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POSITIVELY INVARIANT SETS, MONOTONE SOLUTIONS, AND CONTRACTING RECTANGLES IN NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS *

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Abstract. The asymptotic behaviors are studied for solutions to the following neutral functional differential equation(NFDE)

$$\frac{d}{dt}D(x_t) = f(t, x_t),$$

where D is a quasimonotone operator. General results on positively invariant sets, monotone solutions, and contracting rectangles are obtained and are illustrated by an example.

1. Introduction. There has been remarkable advance in our understanding of the asymptotic behaviors of monotone semiflows generated by retarded functional differential equations on partially ordered spaces. The work of [4-10] shows that "almost every" precompact orbit of solutions of RFDEs converges to the set of equilibria under some quasimonotonicity and irreducibility hypotheses. The theory has been applied to various biological models [11,13], and was extended in Wu and Freedman[12] to a class of neutral functional differential equations.

Let R^n_+ be the space of non-negative vectors in R^n . For any x and y in R^n, we write $x \leq y$ if $x_i \leq y_i$ for each $i \in N = \{1, 2, \cdots, n\}$. Given $r = (r_1, \cdots, r_n) \in R^n_+$, we define $|r| = \max\{r_i : 1 \leq i \leq n\}$,

$$C_r = \Pi_{i=1}^n C([-r_i, 0], R), \quad C_r^+ = \Pi_{i=1}^n C([-r_i, 0], R_+).$$

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Obviously, C_r is a strongly ordered Banach space with the uniform convergence topology defined by the norm

$$||\varphi||_{C_r} = \max_{1 \le i \le n} \sup_{-r_i \le \theta \le 0} |\varphi_i(\theta)|, \quad \varphi \in C_r,$$

and the usual functional ordering $\le_{C_r}$ defined by

$$\varphi \le_{C_r} \psi \quad \text{iff} \quad \varphi_i(\theta) \le \psi_i(\theta) \quad \text{for} \quad 1 \le i \le n \quad \text{and} \quad \theta \in [-r_i, 0], \quad \varphi, \psi \in C_r.$$

We write $\varphi <_{C_r} \psi$ if $\varphi \le_{C_r} \psi$ and $\varphi \neq \psi$; $\varphi \ll_{C_r} \psi$ if $\psi - \varphi \in \operatorname{Int} C_r^+$.

Given a bounded and linear operator $D : C_r \to R^n$, by the Riesz representation theorem, we have

$$D_i(\varphi) = \sum_{j=1}^{n} \int_{-r_j}^{0} [d\mu_{ij}(\theta)]\varphi_j(\theta), \quad 1 \le i \le n, \quad \varphi \in C_r, \tag{1.1}$$

where $\mu_{ij} : [-r_j, 0] \to R$, $i, j \in N$, is of bounded variation on $[-r_j, 0]$. The operator D defined by (1.1) is said to be quasimonotone (Wu and Freedman [12]), if

(i). D is atomic at zero, i.e., the matrix $A = (\mu_{ij}(0) - \mu_{ij}(0^-))$ is nonsingular;

(ii). $b_{ii} > 0$, $b_{ij} \ge 0$ for $i \neq j \in N$, where $(b_{ij}) = A^{-1}$;

(iii). $\mu_{ij} : [-r_j, 0) \to R$, $i, j \in N$, is nonincreasing and continuous from the left.

Now we define an ordering, denoted by $\le_D$, as follows

$$\varphi \le_D \psi \quad \text{iff} \quad \varphi \le_{C_r} \psi \quad \text{and} \quad D(\varphi) \le D(\psi), \quad \text{where} \quad \varphi, \psi \in C_r.$$

Let $C_{r,D}^+ = \{\varphi \in C_r : \varphi \ge_D 0\}$, then $\operatorname{Int} C_{r,D}^+ \neq \phi$ provided that D is quasimonotone. See Wu and Freedman [12] for detailed discussions of C_r and its ordering induced by $C_{r,D}^+$.

In this paper, we consider the following NFDE

$$\frac{d}{dt} D(x_t) = f(t, x_t) \tag{1.2}$$

where the operator D is quasimonotone and D, f satisfy conditions such that the solution of the associated Cauchy initial value problem of (1.2) uniquely exists. We will, in Section 2, formulate a quasimonotone condition (QM) which guarantees the monotonicity of the semiflow Φ_t generated by equations (1.2) and the existence of the positively invariant sets. In Section 3, we will construct the quasimonotone comparison systems related to (1.2) which do not necessarily satisfy the (QM) condition, and we will extablish some convergence results by using a family of contracting rectangles. We will illustrate the general results by a simple example.

