PERGAMON

Nonlinear Analysis 35 (1999) 845–870

Nonlinear Analysis

Theory and applications of Hopf bifurcations in symmetric functional differential equations

Wieslaw Krawcewicz[a,1], Jianhong Wu[b,*,2]

[a] Department of Mathematics, University of Alberta, Edmonton, Alta, Canada
[b] Department of Mathematics and Statistics, York University, North York, Ontario, Canada

Received 24 April 1995; accepted 14 May 1997

1. Introduction

In [15], we provided an analytic construction of an equivariant topological degree for compact fields which preserve certain symmetries. The computation formula for such a degree in some special cases was given in [26], with the help of which a bifurcation theory was developed for multi-parameter equivariant fixed point equations.

One of the purposes of this paper is to apply the results in [15, 26] to establish a general theory of Hopf bifurcations in symmetric functional differential equations. This general theory provides some important bifurcation invariants, called *crossing numbers*, to detect the existence of periodic solutions and to describe their orbits and global continuation. We will show that these crossing numbers can be computed from the linearization around equilibria and from the isotypical decomposition of representation spaces.

In comparison to the general theory of symmetric Hopf bifurcations for ordinary differential equations and some parabolic partial differential equations developed in [17, 32, 13], we establish some local symmetric Hopf bifurcation theorems for retarded functional differential equations without requiring genericity conditions on vector fields, dimension restrictions on some fixed point subspaces and maximality assumptions on a certain isotropy group. Unfortunately, due to the topological nature of our approach, we are unable to study the stability of the obtained branch of periodic solutions. Hopf bifurcation problems have been extensively studied via degree-theoretical approach and our approach is based on the equivariant degree theory developed in [15, 26] and is motivated by the work of [1–5, 7–10, 14, 21, 25, 29, 31] in the Hopf bifurcation theory

* Corresponding author. Tel.: 001 416 736 5250; fax: 001 416 736 5757; e-mail: wujh@mathstat.yorku.ca
[1] This research was partly supported by NSERC-Canada and by Alexander von Humboldt Foundation.
[2] This research was partly supported by NSERC-Canada and by Faculty of Arts (York University) Fellowship.

PII: S0362-546X(97)00711-6

without symmetries. It should be mentioned that Ize, Massabó and Vignoli [22, 23] have developed a competing equivariant bifurcation theory.

Another purpose of this paper is to demonstrate the application of the general theory of Hopf bifurcations of symmetric functional differential equations. Of particular interest is the delay-induced oscillation and the global existence of large-amplitude symmetric periodic oscillations. As an illustrative example, we will consider a ring of identical cells coupled by diffusion along a polygon. Such a Turing ring provides a model for many biological and chemical systems. We incorporate a time delay in the coupling of adjacent cells because in many biological and chemical oscillators the time needed for the transport or processing of chemical components or signals may be of considerable length. To the best of our knowledge, the delay-induced oscillations of a Turing ring has not been investigated. We will illustrate that the time delay provides an important resource for the occurrence and global continuation of oscillations in a Turing ring. In particular, we will show that the delay may give rise to phase-locked oscillations even when the state of each cell is described by a single variable. This is in sharp contrast with the observation of [16, 18] that bifurcations of phase-locked oscillations cannot occur in a Turing ring in which the state of a cell is described by one variable if the delay is not presented.

The remaining part of this paper is organized as follows. In Section 2, we collect some results from [26]. In Section 3, we establish some results on the existence, the minimal period and the global continuation of a branch of periodic solutions for general symmetric functional differential equations. These results are then applied, in Section 4, to a ring of identical cells coupled by delayed diffusion along the sides of a polygon.

2. Preliminary results

We start with a brief review about equivariant degrees and we refer to [15] for details. Suppose that G is a fixed compact Lie group. For a subgroup $H \le G$, we use (H) to denote the conjugacy class of H in G which consists of all subgroups conjugate to H. We will use $O(G)$ to stand for the set of all conjugacy classes of closed subgroups of G. Also, for each nonnegative integer n, define

$$O_n(G) = \{(H) \in O(G);\ \dim WH = n\},$$

where WH is the Weyl group NH/H and NH is the normalizer of H in G. We say a compact Lie group is biorientable if it has an orientation which is invariant under all left and right translations. Define

$$OA_n(G) := \{(H) \in O_n(G);\ WH \text{ is biorientable}\},$$

$$OB_n(G) := \{(H) \in O_n(G);\ WH \text{ is not biorientable}\},$$

$$A_n[G] := \oplus\{\mathbb{Z};\ (H) \in OA_n(G)\},$$

$$B_n[G] := \oplus\{\mathbb{Z}_2;\ (H) \in OB_n(G)\},$$

$$AB_n[G] := A_n[G] \oplus B_n[G].$$

An element of $AB_n[G]$ will be written as $\gamma=\{\gamma_\alpha\}$, where

$$\gamma_\alpha \in \begin{cases} \mathbb{Z} & \text{if } \alpha\in OA_n(G),\\ \mathbb{Z}_2 & \text{if } \alpha\in OB_n(G). \end{cases}$$

Assume that W is a real Banach isometric representation of G. We will consider the product space $W\times\mathbb{R}^n$, where we assume G acts trivially on $\mathbb{R}^n$. Suppose that Ω is an open bounded invariant subset of $W\times\mathbb{R}^n$. A continuous G-equivariant mapping $f: W\times\mathbb{R}^n\to W$ is said to be Ω-admissible if $0\notin f(\partial\Omega)$ and $F:=\pi-f: W\times\mathbb{R}^n\to W$ is compact, where $\pi: W\times\mathbb{R}^n\to W$ is the projection onto W.

It was shown in [15] that for any Ω-admissible $f: W\times\mathbb{R}^n\to W$, we can assign an element $G-\mathrm{Deg}(F,\Omega)\in AB_n[G]$. This element is called the G-(equivariant) degree of the mapping f with respect to Ω and satisfies all standard properties of a degree (existence, homotopy invariance, excision, additivity, etc.). In particular, if $G-\mathrm{Deg}(f,\Omega)=\{\gamma_\alpha\}$ and $\gamma_\alpha\neq 0$ for some $\alpha\in AB_n[G]$ then there exists $x\in\Omega\cap f^{-1}(0)$ such that $(G_x)\le\alpha$, where $G_x:=\{g\in G;\ gx=x\}$ is the isotropy group of x, (G_x) is called the orbit type of x, and the partial ordering in $O(G)$ is defined as follows: $\alpha\le\beta$ for $\alpha,\beta\in O(G)$ iff there exists closed subgroups H and K of G such that $\alpha=(H)$, $\beta=(K)$ and K is conjugate to a subgroup of H.

Throughout the remainder of this section, we suppose that Γ is a compact abelian Lie group, $G=\Gamma\times S^1$ and W is a real Banach isometric representation of G. Denote by W_0 the set of all fixed points of W with respect to the restricted S^1-action, i.e., $W_0=\{x\in W;\ \chi x=x \text{ for all } \chi\in S^1\}$. We consider the nonlinear problem

$$x=F(x,\lambda),\qquad (x,\lambda)\in W\times\mathbb{R}^2, \tag{2.1}$$

where $F: W\times\mathbb{R}^2\to W$ is a given G-equivariant completely continuous map which satisfies the following condition:

(H1) There exists a two-dimensional G-invariant submanifold $M\subset W_0\times\mathbb{R}^2$ such that F is continuously differentiable on M and, for each $(x,\lambda)\in M, x=F(x,\lambda)$ and $Id-DF(x,\lambda)|_{W_0}\in GL(W_0)$, where $DF(x,\lambda)$ denotes the derivative with respect to x.

Each point in M is called a *trivial solution* of Eq. (2.1) and all other solutions are said to be *nontrivial.* A point $(x,\lambda)\in M$ is said to be a *bifurcation point* if (2.1) has nontrivial solutions in every neighborhood of (x,λ).

