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# Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network

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**Abstract**

The existence of a non-trivial phase-locked periodic orbit is established for a system of delay differential equations describing the dynamics of networks of two identical saturating neurons. We discuss the instability and the unstable manifold of this phase-locked orbit. In particular, we give detailed information about the spectrum of the related monodromy operator and establish the connection between the unstable manifold of the phase-locked orbit and the boundary of the global extension of a three-dimensional $C^1$-submanifold of the origin. We obtain a smooth solid spindle, contained in the global attractor, separated by a disk bordered by the phase-locked orbit. Major technical tools include a discrete Lyapunov functional, invariant manifolds and other recently developed geometric theory of delay differential equations. ©1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

We consider the following system of delay differential equations

$$\begin{cases} \dot{U}(t) = -\mu U(t) + f(V(t-\tau)), \\ \dot{V}(t) = -\mu V(t) + f(U(t-\tau)), \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$-smooth increasing function with $f(0) = 0$. Such a system can be regarded as a special example of the following general Hopfield's model [7,8] for artificial neural networks with electronic circuit implementation

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\mu_i u_i(t) + \sum_{j\neq i} T_{ij} f_j(u_j(t-\tau_{ij})), \quad 1 \le i \le n \tag{1.2}$$

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with $\mu_i > 0$ and $\tau_{ij} \in \mathbb{R}$ being given constants and $f_j : \mathbb{R} \to \mathbb{R}$ being usual sigmoid functions for $1 \leq i, j \leq n$, where the delay $\tau_{ij} \geq 0$ was incorporated by Marcus and Westervelt [14] to account for the finite switching speed of amplifiers (neurons). It has been observed that while a network modeled by Eq. (1.2) without delay (namely, $\tau_{ij} = 0$ for $1 \leq i, j \leq n$) relaxed towards the set of equilibria [4,7,8], the presence of large delay $\tau_{ij}$ may cause some stable nonlinear oscillations and lead to a completely different computational performance of the network [1,2,6,14–16,18]. Our ultimate goal is to describe the global attractor of the network, and achieving this seems to be very important in the applications of the model to tasks of classification and associative memory when the global attractor encodes the pattern and memory of the network.

System (1.1) can be regarded as a special case of the network (1.2) with identical neurons ($\mu_i$ and $\tau_{ij}$ are independent of the choices of $i, j$) for two reasons: system (1.2) reduces to Eq. (1.1) when $n = 2$, or the dynamics of Eq. (1.2) are completely described by Eq. (1.1) when the network is divided into two groups of neurons with neurons in each group being convergent to the same states. It is hoped that, with such simplification and idealization, we can have a relatively complete picture about the global attractor which shed some light for our understanding about the general network (1.2).

Rescaling the time variable by $u(t) = U(\tau t)$ and $v(t) = V(\tau t)$, we can rewrite Eq. (1.1) as

$$\begin{cases} \dot{u}(t) = -\tau\mu u(t) + \tau f(v(t-1)), \\ \dot{v}(t) = -\tau\mu v(t) + \tau f(u(t-1)). \end{cases} \tag{1.3}$$

It is easy to see that every continuous $\psi = (\psi_1, \psi_2)^{\mathrm{T}} : [-1, 0] \to \mathbb{R}^2$ uniquely determines a solution $(u^\psi, v^\psi)^{\mathrm{T}} : [-1, \infty) \to \mathbb{R}^2$ of Eq. (1.3) with $(u^\psi, v^\psi)^{\mathrm{T}}|_{[-1,0]} = \psi$. Clearly, if $\psi_1 = \psi_2$ then the uniquely determined solution satisfies $u^\psi = v^\psi$ in $[-1, \infty)$ and this can be characterized by the scalar delay differential equation

$$\dot{w}(t) = -\tau\mu w(t) + \tau f(w(t-1)). \tag{1.4}$$

Such solutions are said to be synchronous and the recent work of Krisztin, Walther and Wu [10] and Krisztin and Walther [9] gives a complete description of the global attractor of Eq. (1.4) as a three-dimensional smooth solid spindle when $\tau$ is in a certain range. It is also interesting to note that, as the first critical value $\tau^*$ of $\tau$ when the trivial solution of Eq. (1.3) loses its stability due to the occurrence of a pair of purely imaginary eigenvalues of the generator of the corresponding linearized system, a Hopf bifurcation of periodic solutions takes place and these periodic solutions are not synchronous but phase-locked in the sense that two neurons oscillate in the same way but in different phases, or more precisely, the periodic solutions $(u, v)^{\mathrm{T}}$ are anti-phase: $u(t + (\omega/2)) = v(t)$ for all $t \in \mathbb{R}$ and for the minimal period $\omega > 0$.

In [5], we proved that when $\tau > \tau^*$ there exists a (locally) unstable manifold tangent to a three-dimensional linearly unstable subspace of the generator of the associated linearization of the trivial solution, the closure of its global forward extension of such a manifold contains 3 equilibria and a closed disk bordered by a phase-locked periodic orbit.

The main tool to get the above results is the discrete Lyapunov functional, introduced by Mallet-Paret and Sell [12,13], for cyclic nearest neighbor systems of differential delay equations. We first note that with the transformation $x(t) = u(2t)$ and $y(t) = v(2t - 1)$, we can rewrite Eq. (1.3) as the following cyclic system of delay differential equations

$$\begin{cases} \dot{x}(t) = -2\tau\mu x(t) + 2\tau f(y(t)), \\ \dot{y}(t) = -2\tau\mu y(t) + 2\tau f(x(t-1)). \end{cases} \tag{1.5}$$

Then using the discrete Lyapunov functional, we get the corresponding results for system (1.5), which imply the above results for system (1.3) by using the aforementioned transformation. For the same reason, in this paper, we only concentrate on system (1.5). The readers can easily state the corresponding results for system (1.3) and derive them as immediate consequences of the above transformation.

The purpose of this paper is to report results about the detailed information of

1. the stability of the phase-locked periodic orbit, including the spectrum analysis of the monodromy operator;
2. the geometric structure of the basins of attraction of this periodic orbit and other equilibria;
3. the geometric structure of the closure of the aforementioned three-dimensional manifold.

It should be remarked that we will only sketch the proofs for most of the results reported. The details are quite long and technical, but parallel to those of Krisztin, Walther and Wu [10] for the scalar equation. All the details are provided in a technical report which is available to all readers upon request.

## 2. The phase-locked orbit and graph representations

Following the work of Smith [17] and Mallet-Paret and Sell [12,13], we let $\mathbb{K} = [-1, 0] \cup \{1\}$ and use the Banach space $C(\mathbb{K}) = \{\varphi : \mathbb{K} \to \mathbb{R};\ \varphi \text{ is continuous}\}$ with the supernorm as the phase space for system (1.5) and we will always tacitly use the identification

$$C(\mathbb{K}) = C([-1, 0]; \mathbb{R}) \times \mathbb{R}.$$

Throughout the paper, we assume

(H1) $f(0) = 0$ and $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$;
(H2) $f'(0) > \mu$;
(H3) there exists $M > 0$ so that $f(\xi)/\xi < \mu$ if $|\xi| > M$;
(H4) $\tau > \tau_{\mathrm{d}} = (\pi - \arccos(\mu/f'(0)))/(\sqrt{[f'(0)]^2 - \mu^2})$.

In the language for networks of neurons, (H1) and (H2) require that the interaction of two neurons is excitatory and the neuron gain, $f'(0)$, is sufficiently large. A standard function employed in the modeling of networks is the sigmoid which clearly satisfies the dissipativeness condition (H3). It is well known (see, for example, [7,8,14]) that the network can possess interesting complicated dynamics only when the delay is large, so is our condition (H4), where $\tau_{\mathrm{d}}$ is the first critical value of the delay $\tau$ where a Hopf bifurcation of periodic solutions takes places from the trivial solution.

Under the above assumptions, for each $\varphi \in C(\mathbb{K})$, there exists a unique pair of continuous maps $x : [-1, \infty) \to \mathbb{R}$ and $y : [0, \infty) \to \mathbb{R}$ such that $(x, y)^{\mathrm{T}} : (0, \infty) \to \mathbb{R}^2$ is continuously differentiable and satisfies system (1.5) for $t > 0$, $x|_{[-1,0]} = \varphi|_{[-1,0]}$ and $y(0) = \varphi(1)$. Let $z^{\varphi} = (x^{\varphi}, y^{\varphi})^{\mathrm{T}}$ denote the above unique pair and define $z_t^{\varphi} = (x_t^{\varphi}, y^{\varphi}(t))^{\mathrm{T}} \in C(\mathbb{K})$ for $t \geq 0$ (Note here the subscript $t$ is used for an element in $C(\mathbb{K})$ other than $C([-1, 0]; \mathbb{R})$). Then the map $\Phi : \mathbb{R}^+ \times C(\mathbb{K}) \ni (t, \varphi) \mapsto z_t^{\varphi} \in C(\mathbb{K})$ is a continuous semiflow with at least three stationary points 0, $z_-$ and $z_+$, where 0, $z_-$ and $z_+$ denote the constant maps on $\mathbb{K}$ with the values 0, $\xi^-$ and $\xi^+$, respectively (where $\xi^-$ and $\xi^+$ are the maximal negative and minimal positive zeros of $f(\xi) = \mu\xi$, respectively. The existence follows from (H2) and (H3)).

