

# Connecting Orbits from Synchronous Periodic Solutions to Phase-Locked Periodic Solutions in a Delay Differential System

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We consider a system of delay differential equations modelling the excitatory interaction of two identical neurons. Assuming the delay is sufficiently large, we show that the closure of the forward extension  $W_5$  of a 5-dimensional leading unstable manifold of the trivial solution contains a phase-locked periodic orbit and a synchronized periodic orbit and we classify the dynamics of the semiflow restricted to  $\overline{W_5}$ . We also obtain the precise information about the Floquet multipliers of the synchronized periodic orbit, which enables us to establish the existence of heteroclinic orbits from the synchronized periodic orbit to the phase-locked orbit.

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## 1. INTRODUCTION

We consider the following system of delay differential equations

$$\begin{cases} \dot{u}(t) = -\mu u(t) + f(v(t-\tau)), \\ \dot{v}(t) = -\mu v(t) + f(u(t-\tau)), \end{cases}$$

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or equivalently (after rescaling the time variable)

$$\begin{cases} \dot{u}(t) = -\mu\tau u(t) + \tau f(v(t-1)), \\ \dot{v}(t) = -\mu\tau v(t) + \tau f(u(t-1)), \end{cases} \quad (1.1)$$

where  $\tau$  and  $\mu$  are given positive constants,  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ -map satisfying the following set of conditions:

$f(0) = 0$ ,  $f'(\xi) > 0$  for all  $\xi \in \mathbb{R}$  (Monotone Positive Feedback);

$f(\xi) = \mu\xi$  has exactly three zeros  $\xi^- < 0 < \xi^+$  and  $\max\{f'(\xi^-), f'(\xi^+)\} < \mu < f'(0)$  (Dissipativeness and Instability of 0);

$f(\xi) = -f(-\xi)$  for all  $\xi \in \mathbb{R}$  (Symmetry);

The function  $(0, \infty) \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$  is monotonically decreasing (Concavity).

It should be mentioned that some results obtained in this paper do not require the symmetry and concavity conditions on  $f$ , and that the function

$$f(\xi) = \arctan(\gamma\xi), \quad \xi \in \mathbb{R}$$

or

$$f(\xi) = \frac{e^{\gamma\xi} - e^{-\gamma\xi}}{e^{\gamma\xi} + e^{-\gamma\xi}}, \quad \xi \in \mathbb{R}$$

satisfies the above set of conditions with  $\gamma > \mu$ .

System (1.1) describes the dynamics of a network (Hopfield [5, 6]) of two identical saturating amplifiers (neurons) with excitatory interaction, where the delay was incorporated (Marcus and Westervelt [12] and Wu [16]) to account for finite switching speed of amplifiers. In the application of the above model to associative information processing where a network is triggered by an appropriate external stimulus and relaxes towards the attractor that encodes previously stored memories (Herz [3]), it is important to describe completely the structure of the global attractor, and in particular, to describe the existence and stability of equilibria and periodic solutions and to describe their connecting orbits and basins of attraction.

Our ultimate goal is to describe the global attractor of system (1.1), and our focus here is on the existence of connecting orbits from a synchronous periodic solution to a phase-locked periodic solution, here a solution  $(u, v)$

of (1.1) is synchronous if  $u \equiv v$  in their domains of definition, and a phase-locked  $T$ -periodic solution of (1.1) is a solution satisfying  $u(t) = v(t - \frac{T}{2})$  for all  $t \in \mathbb{R}$ . The existence of such connecting orbits suggests a mechanism for desynchronization of the network. We refer to Terman, Kopell and Bose [15] and references therein for discussions about synchronization of related models in neurology.

Clearly, a synchronous solution of (1.1) is completely governed by the scalar delay differential equation for  $w (= u = v)$ :

$$\dot{w}(t) = -\mu\tau w(t) + \tau f(w(t-1)), \quad (1.2)$$

which was recently studied by Krisztin, Walther, and Wu [9] and by Krisztin and Walther [8]. Considering Eq. (1.2) in the phase space  $C = C([-1, 0]; \mathbb{R})$  and linearizing it at the stationary point 0, it is obtained that the zeros of the characteristic equation

$$\lambda + \mu\tau - \tau f'(0) e^{-\lambda} = 0 \quad (1.3)$$

form the spectrum of the generator  $A_s$  of the solution semiflow of the linear equation

$$\dot{Z}(t) = -\mu\tau Z(t) + \tau f'(0) Z(t-1). \quad (1.4)$$

All zeros of (1.3) are simple eigenvalues of  $A_s$ . There is one real eigenvalue, the other eigenvalues form complex conjugate pairs with real parts less than the value of the real eigenvalue. Defining

$$\tau_s = \frac{2\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}}, \quad \tau'_s = \frac{4\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}},$$

one finds that for  $\tau > \tau_s$  the realified generalized eigenspace of  $A_s$  associated with the spectral set of the eigenvalues with positive real part is at least 3-dimensional, while for  $\tau_s < \tau < \tau'_s$  it is exactly 3-dimensional. When  $\tau > \tau_s$ , there exists a 3-dimensional local unstable manifold  $\hat{W}_{3,s,loc}$  of the solution semiflow generated by equation (1.2), which is, at zero, tangent to the 3-dimensional realified generalized eigenspace of the generator  $A_s$  associated with the positive real eigenvalue and the pair of complex conjugate eigenvalues with the greatest real part. Krisztin, *et al.* [9] described the fine structure of the closure of the global forward extension  $\overline{\hat{W}_{3,s}}$  of  $\hat{W}_{3,s,loc}$  under the solution semiflow of equation (1.2). In particular, they proved that  $\overline{\hat{W}_{3,s}}$  is a 3-dimensional submanifold of  $C$  which contains a smooth invariant disk bordered by a periodic orbit. The disk separates  $\hat{W}_{3,s}$  into halves: one half contains connecting orbits from the stationary point 0 or the periodic orbit to a positive stationary point, the other half contains

connecting orbits from the stationary point 0 or the periodic orbit to a negative stationary point. Under the above assumptions on  $f$  it is also proved in [8] that if  $\tau \in (\tau_s, \tau'_s)$  then  $\overline{\hat{W}_{3,s}}$  is indeed the global attractor of the solution semiflow of equation (1.2).

It is interesting to note that the characteristic equation of the linearization at 0 of the full system (1.1), namely,

$$\begin{cases} \dot{X}(t) = -\mu\tau X(t) + \tau f'(0) Y(t-1), \\ \dot{Y}(t) = -\mu\tau Y(t) + \tau f'(0) X(t-1) \end{cases} \tag{1.5}$$

can be decoupled as

$$[\lambda + \mu\tau - \tau f'(0) e^{-\lambda}][\lambda + \mu\tau + \tau f'(0) e^{-\lambda}] = 0, \tag{1.6}$$

where the first factor corresponds to the characteristic equation (1.3). Defining

$$\tau_d = \frac{\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}},$$

one sees that for  $\tau > \tau_d$  the zeros of the equation  $\lambda + \mu\tau + \tau f'(0) e^{-\lambda} = 0$  are simple and occur in complex conjugate pairs, at least one pair of complex conjugate zeros has positive real part, while for  $0 < \tau \leq \tau_d$  all zeros have nonpositive real parts. In Chen and Wu [2], it was proved that when  $\tau > \tau_d$ , there exists a complete analogue  $\overline{\hat{W}_{3,d}}$  of  $\overline{\hat{W}_{3,s}}$  for the full system (1.1). In particular, the full system (1.1) has a phase-locked periodic orbit  $\hat{O}_2$ .

Consequently, when  $\tau > \tau_s (> \tau_d)$ , the unstable space of the generator of the  $C_0$ -semigroup generated by (1.5) is at least 5-dimensional and there exists a 5-dimensional local unstable manifold  $\hat{W}_{5,loc}$  tangent, at zero, to the realified generalized eigenspace of the generator associated with the positive real and the two leading pairs of complex conjugate eigenvalues with positive real parts. The global forward extension  $\hat{W}_5$  of  $\hat{W}_{5,loc}$  contains  $\hat{W}_{3,d}$  and  $\widetilde{\hat{W}}_{3,s} = \{(\varphi, \varphi) \mid \varphi \in \hat{W}_{3,s}\}$ , and in particular,  $\overline{\hat{W}_5}$  contains a phase-locked periodic orbit  $\hat{O}_2$  and a synchronous periodic orbit  $\hat{O}_4$ . The purpose of this paper is to describe completely the global dynamics of the semiflow of system (1.1) restricted to  $\overline{\hat{W}_5}$  and to establish the existence of heteroclinic orbits from the synchronous periodic orbit  $\hat{O}_4$  to the phase-locked periodic orbit  $\hat{O}_2$ .

We now briefly describe our main results, technical tools and approach. First of all, we make the following change of variables

$$\begin{cases} x(t) = u(2t), \\ y(t) = v(2t-1) \end{cases}$$

to get an equivalent cyclic system of delay differential equations

$$\begin{cases} \dot{x}(t) = -2\mu\tau x(t) + 2\tau f(y(t)), \\ \dot{y}(t) = -2\mu\tau y(t) + 2\tau f(x(t-1)). \end{cases} \quad (1.7)$$

Powerful technical tools and general results have been developed in the recent work of Mallet-Paret and Sell [10, 11], and our present work provides some evidence to support the statements in the above papers “(that the work [10, 11]) opening the door to a general inquiry into the structure of the attractor of the (cyclic) system” and to support the expectation in the featured review of Smith [14] that “further progress on this interesting class of systems in the future” (due to the work of Mallet-Paret and Sell [10, 11]).

In what follows, we are going to describe our approach only for the transformed system (1.7), with  $O_4$ ,  $O_2$ ,  $\widetilde{W}_{3,d}$ ,  $W_5$ , and  $\widetilde{W}_{3,s}$  denoting the analogues of  $\hat{O}_4$ ,  $\hat{O}_2$ ,  $\hat{W}_{3,d}$ ,  $\hat{W}_5$ , and  $\hat{W}_{3,s}$ , respectively. The fundamental technical tool for (1.7) is an integer-valued Lyapunov functional  $V: C(\mathbb{K}) \setminus \{0\} \rightarrow \{0, 2, \dots, \infty\}$ , where  $\mathbb{K} = [-1, 0] \cup \{1\}$ ,  $C(\mathbb{K}) = \{\varphi: \mathbb{K} \rightarrow \mathbb{R} \mid \varphi \text{ is continuous}\}$  is the phase space of system (1.7) and  $V(\varphi)$  measures, roughly, the number of sign changes of  $\varphi \in C(\mathbb{K}) \setminus \{0\}$ . Important properties of  $V$  have been established in [10], applications of these properties show that solutions of (1.7) can not decay too fast at  $\infty$  (Theorem 4.1 of Chen and Wu [2] for (1.7), but more general results for general cyclic systems were announced in [10, 11]) and lead to the following characterization of  $W_5$ :

$$W_5 \setminus \{0\} = \left\{ \varphi \in C(\mathbb{K}) \setminus \{0\} \left| \begin{array}{l} \text{there exists a solution } z^\varphi: \mathbb{R} \rightarrow \mathbb{R}^2 \\ \text{of (1.7) with } z_0^\varphi = \varphi, V(z_t^\varphi) \leq 4 \\ \text{for all } t \in \mathbb{R} \text{ and } z_t^\varphi \rightarrow 0 \text{ as } t \rightarrow -\infty \end{array} \right. \right\}. \quad (T1)$$

Also

$$V(\varphi - \psi) \leq 4 \quad \text{if } \varphi, \psi \in \overline{W_5} \text{ and } \varphi \neq \psi. \quad (T2)$$

Another important observation we make is that

$$\text{every non-constant periodic solution of (1.1) is either syn-} \quad (T3) \\ \text{chronous or phase locked.}$$

This, together with the concavity and symmetry conditions, enables us to show that

$$\text{there exists at most one periodic orbit in each level set} \quad (T4) \\ V^{-1}(2k), k = 1, 2, \dots, \text{ of } V.$$

One can also apply this Lyapunov functional to show that

there exists no homoclinic connections of  $O_2$  and  $O_4$ . (T5)

The most important application of  $V$  is perhaps the proof of the Poincaré–Bendixson Theorem for cyclic systems (Mallet–Paret and Sell [11]) which, together with the existence of a heteroclinic connection from 0 to non-trivial equilibria of (1.7) (Hirsh [4] or Smith [13]), enables us to completely describe the dynamics of the semiflow on  $W_5$ . To be more precise, we introduce the separatrix

$$S = \{ \varphi \in C(\mathbb{K}) \mid \varphi = 0 \text{ or } V(z_t^\varphi) > 0 \text{ for all } t \geq 0 \}$$

and let  $z_+$  and  $z_-$  denote the nonzero equilibria of (1.7) given by the unique positive and negative zeros of  $f(\xi) = \mu\xi$ ,  $\xi \in \mathbb{R}$ . Then we show that

$$\begin{cases} \text{if } \varphi \in (W_5 \cap S) \setminus \{0\}, \text{ then } \alpha(\varphi) = \{0\}, \omega(\varphi) = O_2 \text{ or } O_4, \\ \text{if } \varphi \in W_5 \setminus S, \text{ then } \alpha(\varphi) = \{0\}, \omega(\varphi) = \{z_+\} \text{ or } \{z_-\}, \\ \text{if } \varphi \in bdW_5 \setminus (S \cup \{z_-, z_+\}), \text{ then } \alpha(\varphi) = O_4 \text{ or } O_2, \omega(\varphi) = \{z_+\} \text{ or } \{z_-\}, \\ \text{if } \varphi \in (bdW_5 \cap S) \setminus \{O_2 \cup O_4\}, \text{ then } \alpha(\varphi) = O_4 \text{ and } \omega(\varphi) = O_2, \end{cases} \quad (T6)$$

where  $bdW_5 = \overline{W_5} \setminus W_5$ . To show that there is indeed a connecting orbit from  $O_4$  to  $O_2$ , we need further information about the Floquet multipliers of  $O_4$ . We apply the general theory of Mallet–Paret and Sell [10] and some type of homotopy argument to show that

there exist exactly 3 Floquet multipliers of  $O_4$ , counting multiplicities, outside the unit circle, and the associated realified generalized eigenspace is contained in  $V^{-1}(\{0, 2\}) \cup \{0\}$ . (T7)

An immediate consequence of (T7) implies that there exists a 3-dimensional  $C^1$ -smooth local unstable manifold  $W_{loc}^u(P)$  of a Poincaré-map  $P$  associated with a certain hyperplane. This manifold contains at least one point of a connecting orbit from  $O_4$  to  $z_+$  and at least one point of a connecting orbit from  $O_4$  to  $z_-$ . Since  $W_{loc}^u(P)$  is 3-dimensional, there exists a continuous curve in  $W_{loc}^u(P) \setminus O_4$  connecting these points, and a continuity argument shows that the curve contains a point  $\varphi \in W_{loc}^u(P) \cap S$ , which gives that

there exists a connecting orbit from  $O_4$  to  $O_2$ . (T8)

There are several questions remaining to be answered. Denoting by  $W^u(O_4)$  the forward extension of  $W_{loc}^u(P)$  we conjecture that

$$\begin{aligned} (bdW_5 \cap S) \setminus \{O_2 \cup O_4\} &= (W^u(O_4) \cap S) \setminus \{O_2 \cup O_4\} \\ &= \{\varphi \mid \alpha(\varphi) = O_4 \text{ and } \omega(\varphi) = O_2\}. \end{aligned} \quad (C1)$$

Also, note that

$$\varphi \in (\overline{W_5} \cap S) \setminus \{0\} \Rightarrow \omega(\varphi) = O_2 \text{ or } O_4.$$

It is thus important to describe

$$H_i = \{\varphi \in (\overline{W_5} S) \setminus \{0\} \mid \omega(\varphi) = O_i\}, \quad i = 2, 4$$

(the basins of attraction of  $O_i$  in  $\overline{W_5}$ ). Finally, we note that the next critical value of  $\tau$  when (1.6) has a pair of purely imaginary zeros is  $\tau_{2,d} = (3\pi - \arccos(\mu/f'(0)))/\sqrt{[f'(0)]^2 - \mu^2}$  ( $> \tau_s$ ). It is then reasonable to expect that  $\overline{W_5}$  is the global attractor of (1.1) if  $\tau \in (\tau_s, \tau_{2,d})$ .

The remaining part of this paper deals with (1.7) only and is organized as follows. Section 2 summarizes some results of Chen and Wu [2] and Krisztin et al. [9] and establishes the existence of periodic orbits in  $\overline{W_5} \cap V^{-1}(2)$  and  $\overline{W_5} \cap V^{-1}(4)$ . In Section 3, we prove that every non-constant periodic solution is either synchronous or phase-locked. This, together with a result of Krisztin and Walther [8] on uniqueness and absence of periodic orbits for scalar positive feedback equations and the analogous result for scalar negative feedback equations, enables us to show the uniqueness of periodic solutions of system (1.7) in each level set of  $V$ . Section 4 gives information about the Floquet multipliers of  $O_4$ , and Section 5 describes the dynamics on  $\overline{W_5}$  and establishes the existence of heteroclinic orbits from  $O_4$  to  $O_2$ .

## 2. EXISTENCE OF PERIODIC ORBITS IN $\overline{W_5}$

In this section, we summarize some results of the monograph of Krisztin et al. [9] and the work of Chen and Wu [2], which will be needed throughout the remaining part of this paper.

Consider the following system of delay differential equations

$$\begin{cases} \dot{x}(t) = -2\mu\tau x(t) + 2\tau f(y(t)), \\ \dot{y}(t) = -2\mu\tau y(t) + 2\tau f(x(t-1)), \end{cases} \quad (2.1)$$

where we assume

(H1)  $f \in C^1(\mathbb{R}; \mathbb{R})$ ,  $f(0) = 0$  and  $f'(\xi) > 0$  for all  $\xi \in \mathbb{R}$ ;

(H2)  $f'(0) > \mu > 0$ ;

(H3) there exists  $M > 0$  so that  $\frac{f(\xi)}{\xi} < \mu$  if  $|\xi| > M$ ;

(H4)  $\tau > \tau_s = (2\pi - \arccos(\mu/f'(0)))/\sqrt{[f'(0)]^2 - \mu^2}$ .

It follows easily from (H3) that there are  $\xi^- < 0 < \xi^+$  so that  $(f(\xi^-)/\xi^-) = \mu = (f(\xi^+)/\xi^+)$  and  $(f(\xi)/\xi) > \mu$  for  $\xi \in (\xi^-, \xi^+) \setminus \{0\}$ .