It follows from (H1) and the implicit function theorem that for every $(x_0,\lambda_0)\in M$ there exist an open neighborhood U_{x_0} of x_0 in W_0, an open neighborhood U_{λ_0} of λ_0 in $\mathbb{R}^2$ and a C^1-map $\eta: U_{\lambda_0}\to U_{x_0}$ so that

$$M\cap(U_{x_0}\times U_{\lambda_0})=\{(\eta(\lambda),\lambda);\ \lambda\in U_{\lambda_0}\}.$$

As a consequence, the isotropy group $\Gamma_{(x_0,\lambda_0)}$ of (x_0,λ_0) with respect to the action of Γ is a closed subgroup of Γ and $dim\,\Gamma_{(x_0,\lambda_0)}=dim\,\Gamma$.

The set

$$\Lambda=\{(x,\lambda)\in M;\ Id-DF(x,\lambda)\notin GL(W)\}$$

will be called the set of M-singular points. We assume that $(x_0, \lambda_0) \in M$ is an isolated M-singular point. Choose $r, \rho > 0$ sufficiently small so that the so-called *special neighbourhood* $U(r, \rho) := U_{(x_0, \lambda_0)}(r, \rho)$ of (x_0, λ_0) defined by

$$U_{(x_0, \lambda_0)}(r, \rho) = \{(x, \lambda) \in W \times \mathbb{R}^2;\ \|x - \eta(\lambda)\| < r, |\lambda - \lambda_0| < \rho\}$$

satisfies

(i) $B_\rho(\lambda_0) = \{\lambda \in \mathbb{R}^2;\ |\lambda - \lambda_0| < \rho\} \subset U_{\lambda_0}$,
(ii) (x_0, λ_0) is the only M-singular point in $U(r, \rho) \cap M$,
(iii) $x \neq F(x, \lambda)$ for all $(x, \lambda) \in \overline{U(r, \rho)}$ with $|\lambda - \lambda_0| = \rho$ and $\|x - \eta(\lambda)\| \neq 0$.

Clearly, $U(r, \rho)$ is H-invariant, where $H = K \times S^1 \leq \Gamma \times S^1$, $K = \Gamma_{(x_0, \lambda_0)}$.

An H-invariant function $\Theta : \overline{U(r, \rho)} \to \mathbb{R}$ is called an *auxiliary function* if
(a) $\Theta(\eta(\lambda), \lambda) = -|\lambda - \lambda_0|$ for $\lambda \in B_\rho(\lambda_0)$,
(b) $\Theta(x, \lambda) = r$ if $\|x - \eta(\lambda)\| = r$ and $\lambda \in B_\rho(\lambda_0)$,
(c) $\Theta(x, \lambda_0) = \|x - \eta(\lambda_0)\|$ if $\|x - \eta(\lambda_0)\| \leq r$.
Such an auxiliary function exists and the system

$$x = F(x, \lambda), \qquad \Theta(x, \lambda) = 0 \tag{2.2}$$

has no solution in $\partial U(r, \rho)$. Therefore, the H-equivariant degree $H - \mathrm{Deg}(F_\Theta, U(r, \rho))$ is well defined, where $F_\Theta : \overline{U(r, \rho)} \subset V \times \mathbb{R} \to V = W \times \mathbb{R}$ is defined by

$$F_\Theta(x, \lambda) = (x - F(x, \lambda), \Theta(x, \lambda)), \quad (x, \lambda) \in \overline{U(r, \rho)}.$$

We need a computational formula for $H - \mathrm{Deg}(F_\Theta, U(r, \rho))$. First, there is an isotypical decomposition $W_\infty = \bigoplus_{n=0}^{\infty} W_n$, $\overline{W_\infty} = W$, of the space W with respect to the restricted action of S^1 on W, where for each $n > 0$ and $x \in W_n \backslash \{0\}$, the isotropy group of x with respect to the restricted S^1-action is $\mathbb{Z}_n$. Since the actions of K and S^1 commutes, W_n, $n = 0, 1, \ldots$, are K-invariant. Moreover, each W_n with $n \geq 1$ has a natural complex structure. We will use the following notations:

W_{0i}, $i \geq 1$: all K-isotypical components of W_0 corresponding to the two-dimensional irreducible subrepresentations of K

$W_{*,j}$, $1 \leq j \leq m$: all K-isotypical components of W_0 corresponding to one-dimensional subrepresentations of K

W_{ni}, $n \geq 1, i \geq 1$: all K-isotypical components of W_n.

Clearly, all the above K-isotypical components are also H-isotypical components. It is known that for each $x \in W_{ni} \backslash \{0\}$ the isotropy group $(K \times S^1)_x$ is exactly the subgroup $H_{ni} = \{(\gamma, \chi); \chi \in \Theta_{ni}(\gamma),\ \gamma \in K\} \leq K \times S^1$ for a group homomorphism $\Theta_{ni} : K \to S^1 \cong S^1/\mathbb{Z}_n$.

Let

$$a(\lambda) = Id - D_x F(\eta(\lambda), \lambda), \quad \lambda \in B_\rho(\lambda_0),$$

$$a_j(\lambda) = a(\lambda)|_{W_{*,j}} : W_{*,j} \to W_{*,j}, \quad 1 \leq j \leq m,$$

$$a_{ni}(\lambda)=a(\lambda)|_{W_{ni}}: W_{ni}\to W_{ni},\quad n\ge 0,\ i\ge 1,$$

$$a_{00}(\lambda)=a(\lambda)|_{W^{K\times S^1}}: W^{K\times S^1}\to W^{K\times S^1},$$

where

$$W^{K\times S^1}=\{x\in W; (\gamma,\chi)x=x \text{ for } (\gamma,\chi)\in K\times S^1\}.$$

Define

$$\mu_{ni}(x_0,\lambda_0)=\Delta([a_{ni}]),$$

where $[a_{ni}]$ denotes the homotopy class of

$$a_{ni}: S^1\cong\partial B_\rho(\lambda_0)\to GL_{\mathbb{C}}(W_{ni}),$$

$GL_{c}(W_{ni})$ is the set of all complex linear isomorphisms of the form $Id-A$ with A being compact, and $\Delta:\pi_1(GL_c(W_{ni}))\to\mathbb{Z}$ is the well-known natural isomorphism.

Let us define the following element of $A_1(K\times S^1)$ by

$$\mu(x_0,\lambda_0)=\sum \mu_{ni}(x_0,\lambda_0)(H_{ni}),$$

where the summation is taken over all subindices (n,i) such that $dim(K\times S^1)/H_{ni}=1$.

For $T\in GL_{\mathbb{C}}(X)$ we denote by $sign\,T$ the number 1 if T belongs to the same connected component as Id, and -1 otherwise. Finally, we define

$$v_j=\begin{cases}1 & \text{if } sign\,a_j=-1,\\ 0 & \text{otherwise}\end{cases}$$

for $1\le j\le m$, and define the following element of the Burnside ring $A(K\times S^1)$:

$$v(x_0,\lambda_0)=\prod_{j=1}^{m}((K\times S^1)-v_j(K_j\times S^1)),$$

where $K_j=(K)_x$, for $x\in W_{*,j}\setminus\{0\}$, is the isotropy group of nontrivial element in $W_{*,j}$ with respect to the restricted K-action. Then we have

Lemma 2.1. *The* $H-\mathrm{Deg}(F_\Theta,U(r,\rho))$ *is given by*

$$H-Deg(F_\Theta,U(r,\rho))=\varepsilon(x_0,\lambda_0)v(x_0,\lambda_0)\mu(x_0,\lambda_0),$$

where

$$\varepsilon(x_0,\lambda_0)=sign\,a_{00}(\lambda),\quad \lambda\in\partial B_\rho(\lambda_0).$$

For the proof, we refer to [26].

