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The Structure of an Attracting Set Defined by Delayed and Monotone Positive Feedback

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In the monograph [21] we study the shape and smoothness properties of an attracting invariant set for a class of delay differential equations

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1}$$

with parameter $\mu \geq 0$ and C^1-smooth nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(0) = 0 \quad \text{and} \quad f'(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}.$$

The present note surveys the results from [21] and explains the major steps towards these results.

Let us first point out that equation (1) models a scalar system governed by instantaneous damping (in case $\mu > 0$) and delayed positive feedback, with at least one rest point given by $\xi = 0$. This is rather general. Particular cases are used to describe, for example, the voltage in single, self-excitatory neurons with graded delayed response. See [16,29] where equation (1) occurs with

$$f(\xi) = f_{\alpha\beta}(\xi) = \alpha \tanh(\beta\xi), \qquad \alpha > 0 \text{ and } \beta > 0.$$

Much applied work has been devoted to models for networks of neurons. Equation (1) also determines a part of the behaviour of such systems. We mention two examples. The system

$$C\dot{x}_i(t) = -\frac{1}{R}x_i(t) + \sum_{j=1}^{n} T_{ij} f_{\alpha\beta}(x_j(t-\tau))$$

with $= 1, \ldots, n$ was used as a model for a network of n identical saturating amplifiers (or neurons) with delayed outputs which are coupled by a resistive interconnection matrix $T = (T_{ij})$. Here, x_i represents the voltage input at the ith neuron which is characterized by input capacitance C, parallel resistance R and activation function $f_{\alpha\beta}$ as in the previous model for single neurons. If the coefficients T_{ij} of the interconnection matrix satisfy

$$\sum_{j=1}^{n} T_{ij} = \sum_{j=1}^{n} T_{kj} \qquad \text{for all } i, k \text{ in } \{1, \ldots, n\},$$

then the system has synchronized solutions. These are solutions satisfying $x_1(t) = x_2(t) = \ldots = x_n(t)$ for all t. Clearly, synchronized solutions are completely characterized by the scalar equation (1) with certain μ and f. The system above without delay ($\tau = 0$) was proposed by HOPFIELD [17,18], a similar model is also due to COHEN and GROSSBERG [11]. MARCUS and WESTERVELT [26] incorporated the time delay in order to account for the finite switching speed of the amplifiers. For studies of the dynamics of such a system, see the work of BÉLAIR [6], BÉLAIR, CAMPBELL and VAN DEN DRIESSCHE [7], BÉLAIR and DUFOUR [8], HERZ [16], OLIEN and BÉLAIR [28], WU [42] and references therein. The system of the $2n$ equations

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} T_{ij} f_{\alpha\beta}(x_j(t - \tau_{ij}) - s_j(t - \tau_{ij})),$$
$$\dot{s}_i(t) = -\mu_s s_i(t) + T f_{\alpha\beta}(x_i(t - \tau) - s_i(t - \tau))$$

with $i = 1, \ldots, n$ describes a network of neurons with internal dynamic threshold [5]. Here, s_i is the threshold in the ith neuron, and it is assumed that the rate of change of the voltage input $x_i(t)$ at the ith neuron does not depend on the past states $x_j(t - \tau_{ij})$ but on their distances from the thresholds $s_j(t - \tau_{ij})$. Again, if one looks for a single neuron or if one looks for the synchronized activities and if $\sum_{j=1}^{n} T_{ij} > T$, $\mu_s = \mu_i$, and $\tau = \tau_{ij}$ for all i and j, then $x_i - s_i$ satisfies equation (1) with positive feedback. See also [3,4,14,19,20,31,32] for problems with delayed monotone positive feedback.