The spectrum of the generator of the $C_0$-semigroup $\{D_2\Phi(t, 0)\}_{t\geq 0}$ coincides with the zero of the characteristic equation

$$\begin{vmatrix} \lambda + 2\tau\mu & -2\tau \\ -2\tau \mathrm{e}^{-\lambda} & \lambda + 2\tau\mu \end{vmatrix} = [(\lambda + 2\tau\mu)^2 - 4\tau^2 \mathrm{e}^{-\lambda}] = 0.$$

It is shown that this spectrum is given by $\alpha_0, \alpha_1 \pm \mathrm{i}\beta_1, \alpha_2 \pm \mathrm{i}\beta_2, \ldots$ with

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots, \qquad \alpha_1 > 0, \quad \beta_1 \in (\pi, 2\pi), \qquad \beta_{2j} \in (2(2j-1)\pi, 4j\pi),$$
$$\beta_{2j+1} \in (4j\pi, 2(2j+1)\pi), \quad j \geq 1.$$

Let $P_0$, $P_1$ and $Q$ be the realified eigenspaces of the generator of the semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$ on $C(\mathbb{K})$ associated with the spectral sets $\{\alpha_0\}$, $\{\alpha_1 + \mathrm{i}\beta_1, \alpha_1 - \mathrm{i}\beta_1\}$ and $\{\alpha_j + \mathrm{i}\beta_j, \alpha_j - \mathrm{i}\beta_j\}_{j\geq 2}$, respectively. Note that $P_0 = \mathbb{R}\chi_0$, where

$$\chi_0(\theta) = \begin{cases} \mathrm{e}^{\alpha_0\theta} & \text{for } \theta \in [-1, 0], \\ \mathrm{e}^{-\alpha_0/2} & \text{for } \theta = 1. \end{cases}$$

For $\gamma \in (\max\{1, \mathrm{e}^{\alpha_2}\}, \mathrm{e}^{\alpha_1})$, there exist convex bounded open neighborhoods $N_0$, $N_1$ and $N_Q$ of 0 in $P_0$, $P_1$ and $Q$, respectively, and a $C^1$-map $w_{\mathrm{loc}} : N_1 \oplus N_0 \to Q$ with $w_{\mathrm{loc}}(0) = 0$, $Dw_{\mathrm{loc}}(0) = 0$, $w_{\mathrm{loc}}(N_1 + N_0) \subseteq N_Q$ and so that the graph $W_{\mathrm{loc}} = \{\chi + w_{\mathrm{loc}}(\chi); \chi \in N_0 + N_1\}$ coincides with the set

$$W_\gamma = \{\varphi \in N_{\mathrm{loc}} = N_0 + N_1 + N_Q; \quad \text{there is a sequence } (\varphi_n)_{-\infty}^{0} \text{ with } \varphi_0 = \varphi, \varphi_n = \Phi(1, \varphi_{n-1}),$$
$$\text{and } \varphi_n\gamma^{-n} \in N_{\mathrm{loc}} \text{ for each integer } n \leq 0 \text{ and } \varphi_n\gamma^{-n} \to 0 \text{ as } n \to -\infty\}.$$

Our focus here is

$$W = \Phi(\mathbb{R}^+ \times W_{\mathrm{loc}})$$

and

$$S = \{\varphi \in C(\mathbb{K}); \varphi = 0 \text{ or } V(z_t^\varphi) > 0 \text{ for } t \geq 0\},$$

where $V : C(\mathbb{K}) \setminus \{0\} \to \{0, 2, 4, \ldots\}$ is the discrete Lyapunov functional introduced by Mallet-Paret and Sell [12]. Namely, for $\varphi \in C(\mathbb{K}) \setminus \{0\}$,

$$V(\varphi) = \begin{cases} \mathrm{sc}(\varphi) & \text{if } \mathrm{sc}(\varphi) \text{ is even or infinite,} \\ \mathrm{sc}(\varphi) + 1 & \text{if } \mathrm{sc}(\varphi) \text{ is odd,} \end{cases}$$

where $\mathrm{sc}(\varphi) = 0$ if either $\varphi \geq 0$ or $\varphi \leq 0$ and otherwise

$$\mathrm{sc}(\varphi) = \sup\{k \geq 1; \quad \text{there exists } \theta^0 < \theta^1 < \cdots < \theta^k \text{ with } \theta^i \in \mathbb{K} \text{ for } i = 0, 1, \ldots, k$$
$$\text{and } \varphi(\theta^{i-1})\varphi(\theta^i) < 0 \text{ for } 1 \leq i \leq k\}.$$

It is easy to see that for each $\varphi \in W$ there is a unique solution defined on $\mathbb{R}$ of system (1.5) passing through $\varphi$, also denoted as $z^\varphi$. Note that $z_t^\varphi \in W$ for $t \in \mathbb{R}$. Thus, we have a flow $\Phi_W : \mathbb{R} \times W \ni (t, \varphi) \mapsto z_t^\varphi \in W$. We have shown in [5] that $S$ is not ordered with respect to the partial ordering on $C(\mathbb{K})$, which is induced by the positive cone

$$K = \{\varphi \in C(\mathbb{K}); \quad \varphi(\theta) \geq 0 \text{ for } \theta \in \mathbb{K}\}.$$

**Theorem 1.**

1. *There is a nontrivial periodic solution* $p = (p^1, p^2)^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^2$ *of system* (1.5) *with the minimal period* $\omega > 1$ *which is determined by three consecutive zeros of* $p^1$ *or* $p^2$ *such that* $p^1(0) = 0$ *and* $p^2(0) > 0$. *Moreover,* $p^1(t) = p^2(t + (\omega+1)/2)$ *and* $p^2(t) = p^1(t + (\omega-1)/2)$ *for* $t \in \mathbb{R}$.
2. $\mathcal{O} = \{p_t; t \in \mathbb{R}\} = bd(W \cap S) = \overline{W \cap S} \setminus (W \cap S)$ *is the only nontrivial periodic orbit in* $\bar{W}$.
3. $\Pi(W \cap S) = \mathrm{int}(\Pi \circ \eta)$, $\Pi(\overline{W \cap S} \setminus (W \cap S)) = \Pi \circ \eta([0, \omega))$, *where* $\eta : [0, \omega] \ni t \mapsto p_t \in C(\mathbb{K})$, *and* $\Pi : C(\mathbb{K}) \ni \varphi \mapsto (\varphi(0), \varphi(1))^{\mathrm{T}} \in \mathbb{R}^2$.
4. $\bar{W} \cap S = \overline{W \cap S}$.

In what follows, we call the periodic orbit $\mathcal{O}$ a phase-locked periodic orbit. This is because if we let $q^1(t) = p^1(t/2), q^2(t) = p^2((t+1)/2)$ for $t \in \mathbb{R}$, then $q = (q^1, q^2)^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^2$ is indeed a phase-locked periodic solution of the origin system (1.3).

Theorem 1 was established in [5]. In the remaining part of this paper, we study the instability of the phase-locked periodic orbit $\mathcal{O}$, the structures of $\bar{W}$ and $bd W = \bar{W} \setminus W$, and the basin of attraction (within $\bar{W}$) of $\mathcal{O}$.

We first give some graph representations of $\bar{W}$ and $\bar{W} \cap S$. Direct calculations lead to, for the projection onto $P_0$ of $C(\mathbb{K})$,

$$\mathrm{Pr}_{P_0}\phi = \frac{1}{2 + 2\tau f'(0)\mathrm{e}^{-\alpha_0/2}} \left[ \phi(0) + \phi(1)\mathrm{e}^{\alpha_0/2} + 2\tau f'(0)\mathrm{e}^{-\alpha_0/2} \int_{-1}^{0} \mathrm{e}^{-\alpha_0\sigma}\phi(\sigma)\,\mathrm{d}\sigma \right] \chi_0$$

for all $\phi \in C(\mathbb{K})$. Define the continuous linear functional

$$c_{P_0} : C(\mathbb{K}) \ni \phi \mapsto \phi(0) + \phi(1)\mathrm{e}^{\alpha_0/2} + 2\tau f'(0)\mathrm{e}^{-\alpha_0/2} \int_{-1}^{0} \mathrm{e}^{-\alpha_0\sigma}\phi(\sigma)\,\mathrm{d}\sigma \in \mathbb{R}.$$

Clearly, $c_{P_0}^{-1}(0) = Q + P_1$.

