2 + b_{0210}\lambda x_2^2 + b_{0201}hx_2^2 + b_{0111}\lambda hx_2 \\
&\quad + (b_{1020} + c_{0120})\,\lambda^2x_1 + (b_{1002} + c_{0102})\,h^2x_1 + (b_{1200} + 3c_{0300})\,x_1x_2^2 \\
&\quad + (b_{1110} + 2c_{0210})\,\lambda x_1x_2 + (b_{1101} + 2c_{0201})\,hx_1x_2 + (b_{1011} + c_{0111})\,\lambda hx_1)\,e_1.
\end{aligned}$$

Note that

$$g_3^1(x_1, x_2, 0, \lambda, h) = (I - P_{I,3}^1)\,\bar{f}_3^1(x_1, x_2, 0, \lambda, h),$$

where

$$\bar{f}_3 = f_3 + \tfrac{3}{2}\,[(D_{x,y}f_2)\,U_2 - (D_{x,y}U_2)\,g_2].$$

Thus, $\bar{f}_3^1 = f_3^1 + \frac{3}{2}[(D_{x,y} f_2^1) U_2 - (D_{x,y} U_2^1) g_2]$. After a routine computation and omitting terms of $(x, \lambda, h)^q$ with $q_3 + q_4 \geqslant 2$, we get

$$g_3^1(x_1, x_2, 0, \lambda, h) = \begin{pmatrix} d_{0300} x_2^3 + d_{0210} \lambda x_2^2 + d_{0201} h x_2^2 + d_{1200} x_1 x_2^2 + d_{1110} \lambda x_1 x_2 + d_{1101} h x_1 x_2 \\ 0 \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned}
d_{0300} &= 12b + \tfrac{64}{3} a^2 - 6a(H_{0200}^{(1)}(-1) + H_{0200}^{(2)}(-1)), \\
d_{0210} &= -\tfrac{16}{3} a - 3(H_{0200}^{(1)}(-1) + H_{0200}^{(2)}(-1)) - 6a(H_{0110}^{(1)}(-1) + H_{0110}^{(2)}(-1)), \\
d_{0201} &= -12a + 3(H_{0200}^{(1)}(0) + H_{0200}^{(2)}(0)) - 3(H_{0200}^{(1)}(-1) + H_{0200}^{(2)}(-1)) \\
&\quad - 6a(H_{0101}^{(1)}(-1) + H_{0101}^{(2)}(-1)), \\
d_{1200} &= 24b + \tfrac{320}{9} a^2 - 4a(H_{0200}^{(1)}(-1) + H_{0200}^{(2)}(-1)) \\
&\quad - 6a(H_{1100}^{(1)}(-1) + H_{1100}^{(2)}(-1)), \\
d_{1110} &= -3(H_{1100}^{(1)}(-1) + H_{1100}^{(2)}(-1)) - 4a(H_{0110}^{(1)}(-1) + H_{0110}^{(2)}(-1)) \\
&\quad - 6a(H_{1010}^{(1)}(-1) + H_{1010}^{(2)}(-1)), \\
d_{1101} &= -16a + 3(H_{1100}^{(1)}(0) + H_{1100}^{(2)}(0)) - 3(H_{1100}^{(1)}(-1) + H_{1100}^{(2)}(-1)) \\
&\quad - 4a(H_{0101}^{(1)}(-1) + H_{0101}^{(2)}(-1)) - 6a(H_{1001}^{(1)}(-1) + H_{1001}^{(2)}(-1)).
\end{aligned}$$