We return to the study of equation (1). Every element ϕ of the Banach space C of continuous real functions on the initial interval $[-1, 0]$ determines a solution $x^{\phi} : [-1, \infty) \to \mathbb{R}$ of equation (1), i.e., a continuous function which is differentiable on $(0, \infty)$ and satisfies equation (1) for all $t > 0$. The relations

$$F(t, \phi) = x_t,\ x = x^{\phi},\ x_t(s) = x(t + s),\ s \in [-1, 0]$$

define a continuous semiflow $F : \mathbb{R}^+ \times C \to C$. All maps $F(t, \cdot)$, $t \geq 0$, are injective and continuously differentiable, and F is monotone with respect to the pointwise ordering on C. The positively invariant set

$$S = \{\phi \in C : (x^{\phi})^{-1}(0) \text{ is unbounded}\}$$

separates the domain of absorption into the interior of the positively invariant cone

$$K = \{\phi \in C : \phi(s) \geq 0 \text{ for all } s \in [-1,0]\}$$

from the domain of absorption into the interior of $-K$. One of the first results in [21] is that the separatrix S is a Lipschitz graph over a closed hyperplane in C. So, we can speak of the parts of C above and below S.

We are interested in a detailed description of the long term behaviour of solutions of equation (1). A natural object to study would be the global attractor of the semiflow, i.e., a compact set $A \subset C$ which is invariant in the sense that $F(t,A) = A$ for all $t \geq 0$ and which attracts every bounded set $B \subset C$ in the sense that for every open set U containing A there exists $T \geq 0$ with $F([T,\infty) \times B) \subset U$ (compare HALE [15]). It is not difficult to show that in case $\mu > 0$ and f bounded the semiflow has a global attractor. However, in case $\mu = 0$ every solution on $[-1,\infty)$ of equation (1) with an initial datum in the interior of $K \cup (-K)$ is unbounded, and thus a global attractor does not exist. So we have to look for substitutes which are present in all the cases we are interested in. These are the closure of the unstable set

$$W^u = \{\phi \in C : \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of equation (1)} \\ \text{with } x_0 = \phi \text{ and } x_t \to 0 \text{ as } t \to -\infty\}$$

of the stationary point $0 \in C$, and subsets thereof. Notice that in case a global attractor A exists, necessarily $\overline{W^u} \subset A$ since for every ball B centered at 0 and for every $t \geq 0$, $W^u \subset F([t,\infty) \times (W^u \cap B))$, which implies that W^u is contained in every neighbourhood of the compact set A, yielding $W^u \subset \overline{A} = A$ and $\overline{W^u} \subset A$.

In order to introduce our further hypotheses on μ and f and the subsets of $\overline{W^u}$ studied in [21] we linearize equation (1) at the stationary point $0 \in C$. The derivatives $D_2F(t,0)$, $t \geq 0$, form a strongly continuous semigroup and satisfy

$$D_2F(t,0)\phi = y_t^\phi,$$

with the solution $y^\phi : [-1,\infty) \to \mathbb{R}$ of the linearized equation

$$\dot{y}(t) = -\mu y(t) + f'(0)y(t-1) \tag{2}$$

given by $y_0^\phi = \phi$. The spectrum σ of the generator of the semigroup consists of simple eigenvalues which coincide with the zeros of the characteristic function

$$\mathbb{C} \ni \lambda \mapsto \lambda + \mu - f'(0)e^{-\lambda} \in \mathbb{C}.$$

There is one real eigenvalue λ_0. The other eigenvalues form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})$ with

$$\text{Re}\lambda_{j+1} < \text{Re}\lambda_j < \lambda_0 \quad \text{and} \quad (2j-1)\pi < \text{Im}\lambda_j < 2j\pi$$

for all integers $j \geq 1$, and $\text{Re}\lambda_j \to -\infty$ as $j \to \infty$. The number of eigenvalues in the open right halfplane depends on μ and $f'(0)$. Any odd integer $2j+1$,

$j \in \mathbb{N}$, can be achieved. Let P, L, and Q denote the reellified generalized eigenspaces of the generator associated with the spectral sets $\{\lambda_0\}$, $\{\lambda_1, \overline{\lambda_1}\}$, and $\sigma \setminus \{\lambda_0, \lambda_1, \overline{\lambda_1}\}$, respectively.