Set  $\mathbb{K} = [-1, 0] \cup \{1\}$  and let  $C(\mathbb{K})$  denote the Banach space of continuous functions  $\varphi: \mathbb{K} \rightarrow \mathbb{R}$  with the supremum norm  $\|\cdot\|$ . Throughout this paper, we will always tacitly use the identification

$$C(\mathbb{K}) = C([-1, 0]; \mathbb{R}) \times \mathbb{R}$$

and write an element  $\varphi \in C(\mathbb{K})$  as  $(\varphi|_{[-1, 0]}, \varphi(1))^T \in C([-1, 0]; \mathbb{R}) \times \mathbb{R}$ . We will also use the identification

$$C^1(\mathbb{K}) = C^1([-1, 0]; \mathbb{R}) \times \mathbb{R}$$

and the  $C^1$ -norm on  $C^1(\mathbb{K})$  is defined as

$$\|\varphi\|_1 = \max \left\{ \sup_{\theta \in [-1, 0]} |\varphi(\theta)|, \sup_{\theta \in [-1, 0]} |\dot{\varphi}(\theta)|, |\varphi(1)| \right\}.$$

Set

$$K = \{ \varphi \in C(\mathbb{R}) \mid \varphi(\theta) \geq 0 \text{ for all } \theta \in \mathbb{K} \}.$$

For any  $\varphi \in C(\mathbb{K}) \setminus (K \cup (-K))$ , define the number of sign changes by

$$sc(\varphi) = \sup \left\{ k \geq 1 \mid \begin{array}{l} \text{there exist } \theta^0 < \theta^1 < \dots < \theta^k \text{ with each} \\ \theta^i \in \mathbb{K} \text{ and } \varphi(\theta^{i-1}) \varphi(\theta^i) < 0 \text{ for } 1 \leq i \leq k \end{array} \right\},$$

and let  $sc(\varphi) = 0$  if either  $\varphi \in K$  or  $-\varphi \in K$ . The Lyapunov functional  $V: C(\mathbb{K}) \setminus \{0\} \rightarrow 2\mathbb{N} \cup \{\infty\}$ , introduced by Mallet-Paret and Sell [10, 11], is defined as

$$V(\varphi) = \begin{cases} sc(\varphi) & \text{if } sc(\varphi) \text{ is even or infinite,} \\ sc(\varphi) + 1 & \text{if } sc(\varphi) \text{ is odd.} \end{cases}$$

The subset

$$R = \left\{ \varphi \in C^1(\mathbb{K}) \mid \begin{array}{l} \varphi(1) = 0 \text{ implies } \varphi(0) \varphi(-1) < 0, \\ \varphi(0) = 0 \text{ implies } \dot{\varphi}(0) \varphi(1) > 0, \\ \varphi(-1) = 0 \text{ implies } \varphi(1) \dot{\varphi}(-1) < 0, \\ \varphi(\theta) = 0 \text{ for some } \theta \in (-1, 0) \text{ implies } \dot{\varphi}(\theta) \neq 0 \end{array} \right\}$$



of  $C(\mathbb{K})$  plays an important role in the evaluation of the Lyapunov functional  $V$ . It is not difficult to see that for each  $\varphi \in R$  there exists  $\varepsilon > 0$  such that

$$V(\psi) = V(\varphi) \quad \text{for all } \psi \in C^1(\mathbb{K}) \quad \text{with} \quad \|\psi - \varphi\|_1 < \varepsilon.$$

Another important property of  $V$  is the following:

**LEMMA 2.1.** *For any  $\mu \geq 0$  and for any given positive constants  $b_1 \geq b_0$  there exists  $k > 0$  such that if  $t_0 \in \mathbb{R}$ , if  $b, c: [t_0 - 6, t_0 + \frac{1}{2}] \rightarrow \mathbb{R}$  are continuous functions with*

$$b_0 \leq b(t), c(t) \leq b_1 \quad \text{for all } t \in [t_0 - 6, t_0 + \frac{1}{2}],$$

and if  $x \in C([t_0 - 7, t_0 + \frac{1}{2}]; \mathbb{R}) \cap C^1([t_0 - 6, t_0 + \frac{1}{2}]; \mathbb{R})$  and  $y \in C^1([t_0 - 6, t_0 + \frac{1}{2}]; \mathbb{R})$  satisfy  $(x_{t_0}, y(t_0))^{\text{tr}} \neq 0$  in  $C(\mathbb{K})$ , and

$$\begin{cases} \dot{x}(t) = -\mu x(t) + b(t) y(t), \\ \dot{y}(t) = -\mu y(t) + c(t) x(t-1) \end{cases}$$

for all  $t \in (t_0 - 6, t_0 + \frac{1}{2}]$ , and  $V((x_{t_0-6}, y(t_0-6))^{\text{tr}}) \leq 4$ , then

$$\|(x_{t_0-1}, y(t_0-1))^{\text{tr}}\| \leq k \|(x_{t_0}, y(t_0))^{\text{tr}}\|.$$

The proof of Lemma 2.1 is similar to that of Corollary 4.4 in [2], and thus is omitted. For other properties of  $V$ , we refer to the work of Mallet-Paret and Sell [10, 11].

Following Smith [13] and Mallet-Paret and Sell [10], for each  $\varphi \in C(\mathbb{K})$  there exists a unique pair of continuous maps  $x: [-1, \infty) \rightarrow \mathbb{R}$  and  $y: [0, \infty) \rightarrow \mathbb{R}$  such that  $(x, y)^{\text{tr}}: (0, \infty) \rightarrow \mathbb{R}^2$  is continuously differentiable and satisfies system (2.1) for  $t > 0$ ,  $x|_{[-1, 0]} = \varphi|_{[-1, 0]}$  and  $y(0) = \varphi(1)$ . Let  $z^\varphi = (x^\varphi, y^\varphi)^{\text{tr}}$  denote the above unique pair and define  $z_t^\varphi = (x_t^\varphi, y^\varphi(t))^{\text{tr}} \in C(\mathbb{K})$  for  $t \geq 0$ . Note that we define  $x_t^\varphi \in C([-1, 0]; \mathbb{R})$  in the usual sense (namely,  $x_t^\varphi(\theta) = x^\varphi(t + \theta)$  for  $\theta \in [-1, 0]$ ) and we use subscripts for either  $z_t^\varphi \in C(\mathbb{K})$  or  $x_t^\varphi \in C([-1, 0]; \mathbb{R})$ , these should be easily distinguished from the context, specially from the spaces involved.

It is easy to see that  $\Phi: \mathbb{R}^+ \times C(\mathbb{K}) \ni (t, \varphi) \mapsto z_t^\varphi \in C(\mathbb{K})$  is a continuous semiflow with at least three stationary points  $0, z_-$  and  $z_+$ , where  $0, z_-$  and  $z_+$  denote the constant maps on  $\mathbb{K}$  with the values given by  $0, \xi^-$  and  $\xi^+$ , respectively.

We now introduce the leading unstable sets  $W_3$  and  $W_5$  of the origin. Recall from Section 3 of [2] that the spectrum of the generator of the  $C_0$ -semigroup  $\{D_2\Phi(t, 0)\}_{t \geq 0}$  is given by a real number  $\lambda_0 > 0$  and a sequence of complex conjugate pairs  $\{\lambda_j, \overline{\lambda_j}\}_{j \geq 1}$  with

$$\lambda_0 > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \dots \rightarrow -\infty,$$

$$\operatorname{Im} \lambda_1 \in (\pi, 2\pi),$$

$$\operatorname{Im} \lambda_{2j} \in (2(2j-1)\pi, 4j\pi), \quad 1 \leq j \in \mathbb{N},$$

$$\operatorname{Im} \lambda_{2j+1} \in (4j\pi, 2(2j+1)\pi), \quad 1 \leq j \in \mathbb{N}.$$

Hypothesis (H4) is equivalent to

$$\operatorname{Re} \lambda_2 > 0.$$

Now, as in Section 3 of [2], let  $E_0, E_1, E_2, Q_1, Q_2$  denote the realified generalized eigenspaces of the generator of the  $C_0$ -semigroup  $\{D_2\Phi(t, 0)\}_{t \geq 0}$  on  $C(\mathbb{K})$  associated with the spectral sets  $\{\lambda_0\}$ ,  $\{\lambda_1, \bar{\lambda}_1\}$ ,  $\{\lambda_2, \bar{\lambda}_2\}$ ,  $\{\lambda_j, \bar{\lambda}_j \mid 2 \leq j \in \mathbb{N}\}$ ,  $\{\lambda_j, \bar{\lambda}_j \mid 3 \leq j \in \mathbb{N}\}$ . Then

$$C(\mathbb{K}) = E_0 \oplus E_1 \oplus Q_1,$$

$$C(\mathbb{K}) = E_0 \oplus E_1 \oplus E_2 \oplus Q_2,$$

$$Q_1 = E_2 \oplus Q_2.$$

Choose  $\beta_j \in (\max\{1, e^{\operatorname{Re} \lambda_{j+1}}\}, e^{\operatorname{Re} \lambda_j})$ ,  $j = 1, 2$ . By Theorem I.4 of [9], we find convex bounded open neighborhoods  $N_{012}$  of 0 in  $E_0 \oplus E_1 \oplus E_2$  and  $Q_{2,5}$  of 0 in  $Q_2$  and a  $C^1$ -map  $w_{5,loc}: N_{012} \rightarrow Q_2$  with  $w_{5,loc}(N_{012}) \subset Q_{2,5}$ ,  $w_{5,loc}(0) = 0$ ,  $Dw_{5,loc}(0) = 0$  and so that the graph  $W_{5,loc} = \{\chi + w_{5,loc}(\chi) \mid \chi \in N_{012}\}$  coincides with the set

$$\left\{ \varphi \in N_{5,loc} = N_{012} + Q_{2,5} \left| \begin{array}{l} \text{there is a sequence } (\varphi_n)_{-\infty}^0 \text{ with} \\ \varphi_0 = \varphi, \varphi_n = \Phi(1, \varphi_{n-1}), \\ \text{and } \varphi_n \beta_2^{-n} \in N_{5,loc} \text{ for each} \\ \text{integer } n \leq 0 \text{ and} \\ \varphi_n \beta_2^{-n} \rightarrow 0 \text{ as } n \rightarrow -\infty \end{array} \right. \right\}.$$

Applying again Theorem I.4 of [9] we find convex bounded open neighborhoods  $N_{01}$  of 0 in  $E_0 \oplus E_1$  and  $Q_{1,3}$  of 0 in  $Q_1$  and a  $C^1$ -map  $w_{3,loc}: N_{01} \rightarrow Q_1$  with  $N_{01} + Q_{1,3} \subset N_{012} + Q_{2,5}$ ,  $w_{3,loc}(N_{01}) \subset Q_{1,3}$ ,  $w_{3,loc}(0) = 0$ ,  $Dw_{3,loc}(0) = 0$  and so that the graph  $W_{3,loc} = \{\chi + w_{3,loc}(\chi) \mid \chi \in N_{01}\}$  coincides with the set

$$\left\{ \varphi \in N_{3,loc} = N_{01} + Q_{1,3} \left| \begin{array}{l} \text{there is a sequence } (\varphi_n)_{-\infty}^0 \text{ with} \\ \varphi_0 = \varphi, \varphi_n = \Phi(1, \varphi_{n-1}), \text{ and } \varphi_n \beta_1^{-n} \in N_{3,loc} \\ \text{for each integer } n \leq 0 \text{ and} \\ \varphi_n \beta_1^{-n} \rightarrow 0 \text{ as } n \rightarrow -\infty \end{array} \right. \right\}$$

Our focus in [2] and this paper is the leading unstable sets  $W_3$  and  $W_5$ , i.e., the forward extensions of  $W_{3,loc}$  and  $W_{5,loc}$ , respectively, defined by

$$W_i = \Phi(\mathbb{R}^+ \times W_{i,loc}), \quad i = 3, 5.$$

It is easy to show that  $W_3 \subseteq W_5$  and that for each  $\varphi \in W_i$ , ( $i = 3, 5$ ), there exists a unique  $C^1$ -map  $z^\varphi = (x^\varphi, y^\varphi)^T: \mathbb{R} \rightarrow \mathbb{R}^2$  with  $z_0^\varphi = \varphi$  satisfying system (2.1) for all  $t \in \mathbb{R}$ . Moreover,  $z_t^\varphi \in W_i$ , ( $i = 3, 5$ ), for all  $t \in \mathbb{R}$  and  $z_t^\varphi \rightarrow 0$  as  $t \rightarrow -\infty$ .

The closed cone  $K$  in  $C(\mathbb{K})$  defines a partial ordering on  $C(\mathbb{K})$ . Thus, we can talk about, for  $\varphi, \psi \in C(\mathbb{K})$ ,  $\varphi \geq \psi$  if  $\varphi(\theta) \geq \psi(\theta)$  for all  $\theta \in \mathbb{K}$ ,  $\varphi \gg \psi$  if  $\varphi(\theta) > \psi(\theta)$  for all  $\theta \in \mathbb{K}$ , and  $\varphi > \psi$  if  $\varphi \geq \psi$  and  $\varphi \neq \psi$ . It is easy to verify that the semiflow  $\Phi$  is monotone. More precisely, we have

$$\begin{aligned} \Phi(t, \varphi) &\geq \Phi(t, \psi) && \text{if } t \geq 0, \varphi \geq \psi \text{ in } C(\mathbb{K}), \\ \Phi(t, \overset{\circ}{K}) &\subseteq \overset{\circ}{K}, \quad \Phi(t, -\overset{\circ}{K}) &\subseteq \overset{\circ}{K} && \text{for } t \geq 0, \\ \Phi(t, \varphi) &\gg \Phi(t, \psi) && \text{if } t \geq 2, \varphi > \psi. \end{aligned}$$

An important subset of  $C(\mathbb{K})$  is the following closed set  $S$ , called separatrix,

$$S = \{\varphi \in C(\mathbb{K}) \mid \varphi = 0 \text{ or } V(z_t^\varphi) > 0 \text{ for all } t \geq 0\}.$$

It is shown in [2] that  $S$  is a nonordered set, that is, if  $\varphi < \psi$  then at least one of them is not in  $S$ . Moreover,  $S$  is the graph of a Lipschitz continuous mapping from  $E_1 \oplus Q_1$  to  $E_0$ , and thus we can speak of  $\varphi$  being above  $S$  or below  $S$  for  $\varphi \in C(\mathbb{K}) \setminus S$ . See [2] for details.

The following results were proved for  $W_3$ , but the same arguments can be utilized for  $W_5$ . We then only state the results:

**LEMMA 2.2.** (i)  $z_- \ll \varphi \ll z_+$  for all  $\varphi \in W_5$  and  $z_- \leq \varphi \leq z_+$  for all  $\varphi \in \overline{W_5}$ .

(ii) There exist  $\chi, \eta \in W_3$  so that both  $x^\chi(t)$  and  $y^\chi(t)$  are positive and increasing for  $t \in \mathbb{R}$ , and both  $x^\eta(t)$  and  $y^\eta(t)$  are negative and decreasing for  $t \in \mathbb{R}$ . Moreover,  $\lim_{t \rightarrow \infty} x^\chi(t) = \lim_{t \rightarrow \infty} y^\chi(t) = \xi^+$  and  $\lim_{t \rightarrow \infty} x^\eta(t) = \lim_{t \rightarrow \infty} y^\eta(t) = \xi^-$ .

(iii)  $\overline{W_i}$  and  $bd W_i = \overline{W_i} \setminus W_i$  are compact and invariant with respect to  $\Phi$ ,  $i = 3, 5$ .

(iv) The map  $\Phi_{W_i}: \mathbb{R} \times W_i \ni (t, \varphi) \mapsto z_t^\varphi \in \overline{W_i}$  is a continuous flow,  $i = 3, 5$ .

(v) For each  $\varphi \in \overline{W_5}$ ,  $\varphi \in C^1(\mathbb{K})$ , and the map  $\overline{W_5} \ni \varphi \mapsto \varphi \in C^1(\mathbb{K})$  is continuous.

(vi) If  $\varphi \in \overline{W_5}$  is above  $S$ , then  $z_t^\varphi \rightarrow z_+$  as  $t \rightarrow \infty$ . If  $\varphi \in \overline{W_5}$  is below  $S$ , then  $z_t^\varphi \rightarrow z_-$  as  $t \rightarrow \infty$ .

(vii) If  $\varphi, \psi \in \overline{W_i}$  and  $\varphi \neq \psi$ , then  $V(\varphi - \psi) \leq i - 1, i = 3, 5$ .

(viii) For  $i = 3, 5$ , we have

$$W_i \setminus \{0\} = \left\{ \varphi \in C(\mathbb{K}) \setminus \{0\} \left| \begin{array}{l} \text{there exists a solution } z^\varphi: \mathbb{R} \rightarrow \mathbb{R}^2 \\ \text{of system (2.1) with } z_0^\varphi = \varphi, V(z_t^\varphi) \leq i - 1 \\ \text{for all } t \in \mathbb{R} \text{ and } z_t^\varphi \rightarrow 0 \text{ as } t \rightarrow -\infty. \end{array} \right. \right\}.$$

(ix) Let  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  be a nontrivial solution of

$$\begin{cases} \dot{x}(t) = -2\mu\tau x(t) + 2\tau f'(0) y(t) \\ \dot{y}(t) = -2\mu\tau y(t) + 2\tau f'(0) x(t-1). \end{cases}$$

Then

$$(x_0, y(0))^{tr} \in E_0 \oplus E_1 \quad \text{if and only if} \quad V((x_t, y(t))^{tr}) \leq 2 \\ \text{for all } t \leq 0,$$

$$(x_0, y(0))^{tr} \in E_0 \oplus E_1 \oplus E_2 \quad \text{if and only if} \quad V((x_t, y(t))^{tr}) \leq 4 \\ \text{for all } t \leq 0.$$

Let us note that (viii) of Lemma 2.2 implies that  $W_i$  does not depend on the choice of  $\beta_i$ . (viii) of Lemma 2.2 also implies that  $W_3 \subseteq W_5$ . Therefore, by Theorem 5.10 of [2], we have the following:

**THEOREM 2.3.** *Under the hypotheses (H1)–(H4), there exists a nontrivial periodic orbit of system (2.1) in  $\overline{W_5} \cap V^{-1}(2)$ .*

To obtain one nontrivial periodic orbit for system (2.1) in  $\overline{W_5} \cap V^{-1}(4)$ , we first recall some results from the monograph of Krisztin *et al.* [9]. Consider the following scalar delay differential equation

$$\dot{u}(t) = -\tau\mu u(t) + \tau f(u(t-1)) \tag{2.2}$$

subject to our hypotheses (H1)–(H4). We use the Banach space  $C([-1, 0]; \mathbb{R})$ , equipped with the supremum norm  $\|\cdot\|_0$ , as the phase space of (2.2). Then  $F: \mathbb{R}^+ \times C([-1, 0]; \mathbb{R}) \ni (t, \phi) \mapsto u_t^\phi \in C([-1, 0]; \mathbb{R})$  is a continuous semiflow, where  $u^\phi: [-1, \infty) \rightarrow \mathbb{R}$  is the unique solution of (2.2) with  $u_0^\phi = \phi$ . The spectrum of the generator of the  $C_0$ -semigroup  $\{D_2 F(t, 0)\}_{t \geq 0}$  is given by  $\lambda_0/2$  and the sequence of complex conjugate pairs  $\{\lambda_{2j}/2, \overline{\lambda_{2j}}/2\}_{j \geq 1}$ .

