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Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation [1]

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Abstract

For the Dirichlet boundary value problem of the diffusive Nicholson's blowflies equation, it was shown in Ref. [17] that in a certain range of the parameter space, there is a unique positive steady state solution. In this paper, we propose a scheme to compute this steady state numerically. In addition, we describe an iterative procedure to locate the critical values of the delay where a Hopf bifurcation of time periodic solutions takes place near the steady state. Some numerical simulations of both schemes are given. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

In this paper, we will report on some numerical schemes and simulations for the positive steady state and Hopf bifurcation analysis of the following normalized Dirichlet boundary problem of diffusive Nicholson's blowflies equation in one spatial dimension:

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$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u}{\partial x^2}(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)\mathrm{e}^{-u(t-1,x)}, \tag{1.1}$$
$$u(t,0) = u(t,1) = 0, \tag{1.2}$$
$$u(\theta,x) = u_0(\theta,x),$$

where $x \in (0,1)$, $t > 0$, and $\theta \in [-1,0]$. The steady state solution satisfies the two-point boundary problem

$$\begin{aligned} &d\phi_{xx} - \tau\phi + \beta\tau\phi\mathrm{e}^{-\phi} = 0,\\ &\phi(0) = \phi(1) = 0. \end{aligned} \tag{1.3}$$

It was shown in Ref. [17] that the boundary value problem (1.3) has a unique positive solution if and only if

$$(\beta - 1)\tau > d\lambda_1, \tag{1.4}$$

where λ_1 is the principle eigenvalue of $-\partial^2/\partial x^2$ with a Dirichlet boundary condition. One also observes that $\phi(x) \leqslant \ln\beta$ for all $x \in [0,1]$ for any positive solution ϕ of (1.3).

Recall that the linearized equation of (1.1)–(1.2) about the positive steady state is

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} = d\frac{\partial^2 v}{\partial x^2}(t,x) - \tau v(t,x) + \beta\tau\mathrm{e}^{-\phi(x)}(1-\phi(x))v(t-1,x),\\ &v(t,0) = v(t,1) = 0 \end{aligned} \tag{1.5}$$

and the corresponding eigenvalue problem is

$$\begin{aligned} &-d\psi_{xx} + \left(\tau + \lambda - \beta\tau\mathrm{e}^{-\phi}(1-\phi)\mathrm{e}^{-\lambda}\right)\psi = 0,\\ &\psi(0) = \psi(1) = 0. \end{aligned} \tag{1.6}$$

The study of the impact of the time delay τ on the existence of Hopf bifurcation of periodic orbits requires to locate the critical values of τ when (1.6) admits nonzero solution for some purely imaginary λ. Locating such critical values seems to be a very difficult problem both theoretically and numerically, since ϕ is not explicitly given. To our knowledge, little work has been done for Hopf bifurcation analysis of Dirichlet boundary value problems with delay (see, for example, Ref. [19]). We note that for the diffusive Hutchinson equation with Dirichlet boundary conditions, Ref. [3] proved the existence of a Hopf bifurcation by using perturbation methods together with the implicit function theorem. Unfortunately, their approach cannot be applied to our case due to the difference in the nonlinearity.

Under condition (1.4), the boundary value problem (1.3) has at least two solutions, one of which is the zero solution. Computing the positive solution of such nonlinear two-point boundary value problems with multiple solutions seems to be a nontrivial task (see, for example, Refs. [1,2,16]). We propose a new approach which employs the Newton's iteration and the evaluation of certain integrals with or without singular points. This approach seems to be quite effective in dealing with problems like (1.3).

Computing the Hopf bifurcation of periodic solutions near the positive steady state requires us to find the critical value of τ, a purely imaginary $\lambda = \mathrm{i}b$ and a nonzero function ψ so that (1.6) is satisfied. Our proposed scheme is motivated by the inverse power method with shift and the inverse iteration method for nonlinear elliptic eigenvalue problem (see, for example, Refs. [10,12]). We establish some a priori estimates for τ and b which proves to be quite useful in restricting the initial guesses for our iteration scheme. The rigorous proof of the convergence of the scheme for a more general problem will be reported in a separate paper, though the provided numerical examples do confirm the convergence for the specific blowflies equation. In particular, for the numerical examples provided, we can locate some critical values of τ and purely imaginary $\lambda = \mathrm{i}b$ such that (1.6) has nonzero solution ψ. Moreover, by substituting these into the original Eq. (1.1), we are able to observe periodic solutions near the positive steady state. These numerical results illustrate that the iteration scheme provides us with a clue to prove the existence of Hopf bifurcation theoretically.

The rest of this paper is organized as follows. In Section 2, we present the numerical methods applied to Eq. (1.3). Following in Section 3 is the numerical Hopf bifurcation analysis. Finally, based on the numerical methods in Sections 2 and 3, some numerical simulations are provided in Section 4.

2. Numerical solutions of the positive steady state

Let $\phi \in H^2(0,1)$ be the unique positive steady state of (1.1)–(1.2). Recall from Ref. [16] that $m = \max_{x\in[0,1]}\phi(x) \leqslant \ln\beta$. Therefore, from

$$d\ddot{\phi} = \tau\phi - \beta\tau\phi\,\mathrm{e}^{-\phi} = \tau\phi[1 - \beta\mathrm{e}^{-\phi}] \tag{2.1}$$

we obtain $d\ddot{\phi} \leqslant 0$ on $[0,1]$. It follows that $\dot{\phi}$ is nonincreasing on $[0,1]$ and $\dot{\phi}(0) > 0$. Let $x_0 > 0$ be the first (smallest) zero of ϕ on $(0,1)$. We claim that $\dot{\phi}(x) < 0$ for $x \in (x_0, 1)$. Otherwise, there exists $\delta > 0$ such that $\dot{\phi}(x) = 0$ for all $x \in (x_0, x_0 + \delta) \subset (0,1)$. Therefore, $\ddot{\phi}(x) = 0$ on $(x_0, x_0 + \delta)$. This, together with (2.1), implies that $1 - \beta\mathrm{e}^{-\phi(x)} = 0$ on $(x_0, x_0 + \delta)$. So, $\phi(x) \equiv \ln\beta$ on $(x_0, x_0 + \delta)$, which contradicts the uniqueness of solution to the initial value problem: $\ddot{z} = \tau z(1 - \beta\mathrm{e}^{-z})$, $z(x_0) = \ln\beta$ and $\dot{z}(x_0) = 0$. This proves the claim. Next, we observe that by (2.1), $\phi(1-x)$, $x \in [0,1]$, also satisfies (1.3). The uniqueness result in Ref. [17] implies that