In the simplest case where $\lambda_0 < 0$, $W^u = \{0\}$. If $\text{Re}\lambda_1 < 0 < \lambda_0$ then one can show that W^u consists of 0 and the segments of two solutions on $\mathbb{R}$, one being positive and the other negative. Interesting structures of W^u appear only when W^u is higher-dimensional. We make an assumption on μ and $f'(0)$ which is equivalent to

$$0 < \text{Re}\lambda_1.$$

Then the topological dimension of the set W^u is at least 3, and there is a 3-dimensional C^1-submanifold $W_{loc} \subset W^u$ of C which is tangent at 0 to the reellified eigenspace $L \oplus P$ of the spectral set $\{\lambda_0, \lambda_1, \overline{\lambda_1}\}$ and has the property that for every $\phi \in W_{loc}$ there exist a solution $x : \mathbb{R} \to \mathbb{R}$ and $T \in \mathbb{R}$ such that $x_0 = \phi$ and $x_s \in W_{loc}$ for all $s \leq T$. The forward extension

$$W = F(\mathbb{R}^+ \times W_{loc})$$

of W_{loc} is an invariant subset of W^u. If $\text{Re}\lambda_2 < 0$ then $W = W^u$. Moreover, in some special cases the set $\overline{W}$ should be the global attractor. We suspect that this is true, for example, if $\mu > 0$, $\text{Re}\lambda_2 < 0$, and if f is bounded with f' being strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$. Compare Walther [39] where a similar situation is studied for equation (1) with negative feedback.

We investigate the structure of $\overline{W}$ under mild additional assumptions on μ and f which are always satisfied by the models for neural networks mentioned before. We require that either

$$\mu = 0 \quad \text{and} \quad -\infty < \inf f \quad \text{or} \quad \sup f < \infty$$

or

$$\mu > 0 \quad \text{and} \quad \frac{f(\xi)}{\xi} < \mu \quad \text{for } \xi \text{ outside a bounded neighbourhood of 0.}$$

The last property and the inequality $\mu < f'(0)$, which follows from $\text{Re}\lambda_1 > 0$, together imply that in case $\mu > 0$ there are a largest negative zero ξ^- of $f - \mu\text{id}$ and a smallest positive zero ξ^+ of $f - \mu\text{id}$, and

$$f'(\xi^-) \leq \mu, \quad f'(\xi^+) \leq \mu.$$

The final assumption is that in case $\mu > 0$

$$f'(\xi^-) < \mu \quad \text{and} \quad f'(\xi^+) < \mu.$$

The main results in [21] concerning $\overline{W}$ begin with the relatively easy facts that $\overline{W}$ is invariant, and that the semiflow defines a continuous flow $F_W : \mathbb{R} \times \overline{W} \to \overline{W}$. For $\overline{W}$ and for the part $\overline{W} \cap S$ of the separatrix S in $\overline{W}$, we then obtain graph representations as follows:

There exist subspaces $G_2 \subset G_3$ of C of dimensions 2 and 3, respectively, a complementary space G_1 of G_2 in G_3, a closed complementary space E of G_3 in C, a compact set $D_S \subset G_2$ and a closed set $D_W \subset G_3$, and continuous mappings $w : D_W \to E$ and $w_S : D_S \to G_1 \oplus E$ such that

$$\overline{W} = \{\chi + w(\chi) : \chi \in D_W\}, \quad \overline{W} \cap S = \{\chi + w_S(\chi) : \chi \in D_S\}.$$

We have

$$D_W = \partial D_W \cup \overset{\circ}{D}_W, \quad W = \{\chi + w(\chi) : \chi \in \overset{\circ}{D}_W\},$$

and the restriction of w to $\overset{\circ}{D}_W$ is C^1-smooth. The restriction of F_W to $\mathbb{R} \times W$ is C^1-smooth. The domain D_S is homeomorphic to the closed unit disk in $\mathbb{R}^2$, and consists of the trace of a simple closed C^1-curve and its interior. The map w_S is C^1-smooth in the sense that $w_S|_{\overset{\circ}{D}_S}$ is C^1-smooth, and for each $\chi \in \partial D_S$ there is an open neighbourhood N of χ in G_2 so that $w_S|_{N \cap D_S}$ extends to a C^1-map on N.

