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Synchronization and Stable Phase-Locking in a Network of Neurons with Memory

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Abstract—We consider a network of three identical neurons whose dynamics is governed by the Hopfield's model with delay to account for the finite switching speed of amplifiers (neurons). We show that in a certain region of the space of (α, β), where α and β are the normalized parameters measuring, respectively, the synaptic strength of self-connection and neighbourhood-interaction, each solution of the network is convergent to the set of synchronous states in the phase space, and this synchronization is independent of the size of the delay. We also obtain a surface, as the graph of a continuous function of $\tau = \tau(\alpha, \beta)$ (the normalized delay) in some region of (α, β), where Hopf bifurcation of periodic solutions takes place. We describe a continuous curve on such a surface where the system undergoes mode-interaction and we describe the change of patterns from stable synchronous periodic solutions to the coexistence of two stable phase-locked oscillations and several unstable mirror-reflecting waves and standing waves.

Keywords—Neuron, Network, Synchronization, Delay, Phase-locking, Wave, Multistability.

1. INTRODUCTION

We consider a network of electronic (artificial) neurons interconnected through nearest neighbourhoods. The mathematical model can be derived from the Kirchhoff's law. With some rescaling and reparametrization, the model for a network of three identical neurons takes the following form:

$$\dot{x}_i(t) = -x_i(t) + \alpha f(x_i(t-\tau)) + \beta \left[f(x_{i-1}(t-\tau)) + f(x_{i+1}(t-\tau)) \right], \tag{1.1}$$

where $i (\bmod 3)$, $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth sigmoid amplification function, normalized so that $f(0) = 0$ and $f'(0) = 1$ (normalized neural gain), α and β measures, respectively, the normalized synaptic strength of self-connection and neighbourhood-interaction, $\tau \geq 0$ represents the relative size of the time delay (the ratio of the absolute size of the delay over the system's

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relaxation time) due to the finite switching speed of amplifiers. See, for example, [1,2]. Among many important results about the dynamics of system (1.1), we mention the convergence to the set of equilibria [1,3–7] in case $\tau = 0$, the nonlinear oscillation caused by large delay [2,8–15], and the desynchronization induced by the large scale of networks with delay [16].

A natural phase space is the Banach space $C = C([-\tau, 0]; \mathbb{R}^3)$ of continuous mappings from $[-\tau, 0]$ into $\mathbb{R}^3$, equipped with the super-norm. Every initial state $\phi \in C$ defines uniquely a solution x^ϕ of (1.1) for all $t \geq -\tau$, and system (1.1) generates a semiflow in C. A phase state (point) $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$ is said to be *synchronous* if $\phi_1 = \phi_2 = \phi_3$. Due to the uniqueness of the Cauchy initial value problem of system (1.1), every synchronous phase point ϕ gives a synchronous solution x^ϕ of (1.1), that is, $x_1^\phi(t) = x_2^\phi(t) = x_3^\phi(t)$ for all $t \geq -\tau$. Another important class of solution of (1.1) is a phase-locked periodic solution which is a periodic solution $x : \mathbb{R} \to \mathbb{R}$ of (1.1) of period p such that $x_1(t) = x_2(t - (p/3))$ and $x_2(t) = x_3(t - (p/3))$ for all $t \in \mathbb{R}$. These are periodic solutions of system (1.1), each component of which oscillates in the same way but in different phases. Other interesting solutions include mirror-reflecting waves and standing waves to be defined later.

Our focus in this paper is on the pattern formation and mode-interaction/change of system (1.1) in different regions of the space of normalized parameters (α, β, τ). The study of such problems is important in various areas, for example, in the theory and applications of content addressable memories where a stable solution can be used as coded information of a memory of the system to be stored and retrieved.

We show that in the region $A_{as} = \{(\alpha, \beta);\ |\alpha - \beta| < 1\}$ of the normalized parameters, where the difference of self-connection and the neighbourhood-interaction is not significant, system (1.1) is absolutely synchronous in the sense that every solution is convergent to the set of all synchronized phase states independent of the size of the time delay. The ω-limit set of a given orbit can be either a synchronized equilibrium or a synchronized periodic solution, depending on the connection topology of the network, the strength of the self-connection and the neighbourhood-interaction and the size of the delay. On the other hand, in the region where $\alpha - \beta < -1$, we obtain a continuous surface $\tau = \tau(\alpha, \beta)$ where Hopf bifurcations of either a stable synchronized periodic solution or two stable phase-locked periodic solutions and six unstable periodic waves (more precisely, three mirror-reflecting waves and three standing waves) take place, and there is a continuous curve on such a surface where we observe the change of stable patterns of system (1.1) from stable synchronized periodic solutions and to stable phase-locked oscillations.

Our main technical tools are Lyapunov functionals, the symmetric local Hopf bifurcation theory of delay differential equations [15], the normal forms on center manifolds of functional differential equations [17], and the stability theory of bifurcated symmetric periodic solutions of ordinary differential equations [18].

The remaining part of this paper is organized as follows. In Section 2, we describe the model equation and formulate the standing assumptions throughout the paper. Section 3 is devoted to the discussion of when system (1.1) is absolutely synchronous and what is the typical ω-limit set of a given synchronized solution. In Section 4, we establish the existence and stability of synchronized or phase-locked periodic solutions. Section 5 provides a brief discussion of our results and some remarks. The Appendix contains the detailed calculations of the normal forms on center manifolds of system (1.1) near Hopf bifurcation points.

2. THE MODEL OF ARTIFICIAL NEURAL NETWORKS

Consider an artificial neural network consisting of a set of electronic neurons interconnected through a matrix of resistors. Here an electronic neuron, the building block of the network, consists of a nonlinear amplifier which transforms an input signal u_i into the output signal v_i, and the input impedance of the amplifier unit is described by the combination of a resistor ρ_i and a capacitor C_i.

We assume the input-output relation is completely characterized by a voltage amplification function

$$v_i = f(u_i), \tag{2.1}$$

which is C^2-smooth and has a sigmoid form.

It is useful to observe that large negative and positive input signals can steer the amplifier into saturation, thus providing for a degree of nonlinearity which is crucial for the operation of the network. A commonly used amplification function is

$$f(x) = \tanh(\gamma x) = \frac{e^{\gamma x} - e^{-\gamma x}}{e^{\gamma x} + e^{-\gamma x}}, \qquad x \in \mathbb{R}, \tag{2.2}$$

which satisfies the following monotonicity and concavity properties:

$$f(0) = 0, \qquad f'(x) > 0, \quad \text{for all } x \in \mathbb{R}; \tag{2.3}$$

$$f''(x)x < 0, \quad \text{for all } x \neq 0. \tag{2.4}$$

$$-\infty < \lim_{x \to \pm\infty} f(x) < +\infty. \tag{2.5}$$

An important parameter is the so-called neuron gain defined by

$$\gamma = f'(0).$$

Note also that for the function f defined in (2.2), we have

$$f : \mathbb{R} \to \mathbb{R} \text{ is } C^3\text{-smooth and } f'''(0) < 0. \tag{2.6}$$

In fact, we have $f'''(0) = -2\gamma^3$.

The synaptic connections of the network are represented by resistors R_{ij} which connect the output terminal of the amplifier j with the input port of the neuron i. In order that the network can function properly, the resistances R_{ij} must be able to take on negative values. To accomplish this, we supply the amplifiers with an inverting output line which produces the signal $-v_j$. The number of rows in the resistor matrix is doubled, and whenever a negative value of R_{ij} is needed, this is realized by using an ordinary resistor which is connected to the inverted output line.

The time evolution of the signals of the network is described by the Kirchhoff's law. Namely, the strength of the incoming and outgoing current at a given amplifier input port must balance. Consequently, we arrive at

$$C_i \frac{du_i}{dt} + \frac{u_i}{\rho_i} = \sum_{j=1}^{n} \frac{1}{R_{ij}} (v_j - u_i). \tag{2.7}$$

Let

$$\frac{1}{R_i} = \frac{1}{\rho_i} + \sum_{j=1}^{n} \frac{1}{R_{ij}}. \tag{2.8}$$

We get

$$C_i R_i \frac{du_i}{dt} + u_i = \sum_{j=1}^{n} \frac{R_i}{R_{ij}} v_j$$

or

$$T_i \frac{du_i}{dt} + u_i = \sum_{j=1}^{n} w_{ij} f(u_j); \tag{2.9}$$

here

$$T_i = C_i R_i, \tag{2.10}$$

and

$$w_{ij} = \frac{R_i}{R_{ij}} \tag{2.11}$$

denote the local relaxation time and the synaptic strength. See [1,19] for more details.

Implicitly assumed in the above model is that neurons communicate and respond instantaneously. Consideration of the finite switching speed of amplifiers requires that the input-output relation (2.1) be modified to

$$v_i = f\left(u_i\left(t-\tau_i\right)\right), \qquad \tau_i > 0, \tag{2.12}$$

and thus, we obtain the following system of delay differential equations:

$$C_i R_i \frac{du_i(t)}{dt} + u_i(t) = \sum_{j=1}^{n} \frac{R_i}{R_{ij}} f\left(u_j\left(t-\tau_j\right)\right). \tag{2.13}$$

In what follows, for the sake of simplicity, we assume that all local relaxation times are the same. More precisely, we assume that

$$C_i = C, \quad R_i = R, \qquad \text{for all } i = 1, \dots n. \tag{2.14}$$

Rescaling the time and the delay with respect to the relaxation time and rescaling the synaptic strength by

$$x_i(t) = u_i(CRt), \qquad \tau_j^* = \frac{\tau_j}{CR}, \qquad J_{ij} = \frac{R}{R_{ij}}, \tag{2.15}$$

we get

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{n} J_{ij} f\left(x_j\left(t-\tau_j^*\right)\right). \tag{2.16}$$

It is now easy to observe that it is the relative size τ_j^* of the delay which determines the dynamics and the computational performance of the network, and designing a network to operate more quickly (reducing RC) will increase the relative size of the delay and may lead to nonlinear oscillations of the network (as numerical simulations and linear analysis of the celebrated work of Marcus and Westervelt [2] and Marcus *et al.* [8] show).