Let  $P_s$ ,  $L_s$  and  $Q_s$  be the realified generalized eigenspaces of the generator of the  $C_0$ -semigroup  $\{D_2F(t, 0)\}_{t \geq 0}$  on  $C([-1, 0]; \mathbb{R})$  associated with the spectral sets  $\{\lambda_0/2\}$ ,  $\{\lambda_2/2, \overline{\lambda_2}/2\}$  and  $\{\lambda_{2j}/2, \overline{\lambda_{2j}}/2\}_{j \geq 2}$ , respectively. Then

$$C([-1, 0]; \mathbb{R}) = P_s \oplus L_s \oplus Q_s.$$

Choose  $\beta_s \in (\max\{1, e^{\operatorname{Re} \lambda_2/2}\}, e^{\lambda_0/2})$ . Then there exist convex bounded open neighborhoods  $N_{P_s}$ ,  $N_{L_s}$  and  $N_{Q_s}$  of 0 in  $P_s$ ,  $L_s$  and  $Q_s$ , respectively, and a  $C^1$ -map  $w_s: N_{P_s} + N_{L_s} \rightarrow Q_s$  with range in  $N_{Q_s}$ ,  $w_s(0) = 0$ ,  $Dw_s(0) = 0$  and so that the graph  $W_{s, \text{loc}} = \{\chi + w_s(\chi) | \chi \in N_{P_s} + N_{L_s}\}$  coincides with the set

$$\left\{ \phi \in N_{s, \text{loc}} = N_{P_s} + N_{L_s} + N_{Q_s} \left| \begin{array}{l} \text{there is a sequence } (\phi_n)_{-\infty}^0 \text{ with } \phi_0 = \phi, \\ \phi_n = F(1, \phi_{n-1}), \text{ and } \phi_n \beta_s^{-n} \in N_{s, \text{loc}} \\ \text{for each integer } n \leq 0 \text{ and } \phi_n \beta_s^{-n} \rightarrow 0 \\ \text{as } n \rightarrow -\infty \end{array} \right. \right\}.$$

Let

$$W_s = F(\mathbb{R}^+ \times W_{s, \text{loc}}).$$

For  $\varphi \in W_s$ , define  ${}^s\varphi \in C(\mathbb{K})$  by

$${}^s\varphi(\theta) = \begin{cases} u^\varphi(2\theta), & \theta \in [-1, 0], \\ u^\varphi(-1) = \varphi(-1), & \theta = 1, \end{cases}$$

where  $u^\varphi: \mathbb{R} \rightarrow \mathbb{R}$  is the unique solution of Eq. (2.2) passing through  $\varphi$ . Let

$$W_{s, s} = \{{}^s\varphi \in C(\mathbb{K}) \mid \varphi \in W_s\}.$$

We want to show that  $W_{s, s} \subseteq W_s$ . For this purpose, we have to use the discrete Lyapunov functional for a scalar delay differential equation with monotone feedbacks.

For  $\phi \in C([-1, 0]; \mathbb{R}) \setminus \{0\}$ , define the number of sign changes

$$sc_s(\phi) = \sup \left\{ k \geq 1 \left| \begin{array}{l} \text{there exist } -1 \leq \theta^0 < \theta^1 < \dots < \theta^k \leq 0 \\ \text{and } \phi(\theta^{i-1}) \phi(\theta^i) < 0 \text{ for } 1 \leq i \leq k \end{array} \right. \right\}.$$

with the convention that  $sc_s(\phi) = 0$  if either  $\phi(\theta) \geq 0$  (or  $\leq 0$ ) for all  $\theta \in [-1, 0]$ . We define the Lyapunov functional  $V_+: C([-1, 0]; \mathbb{R}) \setminus \{0\} \rightarrow 2\mathbb{N} \cup \{\infty\}$  by

$$V_+(\phi) = \begin{cases} sc_s(\phi) & \text{if } sc_s(\phi) \text{ is even or infinite,} \\ sc_s(\phi) + 1 & \text{if } sc_s(\phi) \text{ is odd;} \end{cases}$$

and the Lyapunov functional  $V_- : C([-1, 0]; \mathbb{R}) \setminus \{0\} \rightarrow (2\mathbb{N} + 1) \cup \{\infty\}$  by

$$V_-(\phi) = \begin{cases} sc_s(\phi) & \text{if } sc_s(\phi) \text{ is odd or infinite,} \\ sc_s(\phi) + 1 & \text{if } sc_s(\phi) \text{ is even.} \end{cases}$$

$V_{\pm}$  possess similar properties to those of  $V$ , and we refer to Mallet–Paret and Sell [10] or Krisztin, Walther and Wu [9] for details.

Note that  $W_s \setminus \{0\}$  can be characterized by

$$W_s \setminus \{0\} = \left\{ \varphi \in C([-1, 0]; \mathbb{R}) \left| \begin{array}{l} \text{there is a solution } u^\varphi : \mathbb{R} \rightarrow \mathbb{R} \text{ of} \\ \text{equation (2.2) with } u_0^\varphi = \varphi \text{ and } V_+(u_t^\varphi) \leq 2 \\ \text{for all } t \in \mathbb{R} \text{ and } u_t^\varphi \rightarrow 0 \text{ as } t \rightarrow -\infty \end{array} \right. \right\}, \tag{2.3}$$

according to Proposition 5.3 of [9].

**PROPOSITION 2.4.**  $W_{s,s} \subseteq W_5$ .

*Proof.* For any given  $0 \neq \varphi \in W_s$ , let

$$\begin{cases} x(t) = u^\varphi(2t), \\ y(t) = u^\varphi(2t - 1), \end{cases} \quad t \in \mathbb{R}.$$

Then  $(x, y)^{tr}$  satisfies system (2.1) with  $(x_0, y(0))^{tr} = \varphi_s$  and  $(x_t, y(t))^{tr} \rightarrow 0$  as  $t \rightarrow -\infty$ . To show that  $\varphi_s \in W_5$ , by (viii) of Lemma 2.2, we only need to show that  $V((x_t, y(t))^{tr}) \leq 4$  for all  $t \in \mathbb{R}$ . If not, assume that there is a  $t_0 \in \mathbb{R}$  such that  $V((x_{t_0}, y(t_0))^{tr}) \geq 6$ , then by the nonincreasing property of  $V$ , we have

$$V((x_t, y(t))^{tr}) \geq 6 \quad \text{for all } t \leq t_0.$$

Using the definition of  $V$ , we can find a  $t_1 < t_0$  such that  $y(t_1) = 0$ . Since  $V((x_{t_1}, y(t_1))^{tr}) \geq 6$ ,  $x$  has at least 5 sign changes on  $[t_1 - 1, t_1]$  and hence there is a  $t_2 \in [t_1 - \frac{1}{2}, t_1]$  such that  $x$  has at least 3 sign changes on  $[t_2 - \frac{1}{2}, t_2]$ . Thus,  $V_+(u_{2t_2}^\varphi) \geq 4$ , which contradicts (2.3) since  $\varphi \in W_s \setminus \{0\}$ . This completes the proof.

*Remark 2.5.* From [9], we know that under the hypotheses (H1)–(H4) Eq. (2.2) has a nontrivial periodic orbit in  $\overline{W_s} \cap V_+^{-1}(2)$ . Then it is easy to see that system (2.1) has a nontrivial periodic orbit in  $\overline{W_{s,s}} \cap V^{-1}(4)$ . Indeed, assume the nontrivial periodic orbit in  $\overline{W_s} \cap V_+^{-1}(2)$  is given by the periodic solution  $r^\tau : \mathbb{R} \rightarrow \mathbb{R}$  of (2.2) which is normalized so that  $r^\tau(0) = 0$  and  $r^\tau(-1) > 0$ . Then  $(r_1^\tau, r_2^\tau)^{tr} : \mathbb{R} \ni t \mapsto (r^\tau(2t), r^\tau(2t - 1))^{tr} \in \mathbb{R}^2$  satisfies (2.1) and this gives a periodic orbit of system (2.1) in  $W_{s,s}$ . Now, we claim

that  $V((r_1^\tau)_t, r_2^\tau(t))^{\tau} = 4$  for all  $t \in \mathbb{R}$ . First, there exists a nonnegative integer  $\nu$  such that  $V((r_1^\tau)_t, r_2^\tau(t))^{\tau} = 2\nu$  for all  $t \in \mathbb{R}$  by Lemma 5.7 of [11]. Secondly,  $\nu \geq 1$  since  $r^\tau$  is oscillating. Thirdly,  $\nu \leq 2$  by using Proposition 2.4 and (vii) of Lemma 2.2 and the fact that  $0 \in \overline{W}_5$ . Thus,  $\nu = 1$  or 2. Finally, we show  $\nu = 2$ . If this is true, then the claim is proved. Let us assume  $\nu = 1$ . Particularly,  $V((r_1^\tau)_0, r_2^\tau(0))^{\tau} = 2$ . Note that  $r_2^\tau(0) = r^\tau(-1) > 0$  and when  $\theta < 0$  and sufficiently close to 0 we have  $r_1^\tau(\theta) = r^\tau(2\theta) < 0$ . Thus,  $r_1^\tau$  has at most one sign change on  $[-1, 0]$ , i.e.,  $r^\tau$  has at most one sign change on  $[-2, 0]$ . Then there exists an interval of length 1 in  $[-2, 0]$ , say  $[t_0 - 1, t_0]$ , such that  $r^\tau$  has no sign change on it. This means that  $V_+(r_{t_0}^\tau) = 0$ , a contradiction. Therefore, we have proved the following theorem.

**THEOREM 2.6.** *Under the hypotheses (H1)–(H4), system (2.1) has a nontrivial periodic orbit in  $\overline{W}_5 \cap V^{-1}(4)$ .*

### 3. UNIQUENESS OF PERIODIC ORBITS WITH GIVEN OSCILLATION FREQUENCIES

In this section, we will show that under certain stronger assumptions than (H1)–(H4), every periodic solution of (2.1) is either synchronous or phase-locked. This, coupled with some results of Krisztin and Walther [8] for scalar equations, enables us to derive some results about the uniqueness and absence of periodic solutions in each level set of  $V$ .

Throughout the remaining of this paper, in addition to (H1)–(H4), we assume

- (H5) (i)  $f(\xi) = -f(-\xi)$  for all  $\xi \in \mathbb{R}$ .  
(ii) The function  $h: (0, \infty) \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$  is strictly decreasing.

Observe that (H1)–(H5) imply that  $-\mu \text{id} + f$  has exactly three zeros  $\xi^-, 0, \xi^+$  with  $\xi^- < 0 < \xi^+$ ,  $f'(0) > \mu$ ,  $f'(\xi^-) < \mu$ ,  $f'(\xi^+) < \mu$ .

In order to characterize uniqueness and absence of periodic orbits of Eq. (2.2) and another scalar equation with negative feedback, we consider the characteristic Eq. (1.3) with parameter  $\tau > 0$ . The zeros  $\lambda_0/2, \lambda_2/2, \overline{\lambda}_2/2, \dots$  depend analytically on  $\tau$ . From (H2) we get  $\lambda_0/2 > 0$  for all  $\tau > 0$ . If  $j \in 2\mathbb{N} \setminus \{0\}$  then there is a uniquely determined parameter  $\tau^j$  so that

$$\text{Re } \lambda_j(\tau^j) = 0.$$

We have

$$\tau^j = \frac{j\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}}.$$

Note that  $\tau^2 = \tau_s$  and  $\tau^4 = \tau'_s$ . We compute

$$(\operatorname{Re} \lambda_j)'(\tau^j) > 0$$

for  $0 \neq j \in 2\mathbb{N}$ , and conclude that for every  $j \in 2\mathbb{N} \setminus \{0\}$

$$\tau < \tau^j \quad (> \tau^j) \quad \text{if and only if} \quad \operatorname{Re} \lambda_j(\tau) < 0 \quad (> 0).$$

Hence we obtain that the following two results on the uniqueness and absence of periodic orbits of Eq. (2.2) are equivalent to Theorem 3.5 in [8].

**LEMMA 3.1.** *Let either  $k = 0$  or  $k \in \mathbb{N} \setminus \{0\}$  and  $0 < \tau \leq \tau^{2k}$ . Then there is no nonconstant periodic solution  $u: \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (2.2) so that  $V_+(u_t) = 2k$  for all  $t \in \mathbb{R}$ .*

**LEMMA 3.2.** *Let  $k \in \mathbb{N} \setminus \{0\}$  and  $\tau > 0$ . If  $u^1: \mathbb{R} \rightarrow \mathbb{R}$  and  $u^2: \mathbb{R} \rightarrow \mathbb{R}$  are two periodic solutions of Eq. (2.2) with  $V_+(u_t^1) = V_+(u_t^2) = 2k$  for all  $t \in \mathbb{R}$ , then there exists a  $\sigma \in \mathbb{R}$  so that*

$$u^1(t) = u^2(t + \sigma) \quad \text{for all } t \in \mathbb{R}.$$

Similar results were previously established by Cao [1] for slowly oscillating periodic solutions of a negative feedback differential delay equation without the oddness condition on  $f$ . Analogously to the positive feedback case in Lemmas 3.1–3.2, by using the oddness of  $f$  in (H5), Cao’s results can be extended to periodic solutions with higher oscillation frequencies as well. More precisely, we consider the negative feedback equation

$$\dot{v}(t) = -\tau\mu v(t) - \tau f(v(t-1)) \tag{3.1}$$

for the parameter  $\tau > 0$  and the characteristic equation

$$\lambda + \tau\mu + \tau f'(0) e^{-\lambda} = 0 \tag{3.2}$$

of the linearized equation at 0. Then  $\{\lambda_{2j+1}(\tau)/2, \overline{\lambda_{2j+1}(\tau)}/2 \mid j \in \mathbb{N}\}$  is the zerset of (3.2). Setting

$$\tau^j = \frac{j\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}}$$



for  $j \in 2\mathbb{N} + 1$ , we have  $\operatorname{Re} \lambda_j(\tau^j) = 0$ , and obtain analogously to the positive feedback case that, for every  $j \in 2\mathbb{N} + 1$ ,

$$\tau < \tau^j \quad (> \tau^j) \quad \text{if and only if} \quad \operatorname{Re} \lambda_j(\tau) < 0 \quad (> 0).$$

The proof of the following two lemmas can be carried out similarly to that of Lemmas 3.1–3.2. So, we omit the proofs and refer to Section 3 of [8].

**LEMMA 3.3.** *Let  $k \in \mathbb{N}$  and  $0 < \tau \leq \tau^{2k+1}$ . Then there is no periodic solution  $v: \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (3.1) so that  $V_-(v_t) = 2k + 1$  for all  $t \in \mathbb{R}$ .*

**LEMMA 3.4.** *Let  $k \in \mathbb{N}$  and  $\tau > 0$ . If  $v^1: \mathbb{R} \rightarrow \mathbb{R}$  and  $v^2: \mathbb{R} \rightarrow \mathbb{R}$  are two periodic solutions of Eq. (3.1) with  $V_-(v_t^1) = V_-(v_t^2) = 2k + 1$  for all  $t \in \mathbb{R}$ , then there is a  $\sigma \in \mathbb{R}$  such that*

$$v^1(t) = v^2(t + \sigma) \quad \text{for all } t \in \mathbb{R}.$$

In the remaining part of this section, we show that the above results, together with some special “coupling and decoupling” techniques described below and the consideration of the symmetry properties of (2.1), yield uniqueness and absence of periodic orbits with given oscillation frequency for system (2.1). We start with

**PROPOSITION 3.5.** *If  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  is a nonconstant periodic solution of system (2.1), then there exists a  $k \in \mathbb{N} \setminus \{0\}$  such that*

$$V((x_t, y(t))^{tr}) = 2k \quad \text{for all } t \in \mathbb{R}. \quad (3.3)$$

*Proof.* It is known that there is a  $k \in \mathbb{N}$  such that (3.3) holds and  $(x_t, y(t))^{tr} \in R$  for all  $t \in \mathbb{R}$  (see Lemma 5.7 of [11]). We only need to show that  $k \neq 0$ . If  $k = 0$ ,  $x$  and  $y$  are either both nonpositive or both nonnegative. Without loss of generality, we assume that both  $x$  and  $y$  are nonnegative. Since all zeros of  $x$  must be simple, we conclude that  $x$  is positive. We claim that  $y$  is also positive, since all possible zeros of  $y$  must be simple (otherwise from the second equation of system (2.1), we know that  $x$  has a zero, which is a contradiction to the positiveness of  $x$ ). Let  $m = \min_{t \in \mathbb{R}} \{x(t), y(t)\}$ ,  $M = \max_{t \in \mathbb{R}} \{x(t), y(t)\}$ . As  $(x, y)^{tr}$  is a nonconstant periodic solution of system (2.1), either  $m \in (0, \xi^+)$  or  $M > \xi^+$  holds.

If  $m \in (0, \xi^+)$  holds and  $s_1 \in \mathbb{R}$  is given so that  $x(s_1) = m$  (or  $y(s_1) = m$ ), then  $\dot{x}(s_1) = 0$  (or  $\dot{y}(s_1) = 0$ ). Hence, by system (2.1) and by  $f(\xi) > \mu\xi$ ,  $0 < \xi < \xi^+$ , which follows from (H1) and (H5),

$$\begin{aligned} \dot{x}(s_1) &= -2\mu\tau x(s_1) + 2\tau f(y(s_1)) \geq -2\mu\tau x(s_1) + 2\tau f(x(s_1)) > 0 \\ (\text{or } \dot{y}(s_1) &= -2\mu\tau y(s_1) + 2\tau f(x(s_1 - 1)) \geq -2\mu\tau y(s_1) + 2\tau f(y(s_1)) > 0), \end{aligned}$$

a contradiction.

If  $M > \xi^+$  and  $s_2 \in \mathbb{R}$  is chosen so that  $x(s_2) = M$  (or  $y(s_2) = M$ ), then  $\dot{x}(s_2) = 0$  (or  $\dot{y}(s_2) = 0$ ). On the other hand, the fact  $-\mu\xi + f(\xi) < 0$  for  $\xi > \xi^+$ , which is implied by (H1) and (H5), and the monotonicity of  $f$  combined yield

$$\begin{aligned} \dot{x}(s_2) &= -2\mu\tau x(s_2) + 2\tau f(y(s_2)) \leq -2\mu\tau x(s_2) + 2\tau f(x(s_2)) < 0 \\ (\text{or } \dot{y}(s_2) &= -2\mu\tau y(s_2) + 2\tau f(x(s_2 - 1)) \leq -2\mu\tau y(s_2) + 2\tau f(y(s_2)) < 0), \end{aligned}$$

a contradiction. This completes the proof.

**PROPOSITION 3.6.** *If  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  is a nonconstant periodic solution of system (2.1), then both  $x$  and  $y$  are oscillating (that is, both have zeros).*

*Proof.* It is easy to see from system (2.1) that both  $x$  and  $y$  are oscillating or both have no zeros. If the proposition is not true, then Proposition 3.5 combined with the definition of  $V$  and the periodicity of  $(x, y)^{tr}$  implies that  $x(t)y(t) < 0$  for all  $t \in \mathbb{R}$ . This implies that  $x$  is either strictly increasing or strictly decreasing on  $\mathbb{R}$ , which contradicts the periodicity of  $x$ , and hence the proof is completed.

