Journal of Dynamics and Differential Equations, Vol. 11, No. 2, 1999

Rotating Waves in Neutral Partial Functional Differential Equations

J. Wu[1,3] and H. Xia[2]

Received November 23, 1994

General results on the existence and global continuation of rotating waves are established for partial neutral functional differential equations defined on the unit circle. These results are applied to a class of coupled lossless transmission lines.

KEY WORDS: Neutral equations; partial functional differential equations; rotating waves; global bifurcation; transmission lines.
AMS (MOS) SUBJECT CLASSIFICATIONS: 34K40, 35K57, 35R10.

1. INTRODUCTION

It seems that one of the main motivations for the development of neutral functional differential equations has been the study of a class of linear hyperbolic partial differential equations subject to certain nonlinear boundary conditions arising from the lossless transmission line theory. We refer to Abolinia and Mishkis (1960), Brayton (1966), Brayton and Moser (1964), Cooke and Krumme (1968), Cruz and Hale (1979), Lopes (1975, 1976), Nagumo and Shimura (1961), Slemrod (1971), Hale (1977, 1993), Hale and Lunel (1993), and Wu and Xia (1996) for a detailed account of the history, the list of references, and the current status of the subject.

In a recent work, Wu and Xia (1996) have shown that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling

[1] Department of Mathematics and Statistics, York University, North York, Ontario, Canada M3J 1P3.
[2] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1.
[3] To whom correspondence should be addressed.

1040-7294/99/0400-0209$16.00/0

which exhibits various types of discrete waves. By taking a natural limit, one obtains from this system of neutral equations a scalar partial neutral functional differential equation defined on the unit circle. Such a partial neutral functional differential equation was recently investigated by Hale (1993), where the fundamental existence and uniqueness of the Cauchy initial value problem, set-condensing property of the solution operator, Hopf bifurcations, and stability of periodic orbits have been established. More interestingly, using an argument similar to that in the earlier work of Hale *et al.* (1988), Hale established a result on space discretization which ensures the existence of discrete waves of the system of ordinary neutral equations near a stable hyperbolic rotating wave of the limiting scalar partial neutral equation defined on the unit circle. This provides a useful tool to establish the existence of discrete waves for systems of neutral equations with discrete diffusive coupling, at least for such systems of sufficiently large size.

In the present paper, we study the local existence and global continuation of rotating waves for a class of partial neutral functional differential equations defined on the unit circle. Our approach is as follows. We look at rotating waves bifurcating from a spatially homogeneous equilibrium and reduce the partial neutral equations under consideration into a second order ordinary neutral functional differential equation whose 2π-periodic solutions give rise to rotating waves of the partial neutral equations. We then show that finding a 2π-periodic solution of the second-order ordinary neutral functional differential equation can be formulated as finding a fixed point of an S^1-equivariant set-condensing mapping depending on two papereters, by using the compact resolvent of a Fredholm operator arising from the linear part of the second-order ordinary neutral equation. This abstract formulation allows us to apply the general global Hopf bifurcation theorem of ordinary neutral equations developed by Krawcewicz *et al.* (1993) to investigate the existence and maximal continuation of rotating waves for partial neutral equations.

Our general results will be illustrated by an example arising from a continuous circular array of transmission lines. The global bifurcation theorem will be applied to establish the global existence of slowly oscillatory rotating waves, where slowly oscillatory rotating waves are those whose time periods are larger than twice of the involved time lag, denoted by τ. In order to achieve this, we need to exclude 4τ-periodic rotating waves. This is equivalent to excluding nontrivial 4τ-periodic solutions for a system of four second-order ordinary differential equations. Fortunately, the latter resembles a coupled Liénard equation and naturally suggests a Liénard transformation and a Liapunov function, with the help of which nontrivial 4τ-periodic solutions can be ruled out.

The rest of this paper is organized as follows: general results on global bifurcations of rotating waves of partial neutral equations are in Section 2 and their application to a continuous array of transmission lines is provided in Section 3.

2. GENERAL RESULTS ON GLOBAL BIFURCATIONS OF ROTATING WAVES

A. Rotating Waves and Reductions. Let $\tau \geq 0$ be a given constant, S^1 be the unit circle, and $X = C([-\tau, 0]; C(S^1; R)) \cong C([-\tau, 0] \times S^1; R)$. For any $\varphi \in X$, we write $\varphi(\theta, x)$ or $\varphi(\theta)(x)$ for $\theta \in [-\tau, 0]$ and $x \in S^1$.

Assume that $\tilde{D}, \tilde{f}\colon R \times C([-\tau, 0]; R) \to R$ are completely continuous C^1 mappings and map bounded subsets into bounded subsets with $\tilde{D}(\alpha, 0) = \tilde{f}(\alpha, 0) = 0$ for $\alpha \in R$. Moreover, we assume that

$$\begin{cases} \tilde{D}(\alpha, \tilde{\varphi}) = \tilde{\varphi}(0) - \tilde{b}(\alpha, \tilde{\varphi}), \text{ where } \tilde{b}\colon R \times C([-\tau, 0]; R) \to R \\ \quad \text{satisfies } |\tilde{b}(\alpha, \tilde{\varphi}) - \tilde{b}(\alpha, \tilde{\psi})| \leq k \sup_{\theta \in [-\tau, 0]} |\tilde{\varphi}(\theta) - \tilde{\psi}(\theta)| \\ \quad \text{for } \alpha \in R, \tilde{\varphi}, \tilde{\psi} \in C([-\tau, 0]; R), \text{ where } k \in [0, 1) \text{ is a constant} \end{cases} \tag{D}$$

Define b, D, and $f\colon R \times X \to C(S^1; R)$ by

$$b(\alpha, \varphi)(x) = \tilde{b}(\alpha, \varphi(\,\cdot\,, x))$$

$$D(\alpha, \varphi)(x) = \tilde{D}(\alpha, \varphi(\,\cdot\,, x))$$

$$f(\alpha, \varphi)(x) = \tilde{f}(\alpha, \varphi(\cdot, x))$$

for $(\alpha, \varphi, x) \in R \times X \times S^1$. We now consider the following partial neutral functional differential equation

$$\frac{\partial}{\partial t} D(\alpha, u_t) = d \frac{\partial^2}{\partial x^2} D(\alpha, u_t) + f(\alpha, u_t) \tag{2.1}$$

defined on the unit circle $x \in S^1$, where $d > 0$ is a given constant, and if $u \in C([-\tau, \delta); C(S^1; R))$, $\delta > 0$, $t \in [0, \delta)$, then $u_t \in X$ is defined as $u_t(\theta, x) = u(t + \theta, x)$ for $(\theta, x) \in [-\tau, 0] \times S^1$.

As Eq. (2.1) is imposed on the unit circle, we are looking for Hopf bifurcation of rotating waves from the trivial solution. That is, we seek Hopf bifurcations of time-periodic solutions $u\colon R \times S^1 \to R$ which are continuously differentiable in t and twice continuously differentiable in x, satisfy (2.1), and

$$u(t, x) = u\left(t + \frac{p}{2\pi} x, 0\right), \qquad (t, x) \in R \times S^1 \tag{2.2}$$

$$u(t + p, x) = u(t, x), \qquad (t, x) \in R \times S^1 \tag{2.3}$$

where $p > 0$ is a constant.

Let $v(t) = u(t, 0)$. Substituting (2.2) into (2.1), we get

$$\frac{\partial}{\partial t} D(\alpha, v_{t+(p/2\pi)x}) = d \frac{\partial^2}{\partial x^2} D(\alpha, v_{t+(p/2\pi)x}) + f(\alpha, v_{t+(p/2\pi)x})$$

Making a change of variable $s = t + (p/2\pi)x$, we obtain

$$\frac{d}{ds} D(\alpha, v_s) = d\left(\frac{p}{2\pi}\right)^2 \frac{d^2}{ds^2} D(\alpha, v_s) + f(\alpha, v_s) \tag{2.4}$$

Finally, we normalize the period of $v(t)$ by the transformation

$$y(t) = v\left(t \frac{p}{2\pi}\right) = u\left(t \frac{p}{2\pi}, 0\right) \tag{2.5}$$

Let $y_{s,(2\pi/p)} \in C([-\tau, 0]; R)$ be defined by

$$y_{s,(2\pi/p)}(\theta) = y\left(s + \frac{2\pi}{p} \theta\right), \qquad \theta \in [-\tau, 0] \tag{2.6}$$

Then from (2.4) it follows that

$$\frac{2\pi}{p} \frac{d}{ds} D(\alpha, y_{s,(2\pi/p)}) = d \frac{d^2}{ds^2} D(\alpha, y_{s,(2\pi/p)}) + f(\alpha, y_{s,(2\pi/p)})$$

That is,

$$\frac{d}{ds} D(\alpha, y_{s,(2\pi/p)}) = \frac{p}{2\pi}\left[d \frac{d^2}{ds^2} D(\alpha, y_{s,(2\pi/p)}) + f(\alpha, y_{s,(2\pi/p)})\right] \tag{2.7}$$

Summarizing the above discussions, we have the following.

Lemma 2.1. *The partial neutral function differential equation (2.1) has a rotating wave satisfying (2.2) and (2.3), if and only if the second-order ordinary neutral functional differential equation (2.7) has a 2π-periodic solution.*

Our next goal is to find constants α and p so that (2.7) has a nontrivial 2π-periodic solution.

B. Abstract Formulation. Denote by $C_{2\pi}^k$ the Banach space of 2π-periodic functions from R into R whose kth derivative exists and is continuous. The norm in $C_{2\pi}^k$ is defined by

$$\|x\|_k = \sum_{j=0}^{k} \sup_{t \in R} |x^{(j)}(t)|, \qquad x \in C_{2\pi}^k$$

and $C_{2\pi}^0$ will also be written $C_{2\pi}$.