2. Monotonicity and Positively Invariant Sets. We consider the following NFDE

$$\frac{d}{dt}D(x_t) = f(t, x_t), \tag{2.1}$$

where $D : C_r \to R^n$ is a quasimonotone operator, $f : \Omega \to R^n$ is continuous and Lipschitz with respect to $\varphi \in C_r$ on any compact subset of Ω, where Ω is an open subset of $R \times C_r$. Under these assumptions, the Cauchy initial value problem of (2.1) is well posed. That is, for any $(\sigma, \varphi) \in \Omega$, there exists $\tau(\sigma, \varphi) > \sigma$ and a continuous function, the solution of (2.1) through (σ, φ), $x = (x_1, \cdots, x_n)$ with $x_i : [\sigma - r_i, \tau(\sigma, \varphi)) \to R$, $i \in N$, such that $(t, x_t) \in \Omega$, the mapping $t \to D(x_t) \in R^n$ is differentiable and (2.1) holds for $t \in [\sigma, \tau(\sigma, \varphi))$. Here and in what follows, $x_t = (x_t^1, \cdots, x_t^n)$ with $x_t^i(\theta_i) = x_i(t + \theta_i)$, $\theta_i \in [-r_i, 0]$,$x(t) = x(t; \sigma, \varphi, f)$ denotes the unique solution of (2.1) through (σ, φ). For details, we refer to [1,12].

The following is called the quasimonotone condition:

(QM) if (t, φ), $(t, \psi) \in \Omega$ with $\varphi \leq_D \psi$ and $D_i(\varphi) = D_i(\psi)$ for some $i \in N$, then

$$f_i(t, \varphi) \leq f_i(t, \psi).$$

The next theorem not only establishes the desired monotonicity of the semiflow Φ_t but also gives comparisons between solutions of order-related NFDEs, where a (local) semiflow Φ_t on C_r is defined by $\Phi_t(\varphi) = x_t(\sigma, \varphi, f)$.

THEOREM 1. *Let $f, g : \Omega \to R^n$ be continuous and Lipschitz with respect to $\varphi \in C_r$ on each compact subset of Ω and assume that either f or g satisfies (QM) and $f(t, \varphi) \leq g(t, \varphi)$ for all $(t, \varphi) \in \Omega$. If (σ, φ), $(\sigma, \psi) \in \Omega$ satisfy $\varphi \leq_D \psi$, then $x_t(\sigma, \varphi, f) \leq_D x_t(\sigma, \psi, g)$ holds for all $t \geq \sigma$ for which both are defined.*

Proof. The proof is similar to that in Wu and Freedman[12], we provide some details for the sake of completeness.

Assume that f satisfies (QM), a similar argument holds if g satisfies (QM). Since D is quasimonotone, the conclusions of Lemmas 2.1-2.4 of [12] are valid. By Lemma 2.2 of [12], for any given positive integer m we can choose $\psi^m \in C_r$ such that $\psi^m = \psi + \delta_m$, $\delta_m \in IntC_{r,D}^+$, $D(\psi^m) = D(\psi) + \varepsilon_m$, $\varepsilon_m \in IntR_+^n$ and $\delta_m \to 0$ as $m \to \infty$. Let $g^m(t, \psi) = g(t, \psi) + \frac{1}{m}e$ and $x^m(\cdot, \sigma, \psi^m, g^m)$ be the solution of the following initial value problem

$$\frac{d}{dt}D(x_t) = g(t, x_t) + \frac{1}{m}e = g^m(t, x_t), x_\sigma = \psi^m$$

where $e = (1, 1, \cdots, 1)^T \in R^n$. By the continuous dependence of solutions, it is sufficient to prove that

$$x_t(\sigma, \varphi, f) <<_D x_t^m(\sigma, \psi^m, g^m)$$

for any positive integer m and all $t \in [\sigma, t_1]$, where $t_1 \geq \sigma$ belongs to the common domain where both $x_t(\sigma, \varphi, f)$ and $x_t^m(\sigma, \psi^m, g^m)$ are defined. By way of contradiction, if the above assertion is false, then by Lemma Lemma 2.3 of [12], there exist an integer $m > 0$ and a constant $s \in (\sigma, t_1]$, such that

$$x_t(\sigma, \varphi, f) <<_D x_t^m(\sigma, \psi^m, g^m)$$

for all $t \in [\sigma, s)$ and

$$D_i(x_s(\sigma, \varphi, f)) = D_i(x_s^m(\sigma, \psi^m, g^m))$$

for some index $i \in N$. Clearly,

$$\frac{d}{dt} D_i(x_t^m(\sigma, \psi^m, g^m))|_{t=s} \leq \frac{d}{dt} D_i(x_t(\sigma, \varphi, f))|_{t=s}.$$