Using the existence property of the equivariant degree, we then have the following local bifurcation theorem:

Theorem 2.1. *If $H - \mathrm{Deg}(F_\Theta, U(r,\rho)) \neq 0$, then (x_0,λ_0) is a bifurcation point for Eq. (2.1). More precisely, if $\mu_{ni}(x_0,\lambda_0)\neq 0$ then Eq. (2.1) has a sequence of nontrivial solutions (x_m,λ_m) such that $((K\times S^1)_{x_m}) \leq (H_{ni})$ for $m=1,2,\ldots,$ and $(x_m,\lambda_m)\to(x_0,\lambda_0)$ as $m\to\infty$.*

To describe the global continuation of a local bifurcation of nontrivial solutions, we need to compute a related G-degree. To achieve this, we first extend Θ to a G-invariant function $\bar{\Theta}: GU(r,\rho)\to\mathbb{R}$ and then apply the reduction formula to get

$$G - Deg\,(F_{\bar{\Theta}}, GU(r,\rho)) = \varepsilon(x_0,\lambda_0)\Phi(v(x_0,\lambda_0)\mu(x_0,\lambda_0)),$$

where $\Phi(\sum_\alpha \gamma_\alpha(\alpha)_H) = \sum_\alpha \gamma_\alpha(\alpha)_G, (\alpha)_H$ and $(\alpha)_G$ denote the orbit types of α with respect to H and G, respectively.

Theorem 2.2. *Suppose that M is complete and every M-singular point in M is isolated. Let $\mathscr{S}$ denote the closure of the set of all nontrivial solutions to (2.1). Then for each bounded connected component $\mathscr{C}$ of $\mathscr{S}$, the set $G\mathscr{C}\cap M$ is finite and is composed of a finite number of disjoint Γ-orbits, i.e.*

$$G\mathscr{C}\cap M = \bigcup_{i=1}^{q} \Gamma(x_i,\lambda_i).$$

Moreover,

$$\sum_{i=1}^{q} G - Deg(F_{\bar{\Theta}}, GU_{(x_i,\lambda_i)}(r,\rho)) = 0.$$

In particular, if $M\subset W^G\oplus\mathbb{R}^2$ and S^{ni} denotes the closure of the set of all nontrivial solutions of Eq. (2.1) whose isotropy group contains (G_{ni}), then for each bounded connected component $\mathscr{C}^{ni}$ of S^{ni}, we have that $G\mathscr{C}^{ni}=\mathscr{C}^{ni}$ and $\mathscr{C}^{ni}\cap M=\{(v_1,\lambda_1),\ldots,(v_q,\lambda_q)\}$ is a finite set and

$$\sum_{k=1}^{q} \varepsilon(v_k,\lambda_k)\mu_{ni}(v_k,\lambda_k) = 0.$$

We refer to [26] for the proof.

3. Hopf bifurcation theory for symmetric FDEs

Let $\tau\geq 0$ be a given constant, N a positive integer and $C_{N,\tau}$ the Banach space of continuous functions from $[-\tau,0]$ into $\mathbb{R}^N$ equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|, \quad \varphi\in C_{N,\tau}.$$

In what follows, if $x:[-r,A]\to\mathbb{R}^N$ is a continuous function with $A>0$ and if $t\in[0,A]$, then $x_t\in C_{N,\tau}$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \theta\in[-\tau,0].$$

Consider the following retarded functional differential equation

$$\dot{x}(t) = f(x_t), \tag{3.1}$$

where $f: C_{N,\tau} \to \mathbb{R}^N$ is a continuously differentiable function preserving a certain symmetry described by the following condition: There exists a compact Lie group Γ as well as an orthogonal representation $\rho: \Gamma \to GL(\mathbb{R}^N)$ such that

$$f(\rho(\gamma)\varphi) = \rho(\gamma)f(\varphi), \quad \varphi \in C_{N,\tau}, \ \gamma \in \Gamma,$$

where $\rho(\gamma)\varphi \in C_{N,\tau}$ is defined by

$$(\rho(\gamma)\varphi)(\theta) = \rho(\gamma)\varphi(\theta), \quad \theta \in [-\tau, 0].$$

In what follows, a system satisfying the above condition is said to be *equivariant with respect to the action of* Γ on $\mathbb{R}^N$.

For a given periodic solution $x = x(t)$ of Eq. (3.1) with the minimal period $p > 0$, we use O_x to denote the trajectory of x, i.e.,

$$O_x := \{x_t;\ t \in \mathbb{R}\} \subseteq C_{N,\tau}.$$

Define

$$H := \{\gamma \in \Gamma;\ \gamma O_x = O_x\}, \tag{3.2}$$

$$K := \{\gamma \in \Gamma;\ \gamma x_0 = x_0\}. \tag{3.3}$$

K and H describe the spatial symmetry and the dynamic phase-shift symmetry of the periodic solution. Clearly, H and K are closed subgroups of Γ and $\Gamma_{x_t} = \Gamma_{x_0}$ for all $t \in \mathbb{R}$. From the above definitions of H and K, for every $h \in H$ there exists a unique $\theta(h) \in \mathbb{R}/\mathbb{Z}$ such that $\rho(h)x_0 = x_{\theta(h)p}$. By the uniqueness of solutions to the Cauchy initial value problem of (2.1) (see, cf. [19]), we obtain

$$\rho(h)x_t = x_{t+\theta(h)p}, \quad t \in \mathbb{R}. \tag{3.4}$$

The obtained mapping $\theta: H \to \mathbb{R}/\mathbb{Z}$ is a group homomorphism between H and the additive group $\mathbb{R}/\mathbb{Z}$. Consequently, $K = \operatorname{Ker}\theta$ is a closed normal subgroup of H and

$$H/K \cong \operatorname{Im}\theta \cong \begin{cases} \mathbb{Z}_n := \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\right\} \le \mathbb{R}/\mathbb{Z}, \\ \mathbb{Z}_\infty = \mathbb{R}/\mathbb{Z} \cong S^1. \end{cases} \tag{3.5}$$

Following [13], we call the periodic solution x a *discrete wave* if $H/K \cong \mathbb{Z}_n$ with $n < \infty$, and a *rotating wave* if $H/K \cong \mathbb{Z}_\infty$.

Suppose, on the other hand, two subgroups $K \le H \le \Gamma$ are prescribed such that K is normal in H and $H/K \cong \mathbb{Z}_n$, $n \ge 1$. We want to look for periodic solutions satisfying Eqs. (3.2) and (3.3). It clearly suffices to restrict to the invariant subspace $X = (\mathbb{R}^N)^K$. The action of Γ on $\mathbb{R}^N$ induces a $\mathbb{Z}_n$ action on X by

$$[h]x = \rho(h)x, \quad x \in X,\ h \in [h] \in H/K$$

and $f: C^K_{N,\tau} \to X$ is equivariant with respect to this induced action, where $C^K_{N,\tau} = \{\varphi \in C_{N,\tau};\ \varphi(\theta) \in (\mathbb{R}^N)^K$ for $\theta \in [-\tau, 0]\}$. In this sense we will assume, throughout the remainder of this paper, that $\Gamma = \mathbb{Z}_n$, $n \le \infty$. We should emphasize that we make this assumption mainly for the purpose of simplicity and clarity. However, in order to discuss the global interaction among different branches of periodic solutions, one needs to take into consideration the whole group. We will address this general case in another paper.