Now, let  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  be a nonconstant periodic solution of system (2.1) with minimal period  $T_0$ . Since  $x$  is oscillating (by Proposition 3.6) and since  $f$  is odd, Theorem 7.2 of [11] implies that

$$\begin{cases} x\left(t + \frac{T_0}{2}\right) = -x(t), \\ y\left(t + \frac{T_0}{2}\right) = -y(t), \end{cases} \quad t \in \mathbb{R}. \tag{3.4}$$

Furthermore, let  $(\tilde{x}, \tilde{y})^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  be defined by

$$\begin{cases} \tilde{x}(t) = y(t), \\ \tilde{y}(t) = x(t - 1), \end{cases} \quad t \in \mathbb{R}.$$

Then it is easy to check that  $(\tilde{x}, \tilde{y})$  is also a nonconstant periodic solution of system (2.1) with the same minimal period  $T_0$ . Let  $\eta_1, \eta_2: [0, T_0] \rightarrow C(\mathbb{K})$  be defined by

$$\begin{aligned}\eta_1(t) &= (x_t, y(t))^{\text{tr}}, \\ \eta_2(t) &= (\tilde{x}_t, \tilde{y}(t))^{\text{tr}},\end{aligned}$$

for  $t \in \mathbb{R}$ , respectively. Then by Lemma 5.7 of [11], either  $\Pi |\eta_1| = \Pi |\eta_2|$  or  $\Pi |\eta_1| \cap \Pi |\eta_2| = \emptyset$  holds, where  $\Pi: C(\mathbb{K}) \rightarrow \mathbb{R}^2$  is defined by

$$\Pi(\varphi) = (\varphi(0), \varphi(1))^{\text{tr}} \in \mathbb{R}^2, \quad \varphi \in C(\mathbb{K}). \quad (3.5)$$

We claim that  $\Pi |\eta_1| \cap \Pi |\eta_2| \neq \emptyset$ . By way of contradiction, we assume  $\Pi |\eta_1| \cap \Pi |\eta_2| = \emptyset$ . By Proposition 3.6 and by Proposition 7.3 of [11],  $(0, 0)^{\text{tr}} \in \text{int}(\Pi |\eta_1|)$  and  $(0, 0)^{\text{tr}} \in \text{int}(\Pi |\eta_2|)$ . Therefore, either  $\Pi |\eta_1| \subset \text{int}(\Pi |\eta_2|)$  or  $\Pi |\eta_2| \subset \text{int}(\Pi |\eta_1|)$  holds. We consider the case  $\Pi |\eta_1| \subset \text{int}(\Pi |\eta_2|)$ . Then

$$\max_{t \in \mathbb{R}} x(t) < \max_{t \in \mathbb{R}} \tilde{x}(t) = \max_{t \in \mathbb{R}} y(t) < \max_{t \in \mathbb{R}} \tilde{y}(t) = \max_{t \in \mathbb{R}} x(t),$$

a contradiction. This verifies the claim. Hence,  $\Pi |\eta_1| = \Pi |\eta_2|$ , which implies that there is a  $\sigma \in [0, T_0)$  such that

$$\begin{cases} x(t) = y(t + \sigma), \\ y(t) = x(t - 1 + \sigma), \end{cases} \quad t \in \mathbb{R}.$$

Then  $x(t) = x(t + 2\sigma - 1)$  and  $y(t) = y(t + 2\sigma - 1)$  for  $t \in \mathbb{R}$ . So  $2\sigma - 1 = mT_0$  for some integer  $m$ , or  $\sigma = (1 + mT_0)/2$ . By the periodicity of  $(x, y)^{\text{tr}}$ , we must have either  $\sigma = \frac{1}{2}$  or  $\sigma = (1 + T_0)/2$ . Thus, we have proved the following result:

**PROPOSITION 3.7.** *Let  $(x, y)^{\text{tr}}: \mathbb{R} \rightarrow \mathbb{R}^2$  be a nonconstant periodic solution of system (2.1) with the minimal period  $T_0$ . Then either*

$$\begin{cases} x(t) = y(t + \frac{1}{2}), \\ y(t) = x(t - \frac{1}{2}), \end{cases} \quad t \in \mathbb{R} \quad (3.6)$$

or

$$\begin{cases} x(t) = y\left(t + \frac{1 + T_0}{2}\right), \\ y(t) = x\left(t + \frac{T_0 - 1}{2}\right), \end{cases} \quad t \in \mathbb{R} \quad (3.7)$$

holds.

Note that the case  $\sigma = \frac{1}{2}$  corresponds to synchronized periodic solutions and  $\sigma = (1 + T_0)/2$  corresponds to phase-locked periodic solutions for the untransformed system (1.1). In the following, we will call a periodic solution  $(x, y): \mathbb{R} \rightarrow \mathbb{R}^2$  of (2.1) synchronous or phase-locked if it satisfies (3.6) or (3.7), respectively.

**PROPOSITION 3.8.** (i) For  $k \in \mathbb{N}$ , system (2.1) has no synchronized periodic orbit in  $V^{-1}(4k + 2)$ .

(ii) For  $k \in \mathbb{N} \setminus \{0\}$ , system (2.1) has no phase-locked periodic orbit in  $V^{-1}(4k)$ .

*Proof.* (i) Assume, by way of contradiction, there is a synchronized periodic solution  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  of (2.1) such that  $(x_t, y(t))^{tr} \in V^{-1}(4k + 2)$  for  $t \in \mathbb{R}$ . Then using system (2.1) and (3.6), we have

$$\dot{x}(t) = -2\mu\tau x(t) + 2\tau f(x(t - \frac{1}{2})).$$

Let  $u: \mathbb{R} \ni t \mapsto x(\frac{t}{2}) \in \mathbb{R}$ . Then  $u$  satisfies (2.2). Since  $y$  is oscillating, we can choose a  $t_0 \in \mathbb{R}$  such that  $y(t_0) = 0$ . Then we know that  $x$  has at least  $4k + 1$  sign changes on the interval  $[t_0 - 1, t_0]$ . Thus there is a subinterval of length  $\frac{1}{2}$ , say  $[t_1 - \frac{1}{2}, t_1] \subseteq [t_0 - 1, t_0]$ , such that  $x$  has at least  $2k + 1$  sign changes on  $[t_1 - \frac{1}{2}, t_1]$ , which implies that  $V_+(u_{2t_1}) \geq 2k + 2$ . Since  $u$  is periodic and  $V$  is nonincreasing, we have  $V_+(u_t) \geq 2k + 2$  for all  $t \in \mathbb{R}$ . Note that all zeros of  $x$ , and hence of  $u$ , are simple. Choose  $t_2 \in \mathbb{R}$  such that  $u(t_2) = 0$ . Then  $u$  has a sign change at  $t_2$  and has at least  $2k + 1$  sign changes on each of the intervals  $(t_2 - 1, t_2)$  and  $(t_2, t_2 + 1)$ . Thus  $u$  has at least  $4k + 3$  sign changes on  $[t_2 - 1, t_2 + 1]$ , which means that  $x$  has at least  $4k + 3$  sign changes on  $[(t_2 - 1)/2, (t_2 + 1)/2]$ . Thus  $V((x_{t_2+12}, y((t_2 + 1)2))^{tr}) \geq 4k + 4$ , a contradiction to our assumption.

(ii) This can be proved similarly using (3.4), (3.7), the assumption (H5), and  $V_-$ . This completes the proof.

**PROPOSITION 3.9.** (i) For  $k \in \mathbb{N}$ , if  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  is a phase-locked periodic solution of system (2.1) such that  $V((x_t, y(t))^{tr}) = 4k + 2$  for all  $t \in \mathbb{R}$ , then  $v: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $v(t) = x(\frac{t}{2})$  for all  $t \in \mathbb{R}$  satisfies Eq. (3.1) and  $V_-(v_t) = 2k + 1$  for all  $t \in \mathbb{R}$ .

(ii) For  $k \in \mathbb{N} \setminus \{0\}$ , if  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  is a synchronized periodic solution of system (2.1) such that  $V((x_t, y(t))^{tr}) = 4k$  for all  $t \in \mathbb{R}$ , then  $u: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $u(t) = x(\frac{t}{2})$  for all  $t \in \mathbb{R}$  satisfies (2.2) and  $V_+(u_t) = 2k$  for all  $t \in \mathbb{R}$ .

*Proof.* First note that from (3.4) and (3.7), we have

$$y(t) = x\left(t + \frac{T_0 - 1}{2}\right) = -x\left(t - \frac{1}{2}\right).$$

Thus,  $x$  satisfies

$$\dot{x}(t) = -2\mu\tau x(t) - 2\tau f(x(t - \frac{1}{2}))$$

since  $f$  is odd. Thus it is easy to check that  $v$  satisfies equation (3.1). Now, choose a zero  $t_0 \in \mathbb{R}$  for  $y$ . Then  $x$  has at least  $4k + 1$  but at most  $4k + 2$  sign changes on  $[t_0 - 1, t_0]$ . Then either on  $[t_0 - 1, t_0 - \frac{1}{2}]$  or on  $[t_0 - \frac{1}{2}, t_0]$ ,  $x$  has at least  $2k$  but at most  $2k + 1$  sign changes. This means that  $v$  has at least  $2k$  sign changes either on  $[2t_0 - 2, 2t_0 - 1]$  or on  $[2t_0 - 1, 2t_0]$ . Thus the periodicity of  $v$  and the nonincreasing property of  $V_-$  combined yield that  $V_-(v_t) \geq 2k + 1$  for all  $t \in \mathbb{R}$ . In the same way we find that  $x$  has at most  $2k + 1$  sign changes on one open half of  $[t_0 - 1, t_0]$ . Periodicity of  $v$  and monotonicity of  $V$  imply  $V_-(v_t) \leq 2k + 1$  for all  $t \in \mathbb{R}$ . This completes the proof of (i). Part (ii) can be proved similarly.

Now, as immediate consequences of Lemmas 3.1–3.4 and Propositions 3.7–3.9, we have

**THEOREM 3.10.** (i) *For any given  $k \in \mathbb{N} \setminus \{0\}$ , system (2.1) has at most one periodic orbit in  $V^{-1}(2k)$ .*

(ii) *All periodic solutions of (2.1) are either phase-locked and belong to  $V^{-1}(4k + 2)$  for some nonnegative integer  $k$ , or synchronous and belong to  $V^{-1}(4k)$  for some positive integer  $k$ .*

(iii) *For any given  $l \in \mathbb{N} \setminus \{0\}$ , if  $0 < \tau \leq \tau^l$ , then system (2.1) has no periodic orbit in  $V^{-1}(2k)$  for any integer  $k \geq l$ .*

In particular, by Theorems 2.3 and 2.6, we know that under the hypotheses (H1)–(H5), there are one and only one periodic orbit of system (2.1) in  $V^{-1}(2)$  and also one and only one periodic orbit of system (2.1) in  $V^{-1}(4)$ , and both orbits belong to  $\overline{W_5}$ . We will denote these uniquely determined periodic orbits by  $O_{2,\tau}$  and  $O_{4,\tau}$  (or  $O_2$  or  $O_4$  if dependence on  $\tau$  is not needed to be explicitly mentioned), respectively.

#### 4. THE FLOQUET MULTIPLIERS OF THE SYNCHRONOUS PERIODIC ORBIT

Recall that the system (2.1) has a unique periodic orbit  $O_{4,\tau}$  in  $V^{-1}(4)$  which is synchronized and can be derived from the periodic orbit of the scalar equation (2.2) discussed by Krisztin *et al.* [9].

The purpose of this section is to get some information about the Floquet multipliers of  $O_{4,\tau}$ . In particular, we want to show that  $O_{4,\tau}$  has exactly three Floquet multipliers outside the unit circle. We will need a result about the continuity of  $O_{4,\tau}$  with respect to  $\tau$ . By the results of Sections 2–3 and [9], for  $\tau > \tau_s$ , there is a unique periodic solution  $r^\tau: \mathbb{R} \rightarrow \mathbb{R}$  of (2.2) which has minimal period  $\omega^\tau \in (1, 2)$  and which is normalized so that  $r^\tau(0) = 0, r^\tau(-1) > 0$ . Let

$$H = \{ \varphi \in C(\mathbb{K}) \mid \varphi(0) = 0, \varphi(1) > 0 \}.$$

Then the function  $(p^\tau, q^\tau)^{tr}: \mathbb{R} \ni t \mapsto (r^\tau(2t), r^\tau(2t - 1))^{tr} \in \mathbb{R}^2$  is a periodic solution of (2.1) with minimal period  $T^\tau = \omega^\tau/2$ , and  $(p_0^\tau, q^\tau(0))^{tr} \in H, O_{4,\tau} = \{ (p_t^\tau, q^\tau(t))^{tr} \mid 0 \leq t \leq T^\tau \}$ . Moreover,  $(p^\tau, q^\tau)^{tr}$  is the unique representation of  $O_{4,\tau}$  with  $(p_0^\tau, q^\tau(0))^{tr} \in H$ . We define, for  $\tau = \tau_s, (p^\tau, q^\tau) = (0, 0)$ .

**PROPOSITION 4.1.** *For every  $\tau^0 \geq \tau_s$  and for every sequence  $(\tau_n)_{n=0}^\infty$  in  $(\tau_s, \infty)$  with  $\tau_n \rightarrow \tau^0$  as  $n \rightarrow \infty$ , the sequence of periodic solutions  $\{ (p^{\tau_n}, q^{\tau_n})^{tr} \}_{n=0}^\infty$  converges to  $(p^{\tau^0}, q^{\tau^0})^{tr}$  as  $n \rightarrow \infty$  uniformly on any compact subset of  $\mathbb{R}$ .*

*Proof.* It suffices to show that for any sequence  $(\tau_n)_{n=0}^\infty$  in  $(\tau_s, \infty)$  with  $\tau_n \rightarrow \tau^0$  as  $n \rightarrow \infty$ , there is a subsequence  $(n_k)_{k=0}^\infty$  such that  $(p^{\tau_{n_k}}, q^{\tau_{n_k}})^{tr} \rightarrow (p^{\tau^0}, q^{\tau^0})^{tr}$  as  $k \rightarrow \infty$  uniformly on any compact subset of  $\mathbb{R}$ .

Recall that  $p^{\tau_n}(t), q^{\tau_n}(t) \in [\xi^-, \xi^+]$  for all  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then by system (2.1) and using the Arzèla–Ascoli Theorem and the method of diagonalization, there exist a subsequence  $(n_k)$ , a constant  $T^* \geq 1/2$  and a  $C^1$ -map  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$T^{\tau_{n_k}} \rightarrow T^* \quad \text{as } k \rightarrow \infty,$$

$$(p^{\tau_{n_k}}, q^{\tau_{n_k}}, \dot{p}^{\tau_{n_k}}, \dot{q}^{\tau_{n_k}}) \rightarrow (x, y, \dot{x}, \dot{y}) \quad \text{as } k \rightarrow \infty$$

uniformly on any compact subset of  $\mathbb{R}$

and  $(x, y)$  satisfies (2.1) with  $\tau = \tau^0$ .

If there is a  $t_0 \in \mathbb{R}$  such that  $(x_{t_0}, y(t_0))^{tr} \neq 0$  in  $C(\mathbb{K})$ , then  $(x_t, y(t))^{tr} \neq 0$  for all  $t \in \mathbb{R}$  using system (2.1), then  $T^*$  is a period of  $(x, y)^{tr}$  and hence  $(x_t, y(t))^{tr} \in R$  for all  $t \in \mathbb{R}$ , and therefore by the  $C^1$ -continuity of  $V$ , we have for all  $t \in \mathbb{R}$ ,

$$V((x_t, y(t))^{tr}) = \lim_{k \rightarrow \infty} V((p_t^{\tau_{n_k}}, q^{\tau_{n_k}}(t))^{tr}) = 4.$$

Clearly, we have  $x(0) = 0, y(0) \geq 0$  and  $y(0)$  must be positive since  $(x_t, y(t))^{\tau} \in R$  for all  $t \in \mathbb{R}$ . So  $(x, y)^{\tau} = (p^{\tau^0}, q^{\tau^0})^{\tau}$ .

In case  $\tau^0 = \tau_s$ , we must have  $x \equiv y \equiv 0$ . For otherwise, the above discussion guarantees the existence of a nontrivial periodic solution  $(x, y)$  of (2.1) with  $\tau = \tau_s$  satisfying  $V((x_t, y(t))^{\tau}) = 4$  for all  $t \in \mathbb{R}$ , a contradiction to Theorem 3.10.

In case  $\tau^0 > \tau_s$ , it remains to exclude the possibility of  $x \equiv y \equiv 0$ . Assume  $(\tau_n)$  is such a sequence that  $\tau_n \rightarrow \tau^0$  as  $n \rightarrow \infty$  but  $(p^{\tau_n}, q^{\tau_n})^{\tau} \rightarrow 0$  as  $n \rightarrow \infty$  uniformly on any compact subset of  $\mathbb{R}$ . Let

$$\begin{cases} u^n(t) = \frac{p^{\tau_n}(t)}{\|(p^{\tau_n}, q^{\tau_n})^{\tau}\|_{\infty}}, \\ v^n(t) = \frac{q^{\tau_n}(t)}{\|(p^{\tau_n}, q^{\tau_n})^{\tau}\|_{\infty}}, \end{cases} \quad t \in \mathbb{R}$$

for all  $n \in \mathbb{N}$ , where for a bounded continuous map  $(x, y)^{\tau}: \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $\|(x, y)^{\tau}\|_{\infty} = \sup_{t \in \mathbb{R}} \max\{|x(t)|, |y(t)|\}$ . Then for  $n \in \mathbb{N}$ ,  $|u^n(t)| \leq 1, |v^n(t)| \leq 1$  and  $V((u_t^n, v^n(t))^{\tau}) = 4$  for all  $t \in \mathbb{R}$  and  $(u^n, v^n)^{\tau}$  satisfies

$$\begin{cases} \dot{u}^n(t) = -2\tau_n \mu u^n(t) + 2\tau_n \int_0^1 f'(sq^{\tau_n}(t)) ds v^n(t), \\ \dot{v}^n(t) = -2\tau_n \mu v^n(t) + 2\tau_n \int_0^1 f'(sp^{\tau_n}(t-1)) ds u^n(t-1), \end{cases} \quad t \in \mathbb{R}.$$

Using the Arzèla–Ascoli Theorem we find a subsequence  $(n_k)$ , a constant  $T^* \geq 1/2$  and a  $C^1$ -map  $(u, v)^{\tau}: \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$T^{\tau_{n_k}} \rightarrow T^* \quad \text{as } k \rightarrow \infty,$$

$$(u^{n_k}, v^{n_k}, \dot{u}^{n_k}, \dot{v}^{n_k}) \rightarrow (u, v, \dot{u}, \dot{v}) \quad \text{as } k \rightarrow \infty$$

uniformly on any compact subset of  $\mathbb{R}$

and  $(u, v)^{\tau}$  satisfies

$$\begin{cases} \dot{u}(t) = -2\tau^0 \mu u(t) + 2\tau^0 f'(0) v(t), \\ \dot{v}(t) = -2\tau^0 \mu v(t) + 2\tau^0 f'(0) u(t-1). \end{cases} \quad (4.1)$$

Moreover,  $\|(u, v)^{\tau}\|_{\infty} = 1$ . Thus  $(u, v)$  is nonzero and periodic with a period  $T^*$ , which implies that  $(u_t, v(t))^{\tau} \in R$  and  $V((u_t, v(t))^{\tau}) = \lim_{k \rightarrow \infty} V((p_t^{\tau_{n_k}}, q^{\tau_{n_k}}(t))^{\tau}) = 4$  for all  $t \in \mathbb{R}$ . The boundedness of  $(u, v)^{\tau}$  implies that  $(u_0, v(0))^{\tau}$  is in the center space of the linear system (4.1). But

we know that for  $\tau^0 > \tau_s$  the zeros of the characteristic equation of (4.1), given by

$$(\lambda + 2\tau^0\mu + 2\tau^0 f'(0) e^{-\lambda/2})(\lambda + 2\tau^0\mu - 2\tau^0 f'(0) e^{-\lambda/2}) = 0,$$

are given by a real  $\lambda_0$  and a sequence of complex conjugate pairs  $\{\lambda_j, \bar{\lambda}_j\}_{j \geq 1}$  such that

$$\lambda_0 > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \begin{cases} 0, \\ \operatorname{Re} \lambda_3 > \dots \end{cases}$$

$$\operatorname{Im} \lambda_1 \in (\pi, 2\pi),$$

$$\operatorname{Im} \lambda_{2j} \in (2(2j - 1)\pi, 4j\pi), \quad j \geq 1,$$

$$\operatorname{Im} \lambda_{2j+1} \in (4j\pi, 2(2j + 1)\pi), \quad j \geq 1.$$

Thus, if  $\varphi \neq 0$  is in the center space of (4.1), then  $V(\varphi) \geq 6$  by (ix) of Lemma 2.2, a contradiction. This proves that  $(p^\tau, q^\tau)^{tr} \rightarrow (p^{\tau^0}, q^{\tau^0})^{tr}$  as  $\tau \rightarrow \tau^0$  uniformly on any compact subset of  $\mathbb{R}$ . This completes the proof.