In the remaining part of this paper, we consider a network of three identical neurons with self-connection and nearest-neighbour interaction, measured, respectively, by

$$\alpha = \frac{R}{R_{ii}}, \quad \beta = \frac{R}{R_{ij}}, \qquad \text{for } 1 \le i \ne j \le 3, \quad |j-i| = 1. \tag{2.17}$$

We also assume the time delays in the self-connection and nearest-neighbor interaction are the same

$$\tau_1^* = \tau_2^* = \tau_3^* = \tau. \tag{2.18}$$

This leads to the following system of delay differential equations:

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha f\left(x_i(t-\tau)\right) + \beta\left[f\left(x_{i-1}(t-\tau)\right) + f\left(x_{i+1}(t-\tau)\right)\right], \tag{2.19}$$

where $i \pmod 3$. We will talk about self-inhibitation or self-excitation if $\alpha < 0$ or $\alpha > 0$. Similarly, we can speak of inhibitory interaction ($\beta < 0$) and excitatory interaction ($\beta > 0$).

There will be four important parameters: α, β, γ, and τ. However, with the help of rescaling the unknown x by γx, we can always reduce equation (2.19) to the case where

$$\gamma = f'(0) = 1. \tag{2.20}$$

Of course, the new variables α and β then incorporate the original parameters for self-connection and neighborhood-interaction with the neuron gain (by multiplication). Henceforth, we will assume $\gamma = 1$ and state our results solely in terms of α, β, and τ.

3. ABSOLUTE SYNCHRONIZATION AND MULTISTABILITY

Our focus in the remaining part of this paper is on system (2.19), where $i \pmod 3$, $f : \mathbb{R} \to \mathbb{R}$ is C^3-smooth and satisfies the normalization, monotonicity, concavity, and boundedness conditions (2.3), (2.20), (2.4), and (2.5). In the stability analysis and normal form calculations, we also need assumption (2.6).

As usual, the phase space for system (2.19) is the Banach space $C = C([-\tau, 0]; \mathbb{R}^3)$ of continuous mappings from $[-\tau, 0]$ into $\mathbb{R}^3$, equipped with the super-norm

$$\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|, \qquad \text{for } \phi = (\phi_1, \phi_2, \phi_3)^{\mathsf{T}} \in C.$$

To specify a solution of (2.19), we need to give an initial condition $x|_{[-\tau,0]} = \phi \in C$. Once an initial condition is given, one can then solve (2.19), by using the variation-of-constants formula

$$x_i(t) = e^{-(t-n\tau)} x_i(n) + \int_{n\tau}^{t} e^{-(t-s)} \left[\alpha f\left(x_i(s-\tau)\right) + \beta\left(f\left(x_{i-1}(s-\tau)\right) + f\left(x_{i+1}(s-\tau)\right)\right)\right] ds$$

on intervals $[n\tau, (n+1)\tau]$ inductively for $n = 0, 1, \dots$. A solution so uniquely defined is denoted by x^ϕ and the mapping $[0, \infty) \times C \ni (t, \phi) \mapsto x^\phi(t + \cdot) \in C$ gives a semiflow on C.

We can easily verify that every solution x^ϕ of (2.19) is bounded and thus its ω-limit set $\omega(\phi)$ is nonempty, compact, connected, and invariant (see, for example, [20]). A solution x^ϕ of (2.19) is said to be *asymptotically synchronous* if $\omega(\phi)$ is contained in the set of *synchronous phase points* given by

$$\Delta = \left\{\phi = (\phi_1, \phi_2, \phi_3)^{\mathsf{T}} \in C;\ \phi_1 = \phi_2 = \phi_3\right\}.$$

We say that system (2.19) is *absolutely synchronous* if every solution of (2.19) is asymptotically synchronous, for every fixed nonnegative delay τ.

Note that due to the uniqueness of the Cauchy initial value problem of (2.19), a solution x of (2.19) with an initial value in Δ must be synchronous, that is, $x_1(t) = x_2(t) = x_3(t)$ for all $t \ge -\tau$.

THEOREM 3.1. *If $|\alpha - \beta| < 1$, then every solution of (2.19) with arbitrarily given τ is asymptotically synchronous.*

PROOF. Consider a given solution $x : [-\tau, \infty) \to \mathbb{R}^3$ of (2.19) and let $y(t) = x_1(t) - x_2(t)$. Then from (2.19), we get for all $t \ge 0$,

$$\begin{aligned}\dot{y}(t) &= -y(t) + (\alpha - \beta)\left[f\left(x_1(t-\tau)\right) - f\left(x_2(t-\tau)\right)\right] \\ &= -y(t) + (\alpha - \beta) p(t) y(t-\tau),\end{aligned}$$

where

$$p(t) = \int_0^1 f'\left(s x_1(t-\tau) + (1-s) x_2(t-\tau)\right) ds.$$

Due to the C^1-smoothness of f, the boundedness of $x_1, x_2 : [-\tau, \infty) \to \mathbb{R}$, and the normalization and concavity conditions, we can find $p^* \in (0, 1]$ such that $p(t) \le p^*$ for all $t \ge 0$.

Let

$$V(t) = y^2(t) + |\alpha - \beta| p^* \int_{t-\tau}^{t} y^2(s)\, ds, \qquad t \ge 0.$$

Then

$$\begin{aligned}\frac{d}{dt} V(t) &= 2y(t)[-y(t) + (\alpha - \beta) p(t) y(t-\tau)] + |\alpha - \beta| p^* y^2(t) - |\alpha - \beta| p^* y^2(t-\tau) \\ &\le -2y^2(t) + |\alpha - \beta| p^* \left[y^2(t) + y^2(t-\tau)\right] + |\alpha - \beta| p^* y^2(t) - |\alpha - \beta| p^* y^2(t-\tau) \\ &= -2\left[1 - |\alpha - \beta| p^*\right] y^2(t),\end{aligned}$$

from which it follows that

$$y^2(t) + |\alpha - \beta|p^* \int_{t-\tau}^{t} y^2(s)\,ds + 2\left[1 - |\alpha - \beta|p^*\right] \int_0^t y^2(s)\,ds \le y^2(0) + |\alpha - \beta|p^* \int_{-\tau}^{0} y^2(s)\,ds.$$

Consequently, $\int_0^\infty y^2(s)\,ds < \infty$.

Note that y is bounded on $[-\tau, \infty)$, and thus, $\dot{y}$ is bounded on $[0, \infty)$. This, together with $y \in L^2([0,\infty))$, implies that $\lim_{t\to\infty} y(t) = 0$.

Therefore, $\lim_{t\to\infty}[x_1(t) - x_2(t)] = 0$. Similarly, we can show that

$$\lim_{t\to\infty} [x_2(t) - x_3(t)] = \lim_{t\to\infty} [x_3(t) - x_1(t)] = 0.$$

This completes the proof.

The above result shows that if the difference between the self-connection and the neighbourhood interaction is not so significant that $|\alpha - \beta| < 1$, then system (2.19) is absolutely synchronous and the dynamics of system (2.19) is completely characterized by the scalar equation

$$\dot{z}(t) = -z(t) + (\alpha + 2\beta) f(z(t - \tau)), \tag{3.1}$$

where z is the common component of a synchronous solution of system (2.19). We will call the region $A_{\mathrm{as}} = \{(\alpha, \beta);\ |\alpha - \beta| < 1\}$ the absolutely synchronous region.

It is easy to employ the same argument for Theorem 3.1 to show that every solution z of (3.1) converges to zero as $t \to \infty$ if $|\alpha + 2\beta| < 1$. Consequently, we can show the following.

COROLLARY 3.2. *If $|\alpha - \beta| < 1$ and $|\alpha + 2\beta| < 1$, then every solution of (2.19) converges to the zero solution as $t \to \infty$.*

In other words, in the region

$$A_{\mathrm{asse}} = \{(\alpha, \beta);\ |\alpha - \beta| < 1, |\alpha + 2\beta| < 1\},$$

system (2.19) can store and retrieve information in the form of a single stable equilibrium. We call A_{asse} the absolutely asynchronous single equilibrium zone.

The complement of A_{asse} in A_{as} is the union of two disjoint sets where either $\alpha + 2\beta > 1$ or $\alpha + 2\beta < -1$. In the case where $\alpha + 2\beta > 1$, the scalar equation (3.1) describing all synchronous solutions of (2.19) admits exactly three equilibria $u_- < 0 < u_+$ given by the equation

$$u = (\alpha + 2\beta) f(u).$$

Note that at $u = u_\pm$, one has

$$(\alpha + 2\beta) f'(u) < 1. \tag{3.2}$$

Evidently, $u_\pm$ give rise to two synchronous equilibria for system (2.19)

$$x_\pm = (u_\pm, u_\pm, u_\pm)^\top,$$

which turn out to be asymptotically stable according to the next result.