Define L: Dom $L \subseteq C_{2\pi} \to C_{2\pi}$ by

$$\text{Dom } L = C_{2\pi}^2, \qquad Lx = x + \ddot{x}, \quad x \in C_{2\pi}^2$$

Clearly,

$$\text{Ker } L = \{x \in C_{2\pi}^2;\, x(t) = c_1 \sin t + c_2 \cos t,\, c_1, c_2 \in R,\, t \in R\}$$

$$\text{Ran } L = \left\{x \in C_{2\pi};\, \int_0^{2\pi} x(s) \cos s\, ds = \int_0^{2\pi} x(s) \sin s\, ds = 0\right\}$$

Then L is a closed Fredholm operator of index zero and, hence, has a compact resolvent K: $C_{2\pi} \to C_{2\pi}$. It can be verified that K: $C_{2\pi} \to C_{2\pi}$ is given by

$$(Kx)(t) = \frac{1}{\pi} \int_0^{2\pi} x(s) \cos(s-t)\, ds, \qquad t \in R$$

and this compact resolvent satisfies the following.

$$\begin{cases} \text{There exists a bounded linear operator } P\text{: } C_{2\pi} \to C_{2\pi}^1 \\ \quad \text{such that if } x \in C_{2\pi}^1, \text{ then } (L+K)^{-1}\dot{x} = Px \end{cases} \tag{C}$$

With the above notations, we can state the following.

Lemma 2.2. *Assume $\beta > 0$ is a given constant and $g \in C_{2\pi}$. Then $x \in C_{2\pi}^2$ is a solution of*

$$\dot{x}(t) = \beta \ddot{x}(t) + g(t), \qquad t \in R \tag{2.8}$$

if and only if x is a solution in $C_{2\pi}$ of the operator equation

$$x = \frac{1}{\beta} Px + (L+K)^{-1}\left(-\frac{1}{\beta} g + x + Kx\right) \tag{2.9}$$

Proof. If $x \in C^2_{2\pi}$ satisfies (2.8), then

$$\ddot{x} + x = \frac{\dot{x}}{\beta} + x - \frac{1}{\beta} g$$

from which it follows that

$$x = (L+K)^{-1}\left[\frac{1}{\beta}\dot{x} + x - \frac{1}{\beta} g + Kx\right]$$
$$= \frac{1}{\beta} Px + (L+K)^{-1}\left[-\frac{1}{\beta} g + x + Kx\right]$$

That is, x solves (2.9).

Conversely, if x is a solution of (2.9), then $x \in C^1_{2\pi}$ since $\mathrm{Ran}(L+K)^{-1}$ and Ran P are both subspaces of $C^1_{2\pi}$. By property (C), we then have $Px = (L+K)^{-1}\dot{x}$ and hence

$$x = (L+K)^{-1}\left[\frac{1}{\beta}\dot{x} - \frac{1}{\beta} g + x + Kx\right]$$

That is, $\ddot{x} + x + Kx = (1/\beta)\dot{x} - (1/\beta) g + x + Kx$, from which it follows that x solves (2.8). This completes the proof. □

We now apply Lemma 2.2 to Eq. (2.7). It follows that $y \in C_{2\pi}$ is a solution of (2.7) if and only if

$$D(\alpha, y._{,(2\pi/p)}) = \frac{2\pi}{p} d^{-1} PD(\alpha, y._{,(2\pi/p)}) + (L+K)^{-1}[-d^{-1} f(\alpha, y._{,(2\pi/\beta)}) + D(\alpha, y._{,(2\pi/p)}) + KD(\alpha, y._{,(2\pi/p)})]$$

Therefore,

$$y = b(\alpha, y._{,(2\pi/p)}) + \frac{2\pi}{p} d^{-1} PD(\alpha, y._{,(2\pi/\beta)}) + (L+K)^{-1}[-d^{-1} f(\alpha, y._{,(2\pi/p)}) + (Id+K)\, D(\alpha, y._{,(2\pi/p)})] \triangleq F(\alpha, p, y) \qquad (2.10)$$

Then (D) and (C) imply that $F\colon R \times (0, \infty) \times C_{2\pi} \to C_{2\pi}$ is k-set condensing and the following result holds.

Lemma 2.3. *u is a rotating wave of (2.1) satisfying (2.2) and (2.3) if and only if y defined by (2.5) is a fixed point of*

$$y = F(\alpha, p, y), \qquad y \in C_{2\pi}$$

C. Abstract Multiparameter Bifurcation Theory. We now reformulate some global Hopf bifurcation theorems in our setting.

Let the compact Lie group $G = S^1$ act on $C_{2\pi}$ by shifting arguments. That is,

$$(e^{i\theta} \cdot x)(t) = x(t + \theta), \qquad t \in R, \quad \theta \in [0, 2\pi]$$

Assume that $F: R^2 \times C_{2\pi} \to C_{2\pi}$ is a C^1-mapping which is equivariant with respect to the aforementioned S^1-action, that is,

$$F(\alpha, p, e^{i\theta} \cdot y) = e^{i\theta} \cdot F(\alpha, p, y), \qquad (\alpha, p, y) \in R^2 \times C_{2\pi}, \qquad e^{i\theta} \in S^1$$

Moreover, we assume that there exists a constant $k \in [0, 1)$ such that F is a k-set condensing mapping.

To describe our bifurcation problem, we first assume the following.

$$\left\{\begin{array}{l}\text{There exists a 2-dimensional submanifold } M \subseteq R^2 \times C^G_{2\pi},\\ \text{where } C^G_{2\pi} = \{y \in C_{2\pi};\ y \text{ is a constant function}\}, \text{ satisfying}\\ \text{(i). } y = F(\alpha, p, y) \text{ for } (\alpha, p, y) \in M; \text{ (ii). For any } (\alpha_0, p_0, y_0)\\ \in M \text{ there exists an open neigborhood } U_{(\alpha_0, p_0)} \text{ of}\\ (\alpha_0, p_0) \in R^2, \text{ an open neighborhood } U_{y_0} \text{ of } y_0 \in C^G_{2\pi}, \text{ and a}\\ C^1\text{-mapping } \eta: U_{(\alpha_0, p_0)} \to C^G_{2\pi} \text{ such that } M \cap (U_{(a_0, p_0)} \times U_{y_0})\\ = \{(\alpha, p, \eta(\alpha, p));\ (\alpha, p) \in U_{(a_0, p_0)}\}\end{array}\right. \qquad \text{(M)}$$

A point $(\alpha_0, p_0, y_0) \in M$ is called a bifurcation point if in any neighborhood of (α_0, p_0, y_0) there exists a point $(\alpha, p, y) \notin M$ satisfying $y = F(\alpha, p, y)$. Clearly, every bifurcation point belongs to the set

$$\Lambda = \{(\alpha, p, y);\ Id - F_y(\alpha, p, y) : C_{2\pi} \to C_{2\pi} \text{ is not an isomorphism}\}$$

Let (α_0, p_0, y_0) be an isolated point in Λ. We identify R^2 with $\mathbb{C}$, and for a sufficiently small $\rho > 0$ we define $\beta: D \to M$, where $D = \{z \in \mathbb{C}; |z| \leqslant 1\}$, by

$$\beta(z) = (\lambda_0 + \rho z, \eta(\alpha_0 + \rho z)), \qquad \lambda_0 = (\alpha_0, p_0) \in R^2 \cong \mathbb{C}$$

If $\rho > 0$ is sufficiently small, then $\psi(z) = Id - D_y F(\beta(z))$, $z \in \delta D$, defines a continuous mapping $\psi: S^1 \to GL(C_{2\pi})$. It follows that

$$C_{2\pi} = C^G_{2\pi} \oplus V_1 \oplus \cdots \oplus V_k \oplus \cdots$$

with V_k being the space spanned by e^{ikt} [or $(\cos kt, \sin kt)$ in real space] in the isotypical direct sum decomposition of V with respect to the G-action on $C_{2\pi}$. Consequently, we get

$$\psi = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_k \oplus \cdots$$

where $\psi_k: S^1 \to GL(V_k) = GL(\mathbb{C})$ and $\psi_0: S^1 \to GL(C^G_{2\pi}) = GL(\mathbb{C})$ are defined by

$$\psi_k(z) = \psi(z)|_{V_k}: V_k \to V_k, \qquad k \geqslant 1$$

$$\psi_0(z) = \psi(z)|_{C^G_{2\pi}}: C^G_{2\pi} \to C^G_{2\pi}$$

As $\psi_k(S^1) \subseteq GL(V_k)$ for $k \geqslant 1$, the Brouwer degree

$$\gamma_k(\alpha_0, p_0, y_0) = \deg_B(\psi_k, S^1)$$

is well defined. It turns out that $\gamma_k(\alpha_0, p_0, y_0)$ is a bifurcation invariant. More precisely, we have the following.

Lemma 2.4. *If (α_0, p_0, y_0) is an isolated point in A and if $\gamma_k(\alpha_0, p_0, y_0) \neq 0$ for some $k \geqslant 1$, then (α_0, p_0, y_0) is a bifurcation point. That is, there exists a sequence (α_n, p_n, y_n) of solutions to $y = F(\alpha, p, y)$ such that $(\alpha_n, p_n, y_n) \to (\alpha_0, p_0, y_0)$ as $n \to \infty$ and y_n is $(2\pi/k)$-periodic for each $n \geqslant 1$.*

Lemma 2.5. *Assume that M is complete and every point of A is isolated in*

$$S = cl\{(\alpha, p, y) \in R^2 \times C_{2\pi} \backslash M;\ y = F(\alpha, p, y)\}$$

Then for every bounded, connected component C of S, the set $C \cap M$ is finite and for each $k \geqslant 1$,

$$\sum_{(\alpha, p, y) \in C \cap M} \gamma_k(\alpha, p, y) = 0$$

The above lemmas are taken from Krawcewicz *et al.* (1993). One should also be able to obtain these results by using other multiparameter bifurcation theories. For details, we refer to Krawcewicz *et al.* (1993) and a recent survey paper by Ize (1993).