Concerning the dynamics we have that

the set

$$\mathcal{O} = \overline{W} \cap S \setminus (W \cap S) = \{\chi + w_S(\chi) : \chi \in \partial D_S\}$$

is a periodic orbit, and there is no other periodic orbit in $\overline{W}$. The open annulus $(W \cap S) \setminus \{0\}$ consists of heteroclinic connections from 0 to $\mathcal{O}$. For every $\phi \in W$, $F_W(t, \phi) \to 0$ as $t \to -\infty$.

The further properties of $\overline{W}$ are different in the cases $\mu > 0$ and $\mu = 0$.

For $\mu > 0$, $\overline{W}$ is compact and contains the stationary points ξ_- and ξ_+ in C given by the values ξ^- and ξ^+, respectively. For every $\phi \in \overline{W}$,

$$\xi_- \leq \phi \leq \xi_+.$$

There exist homeomorphisms from $\overline{W}$ and D_W onto the closed unit ball in $\mathbb{R}^3$, which send

$$\mathrm{bd}W = \overline{W} \setminus W = \{\chi + w(\chi) : \chi \in \partial D_W\} \quad \textit{and} \quad \partial D_W$$

onto the unit sphere $S^2 \subset \mathbb{R}^3$. If we define χ_- and χ_+ by $\xi_- = \chi_- + w(\chi_-)$ and $\xi_+ = \chi_+ + w(\chi_+)$, respectively, then the set $\partial D_W \setminus \{\chi_-, \chi_+\}$ is a 2-dimensional C^1-submanifold of G_3, and the restriction of w to $D_W \setminus \{\chi_-, \chi_+\}$ is C^1-smooth (in the sense explained above for w_S). The points $\phi \in \overline{W} \setminus \overline{W} \cap S$ above the separatrix S form a connected set and satisfy $F_W(t, \phi) \to \xi_+$ as $t \to \infty$, and all $\phi \in \overline{W} \setminus \overline{W} \cap S$ below the separatrix S form a connected set and satisfy $F_W(t, \phi) \to \xi_-$ as $t \to \infty$. Finally, for every ϕ in the set bd$W$ *and different from ξ_- and ξ_+, $F_W(t, \phi) \to \mathcal{O}$ as $t \to -\infty$.*

Combining all the results stated so far, one may visualize the invariant set $\overline{W}$ in case $\mu > 0$ as a smooth solid spindle which is split by an invariant disk into the basins of attraction towards the tips ξ_- and ξ_+.

For $\mu = 0$, the sets $\overline{W}$ and D_W are unbounded. There exist homeomorphisms from $\overline{W}$ and D_W onto the solid cylinder $\{z \in \mathbb{R}^3 : z_1^2 + z_2^2 \le 1\}$ which send

$$\text{bd}W = \{\chi + w(\chi) : \chi \in \partial D_W\} \quad \textit{and} \quad \partial D_W$$

onto the cylinder $S^1 \times \mathbb{R}$. The boundary ∂D_W is a 2-dimensional C^1-submanifold of G_3, and w is C^1-smooth. The points $\phi \in \overline{W} \setminus \overline{W} \cap S$ above the separatrix S form a connected set and satisfy $x^\phi(t) \to \infty$ as $t \to \infty$, and the points $\phi \in \overline{W} \setminus \overline{W} \cap S$ below S form a connected set and satisfy $x^\phi(t) \to -\infty$ as $t \to \infty$. Finally, for every $\phi \in \text{bd}W$, $F_W(t, \phi) \to \mathcal{O}$ as $t \to -\infty$.