We need the following projection

$$\Pi_3 : C(\mathbb{K}) \ni \varphi \mapsto (\varphi(0), \varphi(1), c_{P_0}(\varphi))^{\mathrm{T}} \in \mathbb{R}^3.$$

The restrictions of $\Pi_3$ to $\bar{W}$ and $P_0 \oplus P_1$ are injective. Let $\Pi_3^{-1} : \Pi_3\bar{W} \to C(\mathbb{K})$ be the map given by the inverse of $\Pi_3 : \bar{W} \ni \varphi \mapsto \Pi_3\varphi \in \Pi_3\bar{W}$. We can show that $\Pi_3^{-1}$ is Lipschitz continuous. Observe that $\Pi_3$ is surjective since $\Pi_3|_{P_0 \oplus P_1}$ is injective and $\dim P_0 \oplus P_1 = 3$. Choose linearly independent elements $\varphi_j$, $j = 1, 2, 3$, in $C(\mathbb{K})$ with

$$\Pi_3\varphi_j = e_j, \quad j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Let $J_3$ denote the injective linear map from $\mathbb{R}^3$ into $C(\mathbb{K})$ given by $J_3 e_j = \varphi_j$. Then we have a projection

$$P_3 = J_3 \circ \Pi_3 : C(\mathbb{K}) \to C(\mathbb{K}).$$

The space $G_3 = P_3 C(\mathbb{K}) = \mathbb{R}\varphi_1 \oplus \mathbb{R}\varphi_2 \oplus \mathbb{R}\varphi_3$ is three-dimensional, and with $E = P_3^{-1}(0)$ we have

$$C(\mathbb{K}) = G_3 \oplus E.$$

The restriction of $P_3$ to $\bar{W}$ is injective. Let $P_3^{-1} : P_3\bar{W} \to C(\mathbb{K})$ be given by the inverse of $\bar{W} \ni \varphi \mapsto P_3\varphi \in P_3\bar{W}$, and define the map

$$w : P_3\bar{W} \ni \chi \mapsto (\mathrm{id} - P_3) \circ P_3^{-1}(\chi) \in E.$$

Then we have the following graph representation

$$\bar{W} = \{\chi + w(\chi);\ \chi \in P_3\bar{W}\}.$$

Consider also the injective linear map $J$ from $\mathbb{R}^2$ into $C(\mathbb{K})$ given by $J(1,0)^{\mathrm{T}} = \varphi_1$ and $J(0,1)^{\mathrm{T}} = \varphi_2$. As $\Pi\varphi_1 = (1,0)^{\mathrm{T}}$ and $\Pi\varphi_2 = (0,1)^{\mathrm{T}}$, we obtain another projection

$$P = J \circ \Pi : C(\mathbb{K}) \to C(\mathbb{K}).$$

$G_2 = PC(\mathbb{K}) = \mathbb{R}\varphi_1 \oplus \mathbb{R}\varphi_2$ is a two-dimensional subspace of $G_3$, and

$$C(\mathbb{K}) = G_2 \oplus P^{-1}(0).$$

Set $G_1 = \mathbb{R}\varphi_3$, so that

$$G_3 = G_2 \oplus G_1.$$

It is easy to verify that $P^{-1}(0) = G_1 \oplus E$. The injectivity of $\Pi$ to the set $\bar{W} \cap S$ and the injectivity of $J$ combined show that $P|_{\bar{W}\cap S}$ is injective. Let $P^{-1} : P(\bar{W} \cap S) \to C(\mathbb{K})$ be given by the inverse of $\bar{W} \cap S \ni \varphi \mapsto P\varphi \in P(\bar{W} \cap S)$, and define the map

$$w_S : P(\bar{W} \cap S) \ni \chi \mapsto (\mathrm{id} - P) \circ P^{-1}(\chi) \in G_1 \oplus E.$$

We obtain the following graph representation:

$$\bar{W} \cap S = \{\chi + w_S(\chi);\ \chi \in P(\bar{W} \cap S)\}.$$

It follows from the Lipschitz continuity of $\Pi_3^{-1}$ that $w$ and $w_S$ are Lipschitz continuous. The following preliminary smoothness results can be verified:

1. $P_3 W$ is open in $G_3$, and $w|_{P_3 W}$ is $C^1$-smooth;
2. $\Pi_3 W$ is open in $\mathbb{R}^3$, and $\Pi_3$ defines a $C^1$-diffeomorphism from the $C^1$-submanifold $W$ of $C(\mathbb{K})$ onto $\Pi_3 W$;
3. The flow $\mathbb{R} \times W \ni (t, \phi) \mapsto \Phi_W(t, \phi) \in W$ is $C^1$-smooth.

## 3. The minimal linear instability of $\mathcal{O}$

It is known that the Floquet multipliers of the periodic orbit $\mathcal{O}$ are eigenvalues of finite multiplicity forming the spectrum away from 0 of the compact linearized map $M = D_2\Phi(\omega, p_0)$. Let $\sigma$ denote the spectrum of $M$. For $0 \neq \lambda \in \sigma$, let

$$E(\lambda) = \mathrm{Ker}(M_{\mathbb{C}} - \lambda\,\mathrm{id}),$$

$$G(\lambda) = \bigcup_{n=0}^{\infty} \mathrm{Ker}(M_{\mathbb{C}} - \lambda\,\mathrm{id})^n$$

denote the eigenspace and generalized eigenspace associated with $\lambda$, respectively. Let Re $G(\lambda)$ denote the realified generalized eigenspace associated with $\lambda$. If $r > 0$ is given and if there exists $\lambda \in \sigma$ with $r < |\lambda|$, let $C_{\leq r}$ and $C_{r<}$ denote the realified generalized eigenspaces of $M$ associated with the nonempty disjoint spectral sets $\{\lambda \in \sigma; |\lambda| \leq r\}$ and $\{\lambda \in \sigma; r < |\lambda|\}$, respectively, then $C(\mathbb{K}) = C_{\leq r} \oplus C_{r<}$. Analogously, we shall consider realified generalized eigenspaces $C_{<r}$ and $C_{r\leq}$.

The next result establishes the minimal linear instability of $\mathcal{O}$:

**Theorem 2.**
(i) *There exist a Floquet multiplier* $\lambda_u > 1$ *and* $\psi_u \in \overset{\circ}{K}$ *with* $M\psi_u = \lambda_u\psi_u$.
(ii) $\dim C_{1<} = 1$.
(iii) $-1 \notin \sigma$.
(iv) *For every* $\lambda \in \sigma \setminus \{0, 1, \lambda_u\}$, $|\lambda| < 1$.

Theorem 2(i) implies that the periodic orbit $\mathcal{O}$ is linearly unstable. We say that $\mathcal{O}$ is hyperbolic if $dim\ Re\, G(1) = 1$, and nonhyperbolic if otherwise.

**Outline of the proof.**

1. We get a positive $\lambda_u \in \sigma$ with an eigenvector $\psi_u$ in $\overset{\circ}{K} = \{\varphi \in C(\mathbb{K}); \varphi(\theta) > 0 \text{ for } \theta \in \mathbb{K}\}$ from the Krein–Rutman Theorem applied to $M^2$.

2. We show that $\lambda_u > 1$ by iteration of $\psi_u + \varepsilon \dot{p}_0$ under $M$ and using the monotonicity property of $M$.
3. We can show that there exists $r_M > 0$ so that $\sigma \cap \{\lambda \in \sigma; r_M < |\lambda|\} \neq \emptyset, C_{\leq r_M} \cap T = \emptyset, C_{r_M<} \subset \bar{T}$, $\dim C_{r_M \leq} \leq 3$, where $T = V^{-1}(\{0, 2\})$. $r_M < 1$ since $\dot{p}_0 \in E(1) \cap T$. Thus, $1 \leq \dim C_{1<} \leq 2$.