THEOREM 3.3. *If $|\alpha - \beta| < 1$ and $\alpha + 2\beta > 1$, then system (2.1) has three equilibria $(u_-, u_-, u_-)^\top$, $(0,0,0)^\top$, and $(u_+, u_+, u_+)^\top$. The trivial equilibrium is unstable and the other two equilibria are asymptotically stable.*

PROOF. The linearization of (2.19) at an equilibrium $(x^*, x^*, x^*)^\top$ is given by

$$\dot{y}_i(t) = -y_i(t) + \alpha f'(x^*)\, y_i(t - \tau) + \beta f'(x^*)\left[y_{i-1}(t - \tau) + y_{i+1}(t - \tau)\right], \qquad i (\bmod 3).$$

The characteristic matrix is

$$\Delta_\tau(x^*,\lambda) = (\lambda+1)Id - \alpha f'(x^*)\,e^{-\lambda\tau} Id - \beta f'(x^*)\,e^{-\lambda\tau}\delta,$$

where

$$\delta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let

$$\chi = e^{i(2\pi/3)},$$
$$v_j = \left(1, \chi^j, \chi^{2j}\right), \qquad j = 0,1,2.$$

Then

$$\begin{aligned}\Delta_\tau(x^*,\lambda)\,v_j &= \left[(\lambda+1) - \alpha e^{-\lambda\tau} f'(x^*) - \beta f'(x^*)\,e^{-\lambda\tau}\left(\chi^{-j}+\chi^j\right)\right] v_j \\ &= \left[(\lambda+1) - f'(x^*)\,e^{-\lambda\tau}\left(\alpha + 2\beta\cos\frac{2\pi}{3}j\right)\right] v_j,\end{aligned}$$

from which it follows that the characteristic values are given by the zeros of

$$\begin{aligned}\det\Delta_\tau(x^*,\lambda) &= \Pi_{j=0}^2\left[(\lambda+1) - f'(x^*)\,e^{-\lambda\tau}\left(\alpha+2\beta\cos\frac{2\pi}{3}j\right)\right] \\ &= \left[\lambda+1-(\alpha+2\beta)f'(x^*)\,e^{-\lambda\tau}\right]\left[\lambda+1-(\alpha-\beta)f'(x^*)\,e^{-\lambda\tau}\right]^2.\end{aligned}$$

In the case where $x^*=0$, since $(\alpha+2\beta)f'(x^*) = \alpha+2\beta > 1$, we know that the first factor of $\det\Delta_\tau(0,\lambda)$ has a positive real zero, and hence, $(0,0,0)^{\mathrm{T}}$ is unstable.

In the case where $x^* = u_\pm$, we know from (3.2) that $(\alpha+2\beta)f'(u_\pm) < 1$, and also $|(\alpha-\beta)\, f'(x^*)| \le |\alpha-\beta| < 1$. Therefore, all zeros of $\det\Delta_\tau(u_\pm,\lambda)$ have negative real parts (see, for example, [21]), and hence, $(u_\pm,u_\pm,u_\pm)^{\mathrm{T}}$ are asymptotically stable. This completes the proof.

REMARK 3.4. It is interesting to remark that the (local) asymptotic stability of the two nontrivial synchronous equilibria is independent of the size of the delay. Also note that the scalar equation (3.1) with $\alpha+2\beta>1$ generates an eventually strongly monotone semiflow [22], and system (2.19) generates an eventually strongly monotone semiflow with respect to pointwise ordering of the phase space in a subregion

$$I = \{(\alpha,\beta);\ \alpha+2\beta>1,\ |\alpha-\beta|<1,\ \alpha>0,\ \beta>0\}.$$

Consequently, in the subregion I, we can conclude that the generic dynamics of system (2.19) is the convergence to $x_\pm$ [23]. It is reasonable to conjecture that system (2.19) generates a strongly order-preserving semiflow in the whole region $\{(\alpha,\beta);\ \alpha+2\beta>1,\ |\alpha-\beta|<1\}$ with respect to a certain nonstandard ordering of the phase space (see, for example, [24]). Should this conjecture be verified, we can then conclude that the dominant dynamics of the full system (2.19) is the convergence to $x_\pm$ when $\alpha+2\beta>1$ and $|\alpha-\beta|<1$. We call A_{asme} the zone of absolutely synchronous multiple equilibria.

REMARK 3.5. It is also important to emphasize that generic convergence to equilibria does not necessarily imply simple structure of the global attractor for system (2.19), and that the size of the delay does affect the unstable manifold of the trivial solution and the structure of the domain of attraction of each nontrivial equilibria $x_\pm$. As each stable equilibrium represents a memory of the network in the theory and applications of content addressable memories, it is essential to obtain detailed information about the boundary of the domain of attraction for each equilibrium $x_\pm$, and about the dependence of these boundaries on the size of the delay. This is a quite complicated task, though some progress has been achieved in a recent work of Krisztin *et al.* [25], where it was shown that if

$$\tau > \tau_1 = \frac{2\pi - \arccos(1/(\alpha + 2\beta))}{\sqrt{(\alpha + 2\beta)^2 - 1}},$$

then the linearization of the semigroup of (3.1) is a C_0-semigroup whose generator has a positive real eigenvalue and a complex conjugate pair of eigenvalues with positive real part, there exists a leading unstable manifold of the zero solution which is tangent to the three-dimensional eigenspace of the generator associated with the positive real and complex conjugate pair with positive real part of eigenvalues. The closure of the global forward extension of this leading unstable manifold of the zero solution is a three-dimensional smooth spindle with two tips $x_\pm$, this spindle is separated by a smooth two-dimensional invariant manifold with boundary borded by a periodic orbit into two halves such that each half belongs to the basins of attraction of the tips. It is also shown in [26] (together with our Theorem 3.1) that if

$$\tau_1 < \tau < \tau_2 = \frac{4\pi - \arccos(1/(\alpha + 2\beta))}{\sqrt{(\alpha + 2\beta)^2 - 1}},$$

then this spindle is exactly the global attractor of system (2.19) (with f being given by (2.2)). Evidently, as τ passes through each of the critical values

$$\tau_1 < \tau_2 < \tau_3 < \cdots$$

with

$$\tau_n = \frac{2n\pi - \arccos(1/(\alpha + 2\beta))}{\sqrt{(\alpha + 2\beta)^2 - 1}},$$

the dimension of the unstable space of the zero solution increases by two and the boundaries of the basins of attraction of $x_\pm$ become more and more complicated. See also the work of Pakdaman *et al.* [13] and Pakdaman *et al.* [14] for the effect of the delay on the basins of attraction of $u_\pm$ for the scalar equation (2.19).

REMARK 3.6. The remaining part of the absolute synchronization zone $|\alpha - \beta| < 1$ is

$$A_{\text{asso}} = \{(\alpha, \beta);\ |\alpha - \beta| < 1,\ \alpha + 2\beta < -1\}.$$

In this region, the scalar equation (3.1) is the extensively studied scalar delay differential equation with negative feedback. When the delay is sufficiently large, the scalar equation possesses a stable slowly oscillatory periodic solution (here, a slowly oscillatory solution is a solution for which the distance of consecutive zeros is large than the delay τ) and the dominant dynamics is the convergence to slow oscillations. See [21,27–30] and references therein. We thus call A_{asso} the absolutely synchronous slowly oscillatory zone.

4. DESYNCHRONIZATION: STABLE PHASE-LOCKING AND UNSTABLE WAVES

We start with the following simple observation.

LEMMA 4.1. *If $\alpha - \beta < -1$, then every equilibrium of system (2.19) must be synchronous.*

PROOF. For $1 \le i \ne j \le 3$, every equilibrium x^* must satisfy

$$x_i^* - x_j^* = (\alpha - \beta)\left[f\left(x_i^*\right) - f\left(x_j^*\right)\right].$$

Using the monotonicity of f and the assumption $\alpha - \beta < -1$, we get $x_i^* = x_j^*$. This completes the proof.

From Lemma 4.1, it follows that the set E of equilibria of system (2.19) is given by

$$E = \{(0,0,0)^{\top}\}, \qquad \text{if } \alpha-\beta<-1 \text{ and } \alpha+2\beta<1,$$
$$E = \left\{(0,0,0)^{\top}, (u_{\pm},u_{\pm},u_{\pm})^{\top}\right\}, \qquad \text{if } \alpha-\beta<-1 \text{ and } \alpha+2\beta>1.$$

The characteristic equation at $(0,0,0)^{\top}$ and $(u_{\pm},u_{\pm},u_{\pm})^{\top}$ are

$$\det\Delta_{\tau}(0,\lambda) = \left[\lambda+1-(\alpha+2\beta)e^{-\lambda\tau}\right]\left[\lambda+1-(\alpha-\beta)e^{-\lambda\tau}\right]^2$$

and

$$\det\Delta_{\tau}(u_{\pm},\lambda) = \left[\lambda+1-(\alpha+2\beta)f'(u_{\pm})e^{-\lambda\tau}\right]\left[\lambda+1-(\alpha-\beta)f'(u_{\pm})e^{-\lambda\tau}\right]^2,$$

respectively.

In the reminder of this section, we consider system (2.19) where (α,β) is in the zone

$$D = \{(\alpha,\beta);\ \alpha-\beta<-1,\ \alpha+2\beta<1\}.$$

When $\tau=0$, we have

$$\det\Delta_0(0,\lambda) = [\lambda+1-(\alpha+2\beta)][\lambda+1-(\alpha-\beta)]^2$$

and characteristic values are $\alpha+2\beta-1<0$ and $\alpha-\beta-1<0$. Therefore, the trivial solution is asymptotically stable in the absence of delay. However, since $\alpha-\beta<-1$, we can increase the delay to get purely imaginary characteristic values of the factor $\lambda+1-(\alpha-\beta)e^{-\lambda\tau}$. Similarly, when $\alpha+2\beta<-1$, we can obtain purely imaginary characteristic values from the factor $\lambda+1-(\alpha+2\beta)e^{-\lambda\tau}$ by increasing the delay. In fact, it is easy to obtain the first critical value of τ such that

$$\lambda+1-\eta e^{-\lambda\tau}=0, \qquad \text{(for a given } \eta<-1) \tag{4.1}$$

has a pair of purely imaginary zeros when $\tau=\tau_\eta$, with

$$\tau_\eta = \frac{\pi-\arccos(1/|\eta|)}{\sqrt{\eta^2-1}}, \qquad \eta<-1 \tag{4.2}$$

and the associated pair of purely imaginary zeros are $\lambda=\pm i\omega_\eta$ with

$$\omega_\eta=\sqrt{\eta^2-1}. \tag{4.3}$$

Moreover, if $\lambda:(\tau_\eta-\epsilon,\tau_\eta+\epsilon)\to\mathbb{C}$ is a smooth curve of zeros of (4.1) (for some sufficiently small ϵ) such that $\lambda(\tau_\eta)=i\omega_\eta$, then

$$\operatorname{Re}\lambda'(\tau_\eta) = \operatorname{Re}\left.\frac{-\lambda\eta e^{-\lambda\tau}}{1+\eta\tau e^{-\lambda\tau}}\right|_{\tau=\tau_\eta,\lambda=i\omega_\eta} = \frac{\omega^2}{(1+\tau_\eta)^2+(\tau_\eta\omega_\eta)^2}>0. \tag{4.4}$$