D. Global Bifurcation of Rotating Waves. We now return to $F: R \times (0, \infty) \times C_{2\pi} \to C_{2\pi}$ defined by Eq. (2.10) and let $M = \{(\alpha, p, 0); (\alpha, p) \in R \times (0, \infty)\}$. We assume that

$$f_\varphi(\alpha, 0)|_{C^G_{2\pi}}: C^G_{2\pi} \cong R \to C^G_{2\pi} \cong R \text{ is not zero for all } \alpha \in R \qquad \text{(DF)}$$

Under this assumption, M satisfies assumption (M). Using the Fourier series expansion, we can verify that $(\alpha_0, p_0, 0)$ is a point in Λ if and only if there exists an integer $k \geqslant 1$ so that $\lambda = i(2\pi/p_0)k$ and $\alpha = \alpha_0$ satisfy the characteristic equation

$$\lambda[1 - b_\varphi(\alpha, 0)\, e^{\lambda\cdot}] = -dk^2[1 - b_\varphi(\alpha, 0)\, e^{\lambda\cdot}] + f_\varphi(\alpha, 0)\, e^{\lambda\cdot} \qquad (2.11)$$

We also assume the following.

$$\begin{cases} \text{If } \varphi\colon [-\tau, 0] \to R \text{ is a constant mapping and} \\ f(\alpha, \varphi) = 0, \text{ then } \varphi = 0 \end{cases} \qquad \text{(F)}$$

$$\begin{cases} \text{The set } \{(\alpha_0, p_0) \in R \times (0, \infty);\ \lambda = i(2\pi/p_0)k \text{ and } \alpha = \alpha_0 \\ \text{satisfy (2.11) for some positive integer } k\} \text{ is discrete} \end{cases} \qquad \text{(CF)}$$

Let $(\alpha_0, p_0) \in R \times (0, \infty)$ be given so that $\lambda = i(2\pi/p_0)$ and $\alpha = \alpha_0$ satisfy (2.11). By assumption (CF) there exists a sufficiently small $\rho > 0$ such that for $(\alpha, p) \in \partial B_\rho(\alpha_0, p_0) \cong S^1$, where $B_\rho(\alpha_0, p_0) = \{(\alpha, p_0);$ $(\alpha - \alpha_0)^2 + (p - p_0)^2 \leqslant \rho^2\}$, $\psi(\alpha, p) = Id - D_y F(\alpha, p, 0) \in GL(C_{2\pi})$. Define

$$\begin{aligned} \Delta_k(\alpha, p) = {} & i\frac{2\pi}{p} k[1 - b_\varphi(\alpha, 0)\, e^{i(2\pi/p)k\cdot}] \\ & + dk^2[1 - b_\varphi(\alpha, 0)\, e^{i(2\pi/p)k\cdot}] - f_\varphi(\alpha, 0)\, e^{i(2\pi/p)k\cdot} \end{aligned}$$

for $(\alpha, p) \in B_\rho(\alpha_0, p_0)$. Then

$$\begin{aligned} \psi(\alpha, p)\, e^{ikt} &= [Id - D_y F(\alpha, p, 0)]\, e^{ikt} \\ &= [e^{ikt} - b_\varphi(\alpha, 0)\, e^{ik(t + (2\pi/p)\cdot)}] \\ &\quad - \frac{2\pi}{p} d^{-1}(L + K)^{-1} \frac{d}{dt}[e^{ikt} - b_\varphi(\alpha, 0)\, e^{ik(t + (2\pi/p)\cdot)}] \\ &\quad - (L + K)^{-1}[-d^{-1} f_\varphi(\alpha, 0))\, e^{ik(t + (2\pi/p)\cdot)} \\ &\quad + (Id + K)(e^{ikt} - b_\varphi(\alpha, 0)\, e^{ik(t + (2\pi/p)\cdot)})] \\ &= -d^{-1}\Delta_k(\alpha, p)(L + K)^{-1}\, e^{ikt} \\ &= \begin{cases} -d^{-1}\Delta_1(\alpha, p)\, e^{it} & \text{if } k = 1 \\ \dfrac{1}{d(k^2 - 1)} \Delta_k(\alpha, p)\, e^{ikt} & \text{if } k \geqslant 2 \end{cases} \end{aligned}$$

Consequently,

$$\gamma_k(\alpha_0, p_0, 0) = \deg_B(\Delta_k(\cdot, \cdot), B_\rho(\alpha_0, p_0))$$

is well defined. Therefore, by Lemmas 2.4 and 2.5 we get

Theorem 2.1. *Assume that there exists* $(\alpha_0, p_0) \in R \times (0, \infty)$ *so that* $\lambda = i(2\pi/p_0)$ *and* $\alpha = \alpha_0$ *satisfy (2.11) and that* $\lim_{\rho \to 0^+} \deg_B(\Delta_k(\cdot, \cdot), B_\rho(\alpha_0, p_0)) \neq 0$ *for some integer* $k \geqslant 1$. *Then* $(\alpha_0, p_0, 0)$ *is a rotating wave bifurcation point of system (2.1). That is, there exists a sequence* $(\alpha_n, p_n, u_n) \subseteq R \times (0, \infty) \times C(R \times S^1; R)$ *such that each* u_n *is a solution of (2.1) with* $\alpha = \alpha_n$ *and satisfies (2.2)–(2.3) with* $p = p_n$, *and such that* $\alpha_n \to \alpha_0$, $p_n \to p_0$, $u_n(t, x) \to 0$ *as* $n \to \infty$, *uniformly for* $(t, x) \in R \times S^1$. *Furthermore, let*

$$S = cl\{(\alpha, p, u) \in R \times (0, \infty) \times C(R \times S^1; R);$$
$$u \text{ is rotating wave of (2.1) satisfying (2.2)–(2.3)}\}$$

Then for each bounded, connected component C *of* S, *the set* $C \cap M$ *is finite and for each* $k \geqslant 1$,

$$\sum_{(\alpha, p, 0) \in C \cap M} \lim_{\rho \to 0^+} \deg_B(\Delta_k(\cdot, \cdot), B_\rho(\alpha, p)) = 0$$

For the sake of application, we need to relate the computation of $\lim_{\rho \to 0^+} \deg_B(\Delta_k(\cdot, \cdot), B_\rho(\alpha, p))$ to the usual transversality condition. First, by assumption (CF) there exists $\rho_0 > 0$ so that $\Delta(\alpha, p) \neq 0$ for $(\alpha, p) \in B_{\rho_0}(\alpha_0, p_0) \backslash \{(\alpha_0, p_0)\}$. Define

$$\delta_\alpha(\lambda) = \lambda[1 - b_\varphi(\alpha, 0)\, e^{i\lambda \cdot}] + d[1 - b_\varphi(\alpha, 0)\, e^{i\lambda \cdot}] - f_\varphi(\alpha, 0)\, e^{i\lambda \cdot}$$

As $\delta_\alpha(\lambda)$ is analytic in λ, we can find $\rho > 0$ so that $\delta_{\alpha_0}(\lambda) \neq 0$ for $\lambda \neq i(2\pi/p_0)$ and $\lambda = u + iv \in \Omega \cong \{(u, v); 0 \leqslant u \leqslant \rho,\ |v - (2\pi/p_0)| \leqslant \rho\}$. By continuity, $\delta_{\alpha_0 \pm \rho}(\lambda) \neq 0$ on $\partial\Omega$ if $\rho > 0$ is sufficiently small. Therefore, the integer

$$\gamma_\pm(\alpha_0, p_0) = \lim_{\rho \to 0^+} \deg_B(\delta_{\alpha_0 \pm \rho}(\cdot), \Omega)$$

is well defined, and by Lemma 2.5 of Erbe *et al.* (1992), we have

$$\lim_{\rho \to 0^+} \deg_B(\Delta_1(\cdot, \cdot), \beta_\rho(\alpha_0, p_0))$$
$$= \lim_{\rho \to 0^+} \deg_B(\delta_{\alpha_0 - \rho}(\cdot), \Omega) - \lim_{\rho \to 0^+} \deg_B(\delta_{\alpha_0 + \rho}(\cdot), \Omega) \qquad (2.12)$$

Therefore, if $\lambda: (\alpha_0 - \rho, \alpha_0 + \rho) \to \mathbb{C}$ is a smooth curve of solutions of $\delta_\alpha(\lambda) = 0$ with $\lambda(\alpha_0) = i(2\pi/p_0)$, then the transversality condition

$$\frac{d}{d\alpha} Re\lambda(\alpha)|_{\alpha=\alpha_0} \neq 0 \tag{2.13}$$

implies that $\lim_{\rho \to 0^+} \deg_B(\Delta_1(\cdot, \cdot), B_\rho(\alpha, p_0)) \neq 0$. More precisely, we have

$$\lim_{\rho \to 0^+} \deg_B(\Delta_1(\cdot, \cdot), \beta_\rho(\alpha_0, p_0)) = \begin{cases} +1 & \text{if } \dfrac{d}{d\alpha} Re\lambda(\alpha)|_{\alpha=\alpha_0} < 0 \\ -1 & \text{if } \dfrac{d}{d\alpha} Re\lambda(\alpha)|_{\alpha=\alpha_0} > 0 \end{cases} \tag{2.14}$$

In many applications, it happens that for every $(\alpha_0, p_0) \in R \times (0, \infty)$ such that $\delta_{\alpha_0}(i(2\pi/p_0)) = 0$, $(d/d\alpha)\, Re\lambda(\alpha)|_{\alpha=\alpha_0}$ has the same sign. Consequently, the summation formula in Theorem 2.1 rules out the possibility of the occurrence of any bounded connected component of S. This is exactly the case in our later application to neutral equations arising from lossless transmission lines.