The first steps toward these results exploit the monotonicity of the semiflow. Among others we obtain that in case $\mu > 0$ the set $\overline{W}$ is contained in the order interval between the stationary points ξ_- and ξ_+, and that there are heteroclinic connections from 0 to ξ_- and ξ_+, given by monotone solutions $x : \mathbb{R} \to \mathbb{R}$ without zeros.

An important tool for the investigation of finer structures is a version of the discrete Lyapunov functional $V : C \setminus \{0\} \to \mathbb{N}$ counting sign changes, which was introduced by Mallet-Paret [23]. Related are a-priori estimates for the growth and decay of solutions with segments in (sub-)level sets of V, which go back to [23] and in a special case to [35,36]. These tools are first used to characterize the invariant sets $W \setminus 0$ and $W \cap S \setminus 0$ as the sets of segments x_t of solutions $x : \mathbb{R} \to \mathbb{R}$ with α-limit set $\{0\}$ which satisfy $V(x_t) \le 2$ for all $t \in \mathbb{R}$ and $V(x_t) = 2$ for all $t \in \mathbb{R}$, respectively. Moreover, nontrivial differences of segments in $\overline{W}$ and $\overline{W} \cap S$ belong to $V^{-1}(\{0, 2\})$ and $V^{-1}(2)$, respectively. The last facts permit to introduce global coordinates on $\overline{W}$ and $\overline{W} \cap S$: It is not difficult to show that the continuous linear evaluation map

$$\Pi_2 : C \ni \phi \mapsto \big(\phi(0), \phi(-1)\big)^{tr} \in \mathbb{R}^2$$

is injective on $\overline{W} \cap S$, and the continuous linear evaluation map $\Pi_3 : C \to \mathbb{R}^3$ given by

$$\Pi_3 \phi = \big(\phi(0), \phi(-1), c_P(\phi)\big)^{tr}$$

and

$$\big(\text{Pr}_P \phi\big)(t) = \frac{1}{1 + f'(0)e^{-\lambda_0}} c_P(\phi) e^{\lambda_0 t},$$

where Pr_P is the spectral projection onto P along $Q \oplus L$, is injective on $\overline{W}$. The inverse maps of the restrictions of Π_2 and Π_3 to $\overline{W} \cap S$ and $\overline{W}$, respectively, turn out to be locally Lipschitz continuous.

The next step leads to the desired graph representation. Guided by results in [40] on negative feedback equations it seems natural to expect maps from a subset of $L \oplus P = T_0 W_{loc}$ into Q to represent $\overline{W}$, and from a subset of

$L \subset V^{-1}(2) \cup \{0\}$ into $Q \oplus P$ to represent $\overline{W} \cap S$. We mention here that earlier Y. Ammar succeeded to write the set W in case $\mu = 0$ as a C^1-graph over an open set in $L \oplus P$ [1]. On the other hand, our attempts to show that $\overline{W} \cap S$ is given by a map from a subset of L into $Q \oplus P$, for all μ and f considered, were not successful. So we give up the decomposition

$$C = Q \oplus L \oplus P$$

as a framework for graph representations, and embed $\mathbb{R}^3 \supset \Pi_3 \overline{W}$ and $\mathbb{R}^2 \supset \Pi_2(\overline{W} \cap S)$ in a simple way as subspaces $G_3 \supset G_2$ into C, so that representations by maps w and w_S with domains in G_3 and G_2 and ranges in complements E of G_3 in C and $E \oplus G_1$ of G_2 in C, $G_1 \subset G_3$, become obvious. It is not hard to deduce that W is given by the restriction of w to an open set, and that this restriction is C^1-smooth. On $\overline{W}$, the semiflow extends to a flow $F_W : \mathbb{R} \times \overline{W} \to \overline{W}$, and F_W is C^1-smooth on the C^1-manifold $\mathbb{R} \times W$.