With the above preparation, we now consider the following one parameter family of retarded functional differential equations

$$\dot{x} = f(x_t, \alpha), \tag{3.6}$$

where $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$, $f\colon C_{N,\tau} \times \mathbb{R} \to \mathbb{R}^N$ is a continuously differentiable and completely continuous mapping satisfying the following conditions:

(A1) There exists a representation $\rho : \mathbb{Z}_n \to GL(\mathbb{R}^N)$ of $\Gamma := \mathbb{Z}_n$ on $\mathbb{R}^N$ such that

$$f(\rho(\gamma)\varphi, \alpha) = \rho(\gamma) f(\varphi, \alpha), \quad \varphi \in C_{N,\tau}, \ \alpha \in \mathbb{R}, \ \gamma \in \mathbb{Z}_n.$$

(A2) There exists $(x_0, \alpha_0) \in \mathbb{R}^N \times \mathbb{R}$ such that $f(\bar{x}_0, \alpha_0) = 0$ and $D\bar{f}(x_0, \alpha_0) : \mathbb{R}^N \to \mathbb{R}^N$ is an isomorphism, where for each $x \in \mathbb{R}^N$, $\bar{x}$ denotes the constant mapping in $C_{N,\tau}$ with the value x, the mapping $\bar{f}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is defined by

$$\bar{f}(x, \alpha) = f(\bar{x}, \alpha), \quad x \in \mathbb{R}^N, \ \alpha \in \mathbb{R},$$

and $\mathrm{D}\bar{f}(x_0, \alpha_0)$ denotes the derivative of $\bar{f}$ with respect to x, evaluated at (x_0, α_0).

Under the above assumptions, there exists $\delta_0 > 0$ and a C^1-mapping $\chi : (\alpha_0 - \delta_0, \alpha_0 + \delta_0) \to \mathbb{R}^N$ such that $\chi(\alpha_0) = x_0$ and $f(\bar{\chi}(\alpha), \alpha) = 0$ for $\alpha \in (\alpha_0 - \delta_0, \alpha_0 + \delta_0)$. In what follows, $(\chi(\alpha), \alpha)$ will be called a *stationary solution* of Eq. (3.6). Let $\mathbb{C}^N = \mathbb{R}^N + \mathrm{i}\mathbb{R}^N$ and $\{\varepsilon_1, \ldots, \varepsilon_N\}$ denote the standard basis of $\mathbb{R}^N$. For any $\lambda \in \mathbb{C}$ and $1 \le j \le N$, define $\mathrm{e}^{\lambda \cdot}\varepsilon_j$ as a mapping from $[-\tau, 0]$ into $\mathbb{C}^N$ by

$$\mathrm{e}^{\lambda \cdot}\varepsilon_j(\theta) = \mathrm{e}^{\lambda\theta}\varepsilon_j, \quad -\tau \le \theta \le 0.$$

A complex number $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of the stationary solution $(\chi(\alpha), \alpha)$ if

$$\det{}_{\mathbb{C}} \Delta_{(\alpha, \chi(\alpha))}(\lambda) = 0,$$

where

$$\Delta_{(\alpha,\chi(\alpha))}(\lambda) := \lambda\,\mathrm{Id} - \mathrm{D}f(\overline{\chi(\alpha)}, \alpha)(\mathrm{e}^{\lambda \cdot}\,\mathrm{Id}), \tag{3.7}$$

$$\mathrm{D}f(\overline{\chi(\alpha)}, \alpha)(\mathrm{e}^{\lambda \cdot}\mathrm{Id}) = (\mathrm{D}f(\overline{\chi(\alpha)}, \alpha)(\mathrm{e}^{\lambda \cdot}\varepsilon_1), \ldots, \mathrm{D}f(\overline{\chi(\alpha)}, \alpha)(\mathrm{e}^{\lambda \cdot}\varepsilon_N)).$$

We now make the following assumption:

(A3) There exist a constant $\beta_0 > 0$ and sufficiently small constants $b > 0$, $c > 0$, $\delta > 0$ such that

(i) the only eigenvalue $u + \mathrm{i}v$ of the stationary solution (x_0, α_0) such that $(u, v) \in \partial\Omega$ is $\mathrm{i}\beta_0$ where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c)$;

(ii) for $(\alpha,\beta)\in[\alpha_0-\delta,\alpha_0+\delta]\times[\beta_0-c,\beta_0+c]$, $i\beta$ is an eigenvalue of $(\chi(\alpha),\alpha)$ iff $\beta=\beta_0$ and $\alpha=\alpha_0$.

Note that $\Delta_{(\alpha,\chi(\alpha))}(\lambda)$ is analytic in $\lambda\in\mathbb{C}$ and continuous in $\alpha\in[\alpha_0-\delta,\alpha_0+\delta]$. So under the assumption (A3), we may assume that $\det_{\mathbb{C}}\Delta_{(\alpha_0\pm\delta,\chi(\alpha_0+\delta))}(u+iv)\neq 0$ for $(u,v)\in\partial\Omega$.

To give the Hopf bifurcation problem an abstract formulation, we make a change of variable $x(t)=z((\beta/2\pi)t)$ for $t\in\mathbb{R}$ in Eq. (3.6) to obtain

$$\dot z(t)=\frac{2\pi}{\beta}f(z_{t,\beta},\alpha), \tag{3.8}$$

where $z_{t,\beta}\in C_{N,\tau}$ is defined by

$$z_{t,\beta}(\theta)=z\left(t+\frac{\beta}{2\pi}\theta\right),\quad \theta\in[-\tau,0].$$

Evidently, $z(t)$ is an 1-periodic solution of Eq. (3.8) iff $x(t)$ is a $(2\pi/\beta)$-periodic solution of Eq. (3.6).

Let $S^1=\mathbb{R}^1/\mathbb{Z}$, $W=C(S^1;\mathbb{R}^N)$ and define

$$L:C^1(S^1;\mathbb{R}^N)\to W,\qquad Lz(t)=\dot z(t),\quad z\in C^1(S^1;\mathbb{R}^N),\ t\in S^1;$$

$$K:C^1(S^1;\mathbb{R}^N)\to W,\qquad Kz(t)=\int_0^1 z(s)\,ds,\quad z\in C^1(S^1;\mathbb{R}^N),\ t\in S^1.$$

It can be easily shown that $(L+K)^{-1}:W\to C^1(S^1;\mathbb{R}^N)$ exists and the map $F:W\times(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)\to W$ defined by

$$F(z,\alpha,\beta)=(L+K)^{-1}\left[Kz+\frac{2\pi}{\beta}N_f(z,\alpha,\beta)\right]$$

is completely continuous, where $N_f:W\times(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)\to W$ is defined by

$$N_f(z,\alpha,\beta)(t)=f(z_{t,\beta},\alpha),\quad t\in S^1.$$

Moreover, (z,α,β) is an 1-periodic solution of Eq. (3.8) iff $z=F(z,\alpha,\beta)$. For the convenience of later reference, we also point out that

$$\begin{aligned}&(L+K)^{-1}y=y,\\&(L+K)^{-1}\sin(2\pi\cdot)y=-\frac{1}{2\pi}\cos(2\pi\cdot)y,\\&(L+K)^{-1}\cos(2\pi\cdot)y=\frac{1}{2\pi}\sin(2\pi\cdot)y,\end{aligned}\tag{3.9}$$

for every $y\in\mathbb{R}^N$, here and in what follows, for every positive integer k, $\sin 2k\pi\cdot$ and $\cos 2k\pi\cdot$ are mappings from S^1 into $\mathbb{R}$ such that

$$(\sin 2k\pi\cdot)t=\sin 2k\pi t,\qquad(\cos 2k\pi\cdot)t=\cos 2k\pi t,\quad t\in S^1.$$

W is an isometric Banach representation of the group $G=\mathbb{Z}_n\times S^1$ with the action being given by

$$((\gamma,\theta)z)(t)=\rho(\gamma)z(t+\theta), \quad \theta,t\in S^1,\ \ \gamma\in\mathbb{Z}_n,\ \ z\in W.$$

With respect to such an action, F is clearly equivariant. Define $M\subseteq W^{S^1}\times(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)$ by

$$M=\{(\chi(\alpha),\alpha,\beta);(\alpha,\beta)\in(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)\}.$$

M is a two-dimensional submanifold of $W^{S^1}\times(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)$ satisfying (H1) in Section 2. Note that for fixed α_0, the stationary solution (x_0,α_0) has only finitely many eigenvalues on the imaginary axis of the complex plane (see, cf. [19]). So by an continuity argument and the implicit function theorem, assumption (A3) implies that (x_0,α_0,β_0) is an isolated M-singular point.