3. GLOBAL EXISTENCE OF ROTATING WAVES IN NEUTRAL EQUATIONS

E. The Model. Let N be a positive integer. We consider a ring of N mutually coupled lossless transmission lines interconnected by a common resistor R, shown in Fig. 1. Assume that all lines are identical, each of which is a uniformly distributed lossless transmission line with the series inductance L_s and parallel capacitance C_s per unit length of the line. Taking the x-axis in the direction with two ends of the line at $x = 0$ and $x = 1$, we have the following linear hyperbolic partial differential equations subject to nonlinear boundary conditions:

$$\begin{cases} L_s \dfrac{\partial i_k}{\partial t} = -\dfrac{\partial v_k}{\partial x} \\ C_s \dfrac{\partial v_k}{\partial t} = -\dfrac{i_k}{\partial x}, \qquad 0 \leqslant x \leqslant 1 \\ E = v_k(0, t) + R_0 i_k(0, t) \\ i_k(1, t) + I_{k-1} - I_k = C \dfrac{d}{dt} v_k(1, t) + f(v_k(1, t)) + I \\ v_k(1, t) - v_{k+1}(1, t) = R I_k(t) \end{cases} \tag{3.1}$$

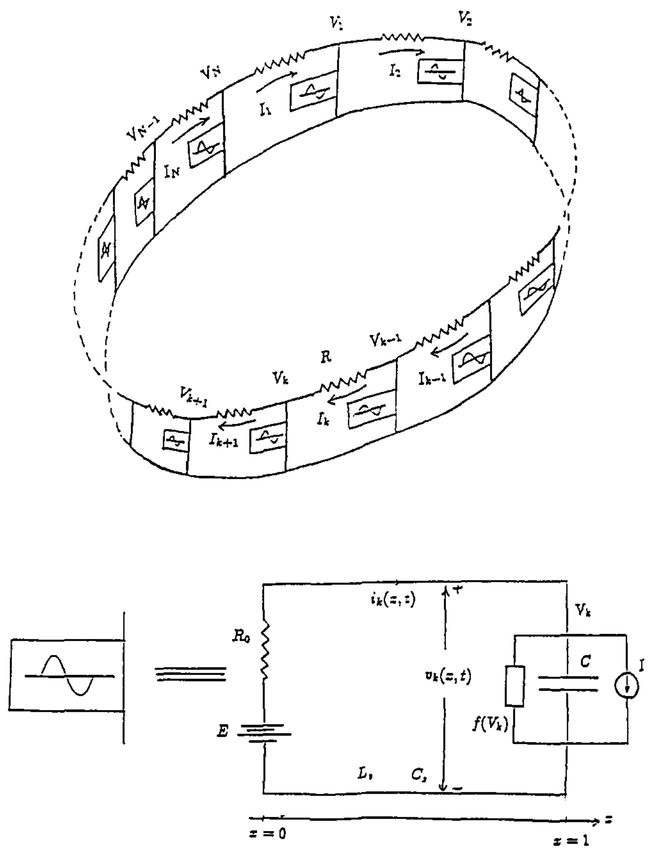

Fig. 1. A ring of mutually coupled lossless transmission lines interconnected by a common resistor.

where $k(\text{mod}\, N)$, $i_k(x, t)$, and $v_k(x, t)$ are the current flowing and the voltage across the kth line at time t and distance x down the line, E is the constant voltage, $f(v_k(1, t))$ is the current through the nonlinear resistor, I_k is the network current coupling term, and I is the constant current indicated in Fig. 1.

It is well known that the linear partial differential Eq. (3.1) has general solutions

$$\begin{cases} v_k(x, t) = \varphi_k(x - \sigma t) + \psi_k(x + \sigma t) \\ i_k(x, t) = \dfrac{1}{Z}\left[\varphi_k(x - \sigma t) - \psi_k(x + \sigma t)\right] \end{cases}$$

where

$$\sigma = \frac{1}{\sqrt{L_s C_s}}, \qquad Z = \sqrt{L_s / C_s}$$

and $\varphi_k \in C^1((-\infty, 1]; \mathbb{R})$, $\psi_k \in C^1([0, \infty); \mathbb{R})$. Substituting this into the first boundary condition in (3.1) gives

$$E = \varphi_k(-\sigma t) + \psi_k(\sigma t) + \frac{R_0}{Z}[\varphi_k(-\sigma t) - \psi_k(\sigma t)]$$

and hence

$$\varphi_k(-\sigma t) = \frac{Z}{Z + R_0} E - q\psi_k(\sigma t), \qquad q = \frac{Z - R_0}{R_0 + Z} \tag{3.2}$$

Replacing t by $t - 1/\sigma$ in (3.2), we obtain

$$\varphi_k(1 - \sigma t) = \frac{Z}{Z + R_0} E - q\psi_k(\sigma t - 1) \tag{3.3}$$

Let

$$u^k(t) = \psi_k(1 + \sigma t) + \frac{ZE}{2R_0}, \qquad \tau = \frac{2}{\sigma}$$

Then

$$\psi_k(\sigma t - 1) = u^k(t - \tau) - \frac{ZE}{2R_0}$$

This gives

$$\begin{cases} v_k(1, t) = \varphi_k(1 - \sigma t) + \psi_k(1 + \sigma t) = u^k(t) - qu^k(t - \tau) \\ i_k(1, t) = \dfrac{1}{Z}[\varphi_k(1 - \sigma t) - \psi_k(1 + \sigma t)] = -\dfrac{1}{Z} u^k(t) - \dfrac{q}{Z} u^k(t - \ ) + \dfrac{E}{R_0} \end{cases} \tag{3.4}$$

Define

$$D(q)\varphi = \varphi(0) - q\varphi(-\tau), \qquad \varphi \in C([-\tau, 0]; R)$$

The third boundary condition in (3.1) implies that

$$I_{k-1} - I_k = \frac{1}{R}[v_{k-1}(1, t) - 2v_k(1, t) + v_{k+1}(1, t)] \tag{3.5}$$

This, together with (3.4) and the second boundary condition in (3.1), gives

$$i_k(1,t)+\frac{1}{R}[v_{k-1}(1,t)-2v_k(1,t)+v_{k+1}(1,t)]$$

$$=C\frac{d}{dt}[u^k(t)-qu^k(t-\tau)]+f[u^k(t)-qu^k(t-\tau)]+I \qquad (3.6)$$

Therefore,

$$C\frac{d}{dt}D(q)\,u_t^k=-\frac{1}{Z}u^k(t)-\frac{q}{Z}u^k(t-\tau)-f(D(q)\,u_t^k)+\frac{E}{R_0}-I$$

$$+\frac{1}{R}D(q)[u_t^{k-1}-2u_t^k+u_t^{k+1}] \qquad (3.7)$$

Let $I=E/R_0$. Then we get

$$\frac{d}{dt}D(q)\,u_t^k=-\frac{1}{ZC}u^k(t)-\frac{q}{ZC}u^k(t-\tau)-\frac{1}{C}f(D(q)\,u_t^k)$$

$$+\frac{1}{RC}D(q)[u_t^{k-1}-2u_t^k+u_t^{k+1}], \qquad k(\text{mod } N) \qquad (3.8)$$

Denote by

$$R^*=NR, \qquad C^*=NC$$

We have

$$\frac{d}{dt}D(q)\,u_t^k=-\frac{1}{ZC}u^k(t)-\frac{q}{ZC}u^k(t-\tau)-\frac{1}{C}f(D(q)\,u_t^k)$$

$$+\frac{N^2}{R^*C^*}D(q)[u_t^{k-1}-2u_t^k+u_t^{k+1}] \qquad (3.9)$$

Taking the limit $N\to\infty$, we then formally obtain

$$\frac{\partial}{\partial t}D(q)\,u_t=-\frac{1}{ZC}u(t)-\frac{q}{ZC}u(t-\tau)-\frac{1}{C}f(D(q)\,u_t)$$

$$+\frac{1}{(2\pi)^2\,R^*C^*}\frac{\partial^2}{x^2}D(q)\,u_t \qquad (3.10)$$

where $x \in S^1$, and $u_t \in C([-\tau, 0]; C(S^1; R))$. In other words, we can regard (3.9) as a space discretization of (3.10), where the unit circle S^1 is parametrized by arc length and the spacing is $h = 2\pi/N$, $u^k(t) = u(t, kh)$, $h = 1, 2,..., N$.

Hale (1994) proved that if (3.10) has a stable hyperbolic rotating wave, then for sufficiently large N, (3.9) has a stable hyperbolic discrete wave, that is, a stable hyperbolic periodic orbit satisfying $u^{k-1}(t) = u^k(t - (m/N)p)$ for some $m(\text{mod } N)$, where p is the period.

To conclude this part, let us point out that the foregoing transformation from a linear hyperbolic partial differential equation subject to nonlinear boundary conditions to a system of neutral equations is not new and has been systematically studied. Also, single transmission lines have been extensively investigated and a qualitative theory for the related (usually scalar and ordinary) neutral functional differential equations has been developed. We refer to the listed references for details.