Phase plane techniques apply to the coordinate curves in $\mathbb{R}^2$ (or G_2) of the flowlines $t \mapsto F_W(t, \phi)$ in the invariant set $\overline{W} \cap S$, and yield the periodic orbit

$$\mathcal{O} = (\overline{W} \cap S) \setminus (W \cap S)$$

as well as the identification

$$\Pi_2(W \cap S) = \text{int}(\Pi_2 \mathcal{O}),$$

which implies that also $W \cap S$ is given by the restriction of w_S to an open subset of G_2.

The investigation of the smoothness of the part $W \cap S$ of the separatrix S in W and of the manifold boundary

$$\text{bd}W = \overline{W} \setminus W$$

starts with a study of the stability of the periodic orbit $\mathcal{O}$. We use the fact that there is a heteroclinic flowline in $W \cap S$ from the stationary point $0 \in C$ to the orbit $\mathcal{O}$, i.e., in the level set $V^{-1}(2)$, in order to show that precisely one Floquet multiplier lies outside the unit circle. It also follows that the center space of the linearized period map, or monodromy operator

$$M = D_2 F(\omega, p_0),$$

given by $p_0 \in \mathcal{O}$ and the minimal period $\omega > 0$, is at most 2-dimensional. The study of the linearized stability is closely related to earlier work in [22] and to a-priori results on Floquet multipliers and eigenspaces for general monotone cyclic feedback systems with delay due to Mallet-Paret and Sell [24].

A first idea to show that the graph $W \cap S \subset V^{-1}(2) \cup \{0\}$ is C^1-smooth might be to look in the family of 2-dimensional locally invariant C^1-submanifolds with tangent space L at the stationary point $0 \in C$ for a member formed by heteroclinics connecting 0 with the periodic orbit $\mathcal{O}$. Our approach is quite different. We identify pieces of $W \cap S$ in a hyperplane Y transversal to $\mathcal{O}$, as open sets in the smooth transversal intersection of W with the center-stable manifold of the Poincaré return map which is associated with Y and a point $p_0 \in \mathcal{O}$. Then we use the C^1-flow F_W to obtain the smoothness of the set $W \cap S \setminus \{0\}$. (Smoothness close to 0 and the relation

$$T_0(W \cap S) = L$$

follow by other arguments.) Of course, this approach relies on the existence of C^1-smooth center-stable manifolds at fixed points of C^1-maps in Banach spaces. The general form of the result we need seems not available elsewhere in the literature, despite the large amount of work devoted to center manifolds. So in [21] we formulate and prove in detail the following easily applicable theorem on existence and smoothness of center-stable manifolds for C^1-maps.

Let $g : U \to E$ be a C^1-map on an open subset U of a Banach space E over $\mathbb{R}$, with a fixed point p. Let $L = Dg(p)$ and assume that E has a decomposition

$$E = E_s \oplus E_c \oplus E_u$$

into a closed subspace $E_s \neq \{0\}$ and finite-dimensional subspaces $E_c \neq \{0\}$ and $E_u \neq \{0\}$ such that $L(E_s) \subset E_s$, $L(E_c) \subset E_c$ and $L(E_u) \subset E_u$, and that the spectra of the induced maps

$$E_s \ni x \mapsto Lx \in E_s, \quad E_c \ni x \mapsto Lx \in E_c, \quad E_u \ni x \mapsto Lx \in E_u$$

are contained in a compact subset of $\{z \in \mathbb{C} : |z| < 1\}$, in $\{z \in \mathbb{C} : |z| = 1\}$ and in $\{z \in \mathbb{C} : |z| > 1\}$, respectively.

Then there exist convex open bounded neighbourhoods N_{sc} of 0 in $E_s \oplus E_c$, N_u of 0 in E_u, N of p in U, and a C^1-map

$$w : N_{sc} \to E_u$$

with $w(0) = 0$, $Dw(0) = 0$ and $w(N_{sc}) \subset N_u$ so that the shifted graph

$$W = p + \{z + w(z) : z \in N_{sc}\}$$

satisfies

$$g(W \cap N) \subset W$$

and

$$\bigcap_{n=0}^{\infty} g^{-n}(p + N_{sc} + N_u) \subset W.$$

The proof employs the method of Vanderbauwhede and van Gils [34].