$$\phi(x) = \phi(1-x) \quad \text{for } x \in [0,1]. \tag{2.2}$$

This shows that ϕ is symmetric about $x = \frac{1}{2}$. Consequently, we must have $x_0 = \frac{1}{2}$. In other words. $\dot{\phi}(x) > 0$ on $[0,\frac{1}{2})$, $\dot{\phi} < 0$ on $(\frac{1}{2},1]$, $\dot{\phi}(\frac{1}{2}) = 0$ and

$$\phi\left(\frac{1}{2}\right) = m = \max_{x\in[0,1]}\phi(x). \tag{2.3}$$

Multiplying (1.3) throughout by $-\phi_x$ and integrate over $[x_0, x]$, we obtain

$$-\frac{d}{2}\phi_x^2 + \tau\int_m^{\phi(x)} w\,\mathrm{d}w - \beta\tau\int_m^{\phi(x)} w\mathrm{e}^{-w}\,\mathrm{d}w = 0.$$

Therefore,

$$\frac{d}{2}\phi_x^2 = \frac{\tau}{2}(\phi^2 - m^2) + \beta\tau\left[(\phi+1)\mathrm{e}^{-\phi} - (m+1)\mathrm{e}^{-m}\right]. \tag{2.4}$$

Consequently,

$$\phi_x = \sqrt{\frac{\tau}{d}\left[\phi^2 + 2\beta(\phi+1)\,\mathrm{e}^{-\phi}\right] - \frac{\tau}{d}[m^2 + 2\beta(m+1)\mathrm{e}^{-m}]} \quad \text{on } \left[0, \frac{1}{2}\right]. \tag{2.5}$$

Define

$$J(w) := \frac{\tau}{d}\left[w^2 + 2\beta(w+1)\mathrm{e}^{-w}\right]. \tag{2.6}$$

Then, solving (2.5) over $[0, \frac{1}{2}]$, we have

$$\int_0^{\phi(x)} \frac{1}{\sqrt{J(w) - J(m)}}\,\mathrm{d}w = x. \tag{2.7}$$

Now letting $x = \frac{1}{2}$ and noting that $\phi(\frac{1}{2}) = m$, we have from (2.7)

$$\int_0^{m} \frac{1}{\sqrt{J(w) - J(m)}}\,\mathrm{d}w = \frac{1}{2}.$$

Making the substitution $w = mt$, we can then write the above equation as

$$\int_0^{1} \frac{1}{\sqrt{K(m,t)}}\,\mathrm{d}t = x_0, \tag{2.8}$$

where

$$K(s,t) := \frac{\tau}{d}\left[(t^2 - 1) + \frac{2\beta(st+1)\mathrm{e}^{-st} - 2\beta(s+1)\mathrm{e}^{-s}}{s^2}\right]. \tag{2.9}$$

It is natural to consider the following function

$$F(s) := \int_0^{1} \frac{1}{\sqrt{K(s,t)}}\,\mathrm{d}t. \tag{2.10}$$

Lemma 2.1. *$F\colon [0, \ln\beta) \to \mathbb{R}$ is monotonically increasing and $\lim_{s\to(\ln\beta)^-}F(s) = \infty$.*

Proof. Note that

$$\int_0^1 \frac{1}{\sqrt{K(s,t)}}\,\mathrm{d}t = \int_0^1 \frac{1}{(1-t)^{1/2}}\sqrt{\frac{1-t}{K(s,t)}}\,\mathrm{d}t, \tag{2.11}$$

and that by Taylor's expansion

$$\frac{K(s,t)}{1-t} = \frac{\tau}{d}[2(\beta \mathrm{e}^{-s}-1) + (\beta(s-1)\mathrm{e}^{-s}+1)(1-t) + 2\beta R(s,t)], \tag{2.12}$$

where

$$\begin{aligned} R(s,t) := &-\frac{1}{2}s\,\mathrm{e}^{-s}(1-t)^2 + s(s+1)\mathrm{e}^{-s}(1-t)^2\sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{(k+3)!} \\ &- s^2\,\mathrm{e}^{-s}(1-t)^3\sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{(k+3)!}. \end{aligned}$$

Using the fact that

$$\sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{(k+3)!} \leqslant \sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{k!} = \mathrm{e}^{s(1-t)},$$

we get

$$|R(s,t)| \leqslant K_1(1-t)^2 + (\ln\beta)^2(1-t)^3, \tag{2.13}$$

for $0 \leqslant t \leqslant 1$ and $0 \leqslant s \leqslant \ln\beta$, where $K_1 = \frac{1}{2}\,\mathrm{e}^{-1} + \ln\beta(\ln\beta+1)$. Hence, for any $0 \leqslant s_0 < \ln\beta$ there exists a constant K_c, depending on s_0 only, such that $(1-t)/K(s,t) \leqslant K_c$ for all $t \in [0,1]$ and $0 \leqslant s \leqslant s_0$. Therefore, the improper integral

$$\int_0^1 \frac{1}{\sqrt{K(s,t)}}\,\mathrm{d}t$$

is uniformly convergent for $0 \leqslant s \leqslant s_0$. This implies that the function $F(s)$ is continuous for $0 \leqslant s < \ln\beta$. Similarly, we can show that the improper integral

$$\int_0^1 \frac{\partial K(s,t)/\partial s}{\sqrt{K^3(s,t)}}\,\mathrm{d}t$$

is uniformly convergent for any fixed $0 \leqslant s \leqslant s_0$, $s_0 \in [0, \ln\beta)$. Hence by Leibniz's rule,

$$F'(s) = -\frac{1}{2}\int_0^1 \frac{\partial K(s,t)/\partial s}{\sqrt{K^3(s,t)}}\,\mathrm{d}t, \quad s \in [0, \ln\beta).$$