Let $r,\rho>0$ be sufficiently small so that

$$U_{(x_0,\alpha_0,\beta_0)}(r,\rho)=\{(z,\alpha,\beta)\in W\times\mathbb{R}^2;\|z-\overline{\rho(\alpha)}\|<r,(\alpha-\alpha_0)^2+(\beta-\beta_0)^2<\rho^2\}$$

is a special neighborhood of (x_0,α_0,β_0) and $\Theta:\overline{U_{(x_0,\alpha_0,\beta_0)}(r,\rho)}\to\mathbb{R}$ is an auxiliary function. Let $K=\Gamma_{x_0}\le\Gamma=\mathbb{Z}_n$. The action of Γ on $\mathbb{R}^N$ induces an action of K on $\mathbb{C}^N=\mathbb{R}^N+\mathrm{i}\mathbb{R}^N$ by

$$\rho(\gamma)(x+\mathrm{i}y)=\rho(\gamma)x+\mathrm{i}\rho(\gamma)y, \quad x+\mathrm{i}y\in\mathbb{C}^N,\ \ \gamma\in K.$$

We have the following isotypical decomposition

$$\mathbb{C}^N=\mathbb{C}^N_0\oplus\mathbb{C}^N_1\oplus\cdots,$$

where $\mathbb{C}^N_j$, $j\ge 0$, is the direct sum of all one-dimensional K-irreducible subspaces V of $\mathbb{C}^N$ such that the restricted action of K on V is isomorphic to the Γ-action on $\mathbb{C}$ defined by

$$\rho_j(\gamma)z=\gamma^j z, \qquad \gamma\in K\le\Gamma=\mathbb{Z}_n\le S^1\subset\mathbb{C},\ \ z\in\mathbb{C}.$$

By assumption (A1), we can easily show that $\Delta_{(\alpha,\chi(\alpha))}(\lambda):\mathbb{C}^N\to\mathbb{C}^N$ is K-equivariant for $\alpha\in(\alpha_0-\delta,\alpha_0+\delta)$ and $\lambda\in\mathbb{C}$. So $\Delta_{(\alpha,\chi(\alpha))}(\lambda)\mathbb{C}^N_j\subset\mathbb{C}^N_j$ for $j\ge 0$. Put

$$\Delta_{(\alpha,\chi(\alpha)),j}(\lambda):=\Delta_{(\alpha,\chi(\alpha))}(\lambda)|_{\mathbb{C}^N_j}, \quad j\ge 0. \tag{3.10}$$

With respect to the restricted S^1-action on W, we have the following isotypical decomposition

$$W=W_0\oplus W_1\oplus\cdots\oplus W_k\oplus\cdots,$$

where W_0 is the space of all constant mappings from S^1 into $\mathbb{R}^N$, and W_k is the vector space of all mappings of the form $x\sin 2k\pi\cdot+y\cos 2k\pi\cdot$, $x+\mathrm{i}y\in\mathbb{C}^N$, for every $k\ge 1$.

W_1 can be endowed with a complex structure by

$$\mathrm{i}\cdot(x\sin 2\pi\cdot+y\cos 2\pi\cdot)=x\cos 2\pi\cdot-y\sin 2\pi\cdot, \quad x+\mathrm{i}y\in\mathbb{C}^N.$$

Clearly, K acts on W_1 and the isotypical decomposition of W_1 with respect to this K-action is given by

$$W_1 = W_{1,0} \oplus W_{1,1} \cdots,$$

where

$$W_{1,j} = \{x \sin 2\pi\cdot + y \cos 2\pi\cdot;\ x + \mathrm{i}y \in \mathbb{C}_j^N\}, \quad j \geq 0.$$

Let

$$a_{1,j}(\chi(\alpha),\alpha,\beta) := \mathrm{Id} - (L+K)^{-1}\left[K + \frac{2\pi}{\beta}\mathrm{D}_z N_f(\chi(\alpha),\alpha,\beta)\right]\bigg|_{W_{1,j}}$$

for $j \geq 0$ $(\alpha,\beta) \in (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$. Since $W_1 \subseteq C^1(S^1;\mathbb{R}^N)$, we have

$$a_{1,j}(\chi(\alpha),\alpha,\beta) = (L+K)^{-1}\left[L - \frac{2\pi}{\beta}\mathrm{D}_z N_f(\chi(\alpha),\alpha,\beta)\right].$$

Lemma 3.1. $a_{1,j}(\chi(\alpha),\alpha,\beta) = (1/\beta i)\Delta_{(\alpha,\chi(\alpha)),j}(\mathrm{i}\beta)$.

Proof. For $z = x\sin 2\pi\cdot + y\cos 2\pi\cdot$, we obtain

$$\begin{aligned}
&a_{1,j}(\chi(\alpha),\alpha,\beta)z\\
&\quad = (L+K)^{-1}\left[\dot z - \frac{2\pi}{\beta}\mathrm{D}f(\chi(\alpha),\alpha)z_{\cdot,\beta}\right]\\
&\quad = (L+K)^{-1}\left[\dot z - \frac{2\pi}{\beta}(\sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha,a)x\cos_\beta + \cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)x\sin_\beta)\right.\\
&\qquad \left. - \frac{2\pi}{\beta}(\cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)x\cos_\beta - \sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)y\sin_\beta)\right],
\end{aligned}$$

where $\cos_\beta$ and $\sin_\beta \in C([-\tau,0];\mathbb{R})$ are defined by

$$\cos_\beta\theta = \cos\beta\theta, \qquad \sin_\beta\theta = \sin\beta\theta, \quad \theta \in [-\tau,0].$$

So, we can apply Eq. (3.9) to obtain

$$\begin{aligned}
&a_{1,j}(\chi(\alpha),\alpha,\beta)(x\sin 2\pi\cdot + y\cos 2\pi\cdot)\\
&\quad = x\sin 2\pi\cdot + y\cos 2\pi\cdot + \frac{1}{\beta}(\cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)x\cos_\beta\\
&\qquad - \sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)x\sin_\beta)\\
&\qquad + \frac{1}{\beta}(-\sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)y\cos_\beta - \cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)y\sin_\beta)\\
&\quad = \sin 2\pi\cdot + y\cos 2\pi\cdot - \frac{1}{\beta i}(\sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)x\cos_\beta\\
&\qquad + \sin 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)(\mathrm{i}x\sin_\beta))
\end{aligned}$$

$$-\frac{1}{\beta \mathrm{i}}(\cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)y\cos_\beta + \cos 2\pi\cdot \mathrm{D}f(\chi(\alpha),\alpha)(\mathrm{i}y\sin_\beta))$$

$$= \frac{1}{\beta \mathrm{i}}[\mathrm{i}\beta\,\mathrm{Id} - \mathrm{D}f(\chi(\alpha),\alpha)((\cos_\beta + \mathrm{i}\sin_\beta)\mathrm{Id})](x\sin 2\pi\cdot + y\cos 2\pi\cdot)$$

$$= \frac{1}{\beta \mathrm{i}}[\mathrm{i}\beta\,\mathrm{Id} - \mathrm{D}f(\chi(\alpha),\alpha)(\mathrm{e}^{\mathrm{i}\beta}\cdot\mathrm{Id})](x\sin 2\pi\cdot + y\cos 2\pi\cdot)$$

$$= \frac{1}{\beta \mathrm{i}}\Delta_{(\alpha,\chi(\alpha))}(\mathrm{i}\beta)(x\sin 2\pi\cdot + y\cos 2\pi\cdot)$$