F. Analysis of Characteristic Equations. Throughout the remainder of this paper, we consider

$$\begin{aligned}\frac{\partial}{\partial t}[u(t, x) - qu(t-\tau, x)] &= d\frac{\partial^2}{\partial x^2}[u(t, x) - qu(t, x)] - au(t, x) \\ &\quad - aqu(t-\tau, x) - g[u(t, x) - qu(t-\tau, x)] \qquad (3.11)\end{aligned}$$

where $x \in S^1$, a, d, τ are positive constants, $g: R \to R$ is continuously differentiable with $g(0) = 0$, $q \in (0, 1)$ is the bifurcation parameter. Note that the range of q is $(0, 1)$, not the whole real line. But Theorem 2.1 can still be applied after an obvious treansformation between $(0, 1)$ and R (see Krawcewicz *et al.*, 1993).

Let

$$g'(0) = -\gamma, \qquad 0 < \gamma < a \qquad (3.12)$$

Then the characteristic equation of (3.11) takes the form

$$\lambda(1 - qe^{-\lambda\tau}) = -dk^2(1 - qe^{-\lambda\tau}) - a - aqe^{-\lambda\tau} + \gamma(1 - qe^{-\lambda\tau}), \qquad k \geq 1 \qquad (3.13)$$

That is,

$$(\lambda + dk^2 + a - \gamma)\, e^{\lambda\tau} - q(\lambda + dk^2 - \gamma - a) = 0 \qquad (3.14)$$

Let $\lambda = i\beta$ in (3.14), we get

$$\begin{cases} -(dk^2 + a - \gamma) \cos \beta\tau + \beta \sin \beta\tau = q(a + \gamma - dk^2) \\ \beta \cos \beta\tau + (dk^2 + a - \gamma) \sin \beta\tau = q\beta \end{cases}$$

or equivalently,

$$\begin{cases} \tan(\beta\tau) = \dfrac{2a\beta}{\beta^2 - (a + dk^2 - \gamma)(a - dk^2 + \gamma)} \\ q^2 = \dfrac{\beta^2 + (a - \gamma + dk^2)^2}{\beta^2 + (a + \gamma - dk^2)^2} \end{cases} \tag{3.15}$$

It is easy to show that for a real number $\beta > 0$, the second equation of (3.15) has a solution $q \in (0, 1)$ only if

$$dk^2 < \gamma \tag{3.16}$$

Therefore, there are only finitely many $k \geqslant 1$ so that (3.15) has a pair of purely imaginary solutions.

For each fix $k \geqslant 1$ so that $dk^2 < \gamma$, we can easily show graphically that there exists a sequence of positive numbers $\beta_{k,1} < \beta_{k,2} < \cdots$ so that the first equation of (3.15) is satisfied by $\beta_{k,j}$, $j = 1, 2, \ldots$. Substituting this $\beta_{k,j}$ into the second equation of (3.15) gives

$$q_{k,j} = \sqrt{\frac{\beta_{k,j}^2 + [a - (\gamma - dk^2)]^2}{\beta_{k,j}^2 + [a + (\gamma - dk^2)]^2}} \tag{3.17}$$

Therefore, we can conclude that the set $\{(q, p) \in (0, 1) \times (0, \infty)$; (3.14) has a solution $i(2\pi/p)m$ for some $m \geqslant 1\}$ is discrete.

Let $\lambda = \lambda(q)$ be a smooth curve of zeros of (3.14) so that $\lambda(q_{k,j}) = i\beta_{k,j}$. Differentiating (3.14) with respect to q, we get

$$\lambda'(q)\, e^{\lambda} + \tau(\lambda + dk^2 + a - \gamma)\, e^{\lambda\tau}\lambda'(q) = \lambda + dk^2 - \gamma - a + q\lambda'(q)$$

That is,

$$\lambda'(q) = \frac{\lambda + dk^2 - \gamma - a}{\tau(\lambda + dk^2 + a - \gamma)\, e^{\lambda\tau} + e^{\lambda\tau} - q}$$

Note that (3.14) gives

$$(\lambda + dk^2 + a - \gamma)\, e^{\lambda\tau} = q(\lambda + dk^2 - \gamma - a)$$

Therefore,

$$e^{\lambda\tau}=\frac{q(\lambda+dk^2-\gamma-a)}{\lambda+dk^2+a-\gamma}$$

and hence

$$\lambda'(q)=\frac{\lambda+dk^2-\gamma-a}{q\tau(\lambda+dk^2-\gamma-a)+\dfrac{q(\lambda+dk^2-\gamma-a)}{\lambda+dk^2+a-\gamma}-a}$$

This leads to

$$\begin{aligned}
&\text{Sign } Re\lambda'(q)|_{q=q_{k,j}}\\
&=\text{Sign } Re\left.\frac{1}{\lambda'(q)}\right|_{q=q_{k,j}}\\
&=\text{Sign } Re\left.\left\{q\tau+\frac{q}{\lambda+dk^2+a-\gamma}-\frac{q}{\lambda+dk^2-\gamma-a}\right\}\right|_{\substack{q=q_{k,j}\\ \lambda=i\beta_{k,j}}}\\
&=\text{Sign } Re\left\{\tau+\frac{1}{i\beta_{k,j}+dk^2+a-\gamma}-\frac{1}{i\beta_{k,j}+dk^2-\gamma-a}\right\}\\
&=\text{Sign}\left\{\tau+\frac{dk^2+a-\gamma}{(dk^2+a-\gamma)^2+\beta_{k,j}^2}-\frac{dk^2-\gamma-a}{(dk^2-\gamma-a)^2+\beta_{k,j}^2}\right\}\\
&=\text{Sign}\left\{\tau+\frac{2a\beta_{k,j}^2}{[(dk^2+a-\gamma)^2+\beta_{k,j}^2][(dk^2-\gamma-a)^2+\beta_{k,j}^2]}\right\}\\
&=1>0
\end{aligned}$$

From the remark following Theorem 2.1, we can see that this will be crucial to rule out bounded connected component of rotating waves of (3.11).

Let us summarize the above discussions for the sake of later reference.

Lemma 3.1. *Assume that (3.12) is satisfied. Then*

(i) The set $\{(q, p); i(2\pi/p)k$ is a solution of (3.14) for some $k\geqslant 1\}$ is discrete in $(0, 1)\times(0, \infty)$.

(ii) For each $k\geqslant 1$ with $dk^2<\gamma$, Eq. (3.14) has a sequence of purely imaginary solutions $\pm i\beta_{k,j}$, $0<\beta_{k,1}<\beta_{k,2}<\cdots$ for $q=q_{k,j}\in(0, 1)$ defined by (3.17).

(iii) If $\lambda(q)$ is a smooth curve of zeros of (3.14) with $\lambda(q_{k,j})=i\beta_{k,j}$, then $Re\,\lambda'(q_{k,j})>0$.

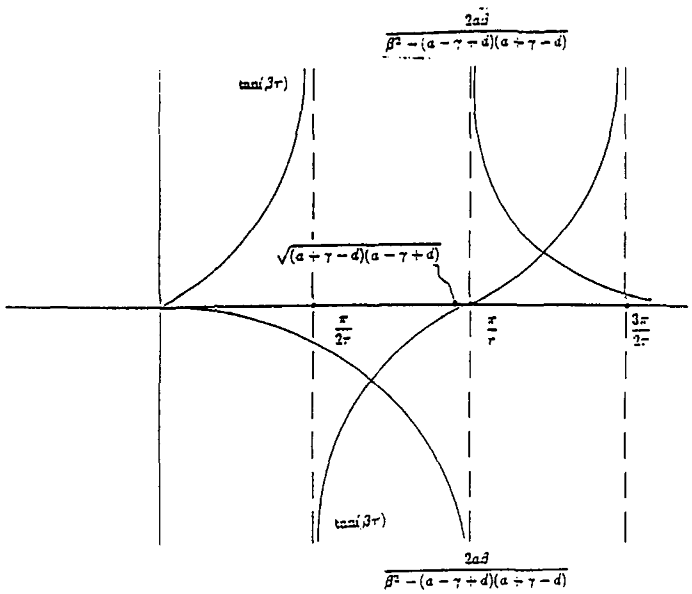

Figure 2

For the sake of later application, let us look at the location of $\beta_0 = \beta_{1,1}$. We assume that

$$0 < d < \gamma \tag{3.18}$$

Then β_0 is the first positive solution of

$$\tan(\beta\tau) = \frac{2a\beta}{\beta^2 - (a-\gamma+d)(a+\gamma-d)} \tag{3.19}$$

and hence $i\beta_0$ is a solution of (3.14) with $k = 1$ and

$$q_0 = q_{1,1} = \sqrt{\frac{\beta_0^2 + (a-\gamma+d)^2}{\beta_0^2 + (a+\gamma-d)^2}} \tag{3.20}$$

It can easily be shown graphically that (see Fig. 2).

Lemma 3.2. *If*

$$\frac{\pi}{2\tau} < \sqrt{(a+\gamma-d)(a-\gamma+d)} \tag{3.21}$$

Then $\pi/2\tau < \beta_0 < \sqrt{(a+\gamma-d)(a-\gamma+d)}$, *and hence*

$$\frac{2\pi}{\sqrt{(a+\gamma-d)(a-\gamma+d)}} < \frac{2\pi}{\beta_0} < 4\tau \tag{3.22}$$

In particular, if

$$\frac{\pi}{2\tau} < \sqrt{(a+\gamma-d)(a-\gamma+d)} < \frac{\pi}{\tau} \tag{3.23}$$

then

$$2\tau < \frac{2\pi}{\beta_0} < 4\tau \tag{3.24}$$

G. A-Priori Bounds for Rotating Waves. In order to apply the global bifurcation theorem to establish the global existence of rotating waves, we need to obtain a priori bounds for rotating waves.