Consider now the system of variational equations

$$\begin{cases} \dot{x}(t) = -2\tau\mu x(t) + 2\tau f'(q^\tau(t)) y(t), \\ \dot{y}(t) = -2\tau\mu y(t) + 2\tau f'(p^\tau(t-1)) x(t-1). \end{cases} \quad (4.2_\tau)$$

The monodromy operator  $M^\tau: C(\mathbb{K}) \rightarrow C(\mathbb{K})$  is defined by  $M^\tau\varphi = (x_{T^\tau}^\varphi, y^\varphi(T^\tau))^{tr}$  for each  $\varphi \in C(\mathbb{K})$ , where the pair  $x^\varphi: [-1, \infty) \rightarrow \mathbb{R}$  and  $y^\varphi: [0, \infty) \rightarrow \mathbb{R}$  is the solution of (4.2 $_\tau$ ) passing through  $\varphi$ . Let  $X(t) = x(\frac{t}{2}) + y(\frac{t+1}{2})$  and  $Y(t) = x(\frac{t}{2}) - y(\frac{t+1}{2})$  for  $t \geq -1$ . Then  $X$  and  $Y$  satisfy

$$\dot{X}(t) = -\tau\mu X(t) + \tau f'(r^\tau(t-1)) X(t-1) \quad (4.3_\tau)$$

and

$$\dot{Y}(t) = -\tau\mu Y(t) - \tau f'(r^\tau(t-1)) Y(t-1), \quad (4.4_\tau)$$

respectively, for  $t \geq 0$ . Let  $M_1^\tau: C([-1, 0]; \mathbb{R}) \ni \phi \mapsto X_{\omega^\tau}^\phi \in C([-1, 0]; \mathbb{R})$  and  $M_2^\tau: C([-1, 0]; \mathbb{R}) \ni \psi \mapsto Y_{\omega^\tau}^\psi \in C([-1, 0]; \mathbb{R})$  be the monodromy operators of (4.3 $_\tau$ ) and (4.4 $_\tau$ ), respectively, where  $X^\phi: [-1, \infty) \rightarrow \mathbb{R}$  is the solution of (4.3 $_\tau$ ) passing through  $\phi$  and  $Y^\psi: [-1, \infty) \rightarrow \mathbb{R}$  is the solution of (4.4 $_\tau$ ) passing through  $\psi$ . As  $\omega^\tau \in (1, 2)$  and  $T^\tau = \omega^\tau/2$ , by using the Arzela–Ascoli Theorem and the variational equations, we obtain that the operators  $(M^\tau)^2$ ,  $M_1^\tau$ ,  $M_2^\tau$ , are compact. Let  $\sigma^\tau$ ,  $\sigma_1^\tau$  and  $\sigma_2^\tau$  denote the spectrum of  $M^\tau$ ,  $M_1^\tau$  and  $M_2^\tau$ , respectively. Then every nonzero element of them is an eigenvalue of finite multiplicity of the corresponding



monodromy operator, which is called the Floquet multiplier of the corresponding system.

**PROPOSITION 4.2.**  $\sigma^\tau = \sigma_1^\tau \cup \sigma_2^\tau$ .

*Proof.* It is easy to see that  $0 \in \sigma^\tau \cap \sigma_1^\tau \cap \sigma_2^\tau$ . Now, let  $0 \neq \lambda \in \sigma^\tau$ . Pick an eigenvector  $0 \neq \varphi \in C(\mathbb{K}; \mathbb{C})$  associated with the eigenvalue  $\lambda$ . Then there is a global solution  $z^\varphi = (x^\varphi, y^\varphi)^{tr}: \mathbb{R} \rightarrow \mathbb{C}^2$  of (4.2 $_\tau$ ) such that  $z_{T^\tau}^\varphi = \lambda\varphi$ . Therefore, we have

$$(x^\varphi(t + T^\tau), y^\varphi(t + T^\tau))^{tr} = \lambda(x^\varphi(t), y^\varphi(t))^{tr} \quad \text{for all } t \in \mathbb{R}. \quad (4.5)$$

Let

$$\begin{aligned} X_0(\theta) &= x^\varphi\left(\frac{\theta}{2}\right) + y^\varphi\left(\frac{\theta+1}{2}\right), \\ Y_0(\theta) &= x^\varphi\left(\frac{\theta}{2}\right) - y^\varphi\left(\frac{\theta+1}{2}\right), \end{aligned} \quad \theta \in [-1, 0].$$

Then  $X_0, Y_0 \in C([-1, 0]; \mathbb{C})$ . We claim that either  $X_0 \neq 0$  or  $Y_0 \neq 0$ . Otherwise, we have

$$\begin{cases} x^\varphi\left(\frac{\theta}{2}\right) = 0, \\ y^\varphi\left(\frac{\theta+1}{2}\right) = 0, \end{cases} \quad \theta \in [-1, 0].$$

Therefore, we get  $\varphi = 0$  by using system (4.2 $_\tau$ ), which is a contradiction to the choice of  $\varphi$ . Thus the claim is proved.

If  $X_0 \neq 0$ , let  $X(t) = x^\varphi(\frac{t}{2}) + y^\varphi(\frac{t+1}{2})$  for all  $t \in \mathbb{R}$ . Then  $X$  satisfies (4.3 $_\tau$ ) and  $X_{\omega^\tau} = \lambda X_0$  from (4.5), which implies that  $\lambda \in \sigma_1^\tau$ . Similarly, if  $Y_0 \neq 0$ , then  $\lambda \in \sigma_2^\tau$ . Hence,  $\sigma^\tau \subseteq \sigma_1^\tau \cup \sigma_2^\tau$ .

Conversely, let  $0 \neq \lambda \in \sigma_1^\tau \cup \sigma_2^\tau$ . If  $\lambda \in \sigma_1^\tau$ , choose an eigenvector  $0 \neq \psi \in C([-1, 0]; \mathbb{C})$  of  $M_1^\tau$  associated with  $\lambda$ , then we have a solution  $X^\psi: \mathbb{R} \rightarrow \mathbb{C}$  of equation (4.3 $_\tau$ ) with  $X_0^\psi = \psi$  and  $X_{\omega^\tau}^\psi = \lambda\psi$ . Therefore, we have

$$X^\psi(t + \omega^\tau) = \lambda X^\psi(t) \quad \text{for all } t \in \mathbb{R}. \quad (4.6)$$

Define  $\varphi \in C(\mathbb{K}; \mathbb{C})$  by

$$\varphi(\theta) = \begin{cases} X^\psi(2\theta), & \theta \in [-1, 0], \\ X^\psi(-1), & \theta = 1. \end{cases}$$

Obviously,  $\varphi \neq 0$  and it is easy to check that

$$\begin{cases} x(t) = X^\psi(2t), \\ y(t) = X^\psi(2t - 1), \end{cases} \quad t \in \mathbb{R}$$

is a solution of (4.2<sub>τ</sub>) passing through  $\varphi$ . (4.6) implies that  $(x_{T^\tau}, y(T^\tau))^{\text{tr}} = \lambda\varphi$ , i.e.,  $\lambda \in \sigma^\tau$ . Similarly, if  $\lambda \in \sigma_2^\tau$ , we have  $\lambda \in \sigma^\tau$ . Thus,  $\sigma_1^\tau \cup \sigma_2^\tau \subseteq \sigma^\tau$ . This completes the proof.

*Remark 4.3.* It is interesting to note that (4.3<sub>τ</sub>) is the variational equation of (2.2). Krisztin *et al.* [9] have got some information about  $\sigma_1^\tau$ . Particularly, there is a positive  $\lambda_u^\tau \in \sigma_1^\tau$  which is larger than 1, and for every  $\lambda \in \sigma_1^\tau \setminus \{0, 1, \lambda_u^\tau\}$ ,  $|\lambda| < 1$ . Moreover, there is a  $\psi_u \in C([-1, 0]; \mathbb{R})$  with  $\psi_u(\theta) > 0$  for  $\theta \in [-1, 0]$  such that  $\psi_u$  is an eigenvector of  $M_1^\tau$  associated with  $\lambda_u^\tau$ . Using the above argument, we can get  $\varphi_u \in \overset{\circ}{K}$  such that  $M^\tau \varphi_u = \lambda_u^\tau \varphi_u$ .

For  $\lambda \in \mathbb{C} \setminus \{0\}$ , let  $G(|\lambda|)$  denote the realified generalized eigenspace of  $M^\tau$  associated with the spectral set  $\{\zeta \in \sigma^\tau \mid |\zeta| = |\lambda|\}$ . Using Theorem 3.1 of [10], the following result can be proved similarly to Proposition 3.8.

**PROPOSITION 4.4.** (i) *If  $0 \neq \lambda \in \sigma_1^\tau$  then the realified generalized eigenspace  $G(|\lambda|) \subseteq V^{-1}(4k) \cup \{0\}$  for some  $k \in \mathbb{N}$ .*

(ii) *If  $0 \neq \lambda \in \sigma_2^\tau$ , then the realified generalized eigenspace  $G(|\lambda|) \subseteq V^{-1}(4k + 2) \cup \{0\}$  for some  $k \in \mathbb{N}$ .*

Proposition 4.4 implies that  $\sigma_1^\tau \cap \sigma_2^\tau = \{0\}$ . Note that  $(\dot{p}_0^\tau, \dot{q}^\tau(0))^{\text{tr}} \in G(1)$  with  $V((\dot{p}_0^\tau, \dot{q}^\tau(0))^{\text{tr}}) = 4$ . These facts combined with Theorem 3.1 of [10] and Remark 4.3 yield that  $O_{4,\tau}$  has at most three Floquet multipliers, counting multiplicities, outside the unit circle and if  $\lambda \neq \lambda_u^\tau$  is such a Floquet multiplier then  $G(|\lambda|) \subset V^{-1}(2) \cup \{0\}$ . In the following, we shall show that  $O_{4,\tau}$  has exactly three Floquet multipliers outside the unit circle.

Recall that  $\omega^\tau$  and the operators  $M_1^\tau, M_2^\tau$  were defined for  $\tau > \tau_s$ . Now we extend their definitions also to  $\tau = \tau_s$ . It is not difficult to see that

$$\lambda_2(\tau^2) = 2 \left( 2\pi - \arccos \frac{\mu}{f'(0)} \right) i.$$

Setting  $\beta_2 = 2\pi - \arccos(\mu/f'(0))$  and using  $\tau^2 = \tau_s$ , we obtain that  $i\beta_2$  is a purely imaginary zero of

$$v + \tau_s \mu - \tau_s f'(0) e^{-v} = 0. \tag{4.7}$$

We define  $\omega^{\tau_s}$  as the minimal period of the function  $t \mapsto e^{i\beta_2 t}$ , that is,

$$\omega^{\tau_s} = \frac{2\pi}{2\pi - \arccos(\mu/f'(0))}.$$

Let  $F_+$  and  $F_-$  be the  $C_0$ -semigroups generated by the solutions of

$$\dot{X}(t) = -\tau_s \mu X(t) + \tau_s f'(0) X(t-1)$$

and

$$\dot{Y}(t) = -\tau_s \mu Y(t) - \tau_s f'(0) Y(t-1),$$

respectively. Define  $M_1^{\tau_s} = D_2 F_+(\omega^{\tau_s}, 0)$  and  $M_2^{\tau_s} = D_2 F_-(\omega^{\tau_s}, 0)$ . Then  $\sigma(M_1^{\tau_s}) \cap \{\zeta \in \mathbb{C} \mid |\zeta| \geq 1\} = \{1, e^{w^{\tau_s} v_0}\}$ ,  $\sigma(M_2^{\tau_s}) \cap \{\zeta \in \mathbb{C} \mid |\zeta| \geq 1\} = \{e^{\omega^{\tau_s} v_1}, \overline{e^{\omega^{\tau_s} v_1}}\}$ , where  $v_0 > 0$  is a simple zero of (4.7), and  $v_1$  with  $\operatorname{Re} v_1 > 0$  and  $\operatorname{Im} v_1 \in (\frac{\pi}{2}, \pi)$  is a simple zero of

$$v + \tau_s \mu + \tau_s f'(0) e^{-v} = 0. \quad (4.8)$$

Note that  $\operatorname{Im}(\omega^{\tau_s} v_1) \in (\frac{2\pi}{3}, 2\pi)$  since  $\omega^{\tau_s} \in (\frac{4}{3}, 1)$ .

**PROPOSITION 4.5.**  $\operatorname{Im}(\omega^{\tau_s} v_1) \neq \pi$ .

*Proof.* Let  $v_1 = \alpha_1 + i\beta_1$  where  $\beta_1 > 0$ . From (4.7) and (4.8) we get

$$\begin{cases} \tau_s \mu = \tau_s f'(0) \cos \beta_2, \\ \beta_2 = -\tau_s f'(0) \sin \beta_2, \end{cases} \quad (4.9)$$

and

$$\begin{cases} \alpha_1 + \tau_s \mu = -\tau_s f'(0) e^{-\alpha_1} \cos \beta_1, \\ \beta_1 = \tau_s f'(0) e^{-\alpha_1} \sin \beta_1. \end{cases} \quad (4.10)$$

If  $\omega^{\tau_s} \beta_1 = \pi$ , then  $\beta_2 = 2\beta_1$ , and therefore from the second equations of (4.9) and (4.10), we have

$$-2\tau_s f'(0) \sin \beta_1 \cos \beta_1 = 2\tau_s f'(0) e^{-\alpha_1} \sin \beta_1.$$

Consequently,

$$\cos \beta_1 = -e^{-\alpha_1}.$$

Then from the first equations of (4.9) and (4.10), we get

$$\begin{aligned} \tau_s \mu &= \tau_s f'(0) \cos \beta_2 = \tau_s f'(0) [2 \cos^2 \beta_1 - 1] \\ &= \tau_s f'(0) [2e^{-2\alpha_1} - 1] = -\alpha_1 + \tau_s f'(0) e^{-2\alpha_1}. \end{aligned} \tag{4.11}$$

The first and the fourth terms imply (4.12) and the last two terms imply (4.13),

$$\tau_s f'(0) e^{-2\alpha_1} = \tau_s f'(0) - \alpha_1. \tag{4.12}$$

$$e^{-2\alpha_1} = \frac{\mu + f'(0)}{2f'(0)}. \tag{4.13}$$

Equations (4.12) and (4.13) imply that

$$\alpha_1 = \frac{\tau_s [f'(0) - \mu]}{2}.$$

Using this, and the first and third terms of (4.11) as well as (4.10), we have

$$\alpha_1 + \tau_s \mu = \frac{\tau_s [f'(0) + \mu]}{2} = \tau_s f'(0) \sqrt{\frac{\mu + f'(0)}{2f'(0)}} e^{-\tau_s [f'(0) - \mu]/2},$$

or, using the definition of  $\tau_s$ ,

$$\frac{f'(0) + \mu}{2f'(0)} = e^{-(2\pi - \arccos(\mu/f'(0))) \sqrt{(f'(0) - \mu)/(f'(0) + \mu)}}.$$

Then  $\mu/f'(0) \in (0, 1)$  is a zero of the equation

$$\frac{1+x}{2} = e^{-g(x)}$$

in  $(0, 1)$ , where  $g(x) = \sqrt{(1-x)/(1+x)} (2\pi - \arccos x)$ . Equivalently, the equation

$$F(x) = \ln(1+x) - \ln 2 + g(x) = 0 \tag{4.14}$$

has a solution in  $(0, 1)$ . On the other hand,  $\lim_{x \rightarrow 1^-} F(x) = 0$  and  $F'(x) = \frac{2}{1+x} - (2\pi - \arccos(\mu/f'(0)))/(1+x) \sqrt{1-x^2} < 0$  for all  $x \in (0, 1)$ . Consequently,  $F(x) > 0$  for all  $x \in (0, 1)$ , a contradiction. This completes the proof.

PROPOSITION 4.6.

$$M_1^\tau \rightarrow M_1^{\tau_1^0} \quad \text{and} \quad M_2^\tau \rightarrow M_2^{\tau_2^0} \quad \text{as} \quad \tau \rightarrow \tau^0 > \tau_s,$$

$$M_1^\tau \rightarrow M_1^{\tau_s} \quad \text{and} \quad M_2^\tau \rightarrow M_2^{\tau_s} \quad \text{as} \quad \tau \rightarrow \tau_s^+.$$

*Proof.* We only show  $M_1^\tau \rightarrow M_1^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$ , the other cases can be discussed similarly. We complete the proof in four steps.

*Step 1.*  $\omega^\tau \rightarrow \omega^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$ . For  $\tau > \tau_s$ , recall that  $r^\tau$  is the periodic solution of (2.2) so that  $V_+(r_t^\tau) = 2$  for all  $t \in \mathbb{R}$  and  $r^\tau(0) = 0$ ,  $r^\tau(-1) > 0$ . Defining the function

$$u^\tau: \mathbb{R} \ni t \mapsto \frac{r^\tau(t)}{|r^\tau|_\infty} \in \mathbb{R}$$

for all  $\tau > \tau_s$ , where  $|x|_\infty = \max_{t \in \mathbb{R}} |x(t)|$  for any bounded and continuous function  $x: \mathbb{R} \rightarrow \mathbb{R}$ . We easily obtain that

$$\dot{u}^\tau(t) = -\mu\tau u^\tau(t) + \tau \int_0^1 f'(sr^\tau(t-1)) ds u^\tau(t-1)$$

for all  $t \in \mathbb{R}$ . By Proposition 4.1,  $r^\tau \rightarrow 0$  as  $\tau \rightarrow \tau_s^+$  uniformly on  $\mathbb{R}$ . Using the above equation and the Arzèla–Ascoli Theorem we find that each sequence  $(\tau_n)_0^\infty$  in  $(\tau_s, \infty)$  with  $\tau_n \rightarrow \tau_s$  as  $n \rightarrow \infty$  has a subsequence  $(\tau_{n_k})_{k=0}^\infty$ , and there exists a  $C^1$ -function  $u: \mathbb{R} \rightarrow \mathbb{R}$  such that  $u^{\tau_{n_k}} \rightarrow u$  and  $\dot{u}^{\tau_{n_k}} \rightarrow \dot{u}$  as  $k \rightarrow \infty$  uniformly on any compact subset of  $\mathbb{R}$ , moreover

$$\dot{u}(t) = -\mu\tau_s u(t) + \tau_s f'(0) u(t-1)$$

and  $V_+(u_t) \leq 2$  for all  $t \in \mathbb{R}$ ,  $u(0) = 0$  and  $|u|_\infty = 1$ . These facts and Proposition 5.1 from [9] yield that for every real  $t$ ,  $u_t$  belongs to the realified generalized eigenspace of  $A_s$  associated with the spectral set  $\{i\beta_2, -i\beta_2\}$ , where  $\beta_2 = 2\pi - \arccos(\mu/f'(0))$ . From  $\beta_2 \in (\pi, 2\pi)$ ,  $u(0) = 0$ ,  $u(-1) \geq 0$  and  $|u|_\infty = 1$ , it is easy to see that  $u(t) = \sin(\beta_2 t)$ . Then from  $\omega^{\tau_{n_k}} \in (1, 2)$  and from the convergence of  $u^{\tau_{n_k}} \rightarrow u$  on any compact subset of  $\mathbb{R}$  it follows that

$$\omega^{\tau_{n_k}} \rightarrow \omega^{\tau_s} \quad \text{as} \quad k \rightarrow \infty.$$

Hence,  $\omega^\tau \rightarrow \omega^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$  also follows.