   If, by way of contradiction, $\dim C_{1<} = 2$, then $\mathcal{O}$ is hyperbolic and $C_{\leq r_M} = C_{<1}$. We would arrive at a contradiction in step 4 and step 5.
4. Let $z = (x, y)^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^2$ be a solution of Eq. (1.5) such that $z_t \in W \cap S$ for all $t \in \mathbb{R}$, $z_t \to 0$ as $t \to -\infty$ and $z_t \to \mathcal{O}$ as $t \to \infty$. Then $p_t - z_s \in T$ for all $t, s \in \mathbb{R}$ and there exists a constant $c > 0$ such that $\|p_t - z_s\| \leq c\|p_{t+1} - z_{s+1}\|$ for all $t, s \in \mathbb{R}$. Now, choose a closed complementary subspace $Y$ of $\mathbb{R}\dot{p}_0$ in $C(\mathbb{K})$. Then there are open neighborhood $U$ of $p_0$ in $C(\mathbb{K})$ and a $C^1$-map $\gamma : U \to \mathbb{R}$ with $\gamma(p_0) = \omega$ and $\Phi(\gamma(\phi), \phi) \in p_0 + Y$ for all $\phi \in U$. The fixed point $p_0$ of the Poincaré-map $P_Y$ is hyperbolic, where $P_Y : U \cap (p_0 + Y) \ni \phi \mapsto \Phi(\gamma(\phi), \phi) \in p_0 + Y$.
5. Let $W^{\mathrm{s}}$ be a local stable manifold of $P_Y$ at $p_0$. Then there exist a neighborhood $U^s$ of $p_0$ in $U$ and a $t_P \in \mathbb{R}$ such that $z_{t_P} \in W^{\mathrm{s}}$ and $W^{\mathrm{s}} \ni P_Y^j(z_{t_P}) \to p_0$ as $j \to \infty$. Using the asymptotic phase theorem of Lani-Wayda and Walther [11], we can find $\alpha > 0$ so that $\phi_0 = \Phi(\alpha, z_{t_P})$ belongs to a local stable manifold $W_\omega^{\mathrm{s}}$ of the periodic map $\Phi(\omega, \cdot)$ at the fixed point $p_0$. Then $\phi_n = \Phi(n\omega, \phi_0) \to p_0$ as $n \to \infty$. It is easy to check that the normalized vectors $(\phi_n - p_0)/(\|\phi_n - p_0\|) \in T$ satisfy the hypotheses of the Arzèla–Ascoli Theorem. Combing this with the fact that

$$\frac{\|\mathrm{Pr}_{C_{r_M<}}(\phi_n - p_0)\|}{\|\mathrm{Pr}_{C_{\leq r_M}}(\phi_n - p_0)\|} \to 0 \quad \text{as } n \to \infty,$$

   we can find a unit vector $\chi \in \bar{T} \cap C_{\leq r_M}$, a contradiction to $C_{\leq r_M} \cap T = \emptyset$.
6. Suppose $-1$ is a Floquet multiplier of $M$. Then there exist solutions $(u^i, v^i)^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^2, i = 1, 2$, of the system of variational equations of Eq. (1.5) with $M(u_0^1, v^1(0))^{\mathrm{T}} = -(u_0^1, v^1(0))^{\mathrm{T}}$ and $(u_0^2, v^2(0))^{\mathrm{T}} = \psi_u$. Fix three consecutive zeros $\alpha_1 < \alpha_2 < \alpha_3$ of $\dot{p}^1$. Then $u_{\alpha_3}^1 = -u_{\alpha_1}^1$. We can show that $V(c_1(u_t^1, v^1(t))^{\mathrm{T}} + c_2\dot{p}_t) = 2$ for all $(c_1, c_2)^{\mathrm{T}} \in \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)^{\mathrm{T}}\}$ and all $t \in \mathbb{R}$. This implies that for such $(c_1, c_2)^{\mathrm{T}}$ the first component of $c_1(u^1, v^1)^{\mathrm{T}} + c_2\dot{p}$ cannot have a double zero. Using this result, we can show the following synchronization property: $\beta_1 < \alpha_1, \beta_2 \in (\alpha_1, \alpha_2), \beta_3 \in (\alpha_2, \alpha_3), \alpha_3 \in (\beta_3, \beta_4)$, where $\beta_1$ is the largest zero of $u^1$ which is less than or equal to $\alpha_1$, $\beta_2$, $\beta_3$ and $\beta_4$ are the next zeros of $u^1$. Therefore, we have

$$\mathrm{sign}\, u^1(\alpha_3) = \mathrm{sign}\, \dot{u}^1(\beta_3) = \mathrm{sign}\, \dot{u}^1(\beta_1) = \mathrm{sign}\, u^1(\alpha_1),$$

   a contradiction to $u^1(\alpha_3) = -u^1(\alpha_1)$. Thus, $-1 \notin \sigma$.
7. Let $\lambda \in \sigma \setminus \{0, 1, \lambda_u\}$. If $|\lambda| \geq 1$ then (i) and (ii) imply that $|\lambda| = 1$. It follows from (iii) that $\lambda \in S_{\mathbb{C}}^1 \setminus \mathbb{R}$. Therefore, $4 \leq \dim C_{1\leq} \leq \dim C_{r_M<} \leq 3$, a contradiction.

   Note that (iii) excludes bifurcation of Möbius strips.

## 4. The unstable set of $\mathcal{O}$

We have noticed that the monodromy operator $M = D_2\Phi(\omega, p_0)$ has exactly one Floquet multiplier $\lambda_u$ outside the unit circle, and

$$\lambda_u \in (1, \infty), \qquad C_{1<} = \mathbb{R}\psi_u, \qquad \psi_u \in \overset{\circ}{K}, \qquad \|\psi_u\| = 1.$$

Choose $\lambda \in (0, 1)$ so that

$$\lambda > \max\left\{\frac{1}{\lambda_u}, \max_{\zeta \in \sigma, |\zeta| < 1} |\zeta|\right\}.$$

Theorem I.3 of [10] guarantees the existence of a local strong unstable manifold of the periodic map $\Phi(\omega,\cdot)$ at its fixed point $p_0$. Namely, there are convex open neighborhoods $N_{1<}$ of 0 in $C_{1<}$ and $N_{\leq 1}$ of 0 in the realified generalized eigenspace $C_{\leq 1}$ of $M$ associated with the spectral set $\{\zeta\in\sigma;\,|\zeta|\leq 1\}$ and a $C^1$-map $w^u: N_{1<}\to C_{\leq 1}$ so that

$$w^u(0)=0,\qquad Dw^u(0)=0,\qquad w^u(N_{1<})\subseteq N_{\leq 1},$$

and with $N^u=N_{1<}+N_{\leq 1}$, the shifted graph

$$W^{\mathrm{u}}(p_0,\Phi(\omega,\cdot),N^u)=\{p_0+\chi+w^u(\chi);\,\chi\in N_{1<}\}$$

$$\{\chi\in p_0+N^u;\quad \text{there is a trajectory } (\chi^n)_{-\infty}^0 \text{ of } \Phi(\omega,\cdot)\text{with } \chi^0=\chi,\ \lambda^n(\chi^n-p_0)\in N^u \text{for all } n\in-\mathbb{N}, \text{and } \lambda^n(\chi^n-p_0)\to 0 \text{ as } n\to-\infty\}.$$

The unstable set $W^{\mathrm{u}}(\mathcal{O})$ of the periodic orbit $\mathcal{O}$ is defined as

$$W^{\mathrm{u}}(\mathcal{O})=\bigcup_{t\geq 0}\Phi(t,W^{\mathrm{u}}(p_0,\Phi(\omega,\cdot),N^u)).$$

The next result identifies the unstable set of the phase-locked periodic orbit with the nonstationary points of the boundary of $W$.

**Theorem 3.** $W^{\mathrm{u}}(\mathcal{O})=bd W\setminus\{z_-,z_+\}$.

**Outline of the proof.** To prove Theorem 3, we first show that $bdW\setminus\{z_-,z_+\}\subseteq W^{\mathrm{u}}(\mathcal{O})$. Let $\phi\in bdW\setminus\{z_-,z_+\}$. Then we can prove $\alpha(\phi)=\mathcal{O}$. Consider the solution $z^\phi=(x^\phi,y^\phi)^{\mathrm{T}}:\mathbb{R}\to\mathbb{R}^2$ of system (1.5). It suffices to show that there exists $t\leq 0$ such that $z_t^\phi\in W^{\mathrm{u}}(p_0,\Phi(\omega,\cdot),N^u)$. Suppose, by way of contradiction, $z_t^\phi\notin W^{\mathrm{u}}(p_0,\Phi(\omega,\cdot),N^u)$ for all $t\leq 0$.

1. Since $\alpha(\phi)=\mathcal{O}$, there exists an $s_0\leq 0$ such that $\{\phi_n=z^\phi_{s_0+n\omega}\}_{n\in-\mathbb{N}}$ has a subsequence converging to $p_0$ as $n\to-\infty$. We can show that $V(\phi_n-p_0)=2$ for all sufficiently large negative integers $n$.
2. We prove that there exists $T_0<0$ such that $V(z_t^\phi-p_0)=2$ for all $t\leq T_0$.
3. We show that $V(z_t^\phi-\psi)=2$ for all $\psi\in\mathcal{O}$ and for all $t\leq T_0-\omega$.
4. We prove that there exists $T_1\leq T_0-\omega$ such that $V(z_t^\phi-\psi)=2$ for all $\psi\in W\cap S$ and for all $t\leq T_1$.
5. The results in 3 and 4 combined yield that $\Pi z_t^\phi\in \mathrm{ext}(\Pi\circ\eta)$ for all $t\leq T_1$. We can choose a sequence $(\chi_n)_0^\infty$ in $W$ such that $\chi_n\to z^\phi_{T_1}$ as $n\to\infty$ and $V(z_t^{\chi_n}-\psi)=2$ for all $t\leq 0$, for all $\psi\in\mathcal{O}$ and for all $n\in\mathbb{N}$. Thus, the curve $\Pi\circ\eta$ and the map $(-\infty,0]\ni t\mapsto\Pi z_t^{\chi_n}\in\mathbb{R}^2$ have disjoint sets of values for any $n\in\mathbb{N}$. Note that $\Pi\chi_n\in\mathrm{ext}(\Pi\circ\eta)$ since $\Pi\chi_n\to\Pi z^\phi_{T_1}\in\mathrm{ext}(\Pi\circ\eta)$ as $n\to\infty$. On the other hand, $z_t^{\chi_n}\to 0$ as $t\to-\infty$. Therefore, $\Pi z_t^{\chi_n}\to 0\in\mathrm{int}(\Pi\circ\eta)$ as $t\to-\infty$. This yields a contradiction and hence we have $\phi\in W^{\mathrm{u}}(\mathcal{O})$.
   Next, we show that $W^{\mathrm{u}}(\mathcal{O})\subseteq bdW$.
6. (Introducing the Poincaré-map $P_H$) Let $H=\{\varphi\in C(\mathbb{K});\varphi(0)=0\}$. Choose $\hat\delta>0$ such that $-2\tau\mu\phi(0)+2\tau f(\phi(1))>0$ for all $\phi\in p_0+C(\mathbb{K})_{\hat\delta}$, where $C(\mathbb{K})_{\hat\delta}=\{\varphi\in C(\mathbb{K});\|\varphi\|<\hat\delta\}$. Select $\tilde\varepsilon>0$ so that $p_t\in p_0+C(\mathbb{K})_{\hat\delta}$ for all $t\in[0,\tilde\varepsilon]$. By the continuous dependence of solutions of system (1.5) on initial data there exists $\tilde\delta>0$ so that