This completes the proof. □

Define

$$a^*_{1,j}(\alpha,\beta) = a_{1,j}(\chi(\alpha),\alpha,\beta),$$

$$r_{1,j}(x_0,\alpha_0,\beta_0) = \deg_B(\det_{\mathbb{C}} a^*_{1,j}(\cdot),(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)),$$
$$c_{1,j}(x_0,\alpha_0,\beta_0) = \deg_B(\det_{\mathbb{C}}\Delta_{(\alpha_0-\delta,\chi(\alpha_0-\delta)),j}(\cdot),\Omega) - \deg_B(\det_{\mathbb{C}}\Delta_{(\alpha_0+\delta,\chi(\alpha_0+\delta)),j}(\cdot),\Omega). \quad (3.11)$$

Then we have

Lemma 3.2. $\gamma_{i,j}(x_0,\alpha_0,\beta_0) = c_{1,j}(x_0,\alpha_0,\beta_0)$.

Proof. By Lemma 3.1, we have

$$\gamma_{1,j}(x_0,\alpha_0,\beta_0) = \deg_B(\det_{\mathbb{C}}\Delta_{1,j}(\cdot),(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta+c)),$$

where

$$\Delta_{1,j}(\alpha,\beta) = \Delta_{(\alpha,\chi(\alpha)),j}(\alpha,\beta).$$

On the other hand, by Lemma 3.1 of [12], we have

$$\deg_B(\det_{\mathbb{C}}\Delta_{1,j}(\cdot),(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)) = c_{1,j}(x_0,\alpha_0,\beta_0)$$

from which the conclusion follows. □

The following assumption will not be required for the local bifurcation theorem, but is useful for the complete description of the degree $D-\mathrm{Deg}(F_\Theta, U_{(x_0,\alpha_0,\beta_0)}(r,\rho))$.

$(A3)_k$ If $\Delta_{(\alpha_0,\chi(\alpha_0))}(\mathrm{i}k\beta_0) = 0$ then there exist sufficiently small $b,c,\delta>0$ such that (i) the only eigenvalue $u+\mathrm{i}kv$ of the stationary solution (x_0,α_0) such that $(u,v)\in\partial\Omega$ is $\mathrm{i}k\beta_0$, where $\Omega := (0,b)\times(\beta_0-c,\beta_0+c)$; (ii) for $(\alpha,\beta)\in[\alpha_0-\delta,\alpha_0+\delta]\times[\beta_0-c,\beta_0+c]$, $\mathrm{i}k\beta$ is an eigenvalue of $(\chi(\alpha),\alpha)$ iff $\beta=\beta_0$ and $\alpha=\alpha_0$.

The isotypical decomposition of W_k with respect to the K-action is given by

$$W_k = W_{k,0}\oplus W_{k,1}\oplus\cdots, \quad (3.12)$$

where

$$W_{k,j}=\{x\sin 2k\pi\cdot + y\cos 2k\pi\cdot\,; x+\mathrm{i}y\in\mathbb{C}_j^N\},\quad j\geq 0.$$

Define

$$\begin{aligned}
&a_{k,j}(\chi(\alpha),\alpha,\beta)=Id-(L+K)^{-1}\left[K+\frac{2\pi}{\beta}D_zN_f(\chi(\alpha),\alpha,\beta)\right]\Bigg|_{W_{k,j}},\\
&a_{k,j}^*(\alpha,\beta)=a_{k,j}(\chi(\alpha),\alpha,\beta),\\
&\gamma_{k,j}(x_0,\alpha_0,\beta_0)=deg_B(det\, a_{k,j}^*(\cdot),(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c)),\\
&\Delta^k_{(\alpha,\chi(\alpha)),j}(\lambda)=\Delta_{(\alpha,\chi(\alpha)),j}(Re\,\lambda+ki\,Im\,\lambda),\quad \lambda\in\mathbb{C}.
\end{aligned}\tag{3.13}$$

Then using a similar argument as those of Lemmas 3.1 and 3.2, we get

Lemma 3.3. *If $k\geq 1$ is given and $(A3)_k$ is satisfied, then*

$$a_{k,j}(\chi(\alpha),\alpha,\beta)=\frac{1}{\beta k\mathrm{i}}\Delta_{(\alpha,\chi(\alpha))}(\mathrm{i}k\beta),$$

$$\gamma_{k,j}(x_0,\alpha_0,\beta_0)=c_{k,j}(x_0,\alpha_0,\beta_0),$$

where

$$c_{k,j}(x_0,\alpha_0,\beta_0)=deg_B(det_{\mathbb{C}}\Delta^k_{(\alpha_0-\delta,\chi(\alpha_0-\delta)),j},\Omega)-\deg_B(\det_{\mathbb{C}}\Delta^k_{(\alpha_0+\delta,\chi(\alpha_0+\delta)),j},\Omega).$$

Moreover, $\gamma_{k,j}(x_0,\alpha_0,\beta_0)=\gamma_{1,j}(x_0,\alpha_0,k\beta_0)$ if both (A3) and $(A3)_k$ are satisfied.

We now state next result in two distinguished cases:

Lemma 3.4. *Assume that $(A3)_k$ is satisfied for each $k\geq 1$ and $K=\mathbb{Z}_m$ with $m<\infty$. Then for $H=K\times S^1$, we have*

$$H-Deg(F\Theta,U_{(x_0,\alpha_0,\beta_0)}(r,\rho))=\varepsilon(x_0,\alpha_0,\beta_0)v(x_0,\alpha_0,\beta_0)\mu(x_0,\alpha_0,\beta_0),$$

where

$$\varepsilon(x_0,\alpha_0,\beta_0)=(-1)^N\,sign\,det\,\mathrm{D}\bar{f}\,(x_0,\alpha_0),$$

$$\mu(x_0,\alpha_0,\beta_0)=\sum_{k\geq 1,\,j\geq 0}c_{k,j}(x_0,\alpha_0,\beta_0)(H_{k,j}),$$

$$H_{k,j}=\left\{\left(\frac{l}{m},\frac{klj}{m}\right)\in\mathbb{Z}_m\times S^1;\ l=0,\ldots,m-1\right\},$$

$$v(x_0,\alpha_0,\beta_0)=(\mathbb{Z}_m\times S^1)-\mu_1(x_0,\alpha_0,\beta_0)(\mathbb{Z}_{m/2}\times S^1).$$

Moreover, $\mu_1(x_0,\alpha_0,\beta_0)=0$ if $\mathbb{R}^N$ has no nontrivial one-dimensional subrepresentation of the $\mathbb{Z}_m$-action, and $\mu_1(x_0,\alpha_0,\beta_0)=\frac{1}{2}[1-(-1)^{\dim W_\alpha^*}\,sign\,det\,\mathrm{D}\bar{f}\,(x_0,\alpha_0)|_{W_0^*}]$ if $\mathbb{R}^N$ has nontrivial one-dimensional subrepresentation of the $\mathbb{Z}_m$-action with W_0^* denoting the corresponding isotypical component.