Also note that

$$-1>\eta>\eta^* \text{ implies } \tau_\eta>\tau_{\eta^*}. \tag{4.5}$$

Note also that if $-1\le\alpha+2\beta\le 1$ and $\alpha-\beta<-1$, then the factor $\lambda+1-(\alpha+2\beta)e^{-\lambda\tau}$ cannot have purely imaginary zeros. Therefore, when $(\alpha,\beta)\in D$, the first critical value of τ where the characteristic equation $\det\Delta_\tau(0,\lambda)=0$ has a pair of purely imaginary zeros is

$$\tau_I = \begin{cases} \tau_{\alpha-\beta} = \dfrac{\pi-\arccos(1/|\alpha-\beta|)}{\sqrt{(\alpha-\beta)^2-1}}, & \text{if } \beta>0, \\[2ex] \tau_{\alpha+2\beta} = \dfrac{\pi-\arccos(1/|\alpha+2\beta|)}{\sqrt{(\alpha+2\beta)^2-1}}, & \text{if } \beta<0 \end{cases} \tag{4.6}$$

and the associated purely imaginary characteristic values are $\pm i\omega_I$, where

$$\omega_I = \begin{cases} \sqrt{(\alpha-\beta)^2-1}, & \text{if } \beta > 0, \\ \sqrt{(\alpha+2\beta)^2-1}, & \text{if } \beta < 0. \end{cases} \tag{4.7}$$

These characteristic values are simple or double as zeros of $\det \Delta_{\tau_I}(0,\lambda) = 0$, depending on whether $\beta < 0$ or $\beta > 0$.

In the case where $\beta < 0$, we can apply the standard Hopf bifurcation theorem of delay differential equations (see, for example, [21,30,31]) to obtain a Hopf bifurcation of synchronous periodic solutions. In the case where $\beta > 0$, the aforementioned standard Hopf bifurcation theorem does not apply since $\pm i\omega_\tau$ are double characteristic values. On the other hand, the considered system (2.19) is equivariant with respect to the D_3-action where the Z_3 subgroup acts by permutation (sending x_i to x_{i+1}) and the flip acts by interchanging (sending x_i to x_{3-i}). This allows us to apply the symmetric Hopf bifurcation theorem for delay differential equations established in [15] (as an extension of the well-known Golubitsky-Stewart Theorem [32] for symmetric ordinary differential equations) to obtain eight branches of asynchronous periodic solutions. More precisely, we have the following theorem.

THEOREM 4.2. *Assume $(\alpha,\beta) \in D$ and define (τ_I,ω_I) as in (4.6) and (4.7). Then*

(i) *in case $\beta < 0$, near $\tau = \tau_I$ there exists a supercritical bifurcation of stable synchronous periodic solutions of period p_τ near $(2\pi/\omega_\tau)$, bifurcated from the zero solution of system (2.19);*

(ii) *in case $\beta > 0$, near $\tau = \tau_I$ there exist eight branches of asynchronous periodic solutions of period p_τ near $(2\pi/\omega_\tau)$, bifurcated simultaneously from the zero solution of system (2.19), and these are*

 (a) *two stable phase-locked oscillations: $x_i(t) = x_{i-1}(t \pm (p_\tau/3))$, for $i(\operatorname{mod} 3)$ and $t \in \mathbb{R}$;*

 (b) *three unstable mirror-reflecting waves: $x_i(t) = x_j(t) \neq x_k(t)$, for $t \in \mathbb{R}$ and for some distinct (i,j,k) in $\{1,2,3\}$;*

 (c) *three unstable standing waves: $x_i(t) = x_j(t + (p_\tau/2))$, for $t \in \mathbb{R}$ and for some pair of distinct elements (i,j) in $\{1,2,3\}$.*

PROOF. (i). The existence is an immediate application of the standard Hopf bifurcation theorem for functional differential equations. Let $\tau^* = \tau_I, \mu = \tau - \tau_I$. According to the calculations in the Appendix, the normal form of (2.19) on the center manifold can be written, in polar coordinates, as

$$\dot{\rho} = \left(a_1\mu + b_1\rho^2\right)\rho + 0\left(\mu^2\rho\right) + 0\left(\rho^4\right),$$

where

$$a_1 = -\frac{\omega_I\tau_I}{\left(1+\tau_I\right)^2 + \omega_I^2} > 0$$

and

$$b_1 = \frac{f'''(0)}{2}\frac{\tau_I\left(1+\tau_I+\omega_I\tau_I\right)}{\left(1+\tau_I\right)^2+\omega_I^2} < 0.$$

Conclusion (i) then follows immediately.

(ii). Let $\tau^* = \tau_I$, $u^* = \omega_\tau/\tau_I$, and $\mu = \tau - \tau_I$. We obtain from the calculations in the Appendix the following normal form of (2.19) on the center manifolds:

$$\begin{aligned}\begin{pmatrix}\dot{w}_1\\ \dot{w}_2\end{pmatrix} &= u^*\tau^*\begin{pmatrix} w_2 \\ -w_1\end{pmatrix} + \mu u^*\begin{pmatrix}\operatorname{Im}\left(a^{-1}\right)w_1 + \operatorname{Re}\left(a^{-1}\right)w_2 \\ -\operatorname{Re}\left(a^{-1}\right)w_1 + \operatorname{Im}\left(a^{-1}\right)w_2\end{pmatrix} \\ &+ 3b\tau^*\left(\rho_1^2+2\rho_2^2\right)\begin{pmatrix}\operatorname{Re}\left(a^{-1}\left(1-iu^*\right)\right)w_1 - \operatorname{Im}\left(a^{-1}\left(1-iu^*\right)\right)w_2 \\ \operatorname{Im}\left(a^{-1}\left(1-iu^*\right)\right)w_1 + \operatorname{Re}\left(a^{-1}\left(1-iu^*\right)\right)w_2\end{pmatrix} \\ &+ O\left(\mu^2|w|\right) + O\left(|w|^4\right),\end{aligned}$$

$$\begin{pmatrix} \dot{w}_3 \\ \dot{w}_4 \end{pmatrix} = u^*\tau^* \begin{pmatrix} w_4 \\ -w_3 \end{pmatrix} + \mu u^* \begin{pmatrix} \operatorname{Im}\left(a^{-1}\right) w_3 + \operatorname{Re}\left(a^{-1}\right) w_4 \\ -\operatorname{Re}\left(a^{-1}\right) w_3 + \operatorname{Im}\left(a^{-1}\right) w_4 \end{pmatrix}$$
$$+ 3b\tau^* \left(\rho_2^2 + 2\rho_1^2\right) \begin{pmatrix} \operatorname{Re}\left(a^{-1}\left(1 - iu^*\right)\right) w_3 - \operatorname{Im}\left(a^{-1}\left(1 - iu^*\right)\right) w_4 \\ \operatorname{Im}\left(a^{-1}\left(1 - iu^*\right)\right) w_3 + \operatorname{Re}\left(a^{-1}\left(1 - iu^*\right)\right) w_4 \end{pmatrix}$$
$$+ O\left(\mu^2|w|\right) + O\left(|w|^4\right),$$

where

$$\rho_1 = \sqrt{w_1^2 + w_2^2},$$
$$\rho_2 = \sqrt{w_3^2 + w_4^2},$$
$$a = 1 + \tau^* - iu^*\tau^*,$$
$$b = \frac{1}{3!} f'''(0).$$

Introducing the periodic-scaling parameter ω and letting

$$z_1(t) = w_1(s) + iw_2(s),$$
$$z_2(t) = w_3(s) + iw_4(s)$$

with

$$s = \left[(1+\omega)u^*\tau^*\right]^{-1} t,$$

we obtain

$$\begin{aligned}
(1+\omega)\dot{z}_1(t) &= w_2(s) - iw_1(s) \\
&\quad + \mu\left(\tau^*\right)^{-1}\left[\operatorname{Im}\left(a^{-1}\right) w_1(s) + \operatorname{Re}\left(a^{-1}\right) w_2(s) - i\operatorname{Re}\left(a^{-1}\right) w_1(s) + i\operatorname{Im}\left(a^{-1}\right) w_2(s)\right] \\
&\quad + 3b\left(u^*\right)^{-1}\left(|z_1|^2 + 2|z_2|^2\right)\left[\operatorname{Re}\left(a^{-1}\left(1 - iu^*\right)\right) w_1(s) - \operatorname{Im}\left(a^{-1}\left(1 - iu^*\right)\right) w_2(s)\right. \\
&\quad \left. + i\operatorname{Im}\left(a^{-1}\left(1 - iu^*\right)\right) w_1(s) + i\operatorname{Re}\left(a^{-1}\left(1 - iu^*\right)\right) w_2(s)\right] \\
&\quad + O\left(\mu^2|z|\right) + O\left(|z|^4\right) \\
&= -iz_1(t) + \mu\left(\tau^*\right)^{-1}\left[\operatorname{Im}\left(a^{-1}\right) - i\operatorname{Re}\left(a^{-1}\right)\right] z_1(t) \\
&\quad + 3b\left(u^*\right)^{-1}\left[\operatorname{Re}\left(a^{-1}\left(1 - iu^*\right)\right) + i\operatorname{Im}\left(a^{-1}\left(1 - iu^*\right)\right)\right] z_1(t)\left(|z_1(t)|^2 + 2|z_2(t)|^2\right) \\
&\quad + O\left(\mu^2|z|\right) + O\left(|z|^4\right) \\
&= -iz_1(t) - i\mu\left(\tau^*\right)^{-1} a^{-1} z_1(t) \\
&\quad + 3b\left(u^*\right)^{-1} a^{-1}\left(1 - iu^*\right) z_1(t)\left(|z_1(t)|^2 + 2|z_2(t)|^2\right) \\
&\quad + O\left(\mu^2|z|\right) + O\left(|z|^4\right).
\end{aligned}$$