On the other hand, we have after direct calculation that

$$\frac{\partial K(s,t)}{\partial s} = \frac{2\beta\tau}{d}\,\frac{T(s,t)}{s^3},$$

where $T(s,t) := [-(st+1)^2 - 1]\,\mathrm{e}^{-st} + [(s+1)^2 + 1]\,\mathrm{e}^{-s}$. Note that $T(s,1) = 0$ and that $\partial T(s,t)/\partial t = s^3 t^2 \mathrm{e}^{-st} \geqslant 0$. Therefore, $T(s,t) \leqslant 0$ for $t \in [0,1]$. This shows that $\partial K(s,t)/\partial s \leqslant 0$ for $t \in [0,1]$ and $s \in [0, \ln\beta)$. This implies that $F(s)$ is an increasing and continuously differentiable function of $s \in [0, \ln\beta)$. We now claim

$$\lim_{s\to(\ln\beta)^-} F(s) = +\infty. \tag{2.14}$$

In fact, for any given $M > 0$, we choose

$$s_1 = \max\left\{0,\ \ln\beta - \ln\left(1 + \frac{1}{2}\,\mathrm{e}^{-M/K_2}\right)\right\},$$

$$K_2 = \sqrt{\frac{d}{\tau}}\sqrt{\frac{1}{\beta + 2 + 2\beta(K_1 + (\ln\beta)^2)}}.$$

Let $\delta = 1 - 2(\beta\mathrm{e}^{-s_1} - 1)$. Then, $0 \leqslant s_1 < \ln\beta$ and $0 < \delta < 1$. So for any $s_1 \leqslant s < \ln\beta$, the monotonicity of F and (2.11)–(2.13) imply

$$\begin{aligned}
F(s) &\geqslant F(s_1)\\
&= \int_0^1 \frac{1}{\sqrt{K(s_1,t)}}\,\mathrm{d}t\\
&= \int_0^1 \frac{1}{(1-t)^{1/2}}\sqrt{\frac{1-t}{K(s_1,t)}}\,\mathrm{d}t\\
&\geqslant \sqrt{\frac{d}{\tau}}\int_0^\delta \frac{1}{\sqrt{1-t}}\\
&\quad\times \frac{1}{\sqrt{\left[|\beta(s_1-1)\mathrm{e}^{-s_1} + 1| + 1 + 2\beta(K_1 + (\ln\beta)^2)\right](1-t)}}\,\mathrm{d}t\\
&\geqslant K_2 \int_0^\delta \frac{1}{(1-t)}\,\mathrm{d}t\\
&= K_2 \ln\left(\frac{1}{1-\delta}\right) \geqslant M.
\end{aligned}$$

This shows $F(s) \to +\infty$ as $s \to (\ln\beta)^-$, and completes the proof of the lemma.

We now describe our numerical scheme to find ϕ. First of all, we note that by (2.2), it is sufficient to compute $\phi(x)$ for $x \in [0, \frac{1}{2}]$. We can use Eq. (2.8) and solve $F(s) = \frac{1}{2}$ for the maximum value m. To do that, Newton's iteration:

$$m_{n+1} = m_n - \frac{F(m_n) - \frac{1}{2}}{F'(m_n)}, \quad n \geqslant 0,$$

is used here. The convergence of this method is guaranteed since $F(s)$ is monotonically increasing for $0 \leqslant s < \ln \beta$. An initial guess is chosen slightly less than $\ln \beta$. Note that $\ln \beta$ cannot be used as an initial guess.

Since $F(s)$ is defined by an integral, one needs to evaluate this integral (as well as the integral for $F'(s)$) in order to iterate. Since $K(s,1) = 0$, $t = 1$ is a singular point for the integrand. Hence, we choose a formula of Gauss type (see, for example, Ref. [9], p. 179) to evaluate this integral:

$$\int_0^1 \frac{H(x)}{(1-x)^{1/2}}\,\mathrm{d}x = \sum_{k=1}^{n} \omega_k H(x_k) + \frac{2^{4n+1}[(2n)!]^3}{(4n+1)[(4n)!]^2} H^{(2n)}(\xi).$$

Here, $0 \leqslant \xi \leqslant 1$, $x_k = 1 - \xi_k^2$, ξ_k is the kth positive zero of the Legendre polynomial $P_{2n}(x)$ and ω_k^{2n} is the weight corresponding to ξ_k in the rule G_{2n}, i.e. a $2n$-point interpolation by Gauss rule (see, for example, Ref. [9], p. 97). To apply this formula to $F(s)$, we simply rewrite $F(s)$ as in (2.11). Using the same argument as for $F(s)$, one obtains

$$F'(s) = -\frac{1}{2}\int_0^1 \frac{1}{(1-t)^{1/2}} \sqrt{\frac{1-t}{K^3(s,t)}} \frac{\partial K(s,t)}{\partial s}\,\mathrm{d}t,$$

where

$$\lim_{t\to 1^-} \sqrt{\frac{1-t}{K^3(s,t)}} \frac{\partial K(s,t)}{\partial s} = \sqrt{\left[\frac{d}{2\tau(\beta \mathrm{e}^{-s} - 1)}\right]^3} \left(-\frac{2\beta\tau}{d}\,\mathrm{e}^{-s}\right)$$

is finite for $0 \leqslant s < \ln \beta$.

Finally, we compute $\phi(x)$ for $x \in (0, \frac{1}{2})$. Let $\alpha = \phi(x)$. Viewed as a function of α with fixed m, Eq. (2.7) can be rewritten as

$$S(\alpha) := \int_0^1 \frac{\alpha\,\mathrm{d}t}{\sqrt{J(\alpha t) - J(m)}} = x.$$

One can also show that for $0 < \alpha < m$, we have $S'(\alpha) > 0$. Therefore, Newton's iteration can be applied again. In order to get a descent initial guess of α, we start with $x < \frac{1}{2}$ but x near x_0. Since $\alpha < m$, the integrand defining $S(\alpha)$ has no singular point on $[0,1]$. Thus, we can use Simpson's formula (see, for example, Ref. [9], pp. 57–58) to evaluate this integral.

3. Numerical analysis of Hopf bifurcations

Recall that the characteristic equation of the linearized equation about a positive steady state takes the form

$$\begin{aligned} &-d\psi_{xx} + \left(\tau + \lambda - \beta\tau \mathrm{e}^{-\phi}(1-\phi))\mathrm{e}^{-\lambda}\right)\psi = 0, \\ &\psi(0) = \psi(1) = 0. \end{aligned} \tag{3.1}$$

To apply the standard Hopf bifurcation theorem for functional differential equations (see, for example, Ref. [19]), we need to find nonzero function ψ) with $\psi(0) = \psi(1) = 0$, such that (3.1) possesses purely imaginary eigenvalue $\lambda = \mathrm{i}b$ for some $b \in \mathbb{R}^+$.