Proof. By Lemma 2.1, we have

$$\varepsilon(x_0,\alpha_0,\beta_0)=sign\,det\,a_{00}(\alpha,\beta),(\alpha-\alpha_0)^2+(\beta-\beta_0)^2=\rho^2,$$

where

$$a_{00}(\alpha,\beta)=Id-(L+K)^{-1}\left[K+\frac{2\pi}{\beta}D_zN_f(\chi(\alpha),\alpha,\beta)\right]\Bigg|_{W_0}.$$

Is can be easily verified that

$$a_{00}(\alpha,\beta)=-\frac{1}{\beta}\mathrm{D}\bar{f}\,(\chi(\alpha),\alpha).$$

Therefore,

$$\varepsilon(x_0,\alpha_0,\beta_0)=(-1)^N\,sign\,det\,\mathrm{D}\bar{f}\,(x_0,\alpha_0).$$

In the case where $\mathbb{R}^N$ has nontrivial one-dimensional subrepresentation with respect to the $\mathbb{Z}_m$-action, we have

$$\mu_1(x_0,\alpha_0,\beta_0)=\frac{1}{2}[1-sign\,a_1(x_0,\alpha_0,\beta_0)],$$

where

$$a_1(x_0,\alpha_0,\beta_0)=Id-(L+K)^{-1}\left[K+\frac{2\pi}{\beta}\mathrm{D}_zN_f(\chi(\alpha),\alpha,\beta)\right]\Bigg|_{W_0^*}.$$

Therefore,

$$a_1(x_0,\alpha_0,\beta_0)=-\frac{1}{\beta}\mathrm{D}\bar{f}\,(\chi(\alpha),\alpha)|_{W_0^*}$$

and

$$\mu_1(x_0,\alpha_0,\beta_0)=\frac{1}{2}[1-(-1)^{\dim W_0^*}\,sign\,det\,\mathrm{D}\bar{f}\,(x_0,\alpha_0)|_{W_0^*}].$$

The conclusion then follows from Lemma 2.1. □

In a similar way, we can prove the following:

Lemma 3.5. *Assume that* $(\mathrm{A3})_k$ *is satisfied for each* $k\geq 1$ *and* $K=\mathbb{Z}_\infty$. *Then for* $H=\mathbb{Z}_\infty\times S^1$,

$$H-Deg(F_\Theta,U_{(x_0,\alpha_0,\beta_0)}(r,\rho))=\varepsilon(x_0,\alpha_0,\beta_0)\mu(x_0,\alpha_0,\beta_0),$$

where

$$\varepsilon(x_0,\alpha_0,\beta_0)=(-1)^N\,sign\,det\,\mathrm{D}\bar{f}\,(x_0,\alpha_0),$$

$$\mu(x_0,\alpha_0,\beta_0)=\sum_{k\geq 1,j\geq 0}c_{k,j}(x_0,\alpha_0,\beta_0)(G_{k,j}),$$

$$H_{k,j}=\{(\theta,kj\theta);\ \theta\in\mathbb{Z}_\infty\cong S^1\}.$$

Note that $z \in W$ has isotropy group containing $H_{k,j}$ means that

$$\rho\left(\frac{1}{m}\right) z(t) = z\left(t - \frac{j}{m}\right), \qquad t \in \mathbb{R} \quad \text{if } K = \mathbb{Z}_m < \infty$$

or

$$\rho(\theta) z(t) = z(t - j\theta), \qquad \theta \in \mathbb{Z}_\infty, \quad t \in \mathbb{R} \text{ if } K = \mathbb{Z}_\infty.$$

Therefore, by using Theorem 2.1, we have the following local Hopf bifurcation theorem:

Theorem 3.1. *Assume that* (A1)–(A3) *are satisfied and* $c_{1,j}(x_0, \alpha_0, \beta_0)$ *defined in* (3.11) *in nonzero. Then there exists a sequence of triples* $\{(x_k, \alpha_k, \beta_k)\}_{k=1}^{\infty}$ *such that*

(i) $\alpha_k \to \alpha_0$, $\beta_k \to \beta_0$ *and* $x_k(t) \to x_0$ *uniformly for* $t \in \mathbb{R}$ *as* $k \to \infty$,

(ii) $x_k(t)$ *is a nontrivial* $2\pi/\beta_k$*-periodic solution of Eq.* (3.1) *with* $\alpha = \alpha_k$ *for* $k = 1, 2, \ldots,$

(iii) *If* $\Gamma_{x_0} = \mathbb{Z}_m$ *with* $m < \infty$, *then* $x_k(t)$ *is a discrete wave satisfying* $\rho(\frac{1}{m}) x_k(t) = x_k(t - (2\pi/\beta_k)(j/m))$ *for* $t \in \mathbb{R}$; *if* $\Gamma_{x_0} = \mathbb{Z}_\infty$ *then* $x_k(t)$ *is a rotating wave with* $\rho(\theta) x_k(t) = x_k(t - (2\pi/\beta_k) j\theta)$ *for* $\theta \in \mathbb{Z}_\infty$ *and* $t \in \mathbb{R}$.

Note that in Theorem 3.1, $2\pi/\beta_k$ is not necessarily the minimal period of $x_k(t)$. To obtain further information about the minimal period of $x_k(t)$, we need the following:

Lemma 3.6. *Suppose that there exists a sequence of real numbers* $\{\alpha_k\}_{k=1}^{\infty}$ *and an integer* $q \geq 0$ *such that*

(i) *for each k, Eq.* (3.1) *with* $\alpha = \alpha_k$ *has a non-constant periodic solution* $x_k(t)$ *with the minimal period* $p_k > 0$.

(ii) $\lim_{k\to\infty} \alpha_k = \alpha_0$, $\lim_{k\to\infty} p_k = p_0 < \infty$ *and* $\lim_{k\to\infty} x_k(t) = x_0 \in \mathbb{R}^N$ *uniformly for* $t \in \mathbb{R}$;

(iii) $\rho(\frac{1}{m}) x_k(t) = x_k(t - p_k q/m)$ *if* $\Gamma x_0 = \mathbb{Z}_m$, $m < \infty$, *or* $\rho(\theta) x_k(t) = x_k(t - p_k q\theta)$ *if* $\Gamma_{x_0} = \mathbb{Z}_\infty$, *where* $t \in \mathbb{R}$, $\theta \in \mathbb{Z}_\infty$ *and* $k = 1, 2, \ldots.$

Then $2\pi/p_0$ *is an eigenvalue of the stationary solution* (x_0, α_0) *and* $\det_{\mathbb{C}} \Delta_{(\alpha_0, x_0), q}(\mathrm{i}\, 2\pi/p_0) = 0$.

Proof. We first show that $p_0 > 0$. There exists $\varepsilon_0 > 0$ and $L_0 > 0$ such that $|f(\varphi, \alpha) - f(\psi, \alpha)| \leq L_0 \|\varphi - \psi\|$ if $\|\varphi - \bar{x}_0\|$, $\|\psi - \bar{x}_0\| < \varepsilon_0$ and $|\alpha_0| < \varepsilon_0$. Therefore, under the assumption (ii), we can apply the argument of [27] to show that $\inf\{p_k;\ k = 1, 2, \ldots,\} > 0$. So, $p_0 > 0$.