On the other hand, $x_s(\sigma, \varphi, f) \leq_D x_s^m(\sigma, \psi^m, g^m)$ due to the continuity of solutions. Consequently, we have

$$\begin{aligned} \frac{d}{dt} D_i(x_t^m(\sigma, \psi^m, g^m))|_{t=s} &= g_i(s, x_s(\sigma, \psi^m, g^m)) + \frac{1}{m} \\ &> f_i(s, x_s^m(\sigma, \psi^m, g^m)) \\ &\geq f_i(x_s(\sigma, \varphi, f)) \\ &= \frac{d}{dt} D_i(x_t(\sigma, \varphi, f))|_{t=s} \end{aligned}$$

where the last inequality follows from (QM). The contradition implies that no such s can exist, completing the proof.

We now consider positively invariant sets under the solution semiflow Φ_t. Let $\Omega_0 \subset C_r$ be open, $D : C_r \to R^n$ quasimonotone, $f : R \times \Omega_0 \to R^n$ continuous and Lipschitz with respect to $\varphi \in C_r$ on each compact subset of $R \times \Omega_0$. Assuming $C_{r,D}^+ \cap \Omega_0$ is nonempty, the following result gives sufficient conditions for $C_{r,D}^+ \cap \Omega_0$ to be positively invariant.

THEOREM 2. *Assume that whenever $\varphi \in \Omega_0$ satisfies $\varphi \geq_D 0, D_i(\varphi) = 0$ for some i, then $f_i(t, \varphi) \geq 0$ for $t \in R$. If $\varphi \in \Omega_0$ satisfies $\varphi \geq_D 0$ and $\sigma \in R$, then $x_t(\sigma, \varphi) \geq_D 0$ for all $t \geq \sigma$ in its maximal interval of existence.*

Proof. The proof is very similar to that of Theorem 1. Let $f_\varepsilon(t,\varphi) = f(t,\varphi)+\varepsilon e$. We will show that $x_t(\sigma,\varphi,f_\varepsilon) \geq_D 0$ whenever $\varphi \geq_D 0$ and $\varepsilon > 0$. The result will then follow by letting $\varepsilon \to 0$ as in the proof of Theorem 1. Now, $x_\sigma = \varphi \geq_D 0$ and if $D_i(x_\sigma(\sigma,\varphi,f_\varepsilon)) = 0$, then, by hypothesis,

$$\frac{d}{dt}D_i(x_t(\sigma,\varphi,f_\varepsilon))|_{t=\sigma} = f_i(\sigma,\varphi)+\varepsilon > 0.$$

Therefore, if the result were false, we can assume that there exists $t_1 > \sigma$ such that $x_t(\sigma,\varphi,f_\varepsilon) \geq_D 0$ for $t \in [\sigma, t_1)$ and $D_i(x_{t_1}(\sigma,\varphi,f_\varepsilon)) = 0$ for some $i \in N$. It would follow that

$$\frac{d}{dt}D_i(x_t(\sigma,\varphi,f_\varepsilon))|_{t=t_1} \leq 0.$$

However, $x_{t_1} \in C^+_{r,D}$ by continuity, and so, by hypothesis,

$$\frac{d}{dt}D_i(x_t(\sigma,\varphi,f_\varepsilon))|_{t=t_1} = f_i(t_1,x_{t_1})+\varepsilon > 0.$$

This contradiction proves the theorem.

COROLLARY 1. . *Under the assumptions of Theorem 2, the following conclusions hold:*

(i). For $h \in R^n$, let $[\hat{h},\infty) = \{\varphi \in C_r | \varphi \geq_D \hat{h}\}$, $[\hat{h},\infty) \cap \Omega_0$ is positively invariant for (2.1) provided that whenever $\varphi \in \Omega_0, \varphi \geq_D \hat{h}$ and $D_i(\varphi) = D_i(\hat{h})$ for some i, then $f_i(t,\varphi) \geq 0$ for $t \in R$.

(ii). For $k \in R^n$, let $(-\infty,\hat{k}] = \{\varphi \in C_r | \varphi \leq_D \hat{k}\}$, $(-\infty,\hat{k}] \cap \Omega_0$ is positively invariant for (2.1) provided that whenever $\varphi \in \Omega_0, \varphi \leq_D \hat{k}$ and $D_i(\varphi) = D_i(\hat{k})$ for some i, then $f_i(t,\varphi) \leq 0$ for $t \in R$.