To get the explicit expression for g_3^1, we need to solve (3.12). By using the fact that $H_q \in Q^1$ for $|q| = 2$, we get

$$\begin{aligned}
H_{0101}^{(1)} &\equiv H_{0101}^{(2)} \equiv 0, \\
H_{0110}^{(1)}(\theta) &= H_{0110}^{(2)}(\theta) = \tfrac{4}{3}\theta + 2\theta^2 + \tfrac{1}{9}, \\
H_{0200}^{(1)}(\theta) &= H_{0200}^{(2)}(\theta) = -a(\tfrac{4}{3}\theta + 2\theta^2 + \tfrac{1}{9}), \\
H_{1001}^{(1)}(\theta) &= H_{1001}^{(2)}(\theta) = \tfrac{4}{3}\theta + 2\theta^2 + \tfrac{1}{9}, \\
H_{1010}^{(1)}(\theta) &= H_{1010}^{(2)}(\theta) = \tfrac{7}{9}\theta + \tfrac{2}{3}\theta^2 - \tfrac{2}{3}\theta^3 + \tfrac{11}{135}, \\
H_{1100}^{(1)}(\theta) &= H_{1100}^{(2)}(\theta) = -\tfrac{14}{9} a\theta - \tfrac{4}{3} a\theta^2 + \tfrac{4}{3} a\theta^3 - \tfrac{22}{135} a,
\end{aligned} \tag{3.14}$$

for $\theta \in [-1, 0]$. Substituting (3.14) into (3.13) gives

$$g_3^1(x_1, x_2, 0, \lambda, h) = \begin{pmatrix} (12b + \frac{92}{3} a^2) x_2^3 - 10a\lambda x_2^2 - 8ahx_2^2 - \frac{56}{3} ahx_1 x_2 \\ + (24b + \frac{856}{15} a^2) x_1 x_2^2 - \frac{56}{9} a\lambda x_1 x_2 \\ 0 \end{pmatrix}, \tag{3.15}$$

Substituting (3.10) and (3.15) into (3.6), we know that the normal form on the center manifold is given by

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}(\frac{4}{3}\lambda+2h)\,x_1+2\lambda x_2-\frac{8}{3}\,ax_1x_2-2ax_2^2+(2b+\frac{46}{9}\,a^2)\,x_2^3-\frac{5}{3}\,a\lambda x_2^2\\ -\frac{4}{3}\,ahx_2^2+(4b+\frac{428}{45}\,a^2)\,x_1x_2^2-\frac{28}{27}\,a\lambda x_1x_2-\frac{28}{9}\,ahx_1x_2\\ -x_1\end{pmatrix}, \tag{3.16}$$

up through terms of $O((\lambda+h)^2\,|x|+(\lambda+h)\,|x|^3+|x|^4)$. This verifies (2.17).

ACKNOWLEDGMENT

The authors greatly appreciate the referee's suggestion which led to the correct limiting profile for the case where $a \neq 0$.

REFERENCES

1. J. Carr, "Applications of Center Manifold Theory," Springer-Verlag, New York, 1981.
2. S.-N. Chow, J. K. Hale and W. Huang, From sine waves to square waves in delay equations, *Proc. Roy. Soc. Edinburgh Sect. A* **120** (1992), 223–229.
3. Y. Chen and J. Wu, Existence and attraction of a phase-locked oscillation in a delayed network of two neurons, *Adv. Differential Equations*, in press.
4. Y. Chen and J. Wu, Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network, *Phys. D* **134** (1999), 185–199.
5. Y. Chen, J. Wu, and T. Krisztin, Connecting orbits from synchronous periodic solutions to phase-locked periodic solutions in a delay differential system, *J. Differential Equations* **163** (2000), 130–173.
6. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, "Delay Equations: Functional-, Complex-, and Nonlinear Analysis," Springer-Verlag, New York, 1995.
7. T. Faria and L. T. Magalhães, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcations, *J. Differential Equations* **122** (1995), 181–200.
8. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
9. J. K. Hale and W. Huang, Period doubling in singularly perturbed delay equations, *J. Differential Equations* **114** (1994), 1–23.
10. J. J. Hopfield, Neurons with graded response have computational properties like two-stage neurons, *Proc. Natl. Acad. Sci. U.S.A.* **81** (1984), 3088–3092.
11. N. Kopell and L. N. Howard, Bifurcations and trajectories joining critical points, *Adv. Math.* **18** (1975), 306–358.
12. C. M. Marcus and R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39** (1989), 347–356.
13. J. Wu, Symmetric functional differential equations and neural networks with memory, *Trans. Amer. Math. Soc.* **350**, No. 12 (1998), 4799–4838.