Similarly, we get an equation for $z_2(t)$. Thus, ignoring the terms $O(\mu^2|z|)$ and $O(|z|^4)$, we get the normal form

$$\begin{aligned}
(1+\omega)\dot{z}_1 &= -iz_1 - i\mu\left(\tau^*\right)^{-1} a^{-1} z_1 + 3b\left(u^*\right)^{-1} a^{-1}\left(1 - iu^*\right)\left(|z_1|^2 + 2|z_2|^2\right) z_1, \\
(1+\omega)\dot{z}_2 &= -iz_2 - i\mu\left(\tau^*\right)^{-1} a^{-1} z_2 + 3b\left(u^*\right)^{-1} a^{-1}\left(1 - iu^*\right)\left(2|z_1|^2 + |z_2|^2\right) z_2.
\end{aligned} \tag{4.8}$$

Let $g : \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R} \to \mathbb{C} \oplus \mathbb{C}$ be given so that $-g(z_1, z_2, \mu)$ is the right-hand side of (4.8). Then (4.8) can be written as

$$(1+\omega)\dot{z} + g(z, \mu) = 0. \tag{4.9}$$

Note that

$$D_z g(0,0)\left(z_1, z_2\right) = i\left(z_1, z_2\right), \qquad z = (z_1, z_2) \in \mathbb{C} \oplus \mathbb{C}.$$

Also note that $g(\cdot,\mu):\mathbb{C}\oplus\mathbb{C}\to\mathbb{C}\oplus\mathbb{C}$ is $D_3\times S^1$-equivariant with respect to the following $D_3\times S^1$-action on $\mathbb{C}\oplus\mathbb{C}$:

$$\begin{aligned}
\gamma\left(z_1,z_2\right) &= \left(e^{i(2\pi/3)}z_1, e^{-i(2\pi/3)}z_2\right), && Z_3=\langle\gamma\rangle\le D_3,\\
\kappa\left(z_1,z_2\right) &= \left(z_2,z_1\right), && Z_2=\langle\kappa\rangle\le D_3,\\
e^{i\theta}\left(z_1,z_2\right) &= \left(e^{i\theta}z_1, e^{i\theta}z_2\right), && e^{i\theta}\in S^1.
\end{aligned}$$

According to [18, pp. 296–297, Theorems 6.3 and 6.5], the bifurcations of small-amplitude periodic solutions of (4.8) are completely determined by the zeros of equation

$$-i(1+\omega)z+g(z,\mu)=0, \tag{4.10}$$

and their (orbital) stability is determined by the signs of three eigenvalues of

$$Dg(z,0)-i(1+\omega)Id \tag{4.11}$$

that are not forced to zero by the group action. To be more precise, we note that (4.10) is equivalent to

$$\begin{aligned}
&i\omega z_1 - i\mu\left(\tau^*\right)^{-1}a^{-1}z_1+3b\left(u^*\right)^{-1}a^{-1}\left(1-iu^*\right)\left(\left|z_1\right|^2+2\left|z_2\right|^2\right)z_1=0,\\
&i\omega z_2 - i\mu\left(\tau^*\right)^{-1}a^{-1}z_2+3b\left(u^*\right)^{-1}a^{-1}\left(1-iu^*\right)\left(2\left|z_1\right|^2+\left|z_2\right|^2\right)z_2=0.
\end{aligned} \tag{4.12}$$

It is known that (4.12) can be written as

$$A\begin{pmatrix} z_1\\ z_2\end{pmatrix}+B\begin{pmatrix} z_1^2\overline{z_1}\\ z_2^2\overline{z_2}\end{pmatrix}=0 \tag{4.13}$$

with

$$\begin{aligned}
A &= A_0+A_N\left(\left|z_1\right|^2+\left|z_2\right|^2\right),\\
B &= B_0
\end{aligned}$$

for some complex numbers A_0, A_N, and B_0 [18, pp. 376] given by

$$\begin{aligned}
A_0 &= iu^*\left(\tau^*\right)^{-1}a^{-1}-i\omega,\\
A_N &= -6b\left(u^*\right)^{-1}a^{-1}\left(1-iu^*\right),\\
B_0 &= 3b\left(u^*\right)^{-1}a^{-1}\left(1-iu^*\right).
\end{aligned}$$

By the results of [18, pp. 376], we know that the bifurcation of phase-locked oscillation is supercritical (respectively, subcritical) depends on whether $\mathrm{Re}(A_N+B_0)>0$ (respectively, $\mathrm{Re}(A_N+B_0)<0$) and these are orbitally asymptotically stable if $\mathrm{Re}(A_N+B_0)>0$ and $\mathrm{Re}\,B_0<0$.

Note that

$$\begin{aligned}
\mathrm{Re}\left(A_N+B_0\right) &= \mathrm{Re}\left(-3b\left(u^*\right)^{-1}a^{-1}\left(1-iu^*\right)\right)\\
&= -3b\left(u^*\right)^{-1}\mathrm{Re}\left[\frac{1+\tau^*+iu^*\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}\left(1-iu^*\right)\right]\\
&= -3b\left(u^*\right)^{-1}\frac{1+\tau^*+\left(u^*\right)^2\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}>0
\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}(B_0) &= \operatorname{Re}\left(3b\,(u^*)^{-1}\,a^{-1}\,(1-iu^*)\right)\\ &= 3b\,(u^*)^{-1}\,\frac{1+\tau^*+(u^*)^2\,\tau^*}{(1+\tau^*)^2+(u^*\tau^*)^2}<0.\end{aligned}$$

Consequently, the bifurcation of phase-locked oscillations is supercritical and orbitally asymptotically stable.

Note also that

$$\begin{aligned}\operatorname{Re}(2A_N+B_0) &= \operatorname{Re}\left(-9b\,(u^*)^{-1}\,a^{-1}\,(1-iu^*)\right)\\ &= -9bu^*\frac{1+\tau^*+(u^*)^2\,\tau^*}{(1+\tau^*)^2+(u^*\tau^*)^2}>0\end{aligned}$$

and

$$\operatorname{Re}B_0<0.$$

We infer from the results of [18, pp. 376] again that the bifurcations of mirror-reflecting waves and standing waves are supercritical and unstable. This completes the proof.

5. CONCLUSIONS AND REMARKS

For the normalized model (2.19) of a network of three identical neurons with self-connection and neighbourhood interaction and with delay τ to account for the finite switching speed of amplifiers, we show that there is a region of the parameters (α,β), measuring, respectively, the strength of self-connection and neighbourhood interaction, given by

$$A_{\text{as}}=\{(\alpha,\beta);\ |\alpha-\beta|<1\},$$

such that for (α,β) in this region, every solution is eventually synchronized and this fact is independent of the size of the delay. The synchronized stable pattern can be in the form of either the trivial equilibrium (if $|\alpha+2\beta|<1$), the multiple nontrivial equilibria $x_\pm$ (if $\alpha+2\beta>1$), or a slowly oscillatory periodic solution (if $\alpha+2\beta<-1$ and if τ is sufficiently large).

When (α,β) moves to the region D where $\alpha-\beta<-1$ and $\alpha+2\beta<1$, we obtain a continuous surface in the (τ,α,β)-space given by

$$\tau=\begin{cases}\dfrac{\pi-\arccos(1/|\alpha-\beta|)}{\sqrt{(\alpha-\beta)^2-1}}, & \text{if } \beta>0,\\[2ex] \dfrac{\pi-\arccos(1/|\alpha+2\beta|)}{\sqrt{(\alpha+2\beta)^2-1}}, & \text{if } \beta<0,\end{cases}$$

where bifurcations of periodic solutions take place. Depending on whether the neighbourhood interaction is inhibitory ($\beta<0$) or excitatory ($\beta>0$), the bifurcated stable periodic solutions are either synchronized or phased-locked. This indicates that *excitation is the main reason for desynchronization* for the model equation considered here, and the coexistence of two stable phase-locked periodic solutions and six other unstable waves suggest possible complicated structure of the global attractor.

Our stability analysis of the bifurcated periodic solutions is based on the normal form calculations developed in [17] and is greatly simplified due to the fact that the usual sigmoid function satisfies $f''(0)=0$. When $\alpha-\beta<-1$ and $\alpha+2\beta>1$, system (2.19) has two nontrivial equilibria $x_\pm$ where Hopf bifurcation can take place when τ crosses a critical value τ^*. We suspect that either the mirror-reflecting waves or the standing waves are stable and the phase-locked are unstable, verification of this will be quite interesting since this provides a single model which exhibits a stable synchronized periodic solution, stable phase-locked oscillations, stable mirror-reflecting waves, and stable standing waves depending on the location of related parameters.

APPENDIX
THE CALCULATION OF NORMAL FORMS ON CENTER MANIFOLDS

In this Appendix, we employ the algorithm and notations of Faria and Magalhães [17] to derive the normal forms of system (2.19) on center manifolds.

We first rescale the time by $t \mapsto (t/\tau)$ to normalize the delay so that (2.19) can be written as

$$\dot{x}(t) = F(x_t, \tau) \tag{A.1}$$

in the phase space $C = C([-1,0];\mathbb{R}^3)$, where for $\phi = (\phi_1, \phi_2, \phi_3)^\top \in C$, we have

$$(F(\phi,\tau))_i = -\tau\phi_i(0) + \alpha\tau f(\phi_i(-1)) + \beta\tau [f(\phi_{i-1}(-1)) + f(\phi_{i+1}(\phi(-1)))]$$

with $i(\bmod 3)$. We also assume that

$$f(x) = x + bx^3 + \text{h.o.t.}$$

with

$$b = \frac{1}{3!}f'''(0)$$

(see (2.3), (2.6), and (2.20)). The linearized equation at zero for system (A.1) is

$$\dot{x}(t) = L(\tau)x_t, \tag{A.2}$$

where

$$L(\tau)(\phi) = -\tau\phi(0) + \tau\alpha\phi(-1) + \beta\tau\delta(\phi(-1)).$$

The characteristic equation of (A.2) at $(0,0,0)^\top$ is $\det \Delta_\tau(0, \lambda/\tau) = 0$, where $\det \Delta_\tau(0,\lambda) = 0$ is the characteristic equation of the linearization of (2.19) at $(0,0,0)^\top$.