To have a good initial guess for our proposed numerical scheme, we first derive some necessary conditions for the existence of nonzero solutions (b, ψ) of the following eigenvalue problem:

$$\begin{aligned} & -d\psi_{xx} + \left(\tau + \mathrm{i}b - \beta\tau\,\mathrm{e}^{-\phi(x)}(1-\phi(x))\mathrm{e}^{-\mathrm{i}b}\right)\psi = 0, \\ & \psi(0) = \psi(1) = 0. \end{aligned} \tag{3.2}$$

where $\mathrm{i}b$ with $b > 0$ is a purely imaginary eigenvalue of (3.1) and ψ is a corresponding eigenfunction. If we multiply both sides of (3.2) by $\overline{\psi}$ and integrate using by parts, the real and imaginary parts of the resulting expression are:

$$\beta\tau \int_0^1 \mathrm{e}^{-\phi(x)}(1-\phi(x))|\psi(x)|^2\,\mathrm{d}x + \frac{b\|\psi\|^2_{L^2(0,1)}}{\sin b} = 0, \tag{3.3}$$

$$d\|\psi_x\|^2_{L^2(0,1)} + (\tau + b\cot b)\|\psi\|^2_{L^2(0,1)} = 0 \tag{3.4}$$

for $\psi \in H_0^1(0,1) \cap H^2(0,1)$. Let

$$U_0 := \{\psi \in H_0^1(0,1) \cap H^2(0,1);\ \|\psi_x\|_{L^2(0,1)} = 1\}.$$

Notice that $\|\psi_x\|^2_{L^2(0,1)} \geqslant \lambda_1\|\psi\|^2_{L^2(0,1)}$. Then on U_0, Eq. (3.4) implies

$$0 \leqslant \tau \leqslant -b\cot b - d\lambda_1. \tag{3.5}$$

On the other hand, Eq. (3.3) gives,

$$\begin{aligned} \frac{b\|\psi\|^2_{L^2(0,1)}}{\sin b} &= \beta\tau \int_0^1 \mathrm{e}^{-\phi(x)}(\phi(x)-1)|\psi(x)|^2\,\mathrm{d}x \\ &= \beta\tau\left[\int_{\Omega_1^\infty} \mathrm{e}^{-\phi(x)}(\phi(x)-1)|\psi(x)|^2\,\mathrm{d}x \right. \\ &\qquad \left. + \int_{\tilde{\Omega}_1^\infty} \mathrm{e}^{-\phi(x)}(\phi(x)-1)|\psi(x)|^2\,\mathrm{d}x\right] \\ &\leqslant \beta\tau \int_{\tilde{\Omega}_1^\infty} \mathrm{e}^{-\phi(x)}(\phi(x)-1)|\psi(x)|^2\,\mathrm{d}x \\ &\leqslant \beta\tau\,\mathrm{e}^{-2} \int_{\tilde{\Omega}_1^\infty} |\psi(x)|^2\,\mathrm{d}x \\ &\leqslant \beta\tau\,\mathrm{e}^{-2}\|\psi\|^2_{L^2(0,1)}, \end{aligned}$$

where $\Omega_1^\infty := \{x \in (0,1);\ \phi(x) < 1\}$ and $\tilde{\Omega}_1^\infty$ is the complement of Ω_1^∞ in $(0,1)$. Therefore, we have

$$\tau \geqslant \frac{1}{\beta e^{-2}} \frac{b}{\sin b}. \tag{3.6}$$

A combination of (3.5) and (3.6) gives

$$\frac{1}{\beta e^{-2}} \frac{b}{\sin b} \leqslant \tau \leqslant -b \cot b - d\lambda_1. \tag{3.7}$$

It was shown in Ref. [17] that $b/\sin b > 0$ and $\cot b < 0$. Hence, we will restrict b to the interval $[\frac{1}{2}\pi, \pi)$. Now we can conclude the following.

Lemma 3.1. *If* (3.2) *has nontrivial solutions, then* $\beta e^{-2} > 1$ *and* $\tau \geqslant \tau_c$, *where*

$$\tau_c = -b_c \cot b_c - d\lambda_1 \tag{3.8}$$

and b_c *is the unique solution of*

$$\frac{\chi}{\beta e^{-2} \sin \chi} = -b \cot \chi - d\lambda_1, \quad \chi \in \left[\frac{1}{2}\pi, \pi\right). \tag{3.9}$$

Proof. Consider two functions

$$f(\chi) := \frac{\chi}{\beta e^{-2} \sin \chi}$$

and

$$g(\chi) := -\chi \cot \chi - d\lambda_1.$$

Both functions are monotonically increasing for $\chi \in [\frac{1}{2}\pi, \pi)$. Notice that

$$h(\chi) := f(\chi) - g(\chi) = \left(\frac{1}{\beta e^{-2}} + \cos \chi\right) \frac{\chi}{\sin \chi} + d\lambda_1.$$

The existence of nontrivial solutions of (3.3) implies that there exists $\chi \in [\frac{1}{2}\pi, \pi)$ such that $f(\chi) < g(\chi)$. This implies

$$\frac{1}{\beta e^{-2}} + \cos \chi < 0$$

and hence $\beta e^{-2} > 1$. Consequently, there always exists $b_c \in [\frac{1}{2}\pi, \pi)$, with $f(b_c) - g(b_c) = 0$ because

$$f\left(\frac{\pi}{2}\right) - g\left(\frac{\pi}{2}\right) > 0$$

and

$$\lim_{\chi \to \pi^-} (f(\chi) - g(\chi)) = -\infty.$$

We now show that b_c is the unique root of $h(\chi) = 0$ in $[\frac{1}{2}\pi, \pi)$. Let b_c^* be the smallest zero of $h(\chi)$ in $[\frac{1}{2}\pi, \pi)$. Since $h(b_c^*) = 0$, we have

$$\cos b_c^* = -\frac{1}{\beta e^{-2}} - \frac{\sin b_c^*}{b_c^*} d\lambda_1 < -\frac{1}{\beta e^{-2}}.$$

We claim that $h(\chi)$ is monotonically decreasing for $x \in [\pi - \arccos(1/\beta e^{-2}), \pi)$. In fact, since $1/\beta e^{-2} + \cos\chi < 0$ and $(\sin\chi - x\cos\chi)/\sin^2\chi > 0$ for $x \in [\pi - \arccos(1/\beta e^{-2}), \pi)$, we have

$$h'(\chi) = -\chi + \left(\frac{1}{\beta e^{-2}} + \cos\chi\right)\frac{\sin\chi - \chi\cos\chi}{\sin^2\chi} < 0.$$

Therefore, $h(\chi) < 0$ for $x \in [b_c^*, \pi)$. This implies that b_c is the unique zero of $h(\chi)$ in $[\frac{1}{2}\pi, \pi)$ and $f(\chi) < g(\chi)$ for $\chi \in [b_c, \pi)$. Furthermore, since both $f(\chi)$ and $g(\chi)$ are increasing, (3.6) implies that

$$\tau \geqslant \frac{b}{\beta e^{-2}\sin b} \geqslant \frac{b_c}{\beta e^{-2}\sin b_c} = \tau_c.$$

This completes the proof.