*Step 2.* Uniform boundedness of  $X^{\tau, \varphi}(t)$  for  $\varphi \in B_1 = \{\varphi \in C([-1, 0]; \mathbb{R}) \mid \|\varphi\|_0 = 1\}$ ,  $\tau \in [\tau_s, \tau_s + 1]$  and  $t \in [-1, 2]$ . Here  $\|\cdot\|_0$  is the supremum norm on  $C([-1, 0]; \mathbb{R})$ ,  $X^{\tau, \varphi}: [-1, \infty) \rightarrow \mathbb{R}$  denotes the solution of (4.3 $_\tau$ )

with  $X_0^{\tau, \varphi} = \varphi \in C([-1, 0]; \mathbb{R})$  and we assume  $r^{\tau} \equiv 0$ . By the variation-of-constants formula, we have

$$X^{\tau, \varphi}(t) = e^{-\tau\mu t} \varphi(0) + \int_0^t \tau f'(r^{\tau}(\zeta - 1)) e^{-\tau\mu(t-\zeta)} X^{\tau, \varphi}(\zeta - 1) d\zeta, \quad t \geq 0.$$

Note that  $r^{\tau}(\zeta) \in [\zeta^-, \zeta^+]$  for all  $\zeta \in \mathbb{R}$  and  $\tau \geq \tau_s$ . Thus there is an  $N > 0$  such that  $|\tau f'(r^{\tau}(\zeta))| \leq N$  for  $\tau \in [\tau_s, \tau_s + 1]$  and  $\zeta \in \mathbb{R}$ . Therefore,

$$\|X_t^{\tau, \varphi}\|_0 \leq 1 + \int_0^t N \|X_{\zeta}^{\tau, \varphi}\|_0 d\zeta, \quad t \geq 0, \quad \varphi \in B_1, \quad \tau \in [\tau_s, \tau_s + 1].$$

Applying the Gronwall inequality, we get

$$\|X_t^{\tau, \varphi}\|_0 \leq e^{Nt} \leq e^{2N}, \quad 0 \leq t \leq 2, \quad \varphi \in B_1, \quad \tau \in [\tau_s, \tau_s + 1].$$

*Step 3.* The estimate of  $\|X_t^{\tau, \varphi} - X_t^{\tau_s, \varphi}\|_0$  for  $t \in [0, 2]$ . Using Proposition 4.1, for any  $\varepsilon > 0$  we can choose  $\delta_1 \in (0, 1)$  such that for  $0 < \tau - \tau_s < \delta_1$ , we have

$$\begin{aligned} |e^{-\tau\mu t} - e^{-\tau_s\mu t}| &< \varepsilon \quad \text{for all } t \in [0, 2], \\ |\tau f'(r^{\tau}(\zeta - 1)) e^{-\tau\mu(t-\zeta)} - \tau_s f'(0) e^{-\tau_s\mu(t-\zeta)}| &< \varepsilon \\ \text{for } 0 \leq \zeta \leq t \leq 2 &\text{ uniformly.} \end{aligned}$$

Then from

$$\begin{aligned} X^{\tau, \varphi}(t) - X^{\tau_s, \varphi}(t) &= (e^{-\tau\mu t} - e^{-\tau_s\mu t}) \varphi(0) + \int_0^t [\tau f'(r^{\tau}(\zeta - 1)) e^{-\tau\mu(t-\zeta)} \\ &\quad - \tau_s f'(0) e^{-\tau_s\mu(t-\zeta)}] X^{\tau, \varphi}(\zeta - 1) d\zeta \\ &\quad + \int_0^t \tau_s f'(0) e^{-\tau_s\mu(t-\zeta)} (X^{\tau, \varphi}(\zeta - 1) - X^{\tau_s, \varphi}(\zeta - 1)) d\zeta, \end{aligned}$$

we get, for all  $t \in [0, 2]$  and  $\varphi \in B_1$ ,

$$\|X_t^{\tau, \varphi} - X_t^{\tau_s, \varphi}\|_0 \leq (1 + e^{2N}) \varepsilon + \int_0^t N \|X_{\zeta}^{\tau, \varphi} - X_{\zeta}^{\tau_s, \varphi}\|_0 d\zeta.$$

Applying Gronwall's inequality again we obtain

$$\|X_t^{\tau, \varphi} - X_t^{\tau_s, \varphi}\|_0 \leq (1 + e^{2N}) \varepsilon e^{Nt}, \quad 0 \leq t \leq 2, \quad \varphi \in B_1. \quad (4.15)$$

Step 4.  $M_1^\tau \rightarrow M_1^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$ . For any  $\varphi \in B_1$ ,

$$\begin{aligned} \|M_1^\tau \varphi - M_1^{\tau_s} \varphi\|_0 &= \sup_{\theta \in [-1, 0]} |X^{\tau, \varphi}(\omega^\tau + \theta) - X^{\tau_s, \varphi}(\omega^{\tau_s} + \theta)| \\ &\leq \sup_{\theta \in [-1, 0]} |X^{\tau, \varphi}(\omega^\tau + \theta) - X^{\tau_s, \varphi}(\omega^\tau + \theta)| \\ &\quad + \sup_{\theta \in [-1, 0]} |X^{\tau_s, \varphi}(\omega^\tau + \theta) - X^{\tau_s, \varphi}(\omega^{\tau_s} + \theta)|. \end{aligned} \quad (4.16)$$

Note that

$$\begin{aligned} &X^{\tau_s, \varphi}(\omega^\tau + \theta) - X^{\tau_s, \varphi}(\omega^{\tau_s} + \theta) \\ &= (e^{-\tau_s \mu(\omega^\tau + \theta)} - e^{-\tau_s \mu(\omega^{\tau_s} + \theta)}) \varphi(0) \\ &\quad + \int_{\omega^{\tau_s} + \theta}^{\omega^\tau + \theta} \tau_s f'(0) e^{-\tau_s \mu(\omega^\tau + \theta - \zeta)} X^{\tau_s, \varphi}(\zeta - 1) d\zeta \\ &\quad + \int_0^{\omega^{\tau_s} + \theta} \tau_s f'(0) [e^{-\tau_s \mu(\omega^\tau + \theta - \zeta)} - e^{-\tau_s \mu(\omega^{\tau_s} + \theta - \zeta)}] X^{\tau_s, \varphi}(\zeta - 1) d\zeta, \end{aligned}$$

since  $\omega^\tau \in (1, 2)$  and  $\omega^{\tau_s} \in (1, 2)$ . By Step 1, we know that  $\omega^\tau \rightarrow \omega^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$ . There exists  $\delta_2 \in (0, 1)$  such that when  $0 < \tau - \tau_s < \delta_2$ , we have

$$|X^{\tau_s, \varphi}(\omega^\tau + \theta) - X^{\tau_s, \varphi}(\omega^{\tau_s} + \theta)| \leq \varepsilon, \quad -1 \leq \theta \leq 0. \quad (4.17)$$

Therefore, when  $0 < \tau - \tau_s < \min\{\delta_1, \delta_2\}$ , (4.15)–(4.17) combined yield

$$\|M_1^\tau \varphi - M_1^{\tau_s} \varphi\|_0 \leq \varepsilon N'$$

for some constant  $N'$ , which implies that  $M_1^\tau \rightarrow M_1^{\tau_s}$  as  $\tau \rightarrow \tau_s^+$ . This completes the proof.

Proposition 4.5 and  $\text{Im } \omega^{\tau_s} v_1 \in (\frac{2\pi}{3}, 2\pi)$  imply that  $e^{\omega^{\tau_s} v_1}$  is a non-real element of  $\sigma(M_2^{\tau_s})$  with  $|e^{\omega^{\tau_s} v_1}| > 1$ . Therefore, it is simple by Theorem 3.1 of [10]. Because of the discrete property of the Floquet multipliers, we can find a connected bounded open set  $O$  in  $\{\zeta \in \mathbb{C} \mid |\zeta| > 1\}$  which contains no other element of  $\sigma(M_2^{\tau_s})$  but  $e^{\omega^{\tau_s} v_1}$ , and for  $\zeta \in O$  we have  $\text{Im } \zeta \neq 0$ . Let  $B$  be the boundary of  $O$  and

$$Q^\tau = \frac{1}{2\pi i} \int_B (zI - M_2^\tau)^{-1} dz.$$

By the spectral theory of perturbations of linear operators (see, for example, [7, pp.213–214]), there exists a  $\varepsilon_0 > 0$  such that when  $0 < \tau - \tau_s < \varepsilon_0$ , we have

$$\begin{aligned} \dim \text{Range } Q^\tau &= \dim \text{Range } Q^{\tau_s} \\ &= \text{the multiplicity of } e^{\omega^{\tau_s} v_1} \text{ as an eigenvalue of } M_{\tau_s}^\tau = 1. \end{aligned}$$

Therefore, if  $\tau \in [\tau_s, \tau_s + \varepsilon_0)$ , then there is a simple complex  $\lambda_2^\tau \in \sigma_2^\tau$  in  $O$ . By Theorem 3.1 of [10] and Proposition 4.4, we get the following result:

**PROPOSITION 4.7.** *There exists a  $\varepsilon_0 > 0$  such that for  $\tau \in (\tau_s, \tau_s + \varepsilon_0)$ ,  $\sigma^\tau \cap \{\zeta \in \mathbb{C} \mid |\zeta| > 1\} = \{\lambda_u^\tau, \lambda_2^\tau, \overline{\lambda_2^\tau}\}$  with  $\lambda_u^\tau \in \sigma_1^\tau$  and  $\lambda_2^\tau \in \sigma_2^\tau$  and  $\lambda_u^\tau > |\lambda_2^\tau|$ .*

Now, let

$$A = \left\{ \tau > \tau_s \left| \begin{array}{l} \text{there exist exactly three Floquet multipliers, counting} \\ \text{multiplicities for } O_{4,\tau} \text{ out side the unit circle} \\ \text{and the associated realified generalized eigenspaces} \\ \text{are contained in } V^{-1}(0) \cup V^{-1}(2) \cup \{0\} \end{array} \right. \right\}.$$

Note that Proposition 4.7 implies  $A \neq \emptyset$ . Using the above spectral theory of perturbations of linear operators similarly to Proposition 4.7, we can show that  $A$  is relatively open in  $(\tau_s, \infty)$ . Now, let  $(\tau_n) \subset A$  with  $\tau_n \rightarrow \tau^0 > \tau_s$  as  $n \rightarrow \infty$ . Let  $\lambda_1^{\tau_n}, \lambda_2^{\tau_n}$  be the other two Floquet multipliers outside the unit circle (not necessarily different) other than  $\lambda_u^{\tau_n}$  for  $O_{4,\tau_n}$  for each  $n \in \mathbb{N}$ . Therefore, by Theorem 3.1 of [10], both  $\lambda_1^{\tau_n}$  and  $\lambda_2^{\tau_n}$  are in  $\sigma_2^{\tau_n}$ . Note that the results in Step 2 of the Proof of Proposition 4.6 ensure that  $(\lambda_1^{\tau_n})$  and  $(\lambda_2^{\tau_n})$  are bounded. Thus, there exist a subsequence  $(n_k)_{k=0}^\infty$  and  $\lambda_1^*$  and  $\lambda_2^*$  such that  $\lambda_1^{\tau_{n_k}} \rightarrow \lambda_1^*$  and  $\lambda_2^{\tau_{n_k}} \rightarrow \lambda_2^*$  as  $k \rightarrow \infty$ . By the aforementioned spectral theory of perturbations of linear operators, we have that both  $\lambda_1^*$  and  $\lambda_2^*$  belong to  $\sigma_2^{\tau^0}$ , and either  $\lambda_1^* \neq \lambda_2^*$  or  $\lambda_1^* = \lambda_2^*$  is a double eigenvalue of  $M_{\tau^0}^{\tau^0}$ . Observe that  $\{\lambda_u^{\tau^0}, 1\} \subset \sigma_1^{\tau^0}$ ,  $G(1) \subseteq V^{-1}(4) \cup \{0\}$  and  $G(\lambda_u^{\tau^0}) \subseteq V^{-1}(0) \cup \{0\}$ . Also recall that  $\sigma_1^{\tau^0} \cap \sigma_2^{\tau^0} = \{0\}$ ,  $1 \leq |\lambda_1^*|$ ,  $1 \leq |\lambda_2^*|$ . If  $|\lambda_1^*| = 1$ , then we have  $G(|\lambda_1^*|) = G(1) \subset V^{-1}(4k+2) \cup \{0\}$  for some  $k \in \mathbb{N}$  by Proposition 4.4(ii) since  $\lambda_1^* \in \sigma_2^{\tau^0}$ , a contradiction. So,  $|\lambda_1^*| > 1$ . Similarly, we can show  $|\lambda_1^*| < \lambda_u^{\tau^0}$  and  $1 < |\lambda_2^*| < \lambda_u^{\tau^0}$ . This means  $\tau^0 \in A$  and hence  $A$  is relatively closed in  $(\tau_s, \infty)$ . Therefore, the connectedness of  $(\tau_s, \infty)$  implies the following main result of this section:

**THEOREM 4.8.**  $A = (\tau_s, \infty)$ .



## 5. DYNAMICS ON $\overline{W_5}$ AND THE CONNECTING ORBITS FROM $O_4$ TO $O_2$

Recall that under the hypotheses (H1)–(H5), system (2.1) has only one periodic orbit in  $V^{-1}(2)$ , denoted by  $O_2$ , and only one periodic orbit in  $V^{-1}(4)$ , denoted by  $O_4$ . Both  $O_2$  and  $O_4$  are in  $\overline{W_5}$ . The purpose of this section is to describe the dynamics of the semiflow on  $\overline{W_5}$ , in particular, we establish the existence of a connecting orbit from  $O_4$  to  $O_2$ .

**PROPOSITION 5.1.** *There exist no homoclinic orbits with respect to the periodic orbits  $O_2$  and  $O_4$  (that is, there exists no solution  $(x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  of (2.1) such that  $(x_0, y(0))^{tr} \notin O_2$  ( $\notin O_4$ ) and  $\alpha((x_0, y(0))^{tr}) = \omega((x_0, y(0))^{tr}) = O_2$  ( $= O_4$ )).*

*Proof.* We only show that  $O_4$  is not homoclinic, the case for  $O_2$  is simpler and can be dealt with similarly. By way of contradiction, let  $z = (x, y)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  be a solution of system (2.1) such that  $\varphi := (x_0, y(0))^{tr} \notin O_4$  and  $\alpha(\varphi) = \omega(\varphi) = O_4$ . Recall that  $O_4$  is determined by  $(p, q)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  with the minimal period  $T \in (\frac{1}{2}, 1)$ . It is easy to show that

$$V(z_t) = 4 \quad \text{with} \quad z_t = (x_t, y(t))^{tr} \quad \text{for all} \quad t \in \mathbb{R}.$$

This yields that both  $x$  and  $y$  are oscillating and all their zeros are simple since  $z_t \in R$  for all  $t \in \mathbb{R}$ . Let  $A = \{z_t \mid t \in \mathbb{R}\}$  and let  $(t_n)_0^\infty$  be such that  $t_n \rightarrow -\infty$  and  $z_{t_n} \rightarrow (p_0, q(0))^{tr}$  as  $n \rightarrow \infty$ . Note that  $z_- \ll (p_0, q(0))^{tr} \ll z_+$ . Therefore, for any  $t \in \mathbb{R}$ , there is a  $t_n < t$  such that  $z_- \ll z_{t_n} \ll z_+$ , from which it follows that  $z_- \ll z_t \ll z_+$  by the monotonicity of  $\Phi$ . So  $A$  is invariant and bounded. Using system (2.1) we know that  $\{\dot{z}_t \mid t \in \mathbb{R}\}$  is also bounded. Consequently,  $A$  is an invariant and relatively compact subset of  $C(\mathbb{K})$  by the Arzèla–Ascoli Theorem.

Note that if  $t \neq s$  in  $\mathbb{R}$ , then  $z_t \neq z_s$ . Otherwise,  $z$  is periodic and thus  $\varphi \in O_4$  since  $\alpha(\varphi) = O_4$ , a contradiction. Now, we want to show that

$$V(z_t - z_s) = 4 \quad \text{for all} \quad t \neq s \text{ in } \mathbb{R}. \quad (5.1)$$

Note that  $z_t \in S$  for all  $t \in \mathbb{R}$ . Therefore,

$$V(z_t - z_s) \geq 2 \quad \text{for all} \quad t \neq s \text{ in } \mathbb{R}. \quad (5.2)$$

To show (5.1), we first show

$$V(z_t - z_s) \leq 4 \quad \text{for all} \quad t \neq s \text{ in } \mathbb{R}. \quad (5.3)$$

Let us consider the two sequences  $(z_{t-nT})_0^\infty$  and  $(z_{s-nT})_0^\infty$ . Using the compactness of  $A$  and  $\alpha(\varphi) = O_4$ , there exist a strictly increasing sequence  $(n_k)_{k=0}^\infty$  and reals  $t^1, s^1$  in  $[0, T)$  so that

$$z_{t-n_k T} \rightarrow (p_{t^1}, q(t^1))^{tr} \quad \text{and} \quad z_{s-n_k T} \rightarrow (p_{s^1}, q(s^1))^{tr} \quad \text{as } k \rightarrow \infty.$$

Now we complete the proof of (5.3) in two cases.

*Case 1.*  $t^1 \neq s^1$ . Then both  $z_{t+1-n_k T} \rightarrow (p_{t^1+1}, q(t^1+1))^{tr}$  and  $z_{s+1-n_k T} \rightarrow (p_{s^1+1}, q(s^1+1))^{tr}$  in  $C^1(\mathbb{K})$  as  $k \rightarrow \infty$ . Recall that  $V_+(r_t - r_s) = 2$  for all  $t, s$  in  $\mathbb{R}$  such that  $r_t \neq r_s$ . Similar arguments to those in the proof of Proposition 3.8 yield that if  $t, s$  in  $\mathbb{R}$  such that  $(p_t, q(t))^{tr} \neq (p_s, q(s))^{tr}$  then  $V((p_t, q(t))^{tr} - (p_s, q(s))^{tr}) = 4$  and hence for such  $t$  and  $s$  we have  $(p_t, q(t))^{tr} - (p_s, q(s))^{tr} \in R$ . By the continuity of  $V$  on  $R$  in the  $C^1$ -topology, we have  $V(z_{t+1-n_k T} - z_{s+1-n_k T}) = 4$  for all sufficiently large  $k \in \mathbb{N}$ . Thus, we have  $V(z_t - z_s) \leq 4$  by the nonincreasing property of  $V$ .