(iii). For h, $k \in R^n$, let $[\hat{h},\hat{k}] = \{\varphi \in C_r | \hat{h} \leq_D \varphi \leq_D \hat{k}\}$, $[\hat{h},\hat{k}] \cap \Omega_0$ is positively invariant for (2.1) provided that whenever $\varphi \in \Omega_0 \cap [\hat{h},\hat{k}]$ and $D_i(\varphi) = D_i(\hat{h})$(resp. $D_i(\varphi) = D_i(\hat{k})$) for some i, then $f_i(t,\varphi) \geq 0$(resp. $f_i(t,\varphi) \leq 0$) for $t \in R$.

Proof. The simple proof follows from Theorem 2 by a change of variables in (2.1).

As an immediate consequence of the above results for the autonomous neutral equation

$$\frac{d}{dt}D(x_t) = f(x_t), \tag{2.2}$$

where $D : C_r \to R^n$ is quasimonotone,$f : \Omega_0 \to R^n$ is continuous and Lipschitz on every compact subset of Ω_0, we have

COROLLARY 2. *Let f in (2.2) satisfy (QM). If $h \in R^n$ is such that $\hat{h} \in \Omega_0$ and $f(\hat{h}) \geq 0$(resp. $f(\hat{h}) \leq 0$),then $x(t, \hat{h})$ is nondecreasing (resp. nonincreasing) in $t \geq 0$. If the positive orbit of $\hat{h}$ has compact closure in Ω_0, then there exists $k \geq h$(resp. $k \leq h$) such that $x_t(\hat{h}) \to \hat{k}$ as $t \to \infty$.*

Proof. Suppose that $f(\hat{h}) \geq 0$, the proof in the case of $f(\hat{h}) \leq 0$ is similar. If $\varphi \geq_D \hat{h}$,$D_i(\varphi) = D_i(\hat{h})$ for some $i \in N$, then by (QM),we conclude that $f_i(\varphi) \geq f_i(\hat{h}) \geq 0$. Therefore by Corollary 1(i), $[\hat{h}, \infty) \cap \Omega_0$ is positively invariant for (2.2). In particular, $x_t(\hat{h}) \geq_D \hat{h}$ for $t \geq 0$. But then Theorem 1 implies that $x_{t+s}(\hat{h}) \geq_D x_s(\hat{h})$ for $t, s \geq 0$. Equivalently, $\hat{h} \leq_D x_s(\hat{h}) \leq_D x_t(\hat{h})$ whenever $0 \leq s \leq t$.Evaluating each function at $\theta = 0$ in the previous inequality yields the conclusion that $x(t, \hat{h})$ is nondecreasing in $t \geq 0$. If the positive orbit of $\hat{h}$ has compact closure in Ω_0,then $x_t(\hat{h})$ must converge to an equilibrium. Therefore,there is a $k \geq h$ such that $x_t(\hat{h}) \to \hat{k}$ as $t \to \infty$.

3. Quasimonotone Comparison Systems and Contracting Rectangles. In the previous section, we have proved that the quasimonotone condition (QM) is sufficient for certain order intervals $[\hat{a}, \hat{b}]$ to be positively invariant where $a, b \in R^n$ or being infinite. In this section, we will show that to any NFDEs with a positively invariant order interval $[\hat{a}, \hat{b}]$, there exist two auxiliary NFDEs $\frac{d}{dt}D(y_t) = h(y_t)$ and $\frac{d}{dt}D(z_t) = H(z_t)$, where h, H satisfy (QM), such that the solutions of (2.2) can be directly compared to those of the auxiliary systems.

Assume that system (2.2) has a positively invariant order interval $\sum = [\hat{a}, \hat{b}]$, where $a, b \in R^n$. That is, we assume that $\varphi \in \sum$ and for some $i \in N$
(3.1)

$$D_i(\varphi) = D_i(\hat{a})(\text{resp. } D_i(\varphi) = D_i(\hat{b})) \quad \text{imply} \quad f_i(\varphi) \geq 0(\text{resp. } f_i(\varphi) \leq 0)$$

and f satisfies a Lipschitz condition on $\sum$. More precisely, there exists $L > 0$ such that

$$|f(\psi) - f(\varphi)| \leq L||\psi - \varphi||_{C_r}, \ \ \varphi, \psi \in \sum. \tag{3.2}$$

Throughout this section, $|a| = \max|a_i|$ for $a \in R^n$. Define comparison functionals $h, H : \sum \to R^n$ for f as follows. Given $\varphi \in \sum$ and $i \in N$, let

$$\begin{aligned} h_i(\varphi) &= \inf\{f_i(\psi) | \varphi \leq_D \psi \leq_D \hat{b} \quad \text{and} \quad D_i(\psi) = D_i(\varphi)\}, \\ H_i(\varphi) &= \sup\{f_i(\psi) | \hat{a} \leq_D \psi \leq_D \varphi \quad \text{and} \quad D_i(\psi) = D_i(\varphi)\} \end{aligned} \tag{3.3}$$

Since f satisfies (3.2), it is bounded on $\sum$ so the infimum and supremum defining h and H are finite. The relationship among h, H and f is revealed in the next result.