Before introducing our numerical scheme, we need to establish an estimate for b. According to Lemma 3.1, b_c and τ_c satisfy (3.8) and (3.9). Note that (3.8) and (3.9) are equivalent to

$$b = \tau\beta e^{-2}\sin b, \tag{3.10}$$

$$\tau = -\tau\beta e^{-2}\cos b - d\lambda_1. \tag{3.11}$$

Therefore,

$$\cos b = -\frac{\tau + d\lambda_1}{\tau\beta e^{-2}}.$$

We solve this equation for b to obtain

$$b = \pi - \arccos\frac{\tau + d\lambda_1}{\tau\beta e^{-2}}. \tag{3.12}$$

Substituting (3.12) into (3.10), we have

$$\pi - \arccos\frac{\tau + d\lambda_1}{\tau\beta e^{-2}} = \sqrt{(\tau\beta e^{-2})^2 - (\tau + d\lambda_1)^2}. \tag{3.13}$$

Hence, τ_c is a root of Eq. (3.13). We should note that (3.12) and (3.13) make sense only when $\beta e^{-2} > 1$ and $\tau + d\lambda_1 < \tau\beta e^{-2}$, i.e.

$$\tau > \frac{d\lambda_1}{\beta e^{-2} - 1} > 0. \tag{3.14}$$

A few more calculations show that τ_c is the unique root of Eq. (3.13). In fact, we denote

$$Y(\tau) := \pi - \arccos\frac{\tau + d\lambda_1}{\tau\beta e^{-2}} - \sqrt{(\tau\beta e^{-2})^2 - (\tau + d\lambda_1)^2}.$$

Then,

$$Y'(\tau) = \frac{-d\lambda_1 - \tau^2(\beta \mathrm{e}^{-2})^2 + \tau(\tau + d\lambda_1)}{\tau\sqrt{(\tau\beta \mathrm{e}^{-2})^2 - (\tau + d\lambda_1)^2}}.$$

Noticing that for $\tau > d\lambda_1/((\beta \mathrm{e}^{-2})^2 - 1)$, we have

$$\tau(1 - (\beta \mathrm{e}^{-2})^2) + d\lambda_1 < 0.$$

Therefore,

$$\tau^2(1 - (\beta \mathrm{e}^{-2})^2) + d\lambda_1\tau - d\lambda_1 < 0.$$

So, we get $Y'(\tau) < 0$ for $\tau > d\lambda_1/((\beta \mathrm{e}^{-2})^2 - 1)$. Furthermore, since $\beta \mathrm{e}^{-2} > 1$, we have

$$\frac{d\lambda_1}{\beta \mathrm{e}^{-2} - 1} > \frac{d\lambda_1}{(\beta \mathrm{e}^{-2})^2 - 1}.$$

Hence, $Y'(\tau) < 0$ for $\tau > d\lambda_1/(\beta \mathrm{e}^{-2} - 1)$. This implies that τ_c is the only one root of Eq. (3.13). We can use Newton iteration to solve the equation $Y(\tau) = 0$ for τ_c. This gives the initial guess of τ satisfying (3.14).

Now, we consider functions $f(\chi)$ and $g(\chi)$ that defined in the proof of Lemma 3.1. As we know, $f(\chi)$ and $g(\chi)$ are monotonically increasing and continuous functions of χ on the interval $(\frac{1}{2}\pi, \pi)$. For any given $\tau \geqslant \tau_c$, we consider the following two equations:

$$\begin{aligned} f(x) &= \tau, \\ g(x) &= \tau. \end{aligned} \tag{3.15}$$

Solving these two equations for $x \in (\frac{1}{2}\pi, \pi)$, we obtain the unique root of the equations, respectively, denoted by b_f and b_g. Clearly, $\frac{1}{2}\pi < b_g \leqslant b_f < \pi$. Moreover, $b_g = b_f$ if and only if $\tau = \tau_c$.

Lemma 3.2. *Let $\tau \geqslant \tau_c$ be given. Suppose $\lambda = \mathrm{i}b$ is a pure imaginary eigenvalue of Eq.* (3.1). *Then,* $b \in [b_g, b_f] \subset (\frac{1}{2}\pi, \pi)$.

Proof. According to (3.7), b and τ satisfy

$$f(b) \leqslant \tau \leqslant g(b).$$

Noting that $\tau = f(b_f) = g(b_g)$, and that $f(b)$ and $g(b)$ are both monotonically increasing continuous functions on $(\frac{1}{2}\pi, \pi)$, we obtain from $f(b) \leqslant \tau = f(b_f)$ that $b \leqslant b_f$, and from $g(b_g) = \tau \leqslant g(b)$ that $b_g \leqslant b$. This completes the proof.

We now describe our numerical scheme to locate the critical values of τ when one can find a purely imaginary number ib so that (3.2) has nonzero solution ψ. Our numerical scheme is motivated by the inverse power method with shift (see Ref. [10]) and the inverse iteration method for nonlinear elliptic eigenvalue problems (see Ref. [12]). Let $\tau \geqslant \tau_c$ be given. We choose initial guesses of b_0 and $\psi^{(0)}$ such that $b_0 \in (b_g, b_f)$ and $\psi^{(0)} \in U_0$. For $k = 1, 2, \ldots$, we update $\psi^{(k)}$ and b_k according to the following iteration scheme:

$$-dW_{xx}^{(k)} - b_f W^{(k)} = -(b_f + \tau)\psi^{(k-1)} + (-\mathrm{i}b_{k-1} + \beta\tau\,\mathrm{e}^{-\mathrm{i}b_{k-1}}\,\mathrm{e}^{-\phi}(1-\phi))\psi^{(k-1)}, \quad (3.16)$$

$$\psi^{(k)} = \|W_x^{(k)}\|_{L^2(0,1)}^{-1} W^{(k)}, \quad (3.17)$$

$$b_k = b_g + \|W_x^{(k)}\|_{L^2(0,1)}^{-1}(b_{k-1} - b_g), \quad \text{if } \|W_x^{(k)}\|_{L^2(0,1)} > 1, \quad (3.18)$$

$$b_k = b_f + \|W_x^{(k)}\|_{L^2(0,1)}(b_{k-1} - b_f), \quad \text{if } \|W_x^{(k)}\|_{L^2(0,1)} < 1. \quad (3.19)$$