A technical aspect of the smoothness proof for $W \cap S$ is that close to a fixed point of a C^1-map with one-dimensional linear center space we have to construct open and positively invariant subsets of the center-stable manifold W^{cs} provided a single trajectory in W^{cs} but not in the strong stable manifold converges to the fixed point.

Having established the smoothness of $W \cap S$ we show in a series of propositions that the manifold boundary bd$W = \overline{W} \setminus W$ (without the stationary points ξ_- and ξ_+ if $\mu > 0$) coincides with the forward extension of a local unstable manifold of the period map $F(\omega, \cdot)$ at a fixed point $p_0 \in \mathcal{O}$. The long proof of this fact involves the charts Π_2 and Π_3, and uses most of the results obtained before.

The next step achieves the smoothness of a piece of $\overline{W}$ in a hyperplane H of C transversal to the periodic orbit $\mathcal{O}$. We construct a C^1-smooth graph over an open set in a plane $X_{12} \subset H$ which extends such a piece of $\overline{W} \cap H$ close to a point $p_0 \in \mathcal{O} \cap H$ beyond the boundary. Using the flow F_W we then derive that $\overline{W}$ and $\overline{W} \cap S$ are C^1-smooth up to their manifold boundaries, in the sense stated before.

The final steps lead to the topological description of $\overline{W}$. First we use the characterization of bd$W$ mentioned above to define homeomorphisms from bd$W$ onto the unit sphere $S^2 \subset \mathbb{R}^3$ in case $\mu > 0$ and onto the cylinder $S^1 \times \mathbb{R}$ in case $\mu = 0$. Then a generalization due to Bing [9] of the Schoenfliess theorem [30] from planar topology is employed to obtain homeomorphisms from $\overline{W}$ onto the closed unit ball in $\mathbb{R}^3$ in case $\mu > 0$, and onto the solid cylinder for $\mu = 0$. In case $\mu > 0$ the application of Bing's theorem requires to identify the bounded component of the complement of the set

$$\Pi_3(\text{bd}W) \cong S^2$$

in $\mathbb{R}^3$ as the set $\Pi_3 W$, and to verify that $\Pi_3 W$ is uniformly locally 1-connected. This means that for every $\epsilon > 0$ there exists $\delta > 0$ so that every closed curve in a subset of $\Pi_3 W$ with diameter less than δ can be continuously deformed to a point in a subset of $\Pi_3 W$ with diameter less than ϵ. We point out that the proof of this topological property relies on the smoothness of the set

$$\overline{W} \setminus \{\xi_-, \xi_+\},$$

and involves subsets of boundaries of neighbourhoods of $0, \xi_-, \xi_+$ in W which are transversal to the flow F_W. In order to construct these smooth boundaries we have to go back to the variation-of-constants formula for retarded functional differential equations in the framework of sun-dual and sun-star dual semigroups [12]. In case $\mu = 0$ the construction of the desired homeomorphism from $\overline{W}$ onto the solid cylinder is different but uses Bing's theorem [9] as well.

Results on unstable sets and attractors for delay differential equations related to ours are also contained in the work of Walther [40], Walther and Yebdri [41], Mallet-Paret and Walther [25] on equations with negative feedback, in the work of Arino and Krisztin [2] on a 2-dimensional attractor for a negative feedback equation with a state-dependent time lag and in Ammar's thesis [1].

Problems

The results described above motivate a variety of questions.

Already mentioned was a conjecture that under certain conditions on $\mu > 0$ and f the set $\overline{W}$ is the global attractor of the semiflow F. A proof would involve a new uniqueness result for periodic solutions of delay differential equations since the global attractor contains all ω-limit sets, in particular, all periodic orbits, while in $\overline{W}$ there is exactly one periodic orbit.