Part 1: Case $(\alpha, \beta) \in D$, $\beta > 0$

In Case (ii) of Theorem 4.2, at $\tau = \tau^*$ the characteristic equation of (A.2) has imaginary zeros $\pm iu^*\tau^*$ which are double, where $\tau^* = \tau_I$ and $u^*\tau^* = \omega_I$ are given by (4.6) and (4.7). Since $\Delta_{\tau^*}(0, iu^*)v_j = [iu^* + 1 - (\alpha - \beta)e^{-iu^*\tau^*}]v_j = 0$, $j = 1,2$, the center space at $\tau = \tau^*$ and in complex coordinates is $X = \operatorname{span}(\phi_1, \phi_2, \phi_3, \phi_4)$, where

$$\begin{aligned}
\phi_1(\theta) &= e^{iu^*\tau^*\theta}v_1, \\
\phi_2(\theta) &= e^{-iu^*\tau^*\theta}\overline{v_1}, \\
\phi_3(\theta) &= e^{iu^*\tau^*\theta}\overline{v_1}, \\
\phi_4(\theta) &= e^{-iu^*\tau^*\theta}v_1, \qquad \theta \in [-1,0],
\end{aligned}$$

and

$$v_1 = \begin{pmatrix} 1 \\ e^{(2i\pi/3)} \\ e^{(-2i\pi/3)} \end{pmatrix}.$$

Let

$$\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$$

and

$$\overline{v_1} = v_2.$$

Note that

$$v_j^\top v_i = 3, \qquad \text{if } j \neq i \in \{1,2\}$$

and

$$v_i^\top v_i = 0, \qquad \text{if } i \in \{1,2\}.$$

It is easy to check that a basis for the adjoint space X^* is

$$\Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \\ \psi_3(s) \\ \psi_4(s) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} (\bar{a})^{-1} e^{-iu^*\tau^* s} \overline{v_1}^\top \\ a^{-1} e^{iu^*\tau^* s} v_1{}^\top \\ (\bar{a})^{-1} e^{-iu^*\tau^* s} v_1{}^\top \\ a^{-1} e^{iu^*\tau^* s} \overline{v_1}^\top \end{pmatrix}, \qquad s \in [0,1],$$

with $(\Psi, \Phi) = Id$ (the 4×4 identity matrix) for the adjoint bilinear form on $C^* \times C$ defined in [30], where

$$a = 1 + \tau^* - iu^*\tau^*. \tag{A.3}$$

It is useful to note the following:

$$\Psi(0) = \frac{1}{3} \begin{pmatrix} (\bar{a})^{-1} \overline{v_1}^\top \\ a^{-1} \bar{v}_1{}^\top \\ (\bar{a})^{-1} v_1{}^\top \\ a^{-1} \overline{v_1}^\top \end{pmatrix}, \tag{A.4}$$

and for $x \in \mathbb{C}^4$, we have

$$\Phi(0)x = [v_1 \; \bar{v}_1 \; \bar{v}_1 \; v_1]\, x = (x_1 + x_4)\, v_1 + (x_2 + x_3)\, \bar{v}_1, \tag{A.5}$$

$$\Phi(-1)x = \left(e^{-iu^*\tau^*} x_1 + e^{iu^*\tau^*} x_4\right) v_1 + \left(e^{iu^*\tau^*} x_2 + e^{-iu^*\tau^*} x_3\right) \bar{v}_1, \tag{A.6}$$

and

$$\delta(\Phi(-1)x) = -\left(e^{-iu^*\tau^*} x_1 + e^{iu^*\tau^*} x_4\right) v_1 - \left(e^{iu^*\tau^*} x_2 + e^{-iu^*\tau^*} x_3\right) \overline{v_1} = -\Phi(-1)x. \tag{A.7}$$

Introducing the new parameter

$$\mu = \tau - \tau^*, \tag{A.8}$$

we can rewrite (A.1) as

$$\dot{z}(t) = L(\tau^*)\, z_t + G(z_t, \mu), \tag{A.9}$$

where

$$G(z_t, \mu) = L(\mu) z_t + b\alpha(\tau^* + \mu) \begin{pmatrix} z_1^3(t-1) \\ z_2^3(t-1) \\ z_3^3(t-1) \end{pmatrix} + b\beta(\tau^* + \mu) \begin{pmatrix} z_2^3(t-1) + z_3^3(t-1) \\ z_1^3(t-1) + z_3^3(t-1) \\ z_2^3(t-1) + z_1^3(t-1) \end{pmatrix} + \text{h.o.t.}$$

Define the 4×4 matrix

$$B = iu^*\tau^* \text{diag}(1, -1, 1, -1).$$

Using the decomposition $z_t = \Phi x(t) + y_t$, we can decompose (A.9) as

$$\begin{aligned} \dot{x} &= Bx + \Psi(0) G(\Phi x + y, \mu), \\ \dot{y} &= A_{Q^1} y + (I - \pi) X_0 G(\Phi x + y, \mu), \end{aligned} \tag{A.10}$$

with $x \in \mathbb{C}^4, y \in Q^1$. Here and throughout this Appendix, we refer to [17] for explanations of several notations involved. We will write the Taylor expansion

$$\Psi(0) G(\Phi x + y, \mu) = \sum_{j \geq 2} \frac{1}{j!} f_j^1(x, y, \mu), \tag{A.11}$$

where $f_j^1(x, y, \mu)$ are homogeneous polynomials of degree j in (x, y, μ) with coefficients in $\mathbb{C}^4$. Then the normal form of (2.19) on the center manifold of the origin at $\mu = 0$ is given by

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, \mu) + \text{h.o.t.},$$

where g_2^1 and g_3^1 will be calculated in the following part of this section.

First of all, using $\delta(\Phi(-1)x) = -\Phi(-1)x$, we get

$$\begin{aligned}
\frac{1}{2}f_2^1(x, 0, \mu) &= \Psi(0)L(\mu)(\Phi x) \\
&= \Psi(0)\mu[-\Phi(0)x + \alpha\Phi(-1)x + \beta\delta(\Phi(-1)x)] \\
&= \Psi(0)\mu\Big\{-(x_1 + x_4)v_1 - (x_2 + x_3)\overline{v_1} \\
&\quad +(\alpha - \beta)\Big[\Big(e^{-iu^*\tau^*}x_1 + e^{iu^*\tau^*}x_4\Big)v_1 + \Big(e^{iu^*\tau^*}x_2 + e^{-iu^*\tau^*}x_3\Big)\overline{v_1}\Big]\Big\} \\
&= \Psi(0)\mu\{-(x_1 + x_4)v_1 - (x_2 + x_3)\overline{v_1} \\
&\quad + [(1 + iu^*)x_1 + (1 - iu^*)x_4]v_1 + [(1 - iu^*)x_2 + (1 + iu^*)x_3]\overline{v_1}\} \\
&= \mu iu^*\Psi(0)[(x_1 - x_4)v_1 + (-x_2 + x_3)\overline{v_1}] \\
&= i\mu u^* \begin{pmatrix} (\bar{a})^{-1}(x_1 - x_4) \\ a^{-1}(-x_2 + x_3) \\ (\bar{a})^{-1}(-x_2 + x_3) \\ a^{-1}(x_1 - x_4) \end{pmatrix}.
\end{aligned}$$

These are the second-order terms in (μ, x) of (A.11) and following Faria and Magalhães [17], we have the second-order terms in (μ, x) of the normal form on center manifold as follows:

$$\frac{1}{2}g_2^1(x, 0, \mu) = \text{Proj}_{\ker(M_2^1)}\frac{1}{2}f_2^1(x, 0, \mu).$$

Here we recall that

$$M_j^1(p)(x, \mu) = D_x p(x, \mu)Bx - Bp(x, \mu), \qquad j \geq 2. \tag{A.12}$$

In particular,

$$M_j^1(\mu x^q e_k) = iu^*\tau^*\mu\left(q_1 - q_2 + q_3 - q_4 + (-1)^k\right)x^q e_k, \qquad |q| = j - 1,$$

where $j \geq 2$, $1 \leq k \leq 4$, and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis for $\mathbb{C}^4$. Therefore, if $|q| = 1$, then

$$\begin{aligned}
&\ker\left(M_2^1\right) \cap \text{span}\{\mu x^q e_k;\ |q| = 1, k = 1, \ldots, 4\} \\
&\qquad = \text{span}\{\mu x_1 e_1, \mu x_3 e_1, \mu x_2 e_2, \mu x_4 e_2, \mu x_1 e_3, \mu x_3 e_3, \mu x_2 e_4, \mu x_4 e_4\},
\end{aligned}$$

and

$$\frac{1}{2}g_2^1(x, 0, \mu) = i\mu u^* \begin{pmatrix} (\bar{a})^{-1}x_1 \\ -a^{-1}x_2 \\ (\bar{a})^{-1}x_3 \\ -a^{-1}x_4 \end{pmatrix}.$$