   $$z_t^\phi\in p_0+C(\mathbb{K})_{\hat\delta}\quad\text{for all }\ \phi\in p_0+C(\mathbb{K})_{\tilde\delta}\ \text{ and for all }\ t\in[0,\tilde\varepsilon].$$

   Note $\dot p_0\notin H$. There exist a convex bounded open neighborhood $N_{p_0}$ of $p_0$ in $p_0+C(\mathbb{K})_{\tilde\delta}$, $\varepsilon_H\in(0,\min\{\omega-1,\tilde\varepsilon/2\})$, and a $C^1$-map $\nu_H: N_{p_0}\to(\omega-\varepsilon_H,\omega+\varepsilon_H)$ such that $\nu_H(p_0)=\omega$, and for every $(t,\phi)\in(\omega-\varepsilon_H,\omega+\varepsilon_H)\times N_{p_0}$, $\Phi(t,\phi)\in H$ if and only if $t=\nu_H(\phi)$, and $D_1\Phi(\nu_H(\phi),\phi)1\notin H$ for all $\phi\in N_{p_0}$. Then the Poincaré-map $P_H: N_{p_0}\cap H\ni\phi\mapsto\Phi(\nu_H(\phi),\phi)\in H$ is $C^1$-smooth, and $p_0$ is a fixed point of $P_H$.

Let $\sigma^{P_H}$ denote the spectrum of the derivative $DP_H(p_0) : H \to H$. Then $\sigma \setminus \{0, 1\} = \sigma^{P_H} \setminus \{0, 1\}$. If $r > 0$ is given and if there exists $\zeta \in \sigma^{P_H}$ with $r < |\zeta|$, then we let $H_{\leq r}$ and $H_{r<}$ denote the realified generalized eigenspaces of $DP_H(p_0)$ associated with the nonempty disjoint compact spectral sets $\{\zeta \in \sigma^{P_H}; |\zeta| \leq r\}$ and $\{\zeta \in \sigma^{P_H}; r < |\zeta|\}$, respectively. For such $r$, $H = H_{\leq r} \oplus H_{r<}$.

For $P_H$ there exist convex open neighborhoods $N_{H,\leq 1}$ of 0 in $H_{\leq 1}$, $N_{H,1<}$ of 0 in $H_{1<}$ and a $C^1$-map $w_H^u : N_{H,1<} \to H_{\leq 1}$ with $w_H^u(0) = 0$, $Dw_H^u(0) = 0$, $w_H^u(N_{H,1<}) \subset N_{H,\leq 1}$, so that for the neighborhood $N_H^u = N_{H,1<} + N_{H,\leq 1}$ of 0 in $H$ we have that the shifted graph

$$W^{\mathrm{u}}(p_0, P_H, N_H^u) = \{p_0 + \chi + w_H^u(\chi); \chi \in N_{H,1<}\}$$

coincides with the set

$$\Big\{\phi \in N_H^u; \text{there is a trajectory } (\phi_n)_{-\infty}^0 \text{ of } P_H \text{with } \phi = \phi_0,$$
$$\lambda^n(\phi_n - p_0) \in N_H^u \text{ for all } n \in -\mathbb{N} \text{ and } \lambda^n(\phi_n - p_0) \to 0 \text{ as } n \to -\infty\Big\}.$$

Furthermore, there exist an open neighborhood $\widetilde{N_H^u}$ of 0 in $N_H^u$, a constant $\alpha_{H,u} \in (0, \lambda)$, and a norm $\|\cdot\|_{H,u}$ on $H$, equivalent to $\|\cdot\|_H$ with

$$\|\phi\|_{H,u} = \max\{\|\mathrm{Pr}_{H,1<}\phi\|_{H,u}, \|\mathrm{Pr}_{H,\leq 1}\phi\|_{H,u}\} \quad \text{for all } \phi \in H,$$

so that the restriction of $P_H$ to $W^{\mathrm{u}}(p_0, P_H, N_H^u) \cap (p_0 + \widetilde{N_H^u})$ defines a $C^1$-diffeomorphism $(P_H)_u$ onto $W^{\mathrm{u}}(p_0, P_H, N_H^u)$ whose inverse satisfies

$$\|(P_H)_u^{-1}(\phi) - (P_H)_u^{-1}(\psi)\|_{H,u} \leq \alpha_{H,u}\|\phi - \psi\|_{H,u} \quad \text{for all } \phi, \psi \in W^{\mathrm{u}}(p_0, P_H, N_H^u).$$

Let $\mathrm{Pr}_{H,\dot{p}_0} : C(\mathbb{K}) \to C(\mathbb{K})$ be the projection onto $H$ along $\dot{p}_0$. Denote $\psi_H = 1/(\|\mathrm{Pr}_{H,\dot{p}_0}\psi_u\|_{H,u})\mathrm{Pr}_{H,\dot{p}_0}\psi_u$. Then $H_{1<} = \mathbb{R}\psi_H$ with $\|\psi_H\|_{H,u} = 1$. Since $N_{H,1<}$ is a convex open neighborhood of 0 in $H_{1<}$, it follows that

$$N_{H,1<} = \{s\psi_H; -\beta_1 < s < \beta_2\}$$

for some constants $\beta_1 > 0$ and $\beta_2 > 0$. Choose $\beta_0 \in (0, \min\{\beta_1, \beta_2\})$ so that $\|Dw_H^u(s\psi_H)\psi_H\|_{H,u} < 1$ for $|s| < \beta_0$. For $0 < \beta < \beta_0$, define

$$W_\beta^{\mathrm{u}}(p_0, P_H) = \{p_0 + s\psi_H + w_H^u(s\psi_H); |s| < \beta\}.$$

Clearly, $W_\beta^{\mathrm{u}}(p_0, P_H) \subseteq W^{\mathrm{u}}(p_0, P_H, N_H^u)$ and $W^{\mathrm{u}}(p_0, P_H, N_H^u) \subseteq \bigcup_{t\geq 0}\Phi(t, W_\beta^{\mathrm{u}}(p_0, P_H))$. If $\phi \in W_\beta^{\mathrm{u}}(p_0, P_H)$, then $(P_H)_u^{-1}(\phi) \in W_\beta^{\mathrm{u}}(p_0, P_H)$ and $\|(P_H)_u^{-1}(\phi) - p_0\|_{H,u} < \alpha_{H,u}\beta$. In addition to the properties of $N_{1<}, N_{\leq 1}, N^u, w^u$ stated before, there exist an open neighborhood $\widetilde{N^u}$ of 0 in $N^u$, a constant $\alpha_u \in (0, \lambda)$ and a norm $\|\cdot\|_u : C(\mathbb{K}) \to \mathbb{R}$, equivalent to $\|\cdot\|$, with $\|\phi\|_u = \max\{\|\mathrm{Pr}_{C_{1<}}\phi\|_u, \|\mathrm{Pr}_{C_{\leq 1}}\phi\|_u\}$ for all $\phi \in C(\mathbb{K})$, so that the restriction of $\Phi(\omega, \cdot)$ to $W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot), N^u) \cap (p_0 + \widetilde{N^u})$ defines a $C^1$-diffeomorphism $\Phi(\omega, \cdot)_u$ onto $W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot), N^u)$, whose inverse satisfies

$$\|(\Phi(\omega, \cdot)_u)^{-1}(\phi) - (\Phi(\omega, \cdot)_u)^{-1}(\psi)\|_u \leq \alpha_u\|\phi - \psi\|_u$$

for all $\phi, \psi \in W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot), N^u)$. Choose $\delta_0 > 0$ such that $\{s\psi_u; |s| < \delta_0\} \subset N_{1<}$ and that $\|Dw^u(s\psi_u)\psi_u\|_u < 1$ for all $|s| \leq \delta_0$. Then, for every $\phi$ in

$$W_\delta^{\mathrm{u}}(p_0, \Phi(\omega, \cdot)) = \{p_0 + s\psi_u + w^u(s\psi_u); |s| < \delta\}, \quad 0 < \delta < \delta_0,$$

we have $(\Phi(\omega, \cdot)_u)^{-1}(\phi) \in W_\delta^{\mathrm{u}}(p_0, \Phi(\omega, \cdot))$ and $\|(\Phi(\omega, \cdot)_u)^{-1}(\phi) - p_0\|_u < \alpha_u\delta$. The unstable set of the fixed point $p_0$ of the periodic map $\Phi(\omega, \cdot)$ is defined by

$$W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot)) = \bigcup_{n\in\mathbb{N}}\Phi(n\omega, W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot), N^u)).$$