LEMMA 1. *Define H and h as in (3.3). Then H and h satisfy (3.1), (3.2) and (QM) on $\sum$, and*

$$h(\varphi) \leq f(\varphi) \leq H(\varphi), \quad \varphi \in \sum . \tag{3.4}$$

If f satisfies (QM), then $f = h = H$ on $\sum$.

Proof. The inequality (3.4) is immediate from the definitions of h and H.

If $\varphi \in \sum$ satisfies $D_i(\varphi) = D_i(\hat{a})$ for some $i \in N$, then $f_i(\psi) \geq 0$ for every ψ satisfying $\varphi \leq_D \psi \leq_D \hat{b}$, $D_i(\psi) = D_i(\varphi) = D_i(\hat{a})$ from (3.1), so $h_i(\varphi) \geq 0$ by the definition of $h_i(\varphi)$. If $\varphi \in \sum$, $D_i(\varphi) = D_i(\hat{b})$, then, by (3.1) and (3.4), $h_i(\varphi) \leq f_i(\varphi) \leq 0$. Similar arguments apply to H, and therefore h and H satisfy (3.1).

Now let $\varphi, \psi \in \sum$ satisfy $\varphi \leq_D \psi$ and $D_i(\varphi) = D_i(\psi)$ for some $i \in N$. Define sets

$$T_\varphi = \{\xi \in \sum | \varphi \leq_D \xi \leq_D \hat{b}; \text{ and } D_i(\xi) = D_i(\varphi)\},$$

$$T_\psi = \left\{\xi \in \sum | \psi \leq_D \xi \leq_D \hat{b} \text{ and } D_i(\xi) = D_i(\varphi)\right\}.$$

Clearly, $T_\psi \subset T_\varphi$. But $h_i(\varphi) = \inf\{f_i(\xi) | \xi \in T_\varphi\}$, $h_i(\psi) = \inf\{f_i(\xi) | \xi \in T_\psi\}$. So $h_i(\varphi) \leq h_i(\psi)$. Again, a similar argument applies to H, and hence both functions h and H satisfy (QM).

Next, to show that H satisfies a Lipschitz condition (3.2), we suppose for convenience that $a = 0$. Fix $\eta, \varphi \in \sum$ and $i \in N$. Let ψ^m satisfy $\hat{0} \leq_D \psi^m \leq_D \varphi$ and $D_i(\psi^m) = D_i(\varphi)$ for $m = 1, 2, \cdots$ such that

$$H_i(\varphi) = \lim_{m \to \infty} f_i(\psi^m).$$

For each $m \geq 1$ and each index j, let

$$\bar{\psi}_j^m(s) = \max\{0, \psi_j^m(s) + \eta_j(s) - \varphi_j(s)\}.$$

Now note that $\bar{\psi}^m$ is continuous, $D_i(\bar{\psi}^m) = D_i(\eta)$ and $\hat{0} \leq_D \bar{\psi}^m \leq_D \eta$ for $m \geq 1$. Hence

$$\begin{aligned} H_i(\varphi) - H_i(\eta) &= \lim_{m \to \infty} f_i(\psi^m) - H_i(\eta) \\ &\leq \lim_{m \to \infty} \sup[f_i(\psi^m) - f_i(\bar{\psi}^m)] \\ &\leq \lim_{m \to \infty} \sup L||\bar{\psi}^m - \psi^m||_{C_r}. \end{aligned}$$

It is easy to see from the definition of $\bar{\psi}^m$ that

$$|\bar{\psi}_j^m(\theta) - \psi_j^m(\theta)| \leq |\eta_j(\theta) - \varphi_j(\theta)|$$

for each j and all $\theta \in [-r_j, 0]$. Therefore,

$$H_i(\varphi) - H_i(\eta) \leq L||\eta - \varphi||_{C_r},$$

and by symmetry

$$|H_i(\varphi) - H_i(\psi)| \leq L||\eta - \varphi||_{C_r}.$$

Since we use the maximum norm and i is arbitrary, H satisfies the Lipschitz condition on Σ. A similar argument applies to h.