To compute $g_3^1(x, 0, \mu)$, we first note that from (12) it follows that

$$\begin{aligned}
g_3^1(x, 0, \mu) &= \text{Proj}_{\ker(M_3^1)}\overline{f_3^1}(x, 0, \mu) \\
&= \text{Proj}_{\ker(M_3^1)}\overline{f_3^1}(x, 0, 0) + O\left(\mu^2|x|\right),
\end{aligned}$$

since $\mu x^q e_j \notin \ker(M_3^1)$, for $|q|=2$, $j=1,\dots,4$. Next, we define

$$\overline{f_3^1}(x,0,\mu) = f_3^1(x,0,\mu) + \frac{3}{2}\left[\left(D_x f_2^1\right) U_2^1 - \left(D_x U_2^1\right) g_2^1\right](x,0,\mu) + \frac{3}{2}\left[\left(D_y f_2^1\right) h\right](x,0,\mu),$$

where U_2^1 is the change of variables associated with the transformation from f_2^1 to g_2^1 and h is such that $M_2^2(h) = g_2^2$, i.e., $h = U_2^2$ is the change of variables associated with the transformation of the second-order terms in the second equation of system (A.10). For $\mu = 0$, $f_2^1(x,0,0) = g_2^1(x,0,0) = 0$, and we have simply

$$\frac{1}{3!}\overline{f_3^1}(x,0,0) = \frac{1}{3!} f_3^1(x,0,0) = b\tau^*\Psi(0)\left[\beta\delta\left((\Phi(-1)x)^3\right) + \alpha\left(\Phi(-1)x\right)^3\right],$$

where we utilized the following notations:

$$(\Phi(-1)x)^3 = \begin{pmatrix} \left[(\Phi(-1)x)_1\right]^3 \\ \left[(\Phi(-1)x)_2\right]^3 \\ \left[(\Phi(-1)x)_3\right]^3 \end{pmatrix}.$$

Let

$$A_1 = e^{-iu^*\tau^*}x_1 + e^{iu^*\tau^*}x_4, \qquad A_2 = e^{iu^*\tau^*}x_2 + e^{-iu^*\tau^*}x_3.$$

We have

$$(\Phi(-1)x)^3 = \begin{pmatrix} (A_1+A_2)^3 \\ \left(e^{i(2\pi/3)}A_1 + e^{-i(2\pi/3)}A_2\right)^3 \\ \left(e^{-i(2\pi/3)}A_1 + e^{i(2\pi/3)}A_2\right)^3 \end{pmatrix} = \left(A_1^3 + A_2^3\right)\begin{pmatrix}1\\1\\1\end{pmatrix} + 3A_1^2A_2v_1 + 3A_1A_2^2\overline{v_1}$$

and

$$\begin{aligned}\delta\left((\Phi(-1)x)^3\right) &= \begin{pmatrix} \left(e^{i(2\pi/3)}A_1 + e^{-i(2\pi/3)}A_2\right)^3 + \left(e^{-i(2\pi/3)}A_1 + e^{i(2\pi/3)}A_2\right)^3 \\ (A_1+A_2)^3 + \left(e^{-i(2\pi/3)}A_1 + e^{i(2\pi/3)}A_2\right)^3 \\ (A_1+A_2)^3 + \left(e^{i(2\pi/3)}A_1 + e^{-i(2\pi/3)}A_2\right)^3 \end{pmatrix} \\ &= 2\left(A_1^3 + A_2^3\right)\begin{pmatrix}1\\1\\1\end{pmatrix} - 3A_1^2A_2v_1 - 3A_1A_2^2\overline{v_1}.\end{aligned}$$

Since

$$v_i^\top\begin{pmatrix}1\\1\\1\end{pmatrix} = 0,$$

we have

$$\frac{1}{3!}f_3^1(x,0,0) = b\tau^*\Psi(0)3(\alpha-\beta)\left[A_1^2A_2v_1 + A_1A_2^2\overline{v_1}\right] = b\tau^*(\alpha-\beta)3\begin{pmatrix} (\bar{a})^{-1}A_1^2A_2 \\ a^{-1}A_1A_2^2 \\ (\bar{a})^{-1}A_1A_2^2 \\ a^{-1}A_1^2A_2 \end{pmatrix}.$$

Note that

$$\begin{aligned} A_1^2A_2 &= \left(e^{-2iu^*\tau^*}x_1^2 + 2x_1x_4 + e^{2iu^*\tau^*}x_4^2\right)\left(e^{iu^*\tau^*}x_2 + e^{-iu^*\tau^*}x_3\right), \\ A_1A_2^2 &= \left(e^{-iu^*\tau^*}x_1 + e^{iu^*\tau^*}x_4\right)\left(e^{2iu^*\tau^*}x_2^2 + 2x_2x_3 + e^{-2iu^*\tau^*}x_3^2\right). \end{aligned}$$

Also note that

$$M_3^1\left(x^q e_j\right) = 0, \qquad \text{with } |q| = 3 \text{ if and only if } q_1 - q_2 + q_3 - q_4 + (-1)^j = 0, \quad j = 1,2,3,4.$$

Then

$$\frac{1}{3!}g_3^1(x,0,0) = \mathrm{Proj}_{\ker(M_3^1)}\frac{1}{3!}f_3^1(x,0,0)$$
$$= 3b\tau^*(\alpha-\beta)\begin{pmatrix} (\bar{a})^{-1}e^{-iu^*\tau^*}\left(x_1^2x_2+2x_1x_3x_4\right) \\ a^{-1}e^{iu^*\tau^*}\left(x_1x_2^2+2x_2x_3x_4\right) \\ (\bar{a})^{-1}e^{-iu^*\tau^*}\left(x_3^2x_4+2x_1x_2x_3\right) \\ a^{-1}e^{iu^*\tau^*}\left(x_3x_4^2+2x_1x_2x_4\right) \end{pmatrix}$$
$$= 3b\tau^*\begin{pmatrix} (\bar{a})^{-1}(1+iu^*)\,x_1\,(x_1x_2+2x_3x_4) \\ a^{-1}(1-iu^*)\,x_2\,(x_1x_2+2x_3x_4) \\ (\bar{a})^{-1}(1+iu^*)x_3(x_3x_4+2x_1x_2) \\ a^{-1}(1-iu^*)\,x_4\,(x_3x_4+2x_1x_2) \end{pmatrix}.$$

Consequently, the normal form on the center manifold becomes

$$\dot{x} = u^*\tau^*\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}x + i\mu u^*\begin{pmatrix} (\bar{a})^{-1}x_1 \\ -a^{-1}x_2 \\ (\bar{a})^{-1}x_3 \\ -a^{-1}x_4 \end{pmatrix} + \frac{1}{3}g_3^1(x,0,0) + O\left(\mu^2|x|\right) + O\left(|x|^4\right) \tag{A.13}$$

for $x \in \mathbb{C}^4$. Changing to real coordinates by the change of variables

$$x = Sw, \qquad \text{with } S = \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix}$$

and letting

$$\rho_1^2 = x_1x_2 = w_1^2 + w_2^2,$$
$$\rho_2^2 = x_3x_4 = w_3^2 + w_4^2,$$

we obtain

$$\begin{aligned}
\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} &= u^*\tau^*\begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} + \mu u^*\begin{pmatrix} \mathrm{Im}\left(a^{-1}\right)w_1 + \mathrm{Re}\left(a^{-1}\right)w_2 \\ -\mathrm{Re}\left(a^{-1}\right)w_1 + \mathrm{Im}\left(a^{-1}\right)w_2 \end{pmatrix} \\
&\quad + 3b\tau^*\left(\rho_1^2+2\rho_2^2\right)\begin{pmatrix} \mathrm{Re}\left(a^{-1}(1-iu^*)\right)w_1 - \mathrm{Im}\left(a^{-1}(1-iu^*)\right)w_2 \\ \mathrm{Im}\left(a^{-1}(1-iu^*)\right)w_1 + \mathrm{Re}\left(a^{-1}(1-iu^*)\right)w_2 \end{pmatrix} \\
&\quad + O\left(\mu^2|w|\right) + O\left(|w|^4\right), \\
\begin{pmatrix} \dot{w}_3 \\ \dot{w}_4 \end{pmatrix} &= u^*\tau^*\begin{pmatrix} w_4 \\ -w_3 \end{pmatrix} + \mu u^*\begin{pmatrix} \mathrm{Im}\left(a^{-1}\right)w_3 + \mathrm{Re}\left(a^{-1}\right)w_4 \\ -\mathrm{Re}\left(a^{-1}\right)w_3 + \mathrm{Im}\left(a^{-1}\right)w_4 \end{pmatrix} \\
&\quad + 3b\tau^*\left(\rho_2^2+2\rho_1^2\right)\begin{pmatrix} \mathrm{Re}\left(a^{-1}(1-iu^*)\right)w_3 - \mathrm{Im}\left(a^{-1}(1-iu^*)\right)w_4 \\ \mathrm{Im}\left(a^{-1}(1-iu^*)\right)w_3 + \mathrm{Re}\left(a^{-1}(1-iu^*)\right)w_4 \end{pmatrix} \\
&\quad + O\left(\mu^2|w|\right) + O\left(|w|^4\right).
\end{aligned} \tag{A.14}$$

If we use double polar coordinates

$$w_1 = \rho_1\cos\chi_1,$$
$$w_2 = \rho_1\sin\chi_1$$

and

$$w_3 = \rho_2 \cos \chi_2,$$
$$w_4 = \rho_2 \sin \chi_2,$$

then we get

$$\begin{aligned}
\dot{\rho}_1 &= \left(a_1\mu + b_1\rho_1^2 + 2b_1\rho_2^2\right)\rho_1 + O\left(\mu^2 \left|(\rho_1,\rho_2)\right|\right) + O\left(\left|(\rho_1,\rho_2)\right|^4\right), \\
\dot{\rho}_2 &= \left(a_1\mu + 2b_1\rho_1^2 + b_1\rho_2^2\right)\rho_2 + O\left(\mu^2 \left|(\rho_1,\rho_2)\right|\right) + O\left(\left|(\rho_1,\rho_2)\right|^4\right), \\
\dot{\chi}_1 &= -u^*\tau^* + c_1\mu + d_1\rho_1^2 + 2d_1\rho_2^2 + O\left(\mu^2 \left|(\rho_1,\rho_2)\right|\right) + O\left(\left|(\rho_1,\rho_2)\right|^4\right), \\
\dot{\chi}_2 &= -u^*\tau^* + c_1\mu + 2d_1\rho_1^2 + d_1\rho_2^2 + O\left(\mu^2 \left|(\rho_1,\rho_2)\right|\right) + O\left(\left|(\rho_1,\rho_2)\right|^4\right),
\end{aligned} \tag{A.15}$$

with

$$\begin{aligned}
a_1 &= \operatorname{Im}\left(a^{-1}\right)u^*, & b_1 &= \operatorname{Re}\left(a^{-1}\left(1-iu^*\right)\right)3b\tau^*, \\
c_1 &= -u^*\operatorname{Re}\left(a^{-1}\right), & d_1 &= 3b\tau^*\operatorname{Im}\left(a^{-1}\left(1-iu^*\right)\right).
\end{aligned}$$