Observe that, for $0 < \delta < \delta_0$, we have

$$W^{\mathrm{u}}(p_0, \Phi(\omega, \cdot)) = \bigcup_{n\in\mathbb{N}} \Phi(n\omega, W^{\mathrm{u}}_{\delta}(p_0, \Phi(\omega, \cdot))).$$

7. We show that there exists a unique continuously differentiable function $z = (x, y)^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^2$ satisfying

$$\begin{cases} \dot{x}(t) = -2\tau\mu x(t) + 2\tau f'(p^2(t))y(t), \\ \dot{y}(t) = -2\tau\mu y(t) + 2\tau f'(p^1(t-1))x(t-1), \\ x(0) = 0, \quad y(0) = 1, \\ V(z_t) = 2 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Let $\phi_H = z_0$. Then we can show that $\Pi_3\psi_H$ and $\Pi_3\phi_H$ span the plane $\Pi_3 H = \{y \in \mathbb{R}^3;\, y_1 = 0\}$ in $\mathbb{R}^3$. Choose $\beta \in (0, \beta_0)$ such that the curve

$$b : (-\beta, \beta) \ni s \mapsto \Pi_3(p_0 + s\psi_H + w^u_H(s\psi_H)) \in \Pi_3 H$$

is $C^1$-smooth and satisfies $b(0) = \Pi_3 p_0$, $Db(0)1 = \Pi_3\psi_H$. It follows that there exist $\beta_3 \in (0, \beta_0)$, $u_1$ and $u_2$ in $(0, \infty)$, and a $C^1$-map $h : (-u_1, u_2) \to \mathbb{R}$ with

$$|h_1'(u)| < 1 \quad \text{for all } u \in (-u_1, u_2)$$

so that

$$\begin{aligned} &b|_{(-\beta_3,\beta_3)} \text{ is injective}, \\ &\Pi_3|_{W^{\mathrm{u}}_{\beta_3}(p_0,P_H)} \text{ is injective}, \\ &h_1(0) = 0 \text{ and } h_1'(0) = 0, \\ &\Pi_3(W^{\mathrm{u}}_{\beta_3}(p_0, P_H)) = b((-\beta_3, \beta_3)) = \{\Pi_3 p_0 + u\Pi_3\psi_H + h(u)\Pi_3\phi_H;\, -u_1 < u < u_2\}. \end{aligned}$$

8. We show that there exists $\beta_4 \in (0, \beta_3)$ such that $W^{\mathrm{u}}_{\beta_4}(p_0, P_H) \subset bdW$. This result combined with the relations $W^{\mathrm{u}}(\mathcal{O}) = \bigcup_{t\geq 0}\Phi(t, W^{\mathrm{u}}(p_0, P_H, N^u_H))$ and $W^{\mathrm{u}}(p_0, P_H, N^u_H) \subset \bigcup_{t\geq 0}\Phi(t, W^{\mathrm{u}}_{\beta_4}(p_0, P_H))$ yields that $W^{\mathrm{u}}(\mathcal{O}) \subset bdW$.

We remark that the above identification enables us to draw a complete picture about the dynamics of the flow on $W^{\mathrm{u}}(\mathcal{O})$, which will be the main ingredient of the constructive proof of the homeomorphism from $W^{\mathrm{u}}(\mathcal{O})$ to the two-dimensional sphere in Section 6.

## 5. Smoothness of $\bar{W}$, $W$, $bdW$ and $\bar{W} \cap S$

Recall from Section 2 that there exist maps $w : P_3\bar{W} \to E$ and $w_S : P(\bar{W} \cap S) \to G_1 \oplus E$ such that

$$\bar{W} = \{\chi + w(\chi);\, \chi \in P_3\bar{W}\}$$

and

$$\bar{W} \cap S = \{\chi + w_S(\chi);\, \chi \in P(\bar{W} \cap S)\},$$

respectively.

The smoothness of the maps $w$ and $w_S$ are described as follows:

**Theorem 4.**
1. *Both* $w|_{P_3W}$ *and* $w_S|_{P(W\cap S)}$ *are* $C^1$*-smooth.*
2. $w$ *and* $w_S$ *are also smooth at* $P_3(bdW)\setminus\{P_3z_-, P_3z_+\}$ *and* $P\mathcal{O}$*, respectively, in the sense that close to these points,* $w$ *and* $w_S$*, respectively, can be extended to* $C^1$*-functions on open subsets in* $G_3$ *and* $G_2$*. Moreover,* $P_3(bdW)\setminus\{P_3z_-, P_3z_+\}$ *is a two-dimensional* $C^1$*-submanifold of* $G_3$*, and* $bdW\setminus\{z_-, z_+\}$ *is a two-dimensional* $C^1$*-submanifold of* $C(\mathbb{K})$*.*

We will only sketch the proof of (i) due to limitation of space. We should, however, mention that the proof of 2 is very technical but similar to that of corresponding results in [10].

*The* $C^1$*-smoothness of* $w|_{P_3W}$: note that for every $\varepsilon>0$, $W=\bigcup_{n\in\mathbb{N}}\Phi(\{n\}\times(W_{\mathrm{loc}}\cap C(\mathbb{K})_\varepsilon))$. Let $D_n=P_3\Phi(\{n\}\times(W_{\mathrm{loc}}\cap C(\mathbb{K})_\varepsilon))$. It is sufficient to find $\varepsilon>0$ so that $w|_{D_n}$ is $C^1$-smooth for every $n\in\mathbb{N}$. Since $P_3$ is $C^1$-smooth and $DP_3(0)=P_3$ defines an isomorphism from $P_0\oplus P_1$ onto $G_3$, there exists $\varepsilon>0$ so that $P_3$ maps $W_{\mathrm{loc}}\cap C(\mathbb{K})_\varepsilon$ one-to-one onto an open neighborhood $U$ of 0 in $G_3$ and $w|_U$ is $C^1$-smooth. Now let $n\in\mathbb{N}$ and $\chi\in D_n$ be given. The point $\phi=\chi+w(\chi)$ satisfies $\phi=\Phi(n,\psi)$ with $\psi\in W_{\mathrm{loc}}\cap C(\mathbb{K})_\varepsilon$. Set $\rho=P_3\psi$. Then $\chi=P_3\Phi(n,\rho+w(\rho))$ and $\rho\in U$. Note that the derivatives of the $C^1$-map $A: U\ni\tilde\rho\mapsto P_3\Phi(n,\tilde\rho+w(\tilde\rho))\in G_3$ are injective. Hence $A$ maps an open neighborhood $V$ of $\rho$ in $U$ one-to-one onto an open neighborhood $N$ of $\chi$ in $G_3$, which belongs to $D_n$, and the inverse $A^{-1}: N\to G_3$ of the map $V\ni\tilde\rho\mapsto A(\tilde\rho)\in N$ is $C^1$-smooth. For every $\tilde\chi\in N$,

$$w(\tilde\chi)=(\mathrm{id}-P_3)\Phi(n, A^{-1}(\tilde\chi)+w(A^{-1}(\tilde\chi))),$$

and it becomes obvious that $w|_N$ is $C^1$-smooth. It follows that $w|_{D_n}$ is $C^1$-smooth.

*The* $C^1$*-smoothness of* $w_S|_{P(W\cap S)}$: we discuss it in two cases.