Finally, if f satisfies(QM) and if $\varphi \leq_D \psi \leq_D \hat{b}$ and $D_i(\psi) = D_i(\varphi)$, then $f_i(\varphi) \leq f_i(\psi)$. Therefore, $h_i(\varphi) = f_i(\varphi)$ for $\varphi \in \Sigma$ and each $i \in N$. A similar argument shows that $f = H$ on Σ. The proof is complete.

The quasimonotone comparison systems related to (2.2) on Σ are given by

$$\frac{d}{dt}D(y_t) = h(y_t), \tag{3.5}$$

$$\frac{d}{dt}D(z_t) = H(z_t), \tag{3.6}$$

According to Lemma 1, Σ is positively invariant for (3.5) and (3.6), so for each $\varphi \in \Sigma$ (3.5), (3.6) have unique solutions $y(t,\varphi)$ and $z(t,\varphi)$ defined for all $t \geq 0$ and satisfying $\hat{a} \leq_D y_t(\varphi) \leq_D z_t(\varphi) \leq_D \hat{b}$ for all $t \geq 0$. Actually, since h and H satisfy(QM), we have the following important relationship among solutions of (3.5), (3.6) and (2.2).

THEOREM 3. *(i). If $\hat{a} \leq_D \xi \leq_D \varphi \leq_D \psi \leq_D \hat{b}$, then*

$$\hat{a} \leq_D y_t(\xi) \leq_D x_t(\varphi) \leq_D z_t(\psi) \leq_D \hat{b} \text{ for all } t \geq 0,$$

where $x(t,\varphi)$ is the solution of (2.2) through $(0,\varphi)$.

(ii). $y(t,\hat{a})$ is nondecreasing and $z(t,\hat{b})$ is nonincreasing for $t \geq 0$.

(iii) There exist $c, d \in R^n$ with $h(\hat{c}) = H(\hat{d}) = 0$ such that $y(t,\hat{a}) \to c$, $z(t,\hat{b}) \to d$ as $t \to \infty$.

(iv). $\Sigma' = [\hat{c}, \hat{d}]$ is positively invariant for (2.2).

Proof. (i). This assertion follows immediately from Theorem 1 and Lemma 1.

(ii). Since h and H satisfy (3.1), $h(\hat{a}) \geq 0$ and $H(\hat{b}) \leq 0$, so $y(t, \hat{a})$ and $z(t, \hat{b})$ are monotone in $t \geq 0$ respectively by Corollary 2.

(iii) The existence of c and d follows from the positive invariance of Σ under (3.5) and (3.6) and (i).

(iv) If $\varphi \in \Sigma'$ and $D_i(\varphi) = D_i(\hat{C})$, then $0 = h_i(\hat{c}) \leq h_i(\varphi) \leq f_i(\varphi)$; if $\varphi \in \Sigma'$ and $D_i(\varphi) = D_i(\hat{d})$, then $f_i(\varphi) \leq H_i(\varphi) \leq H_i(\hat{d}) \leq 0$ by the quasimonotone properties of h and H. Hence the positive invariance of Σ' follows from Corollary 1, Lemma 1 and (iii).

The important implication of Theorem 3 is that we can replace Σ by Σ' and apply the same arguments as above to obtain new comparison functions h' and H' for f relative to Σ'. Clearly, this procedure can be iterated and, under suitable hypotheses, may force solution of (2.2) to converge. We give such conditions in the next result. The order interval Σ is said to be a contracting rectangle for (2.2) provided that $f_i(\varphi) > 0$(resp. $f_i(\varphi) < 0$) holds whenever $\varphi \in \Sigma$ and $D_i(\varphi) = D_i(\hat{a})$ (resp. $D_i(\varphi) = D_i(\hat{b})$) for some $i \in N$.

THEOREM 4. *Suppose there exists $k \in R^n$ such that $\hat{k} \in \Sigma$ and $f(\hat{k}) = 0$. Assume that there exists a one-parameter family of order intervals given by $\Sigma(s) = [\hat{a}(s), \hat{b}(s)], 0 \leq s \leq 1$, such that for $0 \leq s_1 < s_2 \leq 1$,*

$$\hat{a} = \hat{a}(0) \leq_D \hat{a}(s_1) <_D \hat{a}(s_2) \leq_D \hat{a}(1) = \hat{k} = \hat{b}(1)$$

$$\leq_D \hat{b}(s_2) <_D \hat{b}(s_1) \leq_D \hat{b}(0) = \hat{b},$$

where $a(s)$ and $b(s)$ are continuous functions of s. Assume that $\Sigma(s)$ is a contracting rectangle for (2.2) for each $s \in [0, 1)$. If $\varphi \in \Sigma$, then $x(t, \varphi) \to k$ as $t \to \infty$.