*Case 2.*  $t^1 = s^1$ . For  $\varepsilon \in (0, T)$ , we obtain  $z_{t+\varepsilon-n_k T} \rightarrow (p_{t^1+\varepsilon}, q(t^1+\varepsilon))^{tr} \neq (p_{s^1}, q(s^1))^{tr}$  as  $k \rightarrow \infty$ . For  $0 < \varepsilon < \min\{T, |t-s|\}$ , the arguments in Case 1 are applicable for  $t+\varepsilon$  instead of  $t$ . We obtain  $z_{t+\varepsilon} - z_s \neq 0$  and  $V(z_{t+\varepsilon} - z_s) \leq 4$  for  $0 < \varepsilon < \min\{T, |t-s|\}$ . Then the lower semi-continuity of  $V$  produces

$$V(z_t - z_s) \leq \liminf_{\varepsilon \rightarrow 0^+} V(z_{t+\varepsilon} - z_s) \leq 4.$$

Now, we show that

$$V(z_t - z_s) \geq 4 \quad \text{for all } t \neq s \text{ in } \mathbb{R}, \tag{5.4}$$

Using arguments similar to those for of (5.3). We show it by way of contradiction. If (5.4) is not true, then by (5.2) we can assume that there exist  $t_0 \neq s_0$  in  $\mathbb{R}$  such that  $V(z_{t_0} - z_{s_0}) = 2$ . Then, by the nonincreasing property of  $V$  and (5.2), we have

$$V(z_{t_0+t} - z_{s_0+t}) = 2 \quad \text{and} \quad z_{t_0+t} - z_{s_0+t} \in R \quad \text{for all } t \geq 4. \tag{5.5}$$

Consider the sequences  $(z_{t_0+nT})_0^\infty$  and  $(z_{s_0+nT})_0^\infty$ . Using the relative compactness of  $A$  and  $\omega(\varphi) = O_4$ , there exist a strictly increasing sequence  $(n_k)_0^\infty$  and reals  $t^2$  and  $s^2$  in  $[0, T)$  such that

$$z_{t_0+n_k T} \rightarrow (p_{t^2}, q(t^2))^{tr} \quad \text{and} \quad z_{s_0+n_k T} \rightarrow (p_{s^2}, q(s^2))^{tr} \quad \text{as } k \rightarrow \infty.$$

We will arrive at contradictions in two cases.

*Case A.*  $t^2 \neq s^2$ . Then both  $z_{t_0+1+n_k T} \rightarrow (p_{t^2+1}, q(t^2+1))^{tr}$  and  $z_{s_0+1+n_k T} \rightarrow (p_{s^2+1}, q(s^2+1))^{tr}$  in  $C^1(\mathbb{K})$  as  $k \rightarrow \infty$ . Note that  $V((p_{t^2+1}, q(t^2+1))^{tr} - (p_{s^2+1}, q(s^2+1))^{tr}) = 4$  and  $(p_{t^2+1}, q(t^2+1))^{tr} - (p_{s^2+1}, q(s^2+1))^{tr} \in R$ . Thus,  $V(z_{t_0+1+n_k T} - z_{s_0+1+n_k T}) = 4$  for all sufficiently large  $k \in \mathbb{N}$ , which contradicts (5.5).

*Case B.*  $t^2 = s^2$ . For  $\varepsilon \in (0, T)$ , we obtain  $z_{t_0+\varepsilon+n_k T} \rightarrow (p_{t^2+\varepsilon}, q(t^2+\varepsilon))^{tr} \neq (p_{s^2}, q(s^2))^{tr}$  as  $k \rightarrow \infty$ . Thus, for  $0 < \varepsilon < \min\{T, |t_0 - s_0|\}$ , the arguments in Case A are applicable for  $t_0 + \varepsilon$  instead of  $t_0$ . Using the nonincreasing property of  $V$ , we have  $V(z_{t_0+4+\varepsilon} - z_{s_0+4}) \geq 4$  for  $0 < \varepsilon < \min\{T, |t_0 - s_0|\}$ . Note that  $z_{t_0+3+\varepsilon} \rightarrow z_{t_0+3}$  as  $\varepsilon \rightarrow 0^+$ . Thus  $z_{t_0+4+\varepsilon} \rightarrow z_{t_0+4}$  in  $C^1(\mathbb{K})$  as  $\varepsilon \rightarrow 0^+$ . This, combined with  $z_{t_0+4} - z_{s_0+4} \in R$ , implies that

$$V(z_{t_0+4} - z_{s_0+4}) = \lim_{\varepsilon \rightarrow 0^+} V(z_{t_0+4+\varepsilon} - z_{s_0+4}) \geq 4,$$

a contradiction to (5.5). Therefore, (5.1) is proved.

Note that (5.1) implies that  $\Pi|_A$  is injective, where  $\Pi$  is defined by (3.5). Observe that if  $\chi^0 \in \Pi A$  and  $\psi = \Pi^{-1}(\chi^0)$ , then

$$\left\langle (1, 0)^{tr}, \frac{d}{dt} \Pi z_t^\psi \right\rangle = \dot{x}^\psi(t) = 2\tau f(y^\psi(t)) > 0$$

at any  $t \in \mathbb{R}$  with  $\Pi z_t^\psi \in \mathbb{R}_{0,+}^2$ ,

where  $\mathbb{R}_{0,+}^2 = \{(u, v)^{tr} \mid u = 0, v > 0\} \subset \mathbb{R}^2$ . This, combined with the facts that both  $x$  and  $y$  are oscillating and all their zeros are simple, implies that  $\mathbb{R}_{0,+}^2$  is transversal to the curve  $\Pi A = \{\Pi z_t \mid t \in \mathbb{R}\}$ . Fix  $0 \neq \chi^0 \in \Pi A \cap \mathbb{R}_{0,+}^2$ . Then  $\psi = \Pi^{-1}(\chi^0) \in A$ . Let  $\gamma(\psi)$  be the smallest positive zero of  $x^\psi$  with  $\dot{x}^\psi(\gamma(\psi)) > 0$ . We define the first return map  $\rho: \Pi A \cap \mathbb{R}_{0,+}^2 \rightarrow \Pi A \cap \mathbb{R}_{0,+}^2$  by

$$\rho(\chi^0) = \Pi z_{\gamma(\psi)}^\psi, \quad \psi = \Pi^{-1}(\chi^0).$$

By Lemma 5.9 in [11], the map  $\rho$  is continuous and strictly monotone with respect to the natural ordering of  $\mathbb{R}_{0,+}^2$ , which produces a contradiction to  $\alpha(\varphi) = \omega(\varphi) = O_4$ . This completes the proof.

**PROPOSITION 5.2.** *If  $\varphi \in C(\mathbb{K}) \setminus \{0\}$  with  $\omega(\varphi) = \{0\}$ , then  $V(\varphi) \geq 6$ .*

*Proof.* Assume that there exists  $\varphi \in C(\mathbb{K}) \setminus \{0\}$  so that  $\omega(\varphi) = \{0\}$  and  $V(\varphi) \in \{0, 2, 4\}$ . Then  $z_t^\varphi \rightarrow 0$  as  $t \rightarrow \infty$ , where  $z^\varphi$  is the solution of system

(2.1) with  $z_0^\varphi = \varphi$ . The monotonicity of  $V$  implies  $V(z_t^\varphi) \leq 4$  for all  $t \geq 0$ . We can find a sequence  $(t_n)_0^\infty$  in  $\mathbb{R}^+$  such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\max\{|x^\varphi(t_n)|, |y^\varphi(t_n)|\} = \sup_{s \geq 0} \max\{|x^\varphi(t_n + s)|, |y^\varphi(t_n + s)|\}$$

for all  $n \in \mathbb{N}$ .

For  $n \in \mathbb{N}$ , define  $x^n: [-1, \infty) \rightarrow \mathbb{R}$  and  $y^n: [0, \infty) \rightarrow \mathbb{R}$  by

$$\begin{cases} x^n(t) = \frac{x^\varphi(t_n + t)}{\max\{|x^\varphi(t_n)|, |y^\varphi(t_n)|\}}, & t \geq -1, \\ y^n(t) = \frac{y^\varphi(t_n + t)}{\max\{|x^\varphi(t_n)|, |y^\varphi(t_n)|\}}, & t \geq 0. \end{cases}$$

Then  $\max\{|x^n(0)|, |y^n(0)|\} = 1$  and  $|x^n(t)| \leq 1, |y^n(t)| \leq 1$  for  $t \geq 0$  and  $n \in \mathbb{N}$ . Moreover,

$$\begin{cases} \dot{x}^n(t) = -2\tau\mu x^n(t) + 2\tau \int_0^1 f'(sy^\varphi(t_n + t)) ds y^n(t), \\ \dot{y}^n(t) = -2\tau\mu y^n(t) + 2\tau \int_0^1 f'(sx^\varphi(t_n + t - 1)) ds x^n(t - 1) \end{cases}$$

holds for all  $t > 0$  and  $n \in \mathbb{N}$ . Clearly,

$$\begin{aligned} \int_0^1 f'(sx^\varphi(t_n + t - 1)) ds &\rightarrow f'(0), \\ \int_0^1 f'(sy^\varphi(t_n + t)) ds &\rightarrow f'(0) \end{aligned}$$

as  $n \rightarrow \infty$  uniformly on any compact subset of  $\mathbb{R}^+$ . We apply the Arzèla–Ascoli Theorem to find a subsequence  $((x^{n_k}, y^{n_k})^{tr})_{k=0}^\infty$ , a pair of functions  $x \in C([0, \infty); \mathbb{R}) \cap C^1((1, \infty); \mathbb{R})$  and  $y \in C^1([0, \infty); \mathbb{R})$  such that

$$\begin{aligned} (x^{n_k}, y^{n_k}) &\rightarrow (x, y)^{tr} && \text{uniformly on any compact subset of } [0, \infty), \\ (\dot{x}^{n_k}, \dot{y}^{n_k}) &\rightarrow (\dot{x}, \dot{y})^{tr} && \text{uniformly on any compact subset of } [1, \infty) \end{aligned}$$

as  $k \rightarrow \infty$  and

$$\begin{cases} \dot{x}(t) = -2\tau\mu x(t) + 2\tau f'(0) y(t), \\ \dot{y}(t) = -2\tau\mu y(t) + 2\tau f'(0) x(t - 1), \end{cases} \quad t > 1.$$

Obviously,

$$\max\{|x(t)|, |y(t)|\} \leq \max\{|x(0)|, |y(0)|\} = 1 \quad \text{for all } t \geq 0.$$

The lower semicontinuity of  $V$  and  $V(z_{t_n+t}^\varphi) \leq V(z_t^n) \leq 4$  combined yield

$$V((x_t, y(t))^{tr}) \leq 4 \quad \text{for all } t \geq 1. \tag{5.6}$$

Choose  $\alpha \in (0, \operatorname{Re} \lambda_2)$ . There exists  $M > 0$  so that

$$\|D_2\Phi(t, 0) \varphi\| \leq Me^{\alpha t} \|\varphi\| \quad \text{for all } t \leq 0 \quad \text{and} \quad \varphi \in E_0 \oplus E_1 \oplus E_2.$$

Then for  $\sigma \geq 1$ , we have

$$\begin{aligned} \|Pr_{E_0 \oplus E_1 \oplus E_2}(x_1, y(1))^{tr}\| &= \|D_2\Phi(1 - \sigma, 0) Pr_{E_0 \oplus E_1 \oplus E_2}(x_\sigma, y(\sigma))^{tr}\| \\ &\leq Me^{\alpha(1-\sigma)} \|Pr_{E_0 \oplus E_1 \oplus E_2}\| \|(x_\sigma, y(\sigma))^{tr}\| \\ &\leq Me^{\alpha(1-\sigma)} \|Pr_{E_0 \oplus E_1 \oplus E_2}\|. \end{aligned}$$

Letting  $\sigma \rightarrow \infty$ , we obtain that

$$(x_1, y(1))^{tr} \in Q_2 \setminus \{0\}.$$

Lemma 2.1 combined with (5.6) yields a constant  $k > 1$  such that

$$\|(x_{t-1}, y(t-1))^{tr}\| \leq k \|(x_t, y(t))^{tr}\| \quad \text{for } t \geq 7.$$

Then it is to see that there exist constants  $K > 0$  and  $\beta > 0$  so that

$$\|(x_t, y(t))^{tr}\| \geq Ke^{-\beta t} \quad \text{for all } t \geq 1.$$

This estimate implies that there is a minimal  $k \in \mathbb{N}$  such that  $Pr_{E_k}(x_1, y(1))^{tr} \neq 0$ , where  $E_k$  is the realified generalized eigenspace of the generator of  $\{D_2\Phi(t, 0)\}_{t \geq 0}$  associated with the spectral set  $\{\lambda_k, \overline{\lambda_k}\}$ . Since  $Pr_{E_0 \oplus E_1 \oplus E_2}(x_1, y(1))^{tr} = 0$ , we have  $k \geq 3$ . Note that

$$(x_t, y(t))^{tr} = Pr_{E_k}(x_t, y(t))^{tr} + o(Pr_{E_k}(x_t, y(t))^{tr}) \quad \text{as } t \rightarrow \infty$$

and  $V(Pr_{E_k}(x_t, y(t))^{tr}) \geq 6$  for all  $t \geq 1$  since  $k \geq 3$ . Thus, by the lower semi-continuity of  $V$ , there is a sufficiently large  $t_0 > 1$  such that  $V((x_{t_0}, y(t_0))^{tr}) \geq 6$ , which contradicts (5.6). This completes the proof.

**PROPOSITION 5.3.** *Let  $\varphi \in S \setminus \{0\}$  with  $V(\varphi) \leq 4$ . Then  $\omega(\varphi)$  is a nontrivial periodic orbit.*

*Proof.* Let  $\varphi \in S \setminus \{0\}$  with  $V(\varphi) \leq 4$  be given. Then  $\{z_t^\varphi | t \geq 0\}$  is bounded (Theorem 2.9 of [2]). Assume that  $\omega(\varphi)$  is not a nontrivial

periodic orbit. Let  $\psi \in \omega(\varphi) \setminus \{0\}$  and  $z^\psi$  be the solution of system (2.1) with  $z_0^\psi = \psi$ . Observe that  $V(\psi) \leq 4$ . The Poincaré–Bendixson Theorem obtained by Mallet-Paret and Sell [11] tells us that

$$\alpha(\psi) \cup \omega(\psi) \subseteq E,$$

where  $E$  denotes the set of stationary points of system (2.1). We have

$$\alpha(\psi) \cup \omega(\psi) \subseteq \omega(\varphi) \subseteq S,$$

since  $\omega(\varphi)$  is invariant, and  $S$  is closed and positively invariant. Noticing that  $0$  is the only stationary point in  $S$ , we conclude

$$\alpha(\psi) \cup \omega(\psi) \subseteq \{0\}.$$

Particularly,  $\omega(\psi) = \{0\}$  follows. By Proposition 5.2,  $V(\psi) \geq 6$ , which gives a contradiction. Consequently,  $\omega(\varphi) = \{0\}$  which again leads to a contradiction to Proposition 5.2. This completes the proof.

**PROPOSITION 5.4.** *For  $\varphi \in (W_5 \cap S) \setminus \{0\}$ ,  $\alpha(\varphi) = 0$  and  $\omega(\varphi) = O_2$  or  $O_4$  depending upon whether there is a  $t_0 \in \mathbb{R}$  such that  $V(z_{t_0}^\varphi) = 2$  or not.*

*Proof.*  $z_t^\varphi \rightarrow 0$  as  $t \rightarrow -\infty$  follows directly from (viii) of Lemma 2.2, which also implies  $V(\varphi) \leq 4$ . By Proposition 5.3,  $\omega(\varphi)$  is a nontrivial periodic orbit, which is in  $V^{-1}(2) \cup V^{-1}(4)$ . By the uniqueness of periodic orbits in  $V^{-1}(2)$  and  $V^{-1}(4)$ , we get  $\omega(\varphi) = O_2$  or  $O_4$ . The remaining part of Proposition 5.4 is obvious due to the monotonicity of  $V$ . This completes the proof.

**PROPOSITION 5.5.** *For  $\varphi \in W_5 \setminus S$ ,  $\alpha(\varphi) = \{0\}$  and  $\omega(\varphi) = \{z_+\}$  or  $\{z_-\}$  depending upon whether  $\varphi$  is above or below  $S$ .*

*Proof.* The results follow directly from (vi) and (viii) of Lemma 2.2.

**PROPOSITION 5.6.** *If  $\varphi \in bdW_5 \setminus (S \cup \{z_-, z_+\})$ , then  $\alpha(\varphi) = O_4$  or  $O_2$  and  $\omega(\varphi) = \{z_+\}$  or  $\{z_-\}$ .*

*Proof.* The conclusion that  $z_t^\varphi \rightarrow z_+$  or  $z_-$  as  $t \rightarrow \infty$  follows from (vi) of Lemma 2.2. Note that  $\alpha(\varphi) \subset V^{-1}(0) \cup V^{-1}(2) \cup V^{-1}(4) \cup \{0\}$  by (vii) of Lemma 2.2 and the invariance of  $bdW_5$ . Therefore, we only need to show that  $\alpha(\varphi)$  is a nontrivial periodic orbit. If this is true, then  $\alpha(\varphi) \subset V^{-1}(2) \cup V^{-1}(4)$  by Proposition 3.5, and hence the result follows from the uniqueness of periodic orbits in  $V^{-1}(2)$  and  $V^{-1}(4)$ . Let us assume that  $\varphi$  is above  $S$  and hence  $\omega(\varphi) = \{z_+\}$ . The case of  $\varphi$  being below  $S$  is similar.

By way of contradiction, assume  $\alpha(\varphi)$  is not a nontrivial periodic orbit. We prove that  $\alpha(\varphi) = \{z_+\}$ . Let  $\psi \in \alpha(\varphi)$ , then by the Poincaré–Bendixson Theorem,

$$\alpha(\psi) \cup \omega(\psi) \subseteq \{z_+, 0, z_-\}.$$

Note that  $0 \notin \alpha(\psi)$  since  $\alpha(\psi) \subseteq \alpha(\varphi) \subseteq \text{bd}W_5$  and  $0 \notin \text{bd}W_5$ . Hence,  $\alpha(\psi) \subseteq \{z_+, z_-\}$ . By Proposition 5.2 and the monotonicity of  $\Phi$ ,  $\omega(\psi) = \{z_+\}$  or  $\{z_-\}$ . We claim that  $\omega(\psi) = \{z_+\}$ . If not,  $\omega(\psi) = \{z_-\}$ , then there exists a  $t^* \in \mathbb{R}$  such that  $z_{t^*}^\psi \ll 0$  since  $z_- \ll 0$ . Note that  $\psi \in \alpha(\varphi)$ . There exists a sequence  $(t_n)_0^\infty$  such that  $t_n \rightarrow -\infty$  and  $z_{t_n}^\varphi \rightarrow \psi$  as  $n \rightarrow \infty$ . Then  $z_{t_n + t^*}^\varphi \rightarrow z_{t^*}^\psi$  as  $n \rightarrow \infty$  by (iv) of Lemma 2.2. Thus, there is an  $s_{n_0}$  such that  $z_{s_{n_0} + t^*}^\varphi \ll 0$ , which contradicts  $\omega(\varphi) = \{z_+\}$  since  $z_t^\varphi \ll 0$  for  $t \geq s_{n_0} + t^*$  by using the monotonicity of  $\Phi$ . Then  $\alpha(\psi) = \{z_+\}$ , for otherwise  $\alpha(\psi) = \{z_-\}$  implies that there is a  $t_0 \in \mathbb{R}$  such that  $z_{t_0}^\psi \ll 0$ , and hence the monotonicity of  $\Phi$  implies  $z_t^\psi \ll 0$  for all  $t \geq t_0$ , which contradicts  $\omega(\psi) = \{z_+\}$ . From  $\alpha(\psi) = \{z_+\}$  and the monotonicity of  $\Phi$ , we know that  $z_t^\psi \gg 0$  for all  $t \in \mathbb{R}$ . Now we claim that  $\psi = z_+$ .