*Case* 1. $\mathcal{O}$ is hyperbolic: In this case, the closed subspace

$$Y=C_{1<}\oplus C_{<1}$$

has codimension 1 and $D_1\Phi(\omega,p_0)1=\dot p_0\in C(\mathbb{K})\setminus Y$. There exist $\varepsilon_p>0$, a convex open neighborhood $U$ of $p_0$ in $C(\mathbb{K})$, and a $C^1$-map $\nu: U\to\mathbb{R}$ with the following properties:

$$1<\omega-\varepsilon_p,\ \nu(U)\subset(\omega-\varepsilon_p,\omega+\varepsilon_p),\ \nu(p_0)=\omega;$$
$$\text{for every } (t,\phi)\in(\omega-\varepsilon_p,\omega+\varepsilon_p)\times U,\ \Phi(t,\phi)\in p_0+Y \text{ if and only if } t=\nu(\phi);$$
$$D_1\Phi(\nu(\phi),\phi)1\in C(\mathbb{K})\setminus Y \text{ for all } \phi\in U.$$

We can also achieve that $0\notin U$ and $\dot\phi\in C(\mathbb{K})\setminus Y$ for all $\phi\in U\cap\bar W$. Define the Poincaré-map $P_Y: U\cap(p_0+Y)\to(p_0+Y)$ by

$$P_Y(\phi)=\Phi(\nu(\phi),\phi).$$

Let $W^{\mathrm{s}}$ be a local stable manifold of $P_Y$ at $p_0$. Then $W^{\mathrm{s}}\cap W\subset W\cap S$ and for every $\phi\in W\cap S\setminus\{0\}$, there exists $t>0$ with $\Phi_W(t,\phi)\in W^{\mathrm{s}}$. Let $\chi\in P(W\cap S)\setminus\{0\}$. Set $\phi=\chi+w_S(\chi)\in(W\cap S)\setminus\{0\}$. Then there exists $t\in\mathbb{R}$ so that $\psi=\Phi_W(t,\phi)\in W^{\mathrm{s}}\cap W$. We can show that close to $\psi$ the flow extends $W^{\mathrm{s}}\cap W$ to a smooth two-dimensional submanifold $W_\psi$ and that the $C^1$-map $B: W_\psi\ni\rho\mapsto P\Phi_W(-t,\rho)\in G_2$ has an injective derivative at $\psi$. By the Inverse Function Theorem, there exist an open neighborhood $V$ of $P(\psi)=\psi$ in $P(W\cap S)\setminus\{0\}$ and a $C^1$-inverse $B_V^{-1}: V\to W_\psi$ of the restriction of $B$ to $B^{-1}(V)$. Observe that the restriction of $w_S$ to $V$ is given by

$$w_S(\tilde\chi)=(\mathrm{id}-P)\Phi_W(-t,B_V^{-1}(\tilde\chi)).$$

Since $\Phi_W(-t,\cdot)$ defines a $C^1$-diffeomorphism of $W$ onto itself, the $C^1$-smoothness of $w_S|_{P(W\cap S)\setminus\{0\}}$ follows easily.

For the $C^1$-smoothness of $w_S$ at 0, recall that there exits $\varepsilon > 0$, $\eta > 0$ and a Lipschitz continuous map $\mathrm{Sep}_\eta : P_{1,\eta} \to Q + P_0$ such that

$$(W_{\mathrm{loc}} \cap S) \cap (Q + P_{1,\eta} + P_{0,\varepsilon}) = \{\chi + \mathrm{Sep}_\eta(\chi);\ \chi \in P_{1,\eta}\},$$

where $P_{i,\alpha} = \{\chi \in P_i;\ \|\chi\| < \alpha\}$ for $i = 0, 1$ and $\alpha > 0$. We can show that $\mathrm{Sep}_\eta|_{P_{1,\eta}\setminus\{0\}}$ is $C^1$-smooth and that $\mathrm{Sep}_\eta$ is differentiable at 0, $D\,\mathrm{Sep}_\eta(0) = 0$ and $D\,\mathrm{Sep}_\eta$ is continuous at 0. These results imply that

$$a : P_{1,\eta} \ni \chi \mapsto P(\chi + \mathrm{Sep}_\eta(\chi)) \in G_2$$

defines a $C^1$-diffeomorphism from $P_{1,\eta}$ onto an open neighborhood $V$ of 0 in $G_2$. Let $a^{-1}$ denote its inverse. For all $\chi \in V$,

$$w_S(\chi) = (\mathrm{id} - P)(a^{-1}(\chi) + \mathrm{Sep}_\eta(a^{-1}(\chi))),$$

so that $w_S|_V$ is $C^1$-smooth.

*Case* 2. $\mathcal{O}$ is nonhyperbolic: We can choose a unit vector $\xi \in C(\mathbb{K})$ such that the realified generalized eigenspace of $M$ associated with the eigenvalue 1 is $\mathbb{R}\dot{p}_0 \oplus \mathbb{R}\xi$. Set

$$Y = C_{<1} \oplus \mathbb{R}\xi \oplus C_{1<}.$$

Hence $\dot{p}_0 \in C(\mathbb{K}) \setminus Y$. Then there exist $\varepsilon_p > 0$, a convex open neighborhood $U$ of $p_0$ in $C(\mathbb{K})$ and a $C^1$-map $\nu : U \to \mathbb{R}$ with the same properties as in the case where $\mathcal{O}$ is hyperbolic and, in addition, with

$$V(\phi) \geq 2 \quad \text{for all } \phi \in U.$$

Then the Poincaré-map

$$P_Y : U \cap (p_0 + Y) \ni \phi \mapsto \Phi(\nu(\phi), \phi) \in (p_0 + Y)$$

is $C^1$-smooth and has $p_0$ as a fixed point. Let $W^{\mathrm{cs}}$ be a local center-stable manifold of $P_Y$ at $p_0$. Let $\chi \in P(W \cap S)\setminus\{0\}$. Set $\phi = \chi + w_S(\chi) \in (W \cap S) \setminus \{0\}$. We can show that there exist $t \geq 0$, a trajectory $(\phi_n)_0^\infty$ of $P_Y$ in $W^{\mathrm{cs}}$, and a neighborhood $N_t$ of $\phi_0$ in $C(\mathbb{K})$ so that $\phi_0 = \Phi(t, \phi)$, $\phi_n \to p_0$ as $n \to \infty$, and $W^{\mathrm{cs}} \cap N_t \subset S$. Similar arguments as those in the hyperbolic case yield that $w_S|_{P(W\cap S)\setminus\{0\}}$ is $C^1$-smooth. The $C^1$-smoothness of $w_S$ at 0 can be proved in the same way as for the hyperbolic case.

## 6. Homeomorphisms from $(\bar{W}, bdW)$ onto $(D^3, S^2)$

We can now obtain a homeomorphism from $bdW$ onto $S^2$. The approach is constructive and based on the important relation between $bdW$ and $W^{\mathrm{u}}(\mathcal{O})$ described in Section 4. This homeomorphism and the description of dynamics on $\bar{W}$ enables us to apply some powerful results in geometry and topology of three-dimensional manifolds to show that $\bar{W}$ is homeomorphic to $D^3$.

**Theorem 5.**
1. *The set $bdW$ is homeomorphic to $S^2$.*
2. *The set $\bar{W}$ is homeomorphic to $D^3$.*

**Outline of the proof.**

1. Using the notations in the proof of Theorem 3, we can show that there exist $\delta_1 \in (0, \delta_0)$, $\delta_2$ and $\delta_3$ in $(0, \delta_1)$, and a continuous strictly increasing function $g : [-\delta_2, \delta_3] \to [-\delta_1, \delta_1]$ such that $g(-\delta_2) = -\delta_1$, $g(\delta_3) = \delta_1$, $g(0) = 0$, $|g(s)| > |s|$ for all $s \in [-\delta_2, \delta_3] \setminus \{0\}$, and

$$\Phi(\omega, p_0 + s\psi_u + w^u(s\psi_u)) = p_0 + g(s)\psi_u + w^u(g(s)\psi_u) \quad \text{for all } s \in [-\delta_2, \delta_3];$$

moreover, if $-\delta_1 \le s_1 < s_2 \le \delta_1$, then

$$p_0 + s_1\psi_u + w^u(s_1\psi_u) \ll p_0 + s_2\psi_u + w^u(s_2\psi_u).$$

Choose $s_0 \in (0, \min\{\delta_2, \delta_3\})$. Let $s_1^{\pm} = g(\pm s_0)$, $s_0^{\pm} = \pm s_0$. Then the curves

$$\gamma^+ : [s_0^+, s_1^+] \ni s \mapsto p_0 + s\psi_u + w^u(s\psi_u) \in C(\mathbb{K})$$

and

$$\gamma^- : [s_1^-, s_0^-] \ni s \mapsto p_0 + s\psi_u + w^u(s\psi_u) \in C(\mathbb{K})$$

define homeomorphisms onto their images, respectively. Set $\Gamma_\Phi^+ = \{\gamma^+(s);\, s_0^+ \le s \le s_1^+\}$, $\Gamma_\Phi^- = \{\gamma^-(s);\, s_1^- \le s \le s_0^-\}$ and $bd^{\pm}W = \{\phi \in bdW \setminus \{z_-, z_+\};\, z_t^{\phi} \to z_{\pm} \text{ as } t \to \infty\}$. Then $bdW = \{z_-\} \cup bd^-W \cup \mathcal{O} \cup bd^+W \cup \{z_+\}$. We can show that $bd^{\pm}W = \Phi_W(\mathbb{R} \times \Gamma_\Phi^{\pm})$ and that for every $\phi \in bd^{\pm}W$ there is a unique $t \in \mathbb{R}$ such that $\Phi_W(t, \phi) \in \Gamma_\Phi^{\pm}$.