There are a couple of advantages of this iteration scheme. First of all, under this scheme, the interval (b_g, b_f) is invariant in the sense that if $b_0 \in (b_g, b_f)$ then $b_k \in (b_g, b_f)$ for all $k \geqslant 1$. Secondly, suppose that eventually, $\|W_x^{(k)}\|_{L^2(0,1)} > 1$ or $\|W_x^{(k)}\|_{L^2(0,1)} < 1$, then the sequence $\{b_k\}$ is eventually monotone decreasing or increasing, respectively.

A natural and important question is whether this scheme is convergent. We will address this question for a more general problem in a separate paper so that we can focus on our numerical results for the considered Eq. (1.1). Our numerical observations show that eventually either $\|W_x^{(k)}\|_{L^2(0,1)} > 1$ or $\|W_x^{(k)}\|_{L^2(0,1)} < 1$. Hence for k sufficiently large, $\{b_k\}$ is either a monotonically increasing sequence or a monotonically decreasing sequence. In either case, sequence $\{b_k\}$ converges to a certain number $\tilde{b} \in [b_g, b_f]$. If $\tilde{b} \neq b_g, b_f$, then $\|W_x^{(k)}\|_{L^2(0,1)} \to 1$ as $k \to \infty$. Therefore, $\mathrm{i}\tilde{b}$ is a purely imaginary eigenvalue of Eq. (3.1) and $\tilde{\psi} = \lim_{k\to\infty}\psi^{(k)}$ is a corresponding eigenfunction. Unfortunately, when $\tilde{b} = b_g$ or b_f, our scheme fails to find an eigenvalue if $|(\|W_x^{(k)}\|_{L^2(0,1)} - 1)|$ is beyond the tolerance of accuracy. This may imply that Eq. (3.1) has no purely imaginary eigenvalue for the choice of τ, and suggests that we should adjust τ. In our practice, we choose τ bigger than the previously chosen one if $\|W_x^{(k)}\|_{L^2(0,1)} < 1$, and choose τ smaller than the previous one if $\|W_x^{(k)}\|_{L^2(0,1)} > 1$. Our numerical investigations show that the above strategy works well. In the next section, we present some numerical examples of locating Hopf bifurcations to illustrate how one can implement the above strategy.

4. Numerical results

In Section 2, we proposed an approach to solve two-point boundary problem (1.3). It should be mentioned that there are many numerical methods to solve two-point boundary problems, and the associated computer solvers are well-developed (see for instance, Refs. [2,4–8,11,13–15,18]). However, very few of them seem to be applicable to our problems. The difficulties lie in the fact that two solutions (the zero solution and the positive solution) co-exist for

Eq. (1.3). We had tried the solver TWPBVP, which is a Fortran program based on the mono-implicit Runge–Kutta formula and an adaptive mesh refinement for solving two-point boundary problems (see Refs. [4–8], and the references therein for details). Unfortunately, when applied to our equation, this software always ends up with the zero solution, no matter what ranges of parameters are chosen. We also applied other methods, including differences schemes and shooting method, but the results are not as satisfactory as those obtained via our scheme.

In Section 3, we developed an iteration scheme to solve the eigenvalue problem (3.2) for a pure imaginary eigenvalue ib and the corresponding eigenfunction. The difficulties encountered involve the calculation of b and ψ, as well as the location of τ simultaneously. The following tables lists some numerical results that we obtain by running a program based on the iteration scheme, see Eqs. (3.16)–(3.19).

τ	β	b	Iteration
0.1696000E+02	0.9200000E+01	0.2997433E+01	802
0.2890000E+01	0.1920000E+02	0.2737416E+01	1018
0.4703500E+02	0.8000000E+01	0.3080901E+01	5864
0.6845500E+02	0.7800000E+01	0.3098623E+01	8380
0.7744550E+02	0.7750000E+01	0.3103318E+01	8837
0.8176870E+02	0.7730000E+01	0.3105233E+01	7420
0.2006870E+01	0.2420000E+02	0.2711029E+01	467
0.1306870E+01	0.3420000E+02	0.2646918E+01	281
0.7568700E+00	0.5460000E+02	0.2598600E+01	409

Here, the column called "iteration" in the table presents the number of iterations that was needed to locate a eigenvalue ib for a given β and a proper choice of τ. Using the classic finite difference scheme, we obtain a discrete version of Eq. (3.16), that is,

$$
\begin{aligned}
&-\left(d+\frac{h^2 b_f}{12}\right)W^{(k)}(x_{j-1})+\left(2\mathrm{d}+\frac{5h^2 b_f}{6}\right)W^{(k)}(x_j)-\left(d+\frac{h^2 b_f}{12}\right)W^{(k)}(x_{j+1})\\
&=-(b_f+\tau+\mathrm{i}b_{k-1})\frac{h^2}{12}\left[\psi^{(k-1)}(x_{j-1})+10\psi^{(k-1)}(x_j)+\psi^{(k-1)}(x_{j+1})\right]\\
&\quad+\beta\tau\mathrm{e}^{-\mathrm{i}b_{k-1}}\mathrm{e}^{-\phi(x_j)}(1-\phi(x_j))\\
&\quad\times\frac{h^2}{12}\left[\psi^{(k-1)}(x_{j-1})+10\psi^{(k-1)}(x_j)+\psi^{(k-1)}(x_{j+1})\right],
\end{aligned}
$$

for $j=1,2,\ldots,N$, where, $h=1/(N+1)$, $x_j=jh$ for $j=0,1,\ldots,N+1$ and $W^{(k)}(0)=W^{(k)}(1)=0$. The corresponding numerical norm of $W^{(k)}$ is defined as

$$\|W_x^{(k)}\|_h = \left(h^{-1} \sum_{i=1}^{N+1} (W^{(k)}(x_j) - W^{(k)}(x_{j-1})) \overline{(W^{(k)}(x_j) - W^{(k)}(x_{j-1}))} \right)^{1/2}.$$