Assume that $u(t, x)$ is a rotating wave of (3.11) satisfying (2.2)–(2.3). Let $[u(t_0, x_0) - qu(t_0 - \tau, x_0)]^2$ be the maximum value of $[u(t, x) - qu(t - \tau, x)]^2$ over $\times S^1$. Then

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}[u(t_0, x_0) - qu(t_0 - \tau, x_0)]^2 \\
&= 2[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \frac{\partial}{\partial t}[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \\
0 &= \frac{\partial}{\partial x}[u(t_0, x_0) - qu(t_0 - \tau, x_0)]^2 \\
&= 2[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \frac{\partial}{\partial x}[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \\
0 &\leqslant \frac{\partial^2}{\partial x^2}[u(t_0, x_0) - qu(t_0 - \tau, x_0)]^2 \\
&= 2\left\{\frac{\partial}{\partial x}[u(t_0, x_0) - qu(t_0 - \tau, x_0)]\right\}^2 \\
&\quad + 2[u(t_0, x_2) - qu(t_0 - \tau, x_0)] \frac{\partial^2}{\partial x^2}[u(t_0, x_0) - qu(t_0 - \tau, x_0)]
\end{aligned}$$

Without loss of generality, we may assume that $u(t_0, x_0) - qu(t_0 - \tau, x_0) \neq 0$. Therefore, from (3.11) it follows that

$$[u(t_0, x_0) - qu(t_0 - \tau, x_0)]\{-au(t_0, x_0) - aqu(t_0 - \tau, x_0) - g[u(t_0, x_0) - qu(t_0 - \tau, x_0)]\} \geqslant 0$$

That is,

$$\begin{aligned} -2aqu(t_0 - \tau, x_0)&[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \\ &\geqslant \{a[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \\ &\quad + g[u(t_0, x_0) - qu(t_0 - \tau, x_0)]\}[u(t_0, x_0) - qu(t_0 - \tau, x_0)] \end{aligned} \tag{3.25}$$

Note that

$$|u(t, x) - qu(t - \tau, x)| \leqslant |u(t_0, x_0) - qu(t_0 - \tau, x_0)|, \qquad t \in R, \quad x \in S^1$$

Therefore, we get

$$\begin{aligned} |u(t, x)| &\leqslant q\, |u(t - \tau, x)| + |u(t_0, x_0) - qu(t_0 - \tau, x_0)| \\ &\leqslant q^2\, |u(t - 2\tau, x)| + (1 + q)\, |u(t_0, x_0) - qu(t_0 - \tau, x_0)| \\ &\leqslant \cdots \\ &\leqslant q^m\, |(u(t - m\tau, x)| + (1 + q + \cdots + q^{m-1})\, |u(t_0, x_0) \\ &\quad - qu(t_0 - \tau, x_0)| \end{aligned}$$

Letting $m \to \infty$, we have

$$|u(t, x)| \leqslant \frac{1}{1 - q}\, |u(t_0, x_0) - qu(t_0 - \tau, x_0|, \qquad t \in R, \quad x \in S^1 \tag{3.26}$$

Therefore, by (3.25), we obtain

$$a + \frac{g[u(t_0, x_0) - qu(t_0 - \tau, x_0)]}{u(t_0, x_0) - qu(t_0 - \tau, x_0)} \leqslant \frac{2aq}{1 - q} \tag{3.27}$$

If we assume that

$$\lim_{z \to \infty} \frac{g(z)}{z} = \infty \tag{3.28}$$

then (3.27) implies the existence of $Q = Q(2aq/(1-q))$ so that

$$|u(t_0, x_0) - qu(t_0 - \tau, x_0)| \leqslant Q$$

and hence from (3.26) it follows that

$$|u(t, x)| \leqslant \frac{1}{1-q} Q\left(\frac{2aq}{1-q}\right), \qquad t \in R, \quad x \in S^1 \tag{3.29}$$

Summarizing the above discussion, we get the following.

Lemma 3.3. *If (3.28) is satisfied, then there exists a nondecreasing function $Q: (0, \infty) \to (0, \infty)$ such that any rotating wave $u(t, x)$ of (3.11) satisfies $|u(t, x)| \leqslant (1/(1-q))\, Q(2aq/(1-q))$ for $t \in R$ and $x \in S^1$. In particular, for any fixed $q^* \in (0, 1)$ the set of rotating waves of (3.11) corresponding to $q \in [0, q^*)$ is uniformly bounded in sup-norm.*

For a related result in the case of scalar ordinary functional differential equations of neutral type, we refer to Wu (1993).

H. Excluding 4τ-periodic Rotating Waves. The main purpose of this part is to exclude nontrivial 4τ-periodic rotating waves. Assume that $u(t, x)$ is a nontrivial rotating wave of (3.11) satisfying (2.2)–(2.3) with $p = 4\tau$. Then

$$u(t, 0) = u(t + 4\tau, 0)$$

$$u(t, x) = u\left(t - \frac{4\tau}{2\pi} x, 0\right) = u\left(t - \frac{2}{\pi} x, 0\right), \qquad t \in R, \quad x \in S^1$$

So, $v(t) = u(t, 0)$ satisfies

$$\begin{aligned} &\frac{d}{dt}[v(t) - qv(t-\tau)] \\ &\quad = \left(\frac{2\tau}{\pi}\right)^2 d \frac{d^2}{dt^2}[v(t) - qv(t-\tau)] \\ &\qquad - a[v(t) - qv(t-\tau)] - 2aqv(t-\tau) - g[v(t) - qv(t-\tau)] \end{aligned} \tag{3.30}$$

$t \in R$. Let

$$\begin{cases} x_1(t) = v(t) - qv(t-\tau) \\ x_2(t)) = v(t-\tau) - qv(t-2\tau) \\ x_3(t) = v(t-2\tau) - qv(t-3\tau) \\ x_4(t) = v(t-3\tau) - qv(t) \end{cases} \tag{3.31}$$

Then

$$\begin{pmatrix} v(t-\tau) \\ v(t-2\tau) \\ v(t-3\tau) \\ v(t) \end{pmatrix} = \frac{1}{1-q^4} B \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \tag{3.32}$$

where

$$B = \begin{pmatrix} q^3 & 1 & q & q^2 \\ q^2 & q^3 & 1 & q \\ q & q^2 & q^3 & 1 \\ 1 & q & q^2 & q^3 \end{pmatrix}$$

Substituting (3.31) and (3.32) into (3.30), we get

$$\dot{x}_i = \left(\frac{2\tau}{\pi}\right)^2 d\ddot{x}_i - ax_i - \frac{2aq}{1-q^4}(Bx)_i - g(x_i), \qquad 1 \leqslant i \leqslant 4$$

Its similarity to the Liénard equation suggests a transformation which leads to an equivalent system,

$$\begin{cases} \dot{x}_i = y_i + \left(\dfrac{\pi}{2\tau}\right)^2 \dfrac{1}{d} x_i \\ \dot{y}_i = \left(\dfrac{\pi}{2\tau}\right)^2 \dfrac{1}{d} \left[ax_i + \dfrac{2aq}{1-q^4}(Bx)_i + g(x_i) \right], \qquad 1 \leqslant i \leqslant 4 \end{cases} \tag{3.33}$$

and a related Liapunov function,

$$V = \sum_{i=1}^{4} \left[\frac{1}{2} y_i^2 - \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \int_0^{x_i} g(s)\, ds - ax_i y_i - \frac{2aq}{1-q^4} y_i (Bx)_i \right]$$

The derivative of V along solutions of (3.33) is given by

$$
\begin{aligned}
\dot{V} = & \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} \left[ax_i + \frac{2aq}{1-q^4}(Bx)_i + g(x_i)\right] y_i \\
& - \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} \left[y_i + \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} x_i\right] g(x_i) - a \sum_{i=1}^{4} y_i \left[y_i + \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} x_i\right] \\
& - a \sum_{i=1}^{4} \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \left[ax_i + \frac{2aq}{1-q^4}(Bx)_i + g(x_i)\right] x_i \\
& - \frac{2aq}{1-q^4} \sum_{i=1}^{4} \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \left[ax_i + \frac{2aq}{1-q^4}(Bx)_i + g(x_i)\right] (Bx)_i \\
& - \frac{2aq}{1-q^4} \sum_{i=1}^{4} y_i (By)_i - \frac{2aq}{1-q^4} \sum_{i=1}^{4} y_i \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} (Bx)_i \\
= & -a \sum_{i=1}^{4} y_i^2 - \frac{2aq}{1-q^4} \sum_{i=1}^{4} y_i (By)_i \\
& - \left[\left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d}\right]^2 \sum_{i=1}^{4} x_i g(x_i) - \frac{a}{d}\left(\frac{\pi}{2\tau}\right)^2 \sum_{i=1}^{4} x_i g(x_i) \\
& - \frac{a^2}{d} \sum_{i=1}^{4} \left(\frac{\pi}{2\tau}\right)^2 x_i^2 - \frac{4a^2 q}{1-q^4} \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} x_i (Bx)_i \\
& - \left(\frac{2aq}{1-q^4}\right)^2 \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} (Bx)_i (Bx)_i - \frac{2aq}{1-q^4} \left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} g(x_i)(Bx)_i
\end{aligned}
$$

We need the following.