2. The map $G : \mathbb{R} \times D^2 \to D^2$ given by

$$G(t, (0,0)^{\mathrm{T}}) = (0,0)^{\mathrm{T}} \quad \text{for all } t \in \mathbb{R}$$

and by

$$G(t, (x,y))^{\mathrm{T}} = \left( \frac{r}{(1-r)\mathrm{e}^t + r} \cos\left(\theta + \frac{2\pi}{\omega} t\right), \frac{r}{(1-r)\mathrm{e}^t + r} \sin\left(\theta + \frac{2\pi}{\omega} t\right) \right)^{\mathrm{T}},$$
$$r = \sqrt{x^2 + y^2}, \quad 0 \le \theta < 2\pi, \quad x = r\cos\theta, \quad y = r\sin\theta$$

for all $t \in \mathbb{R}$ and $(x, y)^{\mathrm{T}} \in \mathbb{R}^2 \setminus \{(0,0)^{\mathrm{T}}\}$ is a continuous flow on $D^2$. Set $\Gamma_G = \{(x,0)^{\mathrm{T}};\, 1/(1+\mathrm{e}^{\omega}) < x \le 1/2\}$. Then $G(\mathbb{R} \times \Gamma_G) = \{x \in \mathbb{R}^2;\, 0 < x_1^2 + x_2^2 < 1\}$. The maps $i_{\Phi G}^{\pm} : \Gamma_\Phi^{\pm} \to \Gamma_G$ given by

$$i_{\Phi G}^{\pm}(\gamma^{\pm}(s)) = \left( \frac{s - s_0^{\pm}}{s_1^{\pm} - s_0^{\pm}} \frac{1}{1 + \mathrm{e}^{\omega}} + \frac{s_1^{\pm} - s}{s_1^{\pm} - s_0^{\pm}} \frac{1}{2}, 0 \right)^{\mathrm{T}}$$

are homeomorphisms. Using the results of 1 and defining $h^{\pm} : bd^{\pm}W \cup \mathcal{O} \cup \{z_{\pm}\} \to D^2$ by the relations

$$\begin{cases} h^{\pm}(\phi) = G(-t, i_{\Phi G}^{\pm}(\Phi_W(t, \phi))), \quad \phi \in bd^{\pm}W, \quad t \in \mathbb{R}, \quad \Phi_W(t,\phi) \in \Gamma_\Phi^{\pm}, \\ h^{\pm}(z_{\pm}) = 0, \\ h^{\pm}(\phi) = \left( \cos\frac{2\pi}{\omega} t,\, \sin\frac{2\pi}{\omega} t \right)^{\mathrm{T}}, \quad \phi = p_t,\, t \in \mathbb{R}, \end{cases}$$

we can show that $h^+$ and $h^-$ are homeomorphisms, $h^+|_{\mathcal{O}} = h^-|_{\mathcal{O}}$ and $h^+(\mathcal{O}) = S^1$.

3. The map $h^* : bdW \to S^2$ given by

$$h^*(\phi) = h^+(\phi) = h^-(\phi) \quad \text{for } \phi \in \mathcal{O},$$

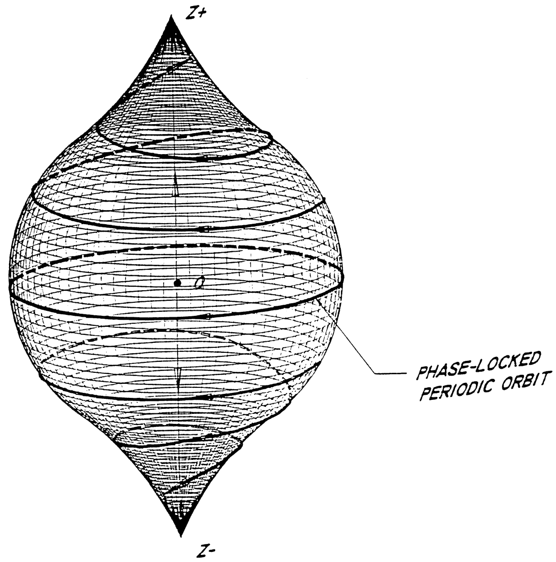


Fig. 1. $\bar{W}$ can be regarded as a smooth spindle, a complete analogue for system (1.5) of the results for scalar equation (1.4) obtained in [10].

$$h^*(\phi) = \begin{pmatrix} x_1 \\ x_2 \\ \sqrt{1 - x_1^2 - x_2^2} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h^+(\phi) \quad \text{for } \phi \in bd^+ W \cup \{z_+\} \setminus \mathcal{O},$$

$$h^*(\phi) = \begin{pmatrix} x_1 \\ x_2 \\ -\sqrt{1 - x_1^2 - x_2^2} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h^-(\phi) \quad \text{for } \phi \in bd^- W \cup \{z_-\} \setminus \mathcal{O},$$

is a homeomorphism from $bdW$ onto $S^2$.

4. Since $\Pi_3$ defines a homeomorphism from $\bar{W}$ onto $\Pi_3\bar{W}$, it suffices to show that there is a homeomorphism from $\Pi_3\bar{W}$ onto $D^3$ such that the homeomorphism sends $\Pi_3(bdW)$ onto $S^2$. The Lipschitz continuity of $\Pi_3^{-1}$ and (i) imply that $\Pi_3(bdW)$ is homeomorphic to $S^2$. Therefore, the Jordan–Brouwer Separation Theorem shows that the set $\mathbb{R}^3 \setminus \Pi_3(bdW)$ has two connected components, one bounded and the other unbounded. Denote the bounded component by $\mathrm{int}(\Pi_3(bdW))$ and the unbounded component by $\mathrm{ext}(\Pi_3(bdW))$. We can show that $\mathrm{int}(\Pi_3(bdW)) = \Pi_3 W$, which implies that

$$\Pi_3\bar{W} = \Pi_3(bdW) \cup \mathrm{int}(\Pi_3(bdW)).$$

5. We show that $\mathrm{int}(\Pi_3(bdW))$ is uniformly locally 1-connected [1]. Then applying Bing's Theorem [3] with $A = \Pi_3(bdW)$, we complete the proof.

[1] The diameter $\mathrm{diam}(X)$ of a set $X \subseteq \mathbb{R}^3$ is defined by $\mathrm{diam}(X) = \sup\{|x - y|_{\mathbb{R}^3};\, x \in X, y \in X\}$. A set $X \subset \mathbb{R}^3$ is called *uniformly locally 1-connected* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous map $a : \partial([0,1]\times[0,1]) \to X$ with $\mathrm{diam}(a(\partial([0,1]\times[0,1]))) < \delta$ can be extended to a continuous map $b : [0,1]\times[0,1] \to X$ with $\mathrm{diam}(b([0,1]\times[0,1])) < \varepsilon$.

## 7. Concluding remarks

We considered a global forward extension $W$ of a leading unstable manifold of the origin. We proved that $\bar{W}$ contains a phase-locked periodic orbit $\mathcal{O}$ and 3 equilibria 0, $z_-$ and $z_+$. The periodic orbit is linear unstable and its basin of attraction within $\bar{W}$ is the disk $\bar{W} \cap S$ minus the trivial equilibrium. Other orbits in $\bar{W} \setminus S$ are heteroclinic orbits from either $\mathcal{O}$ or 0 to the two nontrivial equilibria. The dynamics on $bdW \setminus \{z_-, z_+\}$ and its identification with the unstable set $W^{\mathrm{u}}(\mathcal{O})$ of $\mathcal{O}$ enabled us to construct a homeomorphism from $bdW$ onto $S^2$, which leads to a homeomorphism from $\bar{W}$ onto $D^3$ by using the Bing's theorem and the Jordan–Brouwer Separation Theorem. Smoothness of $\bar{W}$ and $\bar{W} \cap S$ were also established. Vaguely speaking, $\bar{W}$ can be regarded as a smooth solid spindle with two tips $z_-$ and $z_+$ and a separating disk $\bar{W} \cap S$ bordered by the phase-locked orbit, as shown in Fig. 1. Such a spindle was previously observed in [10] for the scalar equation (1.4) and our results here show a complete analogue of their results for the system (1.5).

In the recent work of Krisztin and Walther [9], it was proved that for $\tau$ in a certain range, the closure of the forward extension of a three-dimensional $C^1$-submanifold of the local unstable manifold at the trivial solution for the scalar equation (1.4) is exactly the global attractor. The same result should hold for system (1.5). More precisely, we expect that all periodic solutions of Eq. (1.5) are either synchronous or phase-locked, and that for

$$\tau \in \left( \frac{\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}}, \frac{2\pi - \arccos(\mu/f'(0))}{\sqrt{[f'(0)]^2 - \mu^2}} \right),$$

$\bar{W}$ is the global attractor for system (1.5).

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