Throughout the computation, we first choose $\beta > \mathrm{e}^2$. Then we solve Eq. (3.13) for τ_c. Choosing τ slightly larger than τ_c, we solve (3.15) for b_f and b_g. Next we make initial guesses of $b_0 = \frac{1}{2}(b_f + b_g)$ and $\psi_0(x) = (\sqrt{2}/\pi) \sin \pi x$. Before starting the iteration to update b and ψ, we use the method proposed in Section 2 to compute the positive steady state, since this is required in Eq. (3.16). Figs. 1–4 are graphs of the positive steady state for various values of parameters. These pictures illustrate that, for large τ, the positive steady state has boundary layers which result in difficulties of computation. As τ decreases,

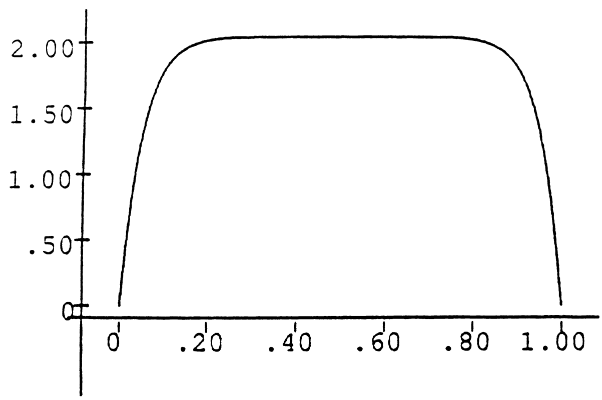

Fig. 1. Plots of the steady solutions with various value of (τ, β, d): $\tau = 81.7687, \beta = 7.73, d = 0.35$.

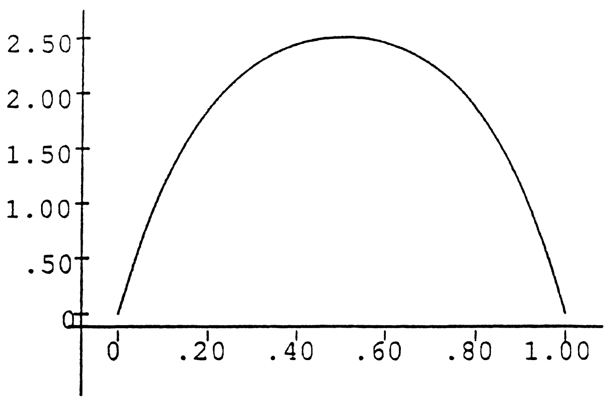

Fig. 2. Plots of the steady solutions with various value (τ, β, d): $\tau = 2.89, \beta = 19.2, d = 0.35$

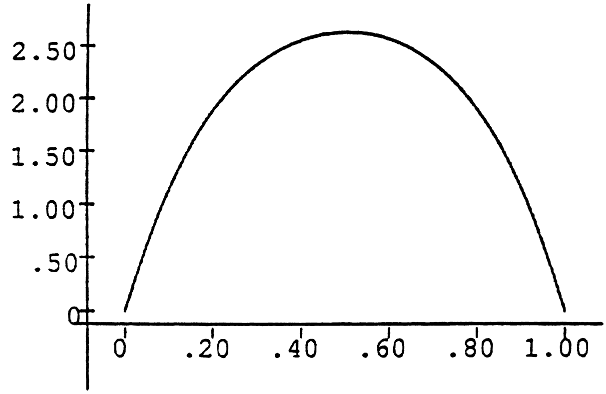

Fig. 3. Plots of the steady solutions with various value (τ, β, d): $\tau = 1.30687, \beta = 34.2, d = 0.35$.

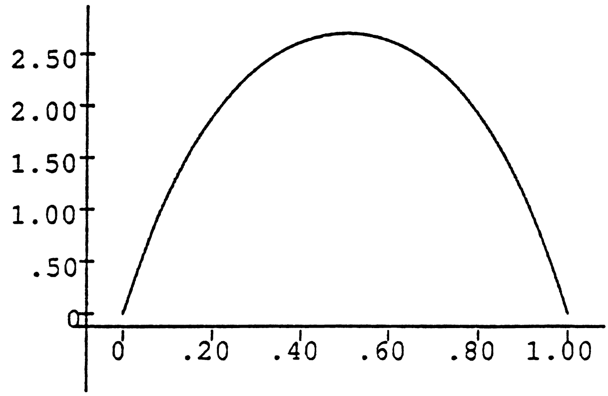

Fig. 4. Plots of the steady solutions with various value (τ, β, d): $\tau = 0.75687, \beta = 54.6, d = 0.35$.

these boundary layer disappear. Note that in these pictures, the value of τ is a critical value near which Hopf bifurcation occurs. Stability of the bifurcated periodic solutions is not clear at this moment. However, with those parameters obtained in these numerical examples, we can solve the original diffusive Nicholson's blowflies equation. Figs. 5–8 are numerical solutions (time evolution) that we obtained. In Figs. 5 and 6, we do observe the time periodic behaviors of the solutions to the equation, which is an indication that the periodic solutions are indeed stable. In Figs. 7 and 8, however, we are not so sure about the stability of the periodic solutions.

To close this section, we compare our iteration scheme with the existing inverse power method with shift, which can be describe as follows:

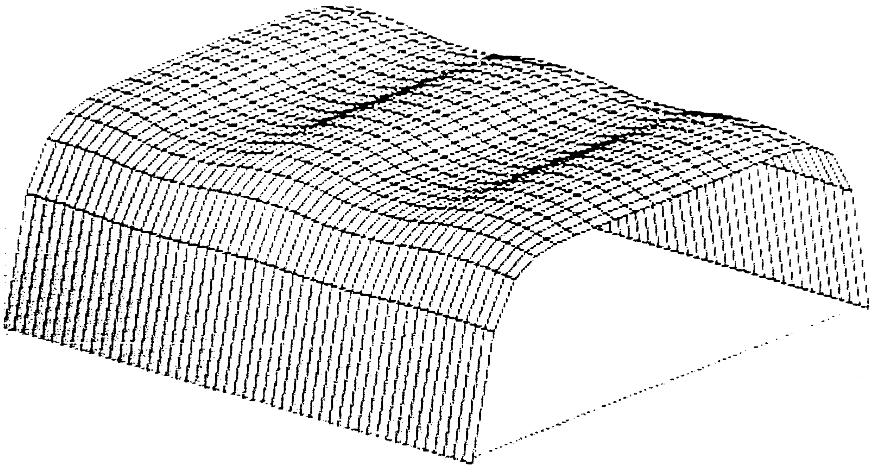

Fig. 5. Time evolution of a solution with the same parameters as Figs. 1 and 2, respectively.