Proof. If $\varphi \in \Sigma$, then, by Theorem 3(i), $x_t(\varphi) \in \Sigma$ for all $t \geq 0$ and $\omega(\varphi)$ is a compact invariant subset of Σ so that the set $\{\psi(s) | \varphi \in \omega(\varphi), s \in [-r, 0]\}$ is a compact subset of the order interval $[a, b] \subset R^n$. From this and the continuity of $a(s)$ and $b(s)$, it is easy to see that there exists a maximum s with the property that $\omega(\varphi) \subset \sum(s)$. We label this value of s as s_0. If $s_0 = 1$, then we are done so we assume that $s_0 < 1$. By the maximality of s_0 and the invariance of the omega limit set, there must exist $\psi \in \omega(\varphi)$ and an index i such that $D_i(\psi) = D_i(\hat{a}(s_0))$ or $D_i(\psi) = D_i(\hat{b}(s_0))$. Suppose the latter holds as the argument is similar if the former holds. Again,by the invariance of

$\omega(\varphi)$, there exists $\eta \in \omega(\varphi)$ such that $x_1(\eta) = \psi$ where $x(t,\eta)$ is the solution of (2.2) thought $(0,\eta)$. Therefore

$$\frac{d}{dt}D_i(x_t(\psi))|_{t=0} = \frac{d}{dt}D_i(x_t(\eta))|_{t=1} = f_i(\psi) < 0,$$

since $\Sigma(s_0)$ is a contracting rectangle. But $D_i(x_1(\eta)) = D_i(\psi) = D_i(\hat{b}(s_0))$, so it follows that $D_i(x_t(\eta)) > D_i(\hat{b}(s_0))$ for $t < 1$ sufficiently near to 1. This implies that $x_t(\eta) \notin \Sigma(s_0)$ for such $t < 1$ sufficiently near to 1. Since $\eta \in \omega(\varphi)$ and $\omega(\varphi)$ is positively invariant for (2.2), we have $x_t(\eta) \in \omega(\varphi)$ for all $t \geq 0$. This contradicts that $\omega(\varphi) \subset \Sigma(s_0)$ and proves the theorem.

4. An Example. Consider the scalar neutral delay differential equation

$$\text{(4.1)}\quad \frac{d}{dt}[x(t) - qx(t-\tau)] = ax(t) + bx(t-\tau) - c[x(t) - qx(t-\tau)]^2 + 1,$$

where $0 < q < 1, \tau \geq 0$, a, b and c are positive numbers. The generalized difference operator $D : C \to R$ is quasimonotone, where $C = C([-\tau, 0], R)$, $D\varphi = \varphi(0) - q\varphi(-\tau)$. Obviously, equation (4.1) has two equilibria

$$x_- = \frac{(a+b) - \sqrt{(a+b)^2 + 4c(1-q)^2}}{2c(1-q)^2} < 0,$$

and

$$x_+ = \frac{(a+b) + \sqrt{(a+b)^2 + 4c(1-q)^2}}{2c(1-q)^2} > 0.$$

PROPOSITION 1. *Equilibrium x_- is unstable.*

Proof. Consider the following linear variational equation about equilibrium x_-

$$\text{(4.2)}\quad \frac{d}{dt}[y(t) - qy(t-\tau)] = ay(t) + by(t-\tau) - 2cx_-(1-q)[y(t) - qy(t-\tau)].$$

The characteristic values associated with (4.2) are roots of

$$g(l) := a + be^{-l\tau} - 2cx_-(1-q)(1-qe^{-l\tau}) - l(1-qe^{-l\tau}) = 0.$$

Clearly,

$$g(0) = a + b - 2cx_-(1-q)^2 = \sqrt{(a+b)^2 + 4c(1-q)^2} > 0$$

and

$$\lim_{l \to +1} g(l) = -1.$$

Therefore, there exists at least one positive characteristic value. This completes the proof.

Nextly, we prove that x_+ is globally attractive in $C^+ = \{\varphi \in C : \varphi \geq_D \hat{0}\}$.

LEMMA 2. *Equation (4.1) satisfies condition (QM).*

Proof: In (4.1), $f(\varphi) = a\varphi(0) + b\varphi(-\tau) - c(D\varphi)^2 + 1$. For any $\varphi, \psi \in C$ with $\varphi \leq_D \psi$ and $D\varphi = D\psi$, we have

$$\psi(-\tau) - \varphi(-\tau) = \frac{1}{q}[\psi(0) - \varphi(0)]$$

and

$$f(\psi) - f(\varphi) = (a + \frac{b}{q})[\psi(0) - \varphi(0)] \geq 0.$$

Hence, $f(\varphi) \leq f(\psi)$. This completes the proof.