Lemma 3.4.

$$
\sum_{i=1}^{4} z_i (Bz)_i \geqslant -(1-q)(1+q^2) \sum_{i=1}^{4} z_i^2, \qquad z_i \in R, \quad 1 \leqslant i \leqslant 4
$$

Proof. By using a result of Nussbaums (1985), we have $\sum_{i=1}^{4} z_i (Bz)_i \geqslant \Gamma \sum_{i=1}^{4} z_i^2$, $z_i \in R$, $1 \leqslant i \leqslant 4$, where

$$\Gamma = \min\left\{Re\sum_{j=1}^{4} a_j z^{j-1};\ z^4=1,\ a_1=q^3,\ a_2=1,\ a_3=q,\ a_4=q^2\right\}$$

$$= \min\{Re(q^3 + e^{i(2\pi/4)j} + qe^{i(4\pi/4)j} + q^2 e^{i(6\pi/4)j});\ j=0,1,2,3\}$$

$$= \min\{Re[e^{i(\pi/2)j}(1 + qe^{i(\pi/2)j} + q^2 e^{i(2\pi/2)j} + q^3 e^{i(3\pi/2)j})];\ j=0,1,2,3\}$$

$$= \min\left\{Re\left[e^{i(\pi/2)j}\left(\frac{1-q^4}{1-qe^{i(\pi/2)j}}\right)\right];\ j=0,1,2,3\right\}$$

$$= \min\left\{Re(1-q^4)\frac{\left[\cos\left(\frac{\pi}{2}j\right)+i\sin\left(\frac{\pi}{2}j\right)\right]\left[1-q\cos\left(\frac{\pi}{2}j\right)+iq\sin\left(\frac{\pi}{2}j\right)\right]}{\left[1-q\cos\left(\frac{\pi}{2}j\right)\right]^2+q^2\sin^2\left(\frac{\pi}{2}j\right)};\quad j=0,1,2,3\right\}$$

$$= \min\left\{\frac{(1-q^4)\left[\cos\left(\frac{\pi}{2}j\right)-q\right]}{1-2q\cos\left(\frac{\pi}{2}j\right)+q^2};\ j=0,1,2,3\right\}$$

$$= \min\left\{\frac{(1-q^4)(1-q)}{1-2q+q^2},\ \frac{(1-q^4)(-q)}{1+q^2},\ \frac{(1-q^4)(-1-q)}{1+2q+q^2}\right\}$$

$$= \min\{(1+q)(1+q^2),\ -q(1-q^2),\ -(1-q)(1+q^2)\}$$

$$= -(1-q)(1+q^2)$$

This completes the proof. □

We also need to compute the eigenvalues of B^TB. While this can be done directly, we present an approach which could be extended to general circular matrices.

Lemma 3.5. *The minimal eigenvalue of B^TB is $\lambda_{\min}(B^TB) = (1-q^4)^2/(1+q)^2$ and the maximal eigenvalue of B^TB is $\lambda_{\max}(B^TB) = (1-q^4)^2/(1-q)^2$.*

Proof. Let $v_j = (1, e^{i(\pi/2)j}, e^{i(2\pi/2)j}, e^{i(3\pi/2)j})$, $j = 0, 1, 2, 3$. It can be shown that v_j is an eigenvector of B corresponding to the eigenvalue

$$\alpha_j = e^{i(\pi/2)j}(1 + qe^{i(\pi/2)j} + q^2 e^{i(2\pi/2)j} + q^3 e^{i(3\pi/2)j}) = e^{i(\pi/2)j} \frac{1 - q^4}{1 - qe^{i(\pi/2)j}}$$

and an eigenvector of B^T corresponding to the eigenvalue

$$\beta_j = e^{-i(\pi/2)j}(1 + qe^{-i(\pi/2)j} + q^2 e^{-i(2\pi/2)j} + q^3 e^{-i(3\pi/2)j})$$
$$= e^{-i(\pi/2)j} \frac{1 - q^4}{1 - qe^{-i(\pi/2)j}}$$

Assume that $x \in \mathbb{C}^4$ is an eigenvector of $B^T B$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$. Then $x = a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3$ and $B^T Bx = \lambda x$ is equivalent to

$$\alpha_0 \beta_0 a_0 v_0 + \alpha_1 \beta_1 a_1 v_1 + \alpha_2 \beta_2 a_2 v_2 + \alpha_3 \beta_3 a_3 v_3 = \lambda(a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3)$$

from which it follows that $\lambda = \alpha_j \beta_j$ for some $j = 0, 1, 2, 3$. Therefore, all eigenvalues of $B^T B$ are given by

$$\left\{ \frac{(1 - q^4)^2}{(1 - qe^{i(\pi/2)j})(1 - qe^{-i(\pi/2)j})}; j = 0, 1, 2, 3 \right\}$$

from which the conclusion follows.

Lemma 3.6. *Assume that*

$$-K \leqslant \frac{g(x)}{x}, \qquad g(-x) = -g(x) \qquad \text{for} \quad x \neq 0 \tag{3.34}$$

$$\frac{g(x)}{x} \qquad \text{is nondecreasing in} \quad x \in (0, \infty) \tag{3.35}$$

Let $x_i(t)$, $i = 1, ..., 4$, *be given by (3.31). Then*

$$\left| \frac{g(x_i(t))}{x_i(t)} \right| \leqslant \max \left\{ K, \frac{a(3q - 1)}{1 - q} \right\} \tag{3.36}$$

Proof. Let $t_0 \in \mathbb{R}$ be given so that $[v(t_0) - qv(t_0-\tau)]^2 = \max_{t\in\mathbb{R}}[v(t) - qv(t-\tau)]^2$. Therefore, $|x_i(t)| \leqslant |v(t_0) - qv(t_0-\tau)|$ for $i = 1, ..., 4$, and $t \in \mathbb{R}$. From the same argument as that of (3.27) we have

$$\frac{g[v(t_0) - qv(t_0-\tau)]}{v(t_0) - qv(t_0-\tau)} \leqslant \frac{a(3q-1)}{1-q}$$

Therefore, under assumptions (3.34) and (3.35), we get

$$-K \leqslant \frac{g[x_i(t)]}{x_i(t)} \leqslant \frac{g[v(t_0) - qv(t_0-\tau)]}{v(t_0 - qv(t_0-\tau)} \leqslant \frac{a(3q-1)}{1-q}$$

from which (3.36) follows. □

We now return to the estimation of $\dot{V}$. Using Lemma 3.5, we get

$$\sum_{i=1}^{4} (Bx)_i (Bx)_i \geqslant \lambda_{\min}(B^T B) \sum_{i=1}^{4} x_i^2 = \frac{(1-q^4)^2}{(1+q)^2} \sum_{i=1}^{4} x_i^2$$

By Lemma 3.6, we have

$$\left| \sum_{i=1}^{4} g(x_i)(Bx)_i \right| \leqslant \sqrt{\sum_{i=1}^{4} g^2(x_i)} \sqrt{\sum_{i=1}^{4} x_i^2 \lambda_{\max}(B^T B)}$$
$$\leqslant \frac{(1-q^4)^2}{(1-q)^2} \max\left\{K, \frac{a(3q-1)}{1-q}\right\} \sum_{i=1}^{4} x_i^2$$

Therefore, using Lemma 3.4 we get

$$\dot{V} \leqslant -a \sum_{i=1}^{4} \left[1 - \frac{2q(1-q)(1+q^2)}{1-q^4}\right] y_i^2$$
$$-\left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \left\{\left(\frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2 + a\right) \sum_{i=1}^{4} x_i g(x_i)\right.$$
$$-\sum_{i=1}^{4} \frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2 \left\{a^2\left[1 - \frac{4q}{1-q^4}(1-q)(1+q^2)\right.\right.$$
$$\left.+\frac{4q^2}{(1-q^4)^2} \frac{(1-q^4)^2}{(1+q)^2}\right] x_i^2$$
$$\left.-\frac{2aq}{1-q^4} \frac{(1-q^4)^2}{(1-q)^2} \max\left\{K, \frac{a(3q-1)}{1-q}\right\} \sum_{i=1}^{4} x_i^2\right\}$$

$$\leqslant -a \sum_{i=1}^{4} \left(\frac{1-q}{1+q}\right) y_i^2$$

$$-\left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} \left\{\left[\frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2 + a\right] \inf_{x \neq 0} \frac{g(x)}{x} + a^2 \left(\frac{1-q}{1+q}\right)^2\right.$$

$$\left. - \frac{2aq(1+q)(1+q^2)}{1-q} \max\left\{K, \frac{a(3q-1)}{1-q}\right\}\right\} x_i^2$$

$$\leqslant -a \sum_{i=1}^{4} \left(\frac{1-q}{1+q}\right) y_i^2$$

$$-\left(\frac{\pi}{2\tau}\right)^2 \frac{1}{d} \sum_{i=1}^{4} \left[a^2 \left(\frac{1-q}{1+q}\right)^2 - \left(\frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2 + a\right) K\right.$$

$$\left. - \frac{2aq(1+q)(1+q^2)}{1-q} \max\left\{K, \frac{a(3q-1)}{1-q}\right\}\right] x_i^2$$

Consequently, if we assume that

$$0 \leqslant q < 1 - \delta \qquad \text{for some} \quad \delta \in (0, 1) \tag{3.37}$$

$$\frac{1}{4} a^2 \delta^2 > \left[\frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2 + a\right] K + \frac{8a(1-\delta) \max\left\{K, \frac{4\alpha}{\delta}\right\}}{\delta} \tag{3.38}$$

then $\dot{V}$ is a strictly negative function of $(x_1, ..., x_4, y_1, ..., y_4)$ unless $x_i = y_i = 0$ for $1 \leqslant i \leqslant 4$. Therefore, under assumptions (3.34), (3.35), (3.37), and (3.38), system (3.33) has no nontrivial periodic solution. This implies that system (3.11) has no nontrivial rotating wave of period 4τ. That is, we have proved the following

Lemma 3.7. *Under assumptions (3.34), (3.35), (3.37) and (3.38), the partial neutral functional differential equation (3.11) has no nontrivial 4π-periodic rotating wave for $q \in [0, 1-\delta)$.*

I. Global Existence of Rotating Waves. We can now state our main result.

Theorem 3.1. *Assume that*

(i) $g'(0) = -\gamma$, $d < \gamma < a$, $\pi/2 < \sqrt{(a+\gamma-d)(a-\gamma+d)}$.