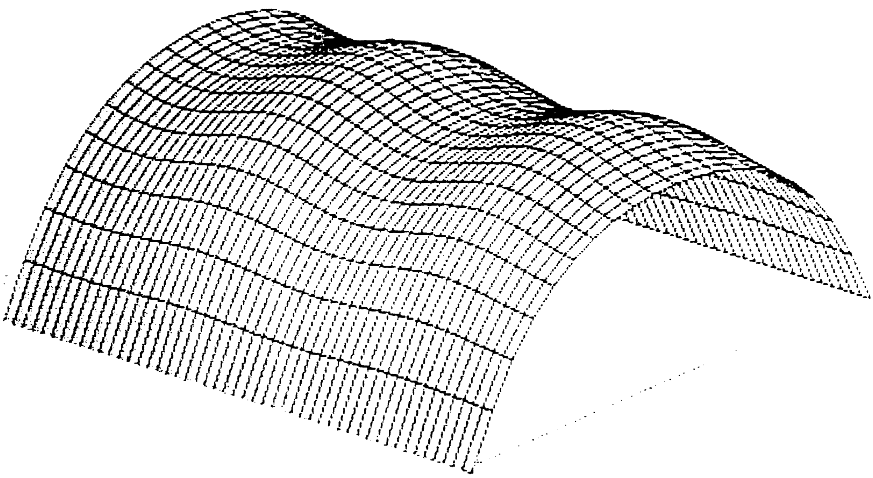

Fig. 6. Time evolution of a solution with the same parameters as Figs. 1 and 2, respectively.

$$\begin{aligned}
&-\mathrm{d}W_{xx}^{(k)} - b_g W^{(k)} \\
&\qquad = -(b_g+\tau)\psi^{(k-1)} + (-\mathrm{i}b_{k-1} + \beta\tau\,\mathrm{e}^{-\mathrm{i}b_{k-1}}\,\mathrm{e}^{-\phi}(1-\phi))\psi^{(k-1)}, \\
&\psi^{(k)} = \|W_x^{(k)}\|_{L^2(0,1)}^{-1} W^{(k)}, \\
&\quad b_k = b_g + \|W_x^{(k)}\|_{L^2(0,1)}^{-1}(b_{k-1} - b_g).
\end{aligned}$$

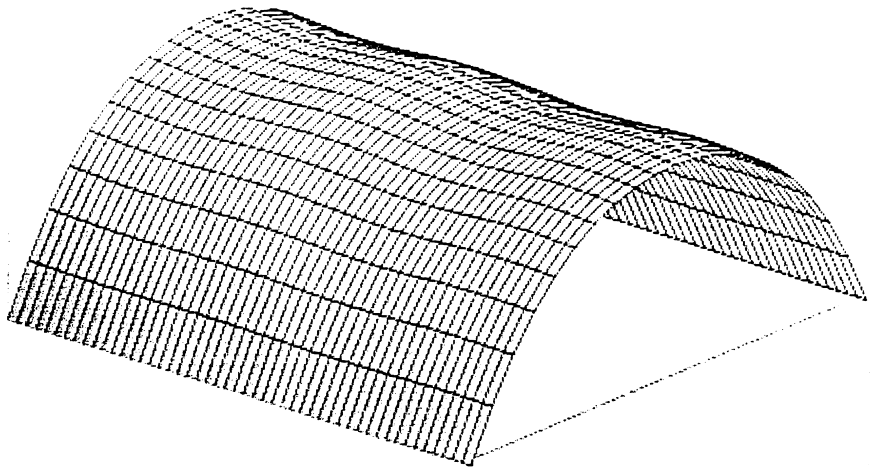

Fig. 7. Time evolution of a solution with the same parameters as Figs. 3 and 4, respectively.

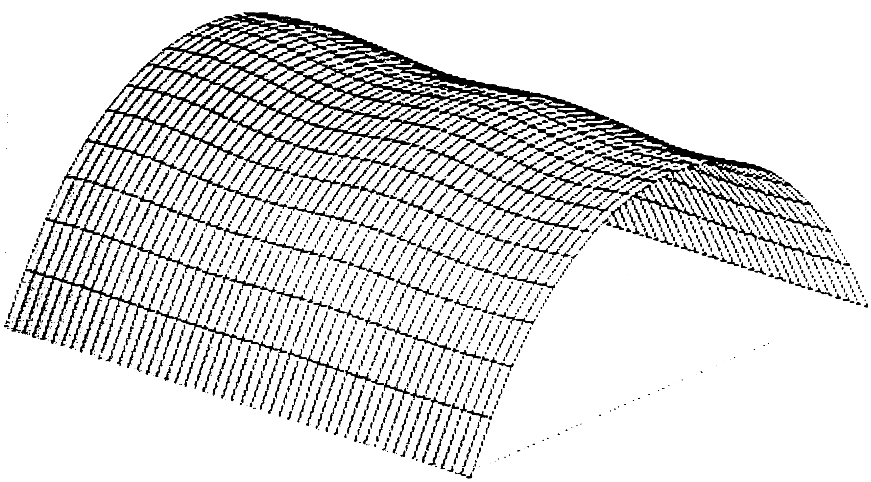

Fig. 8. Time evolution of a solution with the same parameters as Figs. 3 and 4, respectively.

Using the same discrete scheme as on page 50, we obtain the following results:

τ	β	b	Our scheme	Power method
0.8176870E+02	0.7730000E+01	0.3105233E+01	7420	8955
0.7744550E+02	0.7750000E+01	0.3103318E+01	8837	>9999
0.2006870E+01	0.2420000E+02	0.2711029E+01	467	90

We can see that, for given β, if the same τ is used for two methods and β is close to e^2, then our scheme requires less number of iterations than the existing scheme. On the other hand, we should also point out that if β is far away from e^2, then the existing scheme need less number of iterations than ours. Nevertheless, even in this case, our scheme has its own advantage since the classic scheme cannot locate τ so that the eigenvalue lies exactly in the interval of interest to us. For example, when $\beta = 8.55$ the classic scheme ends up with $b = 3.046215$ if τ happens to be chosen as $\tau = 24.445$. While, for such β and τ, the interval $(b_g, b_f) = (3.033295, 3.034119)$. Clearly, the value of b is beyond the interval (b_g, b_f)!

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