We also addressed the question whether $\overline{W} \cap S$ can be represented by a smooth map from a subset of L into the complementary space $Q \oplus P$.

One may ask for diffeomorphisms instead of homeomorphisms from D_W onto the solid cylinder in case $\mu = 0$, and from $D_W \setminus \{\chi_-, \chi_+\}$ onto the unit ball without its north and south poles if $\mu > 0$.

In the sequel, we focus on the case $\mu > 0$. What can be said about the shape of $\overline{W}$ close to the singularities ξ_-, ξ_+ ? At ξ_+, we expect $T_{\xi_+}\overline{W} = \{0\}$. For every C^1-curve $c : (-1, 0] \to C$ in $\overline{W}$ ending at ξ_+, i.e., $c(0) = \xi_+$, the tangent vector $c'(0)$ should be either zero or an eigenvector of the leading eigenvalue of the generator of the linearized semiflow $(D_2F(t, \xi_+))_{t\geq 0}$.

The hyperbolicity assumption $f'(\xi_-) < \mu$, $f'(\xi_+) < \mu$ should be removed.

How is the global attractor organized in case $\overline{W}$ is a proper subset of A ? Consider situations where A is a subset of the order interval $[\xi_-, \xi_+] = \{\phi \in C : \xi_- \leq \phi \leq \xi_+\}$. The segments of the solutions $x : \mathbb{R} \to \mathbb{R}$ with $V(x_t) = 2$ for all $t \in \mathbb{R}$ may form, together with the stationary point 0, a smooth disk-like submanifold A_2 in A which extends $\overline{W} \cap S$ beyond $\mathcal{O}$ and contains at least one additional periodic orbit, forming the manifold boundary. In this case, the unstable sets of the periodic orbits in A_2 should contain analogues of $\text{bd}W \setminus \{\xi_-, \xi_+\}$, namely 2-dimensional invariant submanifolds given by heteroclinic connections from the periodic orbit to ξ_- and ξ_+. These submanifolds should subdivide the 3-dimensional subset $A_{\leq 2}$ of A formed by 0 and the segments of all solutions $x : \mathbb{R} \to \mathbb{R}$ with $\xi^- \leq x(t) \leq \xi^+$ and $V(x_t) \leq 2$ for all $t \in \mathbb{R}$ into invariant layers; a section of $A_{\leq 2}$ containing ξ_- and ξ_+ might have the structure shown by a sliced onion. It may be that $A = A_{\leq 2}$. If A is strictly larger than $A_{\leq 2}$ then we have to expect a finite number of analogues of the smooth disk A_2, given by higher even values of V, and a more complicated variety of heteroclinic connections, between periodic orbits in the same disk, between periodic orbits in different disks, from periodic orbits and 0 to ξ_- and ξ_+, from 0 to periodic orbits and to ξ_- and ξ_+. Also connections from periodic orbits to 0 become possible. A Morse decomposition of A similar to the one constructed by Mallet-Paret [23] should be useful to describe a part of these heteroclinic connections.

Suppose now that A is not confined to the order interval $[\xi_-, \xi_+]$. Then there may exist zeros ξ^* of $f - \mu\text{id}$ below ξ^- and above ξ^+ with $f'(\xi^*) > \mu$; one out of many possibilities is that the part of A in a certain neighbourhood of a stationary point $\xi_* \in C$ given by such a value ξ^* looks just as we began to sketch it for the case $A \subset [\xi_-, \xi_+]$.

Let us emphasize that the main results obtained in [21] and discussed here rely on the hypothesis that the feedback function f in equation (1) is monotone. In the class of equations of the same form, with $f(0) = 0$ and $f'(0) > 0$, but f not necessarily increasing everywhere, other phenomena are known to exist – for example, chaotic motion [37,38]. There is recent evidence that non-monotone feedback occurs in single neurons of living beings and small assemblies of neurons [10]. See also [13,27,33] for problems with non-monotone feedback.

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