To prove the claim, we first assert that either  $x^\psi$  or  $y^\psi$  is monotone. If not, then as  $x^\psi(t) \rightarrow \xi^+$  for  $t \rightarrow \pm\infty$ , both  $x^\psi$  and  $y^\psi$  have minimal values which are less than  $\xi^+$ . Let  $t_0 \in \mathbb{R}$  be such that

$$x^\psi(t_0) = \inf_{t \in \mathbb{R}} x^\psi(t) < \xi^+.$$

Then from

$$0 = \dot{x}^\psi(t_0) = -2\tau\mu x^\psi(t_0) + 2\tau f(y^\psi(t_0)),$$

we have  $y^\psi(t_0) < x^\psi(t_0)$  using the remarks following (H4). Thus,

$$\inf_{t \in \mathbb{R}} y^\psi(t) < \inf_{t \in \mathbb{R}} x^\psi(t). \tag{5.7}$$

On the other hand, let  $t_1 \in \mathbb{R}$  be such that

$$y^\psi(t_1) = \inf_{t \in \mathbb{R}} y^\psi(t) < \xi^+.$$

Then, from

$$0 = \dot{y}^\psi(t_1) = -2\tau\mu y^\psi(t_1) + 2\tau f(x^\psi(t_1 - 1)),$$

we have  $x^\psi(t_1 - 1) < y^\psi(t_1)$ , which contradicts (5.7). This shows that either  $x^\psi$  or  $y^\psi$  is monotone. Consequently, either  $x^\psi \equiv \xi^+$  or  $y^\psi \equiv \xi^+$  holds due to  $\alpha(\psi) = \{z_+\}$  and  $\omega(\psi) = \{z_+\}$ . Then system (2.1) and the remarks following (H4) combined yield  $x^\psi \equiv y^\psi \equiv \xi^+$ . Therefore,  $\psi = z_+$ .

Now  $\alpha(\varphi) = \omega(\varphi) = \{z_+\}$ . The same argument as above yields  $\varphi = z_+$ , which contradicts our assumption. The proof is complete.

**PROPOSITION 5.7.**  $\alpha(\varphi) = O_4$  and  $\omega(\varphi) = O_2$  for every  $\varphi \in (bdW_5 \cap S) \setminus (O_2 \cup O_4)$ .

*Proof.* By (vii) of Lemma 2.2 and Proposition 5.3,  $\omega(\varphi)$  is a nontrivial periodic orbit. If we can show that  $\alpha(\varphi)$  is also a nontrivial periodic orbit, then by Proposition 5.1 and the uniqueness of periodic orbits in  $V^{-1}(4)$  and  $V^{-1}(2)$  we get  $\alpha(\varphi) = O_4$  and  $\omega(\varphi) = O_2$ , since both  $\alpha(\varphi)$  and  $\omega(\varphi)$  are in  $V^{-1}(2) \cup V^{-1}(4)$ .

If  $\alpha(\varphi)$  is not a nontrivial periodic orbit and  $\psi \in \alpha(\varphi)$ , then we can apply the Poincaré–Bendixson Theorem again to get

$$\alpha(\psi) \cup \omega(\psi) \subseteq \{z_+, 0, z_-\}.$$

As  $\alpha(\psi) \cup \omega(\psi) \subseteq \alpha(\varphi) \subseteq S$ , we must have  $\alpha(\psi) = \{0\}$ . Then (viii) of Lemma 2.2 implies that  $\psi \in W_5$ , which is a contradiction since  $\psi \in \alpha(\varphi) \subseteq bdW_5$  by the invariance of  $bdW_5$ . This completes the proof.

In the remaining part of this paper, we shall show the existence of connecting orbits from  $O_4$  to  $O_2$ .

Let  $(p, q)^{tr}: \mathbb{R} \rightarrow \mathbb{R}^2$  be the solution of (2.1) with minimal period  $T > 0$  determining the orbit  $O_4$  as in Section 4. For the sake of convenience, denote  $\varphi_0 = (p_0, q(0))^{tr} \in C(\mathbb{K})$ . For  $r > 0$ , let  $C(\mathbb{K})_{\leq r}$  and  $C(\mathbb{K})_{r <}$  denote the realified generalized eigenspaces of the monodromy operator  $M = D_2\Phi(T, \varphi_0)$  associated with the spectral sets  $\{\zeta \in \sigma \mid |\zeta| \leq r\}$  and  $\{\zeta \in \sigma \mid |\zeta| > r\}$ , respectively. Recall that  $M$  has exactly three Floquet multipliers  $\lambda_u, \lambda_1, \lambda_2$  with  $\lambda_u > |\lambda_1| \geq |\lambda_2|$  outside the unit circle. Choose  $\lambda \in (0, 1)$  so that

$$\lambda > \max \left\{ \frac{1}{|\lambda_2|}, \max_{\zeta \in \sigma, |\zeta| < 1} |\zeta| \right\}.$$

Then, by Theorem I.3 of [9], there exist convex open neighborhoods  $N_{1 <}$  of 0 in  $C(\mathbb{K})_{1 <}$ ,  $N_{\leq 1}$  of 0 in  $C(\mathbb{K})_{\leq 1}$ , and a  $C^1$ -map

$$w^u: N_{1 <} \rightarrow C(\mathbb{K})_{\leq 1}$$

so that

$$w^u(0) = 0, \quad Dw^u(0) = 0, \quad w^u(N_{1 <}) \subseteq N_{\leq 1},$$

and with  $N^u = N_{\leq 1} + N_{1 <}$  the shifted graph

$$W^u(\varphi_0, N^u) = \{ \varphi_0 + \chi + w^u(\chi) \mid \chi \in N_{1 <} \}$$



is equal to the set

$$\left\{ \chi \in \varphi_0 + N^u \left| \begin{array}{l} \text{there is a trajectory } (\chi^n)_{-\infty}^0 \text{ of } \Phi(T, \cdot) \\ \text{with } \chi^0 = \chi, \lambda^n(\chi^n - \varphi_0) \in N^u \\ \text{for all } n \in -\mathbb{N} \text{ and } \lambda^n(\chi^n - \varphi_0) \rightarrow 0 \\ \text{as } n \rightarrow -\infty \end{array} \right. \right\}.$$

The unstable set  $W^u(O_4)$  of the periodic orbit  $O_4$  is defined as

$$W^u(O_4) = \Phi(\mathbb{R}^+ \times W^u(\varphi_0, N^u)).$$

Now let  $O_s = \{r_t \mid t \in [0, \omega]\}$  be the orbit of (2.2) with minimal period  $\omega > 0$  in  $\overline{W_s} \cap V_+^{-1}(2)$  and such that  $p(t) = r(2t)$  and  $q(t) = r(2t - 1)$ . We know that the spectrum of the monodromy operator  $M_1 = D_2F(\omega, r_0)$  is contained in  $\sigma$  and it has exactly one Floquet multiplier  $\lambda_u$  outside the unit circle. Similarly, for the  $\lambda$  chosen above, there exist convex open neighborhoods  $\tilde{N}_{1<}$  of 0 in  $C_{1<}$  and  $\tilde{N}_{\leq 1}$  of 0 in  $C_{\leq 1}$  (where  $C_{1<}$  and  $C_{\leq 1}$  are the realified generalized eigenspaces of  $M_1$  associated with the spectral sets  $\{\zeta \in \sigma_1 \mid |\zeta| > 1\}$  and  $\{\zeta \in \sigma_1 \mid |\zeta| \leq 1\}$ , respectively) and a  $C^1$ -map

$$w_s^u: \tilde{N}_{1<} \rightarrow C_{\leq 1},$$

so that

$$w_s^u(0) = 0, \quad Dw_s^u(0) = 0, \quad w_s^u(\tilde{N}_{1<}) \subseteq \tilde{N}_{\leq 1},$$

and with  $\tilde{N}^u = \tilde{N}_{1<} + \tilde{N}_{\leq 1}$  the shifted graph

$$W^u(r_0, \tilde{N}^u) = \{r_0 + \chi + w_s^u(\chi) \mid \chi \in \tilde{N}_{1<}\}$$

is equal to the set

$$\left\{ \chi \in r_0 + \tilde{N}^u \left| \begin{array}{l} \text{there is a trajectory } (\chi^n)_{-\infty}^0 \text{ of } F(\omega, \cdot) \\ \text{with } \chi^0 = \chi, \lambda^n(\chi^n - r_0) \in \tilde{N}^u \\ \text{for all } n \in -\mathbb{N} \text{ and } \lambda^n(\chi^n - r_0) \rightarrow 0 \\ \text{as } n \rightarrow -\infty \end{array} \right. \right\}.$$

LEMMA 5.8.  $(W^u(r_0, \tilde{N}^u))_s := \{^s\varphi \mid \varphi \in W^u(r_0, \tilde{N}^u)\} \subseteq W^u(O_4)$ .

*Proof.* Let  $\psi \in W^u(r_0, \tilde{N}^u)$ . Then there exists a solution  $x^\psi: \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (2.2) such that

$$x_0^\psi = \psi \quad \text{and} \quad \lambda^n(x_{no}^\psi - r_0) \rightarrow 0 \quad \text{as } n \rightarrow -\infty. \quad (5.8)$$

Define

$$\begin{cases} x(t) = x^\psi(2t), \\ y(t) = x^\psi(2t - 1), \end{cases} \quad t \in \mathbb{R}.$$

Then  $(x, y)^{tr}$  satisfies system (2.1) and  $(x_0, y(0))^{tr} = {}^s\psi$ .

From  $\lambda^n(x_{n\omega}^\psi - r_0) \rightarrow 0$  as  $n \rightarrow -\infty$  and  $\lambda \in (0, 1)$ , we know  $x_{n\omega}^\psi \rightarrow r_0$  as  $n \rightarrow -\infty$ . Then it is easy to get

$$V_+(x_t^\psi) \leq 2 \quad \text{and} \quad \xi_- \ll x_t^\psi \ll \xi_+ \quad \text{for all } t \in \mathbb{R}. \quad (5.9)$$

Note that  $x^\psi$  satisfies

$$\dot{x}^\psi(t) = -\tau\mu x^\psi(t) + \tau \int_0^1 f'(sx^\psi(t-1)) ds x^\psi(t-1), \quad t \in \mathbb{R}.$$

Let  $b(t) = \tau \int_0^1 f'(sx^\psi(t-1)) ds$ ,  $b_0 = \min_{\xi_- \leq \xi \leq \xi_+} \tau \int_0^1 f'(s\xi) ds$  ( $> 0$ ) and  $b_1 = \max_{\xi_- \leq \xi \leq \xi_+} \tau \int_0^1 f'(s\xi) ds$  ( $> 0$ ). Then  $b_0 \leq b(t) \leq b_1$  for all  $t \in \mathbb{R}$ . Thus, Lemma VI.3 of [9] and (5.9) imply that there is a constant  $k_0 > 0$  such that

$$\|x_{t-1}^\psi\|_0 \leq k_0 \|x_t^\psi\|_0 \quad \text{for all } t \in \mathbb{R}. \quad (5.10)$$

Recall that  $\|\varphi\|_0 = \sup_{-1 \leq \theta \leq 0} |\varphi(\theta)|$  for  $\varphi \in C([-1, 0]; \mathbb{R})$ .

Recall that  $T = \frac{\omega}{2}$ . Using (5.10), we have for  $n \in -\mathbb{N}$ ,

$$\begin{aligned} & \|\lambda^n((x_{nT}, y(nT))^{tr} - \varphi_0)\| \\ &= \lambda^n \max \left\{ \sup_{-1 \leq \theta \leq 0} |x(nT + \theta) - p(\theta)|, |y(nT) - q(0)| \right\} \\ &= \lambda^n \max \left\{ \sup_{-1 \leq \theta \leq 0} |x^\psi(n\omega + 2\theta) - r(2\theta)|, |x^\psi(n\omega - 1) - r(-1)| \right\} \\ &\leq \lambda^n \max \{ \|x_{n\omega-1}^\psi - r_{-1}\|_0, \|x_{n\omega}^\psi - r_0\|_0 \} \\ &\leq \lambda^n \max \{ 1, k_0 \} \|x_{n\omega}^\psi - r_0\|_0, \end{aligned}$$

i.e.,

$$\|\lambda^n((x_{nT}, y(nT))^{tr} - \varphi_0)\| \leq \lambda^n \max \{ 1, k_0 \} \|x_{n\omega}^\psi - r_0\|_0 \quad \text{for all } n \in -\mathbb{N}. \quad (5.11)$$

Therefore,  $\lambda^n((x_{nT}, y(nT))^{tr} - \varphi_0) \rightarrow 0$  as  $n \rightarrow -\infty$  by (5.8). Then there is an  $n_0 \in -\mathbb{N}$  such that

$$\lambda^n((x_{nT}, y(nT))^{tr} - \varphi_0) \in N^u \quad \text{for all } n \leq n_0. \quad (5.12)$$

Especially,  $(x_{n_0 T}, y(n_0 T))^{\text{tr}} - \varphi_0 \in N^u$  since  $\lambda \in (0, 1)$ . Let  $\chi^n = (x_{(n+n_0) T}, y((n+n_0) T))^{\text{tr}}$  for all  $n \in -\mathbb{N}$ . Then  $(\chi^n)_-\infty^0$  is a trajectory of  $\Phi(T, \cdot)$  with  $\chi^0 \in \varphi_0 + N^u$ . Moreover, for all  $n \in -\mathbb{N}$ ,  $\lambda^n(\chi^n - \varphi_0) = \lambda^n((x_{(n+n_0) T}, y((n+n_0) T))^{\text{tr}} - \varphi_0) = \lambda^{-n_0}[\lambda^{(n+n_0)}(x_{(n+n_0) T}, y((n+n_0) T))^{\text{tr}} - \varphi_0]$ . Using (5.8), (5.11), (5.12), and  $\lambda \in (0, 1)$ , we have  $\lambda^n(\chi^n - \varphi_0) \in N^u$  for all  $n \in -\mathbb{N}$  and  $\lambda^n(\chi^n - \varphi_0) \rightarrow 0$  as  $n \rightarrow -\infty$ . Therefore,  $\chi^0 \in W^u(\varphi_0, N^u)$  by definition. Note that  ${}^s\psi = \Phi(-n_0 T, \chi^0)$ . This implies  ${}^s\psi \in W^u(O_4)$ , and hence the proof is completed.

By Proposition 12.1 of [9], we can find  $\phi_{\pm} \in W^u(r_0, \tilde{N}^u)$  such that  $\phi_- \ll r_0 \ll \phi_+$  with respect to the natural partial ordering on  $C([-1, 0]; \mathbb{R})$  and  $\alpha(\phi_+) = \alpha(\phi_-) = O_s$ .  $\phi_{\pm} \notin S_s := \{\psi \in C([-1, 0]; \mathbb{R}) \mid (x^\psi)^{-1}(0) \text{ is not bounded from above}\}$ , since  $S_s$  is a nonordered set and  $r_0 \in S_s$ . If  $\phi_-$  is above  $S_s$ , then  $x^{\phi_-}(t) \rightarrow \xi^+$  as  $t \rightarrow \infty$  (Proposition 4.5 of [9]), where  $x^{\phi_-}: [-1, \infty) \rightarrow \mathbb{R}$  is the solution of equation (2.2) passing through  $\phi_-$ . Thus,  $x_{t^*}^{\phi_-} \gg 0$  for some  $t^* \geq 0$  and hence for  $t \geq t^*$  we have  $0 \ll x_{t^*}^{\phi_-} \ll r_{t^*}$ , a contradiction. So  $\phi_-$  is below  $S_s$  and  $x_{t^*}^{\phi_-} \rightarrow \xi_-$  as  $t \rightarrow \infty$ . Similarly,  $x_{t^*}^{\phi_+} \rightarrow \xi_+$  as  $t \rightarrow \infty$ . Let  $\varphi_{\pm} = {}^s\phi_{\pm}$ . Then  $\varphi_{\pm} \in W^u(O_4)$  by Lemma 5.8, and  $\alpha(\varphi_{\pm}) = O_4$ ,  $\omega(\varphi_{\pm}) = \{z_{\pm}\}$  and  $\varphi_{\pm} \notin S$ . Thus, by Theorem 4.12 of [2],  $\varphi_-$  is below  $S$  and  $\varphi_+$  is above  $S$ . Let  $\varphi_{\pm} = F(s_{\pm}, \psi_{\pm})$  for some  $s_{\pm} \geq 0$  and  $\psi_{\pm} \in W^u(\varphi_0, N^u)$ . Then, obviously,  $\alpha(\psi_{\pm}) = O_4$ ,  $\omega(\psi_{\pm}) = \{z_{\pm}\}$ ,  $\psi_-$  is below  $S$  and  $\psi_+$  is above  $S$ . Note that  $W^u(\varphi_0, N^u)$  is the shifted graph of a  $C^1$ -smooth function from  $C(\mathbb{K})_{1<}$  into  $C(\mathbb{K})_{\leq 1}$  defined on an open neighborhood of 0 in  $C(\mathbb{K})_{1<}$ , we can construct a continuous curve  $\gamma: [-1, 1] \ni s \mapsto \gamma(s) \in W^u(\varphi_0, N^u) \setminus O_4 \subseteq W^u(O_4) \setminus O_4$  with  $\gamma(\pm 1) = \psi_{\pm}$ . Let

$$A_+ = \{s \in [-1, 1] \mid \gamma(s) \text{ is above } S\},$$

$$A_- = \{s \in [-1, 1] \mid \gamma(s) \text{ is below } S\}.$$

Then both  $A_+$  and  $A_-$  are nonempty and relatively open in  $[-1, 1]$  since  $S$  is closed. By the connectedness of  $[-1, 1]$ , there is an  $s^* \in (-1, 1)$  such that  $s^* \notin A_+ \cup A_-$ . Set  $\psi = \gamma(s^*)$ . Then  $\psi \in (W^u(O_4) \cap S) \setminus O_4$ . Since  $\psi \in W^u(O_4)$ , we have  $\alpha(\psi) = O_4$ . Then there exists a sequence  $(t_n)_0^\infty$  such that  $t_n \rightarrow -\infty$  and  $z_{t_n}^\psi \rightarrow \varphi_0$  as  $n \rightarrow \infty$  and hence  $z_{t_n+1}^\psi \rightarrow (p_1, q(1))^{\text{tr}}$  in  $C^1(\mathbb{K})$  as  $n \rightarrow \infty$ . Thus,  $V(z_{t_n+1}^\psi) = 4$  for all sufficiently large negative integers  $n$  by the  $C^1$ -continuity of  $V$  on  $R$  and  $V((p_t, q(t))^{\text{tr}}) = 4$  and  $(p_t, q(t))^{\text{tr}} \in R$  for all  $t \in \mathbb{R}$ . This implies that  $V(\psi) \leq 4$  by the nonincreasing property of  $V$ . Using Propositions 5.1 and 5.3, and the uniqueness of periodic orbits on each level set of  $V$ , we have the following result about the connection from  $O_4$  to  $O_2$ .

**THEOREM 5.9.** *There exists a connecting orbit from  $O_4$  to  $O_2$ .*

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