(ii) $\text{ing}_{y \neq 0}\, g(y)/y > -a$, $\lim_{y \to \infty} g(y)/y = \infty$.

(*iii*) $g(-y) = -g(y)$ *for* $y \in \mathbb{R}$ *and* $g(y)/y$ *is nondecreasing in* $y \in (0, \infty)$.

(*iv*) *There exist constants* $\delta \in (0, 1)$ *and* $K \geqslant 0$ *so that* $-K \leqslant g(x)/x$ *for* $x \neq 0$, $\frac{1}{4}a^2\delta^2 > [(1/d)(\pi/2)^2 + a]\, K + (8a(1-\delta) \max\{K, 4a/\delta\})/\delta$, *and* $q_0 := \sqrt{(\beta_0^2 + (a-\gamma+d)^2)/(\beta_0^2 + (a+\gamma-d)^2)} < 1-\delta$, *where* β_0 *is the first solution in* $((\pi/2\tau), \sqrt{(a+\gamma-d)(a-\gamma+d)})$ *of the equation* $\tan(\beta\tau) = 2a\beta/(\beta^2 - (a-\gamma+d)(a+\gamma-d))$.

Then for each $q \in (q_0, 1-\delta)$, *system (3.11) has a rotating wave with a period less than 4. If, in addition, we assume*

(*iv*) $$\frac{\pi}{2} < \sqrt{(a+\gamma-d)(a-\gamma+d)} < \frac{\pi}{\tau}$$

then for each $q \in (q_0, 1-\delta)$, *system (3.11) has a slowly oscillating rotating wave, that is, a rotating wave with a period in* $(2\tau, 4\tau)$.

Proof. For a sufficiently small $\varepsilon > 0$, we reparametrize the system by

$$Q_\varepsilon(\alpha) = \frac{1-\delta}{\pi}\left[\arctan\alpha + \frac{\pi}{2}\right] - \varepsilon$$

Then $Q_\varepsilon(\alpha)$ is strictly increasing, $Q_\varepsilon(\alpha) \to -\varepsilon$ as $\alpha \to -\infty$ and $Q_\varepsilon(\alpha) \to 1-\delta-\varepsilon$ as $\alpha \to \infty$. So, $|Q_\varepsilon(\alpha)| \leqslant \max\{\varepsilon, 1-\delta-\varepsilon\} < 1$ for $\alpha \in R$. We consider the system

$$\begin{aligned}
&\frac{\partial}{\partial t}[u(t,x) - Q_\varepsilon(\alpha)\, u(t-\tau, x)] \\
&\quad = d\frac{\partial^2}{\partial x^2}[u(t,x) - Q_\varepsilon(\alpha)\, u(t-\tau,x)] - au(t,x) - aQ_\varepsilon(\alpha)\, u(t-\tau,x) \\
&\qquad - g[u(t,x) - Q_\varepsilon(\alpha)\, u(t-\tau,x)]
\end{aligned} \tag{3.39}$$

Comparing with (2.1), we have

$$f(\alpha, \varphi) = -a\varphi(0) - aQ_\varepsilon(\alpha)\, \varphi(-\tau) - g[\varphi(0) - Q_\varepsilon(\alpha)\, \varphi(-\tau)]$$

$$f_\varphi(\alpha, 0)|_{C^G_{2\pi}} = -a[1 + Q_\varepsilon(\alpha)] + \gamma[1 - Q_\varepsilon(\alpha)] \neq 0$$

by using assumption (i). Moreover, it is easy to verify that assumption (ii) implies that if φ is a constant mapping and $\alpha \in R$ is given so that $f(\alpha, \varphi) = 0$, then $\varphi = 0$.

By Lemma 3.1, for $\alpha_0 = Q_\varepsilon^{-1}(q_0)$, $i\beta_0$ is a purely imaginary solution of the characteristic equation and

$$\lim_{p \to 0^+} \deg_B \left(\Delta_1(\cdot, \cdot), B_p \left(\alpha_0, \frac{2\pi}{\beta_0} \right) \right) = -1$$

[recall formula (2.14)]. Therefore, there exists a nonempty connected component C_1 containing $(\alpha_0, 2\pi/\beta_0, 0)$ of the set

$$S = cl\{(\alpha, p, u);\ u \text{ is a } p\text{-periodic rotating wave of } (3.11) \text{ satisfying } (3.2)\text{–}(3.3)\}$$

in $R \times [0, \infty) \times C(R \times S^1; R)$.

Again, by using Lemma 3.1 and Theorem 2.1, C_1 must be unbounded (recall the remark at the end of last section).

Assume, by way of contradiction, that the conclusion of the theorem is false. Then there exists $\varepsilon > 0$ so that the projection $\pi_\alpha C_1$ of C_1 onto the α-axis is contained in $(-\infty, 1-\delta-\varepsilon)$. On the other hand, $Q_\varepsilon^{-1}(0) \notin \pi_\alpha C_1$ as system (3.11) at $q = 0$ has no nontrivial rotating wave and its related characteristic equation has no purely imaginary solution. Therefore, $\pi_\alpha C_1 \subseteq [0, 1-\delta-\varepsilon)$.

Therefore, we know from Lemma 3.3 that assumptions (ii) and (iii) guarantee the boundedness of $\pi_u C_1$, the projection of C_1 onto the space $C(R \times S^1; R)$.

On the other hand, condition (i) and Lemma 3.2 imply that $\pi_p C_1$ has nonempty intersection with $(4\pi/m, 4\tau)$, where m is a sufficiently large integer. Lemma 3.7 implies that, under assumption (iii), $\pi_p C_1$ is actually contained in $(4\tau/m, 4\tau)$ as it cannot touch the boundary of $(4\tau/m, 4\tau)$. Therefore, $\pi_p C_1 \subseteq (4\tau/m, 4\tau)$.

Summarizing the above discussion, we have the boundedness of $\pi_\alpha C_1$, $\pi_p C_1$, and $\pi_u C_1$ and hence C_1 must be bounded. This contradicts to the established unboundedness of C_1.

In the case where (iv) is satisfied, we can take $m = 2$ in the above argument and hence establish the required global existence of slowly oscillatory rotating waves. This completes the proof. □

ACKNOWLEDGMENTS

J. Wu's research was partially supported by Natural Sciences and Engineering Research of Canada and by a Faculty of Arts (York University) Research Grant.

REFERENCES

Abolinia, V. E., and Mishkis, A. D. (1960). Mixed problems for quasilinear hyperbolic systems in the plane. *Mat. Sb.* **50**, 423–442.

Brayton, R. K. (1966). Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type. *Q. Appl. Math.* **24**, 215–224.

Brayton, R. K., and Moser, J. K. (1964). A theory of nonlinear networks. *Q. Appl. Math.* **24**, 215–224.

Cooke, K. L., and Krumme, D. W. (1968). Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations. *J. Math. Anal. Appl.* **24**, 372–387.

Cruz, M. A., and Hale, J. K. (1979). Stability of functional differential equations of neutral type. *J. Diff. Eqs.* **7**, 334–355.

Erbe, L., Geba, K., Krawcewicz, W., and Wu, J. (1992). S^1-degree and global Hopf bifurcation theory in functional differential equations. *J. Diff. Eqs.* **98**, 277–298.

Golubitsky, M., Stewart, I., and Schaeffer, D. G. (1988). *Singularities and Groups in Bifurcation Theory, Vol. 2*, Springer-Verlag, New York.

Hale, J. K. (1977). *Theory of Functional Differential Equations*, Springer-Verlag, New York.

Hale, J. K. (1994), Partial neutral functional differential equations. *Rev. Roumaine Mat. Pures Appl.* **39**, 39–344.

Hale, J. K., and Lunel, S. (1993). *Introduction to Functional Differential Equations*, Springer-Verlag, New York.

Hale, J. K., Lin, X. B., and Raugel, G. (1988). Upper semicontinuity of attractors for approximations of semigroups and partial differential equations. *Math. Comp.* **50**, 89–123.

Ize, J. (1993). Topological bifurcations, preprint.

Krawcewicz, W., Xia, H., and Wu. J. (1993). S^1-equivariant degree and global bifurcation theory for condensing fields and neutral equations. *Can. Appl. Math. Q.* **1**, 167–220.

Lopes, O. (1975). Forced oscillations in nonlinear neutral differential equations. *SIAM J. Appl. Math.* **29**, 196–201.

Lopes, O. (1976). Stability and forced oscillations. *J. Math. Anal. Appl.* **55**, 686–698.

Nagumo, J. and Shimura, M. (1961). Self-oscillations in a transmission line with a tunnel diode. *Proc. IRE* **49**, 1281–1291.

Nussbaum, R. D. (1985). Circulant matrices and differential-delay equation. *J. Diff. Eqs.* **60**, 201–217.

Slemrod, M. (1971). Nonexistence of oscillations in a distributed network. *J. Math. Anal. Appl.* **36**, 22–40.

Wu, J. (1993). Global continua of periodic solutions to some difference-differential equations of neutral type. *Tohoku Math. J.* **45**, 67–88.

Wu, J., and Xia, H. (1996). Self-sustained oscillations in a ring array of coupled lossless transmission lines. *J. Diff. Eqs.* **124**, 247–278.