LEMMA 3. *Let $\sum = [\hat{0}, \hat{r}]$, where $r > x_+$, then $\sum$ is positively invariant Eq. (4.1).*

Proof. If $\varphi \geq_D \hat{0}$, $D(\varphi) = D(\hat{0})$, then $\varphi(-\tau) = \frac{1}{q}\varphi(0)$ and

$$f(\varphi) = (a + \frac{b}{q})\varphi(0) + 1 > 0.$$

If $\varphi \leq_D \hat{r}, D(\varphi) = D(\hat{r})$, then

$$\varphi(-\tau) = \frac{1}{q}[\varphi(0) - (1-q)r],$$

and

$$\begin{aligned} f(\varphi) &= (a + \frac{b}{q})\varphi(0) - \frac{b}{q}(1-q)r - c(1-q)^2 r^2 + 1 \\ &\leq (a + \frac{b}{q})r - \frac{b}{q}(1-q)r - c(1-q)^2 r^2 + 1 \\ &= (a+b)r - c(1-q)^2 r^2 + 1 < 0. \end{aligned}$$

By Corollary 1 (iii), $\sum$ is positively invariant for Eq. (4.1). This completes the proof.

LEMMA 4. *Let* $\sum(s) = [s\hat{x}_+, s\hat{x}_+ + (1-s)\hat{r}]$, $s \in [0,1)$, *then* $\sum(s)$ *is a contracting rectangle for Eq. (4.1) for each* $s \in [0,1)$.

Proof. If $\varphi \geq_D s\hat{x}_+, D(\varphi) = D(s\hat{x}_+)$, then

$$\varphi(-\tau) = \frac{1}{q}[\varphi(0) - (1-q)s\hat{x}_+],$$

and

$$\begin{aligned} f(\varphi) &= (a+\frac{b}{q})\varphi(0) - \frac{b}{q}(1-q)sx_+ - c(1-q)^2(sx_+)^2 + 1 \\ &\geq (a+\frac{b}{q})sx_+ - \frac{b}{q}(1-q)sx_+ - c(1-q)^2(sx_+)^2 + 1 \\ &= (a+b)sx_+ - c(1-q)^2(sx_+)^2 + 1 > 0. \end{aligned}$$

If $\varphi \leq_D s\hat{x}_+ + (1-s)\hat{r}$, $D(\varphi) = D(s\hat{x}_+ + (1-s)\hat{r})$, then

$$\varphi(-\tau) = \frac{1}{q}[\varphi(0) - (1-q)(sx_+ + (1-s)r)]$$

and

$$f(\varphi) = (a+\frac{b}{q})\varphi(0) - \frac{b}{q}(1-q)[sx_+ + (1-s)r]$$

$$-c(1-q)^2[sx_+ + (1-s)r]^2 + 1$$

$$\leq (a+\frac{b}{q})[sx_+ + (1-s)r] - \frac{b}{q}(1-q)[sx_+ + (1-s)r]$$

$$- c(1-q)^2[sx_+ + (1-s)r]^2 + 1$$

$$= (a+b)[sx_+ + (1-s)r] - c(1-q)^2[sx_+ + (1-s)r] + 1 < 0.$$

Therefore, $\sum(S)$ is positively invariant for equation (4.1) for each $s \in [0,1)$. This completes the proof.

PROPOSITION 2. *For any $\varphi \in C^+$, $x(t,\varphi) \to x_+$ as $t \to +\infty$.*

Proof. $\hat{x}_+ \in \sum$, $f(\hat{x}_+) = 0$. Let $\sum(s) = [\hat{a}(s), \hat{b}(s)]$, $0 \le s \le 1$, where $a(s) = sx_+$, $b(s) = sx_+ + (1-s)r$, $r > r_+$. Obviously, $a(s)$ and $b(s)$ are continuous functions of s, $a(1) = b(1) = x_+$, and $\sum(0) = \sum$. Because of $r > x_+$, we have $\sum(s_2) \subset \sum(s_1)$ whenever $0 \le s_1 < s_2 < 1$. By Lemmas 2-4 and Theorem 4, for any $\varphi \in \sum$, $x(t,\varphi) \to x_+$ as $t \to +\infty$. But $r > x_+$ is arbitrary, so the conclusion of this proposition is true. This completes the proof.

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