As

$$a^{-1} = \frac{1+\tau^*+iu^*\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2},$$

we get

$$\begin{aligned}
a_1 &= \frac{\left(u^*\right)^2\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}, \\
b_1 &= 3b\tau^*\frac{1+\tau^*+\left(u^*\right)^2\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}, \\
c_1 &= -\frac{u^*\left(1+\tau^*\right)}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}, \\
d_1 &= -3b\frac{u^*\tau^*}{\left(1+\tau^*\right)^2+\left(u^*\tau^*\right)^2}.
\end{aligned} \tag{A.16}$$

Part 2: Case $(\alpha,\beta)\in D,\ \beta<0$

In Case (i) of Theorem 4.2, at $\tau=\tau^*$ the characteristic equation of (A.2) has imaginary zeros $\pm iu^*\tau^*$ which are simple, where τ^* and iu^* are given by (4.6) and (4.7). Since $\Delta_{\tau^*}(0,iu^*)v_0 = [iu^*+1-(\alpha+2\beta)e^{-iu^*\tau^*}]v_0 = 0$, where $v_0=(1,1,1)^\top$, the center space at $\tau=\tau^*$ and in complex coordinates is now $X=\operatorname{span}(\phi_1,\phi_2)$, where

$$\begin{aligned}
\phi_1(\theta) &= e^{iu^*\tau^*\theta}v_0, \\
\phi_2(\theta) &= e^{-iu^*\tau^*\theta}v_0, \qquad \theta\in[-1,0].
\end{aligned}$$

Let

$$\Phi = (\phi_1,\phi_2).$$

From $v_0^\top v_0 = 3$, one can easily show that the adjoint basis satisfying $(\Psi,\Phi)=I_2$ (the 2×2 identity matrix) is

$$\Psi(s) = \begin{pmatrix}\psi_1(s)\\ \psi_2(s)\end{pmatrix} = \frac{1}{3}\begin{pmatrix}(\bar{a})^{-1}e^{-iu^*\tau^*s}v_0^\top \\ a^{-1}e^{iu^*\tau^*s}v_0{}^\top\end{pmatrix}, \qquad s\in[0,1],$$

with a given by (A.3), so

$$\Psi(0) = \frac{1}{3}\begin{pmatrix}(\bar{a})^{-1}v_0^\top \\ a^{-1}v_0^\top\end{pmatrix}.$$

Note that

$$\begin{aligned}\Phi(0)x &= [v_0\ v_0]\,x = (x_1+x_2)\,v_0,\\ \Phi(-1)x &= \left(e^{-iu^*\tau^*}x_1+e^{iu^*\tau^*}x_2\right)v_0,\\ \delta(\Phi(-1)x) &= 2\left(e^{-iu^*\tau^*}x_1+e^{iu^*\tau^*}x_2\right)v_0, \qquad x\in\mathbb{C}^2.\end{aligned}$$

For the new parameter $\mu=\tau-\tau^*$ and decomposition $z_t=\Phi x(t)+y_t$, $x\in\mathbb{C}^2$, $y\in Q^1$, and with

$$B=\operatorname{diag}\left(iu^*\tau^*,-iu^*\tau^*\right),$$

the normal form of (2.19) on the center manifold of the origin at $\mu=0$ is

$$\dot{x}=Bx+\frac{1}{2}g_2^1(x,0,\mu)+\frac{1}{3!}g_3^1(x,0,\mu)+\text{h.o.t.},$$

and we will compute the second- and third-order terms, i.e., $g_2^1(x,0,\mu)$ and $g_3^1(x,0,\mu)$, as we have done above for Case (ii) of Theorem 4.2. We have

$$\begin{aligned}\frac{1}{2}f_2^1(x,0,\mu) &= \Psi(0)L(\mu)(\Phi x)\\ &= \Psi(0)\mu[-\Phi(0)x+\alpha\Phi(-1)x+\beta\delta(\Phi(-1)x)]\\ &= \Psi(0)\mu\left[-(x_1+x_2)+(\alpha+2\beta)\left(e^{-iu^*\tau^*}x_1+e^{iu^*\tau^*}x_2\right)\right]v_0\\ &= \Psi(0)\mu\left[-(x_1+x_2)+(1+iu^*)\,x_1+(1-iu^*)\,x_2\right]v_0\\ &= i\mu u^*\begin{pmatrix}(\bar{a})^{-1}(x_1-x_2)\\ a^{-1}(x_1-x_2)\end{pmatrix}.\end{aligned}$$

Since

$$\frac{1}{2}g_2^1(x,0,\mu)=\operatorname{Proj}_{\ker(M_2^1)}\frac{1}{2}f_2^1(x,0,\mu)$$

and

$$M_j^1\left(\mu x^q e_k\right)=iu^*\tau^*\mu\left(q_1-q_2+(-1)^k\right)x^q e_k,\qquad |q|=j-1,\quad k=1,2,\quad j\ge 2$$

for the canonical basis $\{e_1,e_2\}$ for $\mathbb{C}^2$, then

$$\ker(M_2^1)\cap\operatorname{span}\{\mu x^q e_k;\ |q|=1,\ k=1,2\}=\operatorname{span}\{\mu x_1e_1,\ \mu x_2e_2\},$$

and

$$\frac{1}{2}g_2^1(x,0,\mu)=i\mu u^*\begin{pmatrix}(\bar{a})^{-1}x_1\\ -a^{-1}x_2\end{pmatrix}.$$

As for the previous case and for similar reasons,

$$\begin{aligned}g_3^1(x,0,\mu) &= \operatorname{Proj}_{\ker(M_3^1)}\overline{f_3^1}(x,0,\mu)\\ &= \operatorname{Proj}_{\ker(M_3^1)}f_3^1(x,0,0)+O\left(\mu^2|x|\right),\end{aligned}$$

where

$$\begin{aligned}\frac{1}{3!}f_3^1(x,0,0) &= b\tau^*\Psi(0)\left[\beta\delta\left((\Phi(-1)x)^3\right)+\alpha\left(\Phi(-1)x\right)^3\right]\\ &= b\tau^*\Psi(0)(\alpha+2\beta)\left(e^{-iu^*\tau^*}x_1+e^{iu^*\tau^*}x_2\right)^3v_0\\ &= b\tau^*(\alpha+2\beta)\left(e^{-iu^*\tau^*}x_1+e^{iu^*\tau^*}x_2\right)^3\begin{pmatrix}(\bar{a})^{-1}\\ a^{-1}\end{pmatrix}.\end{aligned}$$

Also note that

$$M_3^1\left(x^q e_j\right) = iu^*\tau^*\left(q_1 - q_2 + (-1)^k\right) x^q e_j, \qquad |q| = 3, \quad j = 1,2,$$

which implies

$$\ker\left(M_3^1\right) \cap \operatorname{span}\left\{x^q e_j;\ |q| = 3,\ j = 1,2\right\} = \operatorname{span}\left\{x_1^2 x_2 e_1,\ x_1 x_2^2 e_2\right\}.$$

We can then derive

$$\frac{1}{3!} g_3^1(x,0,0) = 3b\tau^*(\alpha + 2\beta)\begin{pmatrix} (\bar{a})^{-1}(1 + iu^*)\, x_1^2 x_2 \\ a^{-1}(1 - iu^*)\, x_1 x_2^2 \end{pmatrix}.$$

Consequently, the normal form on the center manifold becomes

$$\dot{x} = u^*\tau^*\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} x + i\mu u^*\begin{pmatrix} (\bar{a})^{-1} x_1 \\ -a^{-1} x_2 \end{pmatrix} + \frac{1}{3!} g_3^1(x,0,0) + O\left(\mu^2|x|\right) + O\left(|x|^4\right)$$

for $x \in \mathbb{C}^2$. Changing to real coordinates by the change of variables

$$x = Sw, \qquad \text{with } S = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and letting

$$\rho^2 = x_1 x_2 = w_1^2 + w_2^2,$$

we obtain

$$\begin{aligned}\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} &= u^*\tau^*\begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} + \mu u^*\begin{pmatrix} \operatorname{Im}\left(a^{-1}\right) w_1 + \operatorname{Re}\left(a^{-1}\right) w_2 \\ -\operatorname{Re}\left(a^{-1}\right) w_1 + \operatorname{Im}\left(a^{-1}\right) w_2 \end{pmatrix} \\ &\quad + 3b\tau^*\rho^2\begin{pmatrix} \operatorname{Re}\left(a^{-1}(1 - iu^*)\right) w_1 - \operatorname{Im}\left(a^{-1}(1 - iu^*)\right) w_2 \\ \operatorname{Im}\left(a^{-1}(1 - iu^*)\right) w_1 + \operatorname{Re}\left(a^{-1}(1 - iu^*)\right) w_2 \end{pmatrix} \\ &\quad + O\left(\mu^2|w|\right) + O\left(|w|^4\right).\end{aligned}$$

If we use polar coordinates

$$\begin{aligned} w_1 &= \rho\cos\chi, \\ w_2 &= \rho\sin\chi, \end{aligned}$$

then we get

$$\begin{aligned} \dot{\rho} &= \left(a_1\mu + b_1\rho^2\right)\rho + O\left(\mu^2\rho\right) + O\left(\rho^4\right), \\ \dot{\chi} &= -u^*\tau^* + c_1\mu + d_1\rho^2 + O\left(\mu^2\rho\right) + O\left(\rho^4\right), \end{aligned}$$

where a_1, b_1, c_1, d_1 as in (16).

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