We next use the idea of [29] to show that the linear retarded equation

$$\dot{y}(t) = \mathrm{D}f(\bar{x}_0, \alpha_0) y_t \tag{3.14}$$

has a periodic solution which is of the minimal period p_0 and satisfies $\rho(1/m) y(t) = y(t - p_0 q/m)$ if $\Gamma_{x_0} = \mathbb{Z}_m$, $m < \infty$ or $\rho(\theta) y(t) = y(t - p_0 q\theta)$ if $\Gamma_{x_0} = \mathbb{Z}_\infty$, $\theta \in \mathbb{Z}_\infty$. For any $\tau \in (0,1)$, define

$$\varepsilon_{k,\tau} = \max_{t\in\mathbb{R}} |x_k(t+\tau p_k) - x_k(t)|,$$

$$y^{k,\tau}(t) = \varepsilon_{k,\tau}^{-1}[x_k(t+\tau p_k) - x_k(t)], \quad t\in\mathbb{R}.$$

Then $y^{k,\tau}(t)$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} y^{k,\tau}(t) = \mathrm{D}f(\bar{x}_0,\alpha_0) y_t^{k,\tau} + \delta_{k,\tau}(t)$$

with

$$\begin{aligned}|\delta_{k,\tau}(t)| &= \varepsilon_{k,\tau}^{-1} |f((x_k)_{t+\tau p_k},\alpha_k) - f((x_k)_t,\alpha_k) - \mathrm{D}f(\bar{x}_0,\alpha_0)((x_k)_{t+\tau p_k} - (x_k)_t)| \\ &\to 0 \quad \text{as } k\to\infty, \quad \text{uniformly for } t\in\mathbb{R}.\end{aligned}$$

Since $\sup_{t\in\mathbb{R}} |y^{k,\tau}(t)| = 1$, $\{y^{k,\tau}\}_{k=1}^{\infty}$ has a convergent subsequence, denoted again by $\{y^{k,\tau}\}_{k=1}^{\infty}$ for simplicity. Let $y^{\tau}(t) = \lim_{k\to\infty} y^{k,\tau}(t)$. Then $y^{\tau}(t)$ is a non-constant periodic solution of Eq. (3.14) and

$$\left.\begin{aligned} &\rho\left(\frac{1}{m}\right) y^{\tau}(t) = y^{\tau}\left(t - \rho_0 \frac{q}{m}\right) \quad \text{if } \Gamma_{x_0} = \mathbb{Z}_m,\ m<\infty \\ &\text{or} \\ &\rho(\theta) y^{\tau}(t) = y^{\tau}(t - \rho_0 q\theta) \qquad \text{if } \Gamma_{x_0} = \mathbb{Z}_{\infty},\ \theta\in\mathbb{Z}_{\infty}. \end{aligned}\right\} \tag{3.15}$$

Denote by T_τ the minimal period of $y^{\tau}(t)$. Then $p_0 = mT_\tau$ for some positive integer m. If $m=1$, then we are done. If $m\neq 1$, then τp_0 is not an integer multiple of T_τ. For, otherwise, the equality $x_k(t+m\tau p_k) = x_k(t)$ implies that

$$\begin{aligned}0 &= \sum_{j=1}^{m} [x_k(t+j\tau p_k) - x_k(t+(j-1)\tau p_k)]\varepsilon_{k,\tau}^{-1} \to \sum_{j=0}^{m-1} y^{\tau}(t+j\tau p_0) \\ &= m y^{\tau}(t),\end{aligned}$$

a contradiction to the fact that $y^{\tau}(t)$ is non-constant.

Since for every rational $\tau\in(0,1)$ there is a periodic solution of Eq. (3.14) satisfying Eq. (3.15) whose minimal period divides p_0 but does not divide τp_0, we may choose some collection $\{z_j\}$ of solutions of Eq. (3.14) satisfying Eq. (3.15) and such that p_0 is the smallest number which is a multiple of their minimal periods. It follows that for almost any choice of real numbers $\{c_j\}$, $\sum c_j z_j$ is a periodic solution of Eq. (3.14) satisfying Eq. (3.15) with the minimal period p_0. This completes the proof. □

Now we can state the following refinement of Theorem 3.1, controlling the minimal period of the branch of periodic solutions.

Theorem 3.2. *Suppose that all assumptions in Theorem* 3.1 *are satisfied. Let* p_k *denote the minimal period of* $x_k(t)$. *Then for every convergent subsequence* $\{p_{k_j}\}$ *of* $\{p_k\}$ *there exists a positive integer m such that* $2\pi/\beta_0 = mp_0$, $\mathrm{i}mp_0$ *is an eigenvalue of*

(x_0, α_0) *and* $\det_{\mathbb{C}} \Delta_{(x_0,\alpha_0),mr}(\mathrm{i}m\beta_0) = 0$, *where* $p_0 = \lim_{j\to\infty} p_{k_j}$. *In particular, if other purely imaginary eigenvalues of* (x_0, α_0) *are not integer multiple of* $\pm\mathrm{i}\beta_0$, *then* $2\pi/\beta_k$ *is the minimal period of* $x_k(t)$.

Proof. Since p_{k_j} divides $2\pi/\beta_{k_j}$, $\lim_{j\to\infty} p_{k_j} = p_0$ and $\lim_{j\to\infty} 2\pi/\beta_{k_j} = 2\pi/\beta_0$, there exists a positive integer m such that $2\pi/\beta_0 = mp_0$ and $2\pi/\beta_{k_j} = mp_{k_j}$, except for a finite number of terms. Then our conclusion follows easily from Lemma 3.6. □

We now consider global continuation of local bifurcation of discrete/rotating waves. We need the following assumption:

(A4) M^* defined in (A2) is complete and the set M of $(x, \alpha, \beta) \in M^* \times (0, \infty)$ such that $\mathrm{i}\beta$ is an eigenvalue of the stationary solution of (x, α) is discrete in $\mathbb{R}^N \times \mathbb{R} \times (0, \infty)$.

Theorem 3.3. *Assume that* (A1), (A2), $(\mathrm{A3})_k$ *and* (A4) *are satisfied for every* $k \geq 1$. *Let* $\mathscr{S}$ *denote the closure in* $C(S^1; \mathbb{R}^N) \times \mathbb{R}^2$ *of the set of all* $(z, \alpha, \beta) \notin M$ *such that* $z((\beta/2\pi)t)$ *is a* $2\pi/\beta$*-periodic solution of Eq.* (3.1). *Then for each bounded connected component* $\mathscr{C}$ *of* $\mathscr{S}$, *the set* $(\Gamma \times \mathscr{S}^1)\mathscr{C} \cap M$ *is finite and is composed of a finite number of disjoint* Γ*-orbits*

$$(\Gamma \times S^1)\mathscr{C} \cap M = \bigcup_{i=1}^{q} (\Gamma x_i, \alpha_i, \beta_i).$$

Moreover,

$$\sum_{i=1}^{q} \varepsilon(x_i, \alpha_i, \beta_i) \nu(x_i, \alpha_i, \beta_i) \mu(x_i, \alpha_i, \beta_i) = 0.$$

In particular, if $M^* \subset (\mathbb{R}^N)^\Gamma \times \mathbb{R}$ *and* S_q *denotes the closure in* $C(S^1; \mathbb{R}^N) \times \mathbb{R}^2$ *of the set of all* $(z, \alpha, \beta) \notin M$ *such that* $x(t) = z((\beta/2\pi)t)$ *is a* $2\pi/\beta$*-periodic solution of Eq.* (3.1) *satisfying* $\rho(1/m)x(t) = x(t - (2\pi/\beta)(q/m))$ *if* $\Gamma = \mathbb{Z}_m$, $t \in \mathbb{R}$ *or* $\rho(\theta)x(t) = x(t - (2\pi/\beta)q\theta)$ *if* $\Gamma = \mathbb{Z}_\infty$, $t \in \mathbb{R}$, *then for each bounded connected component* C_q *of* S_q, *we have* $(\Gamma \times S^1)C_q \cap M$ *is a finite set* $\bigcup_{i=1}^{q} (x_i, \alpha_i, \beta_i)$ *and* $\sum_{i=1}^{q} \varepsilon(x_i, \alpha_i, \beta_i) c_{1,q}(x_i, \alpha_i, \beta_i) = 0$.

Proof. For each bounded connected component $\mathscr{C}$ of $\mathscr{S}$, we can apply the argument of [27] to show that $\inf\{2\pi/\beta; (z, \alpha, \beta) \in \mathscr{C}\} > 0$. Consequently, the conclusion follows from Theorem 2.2. □

4. An example: discrete waves caused by delays in identical cells coupled in a ring

In this section, we illustrate our main result for discrete waves with a ring of identical oscillators with identical coupling between adjacent cells. Such a ring was modelled in

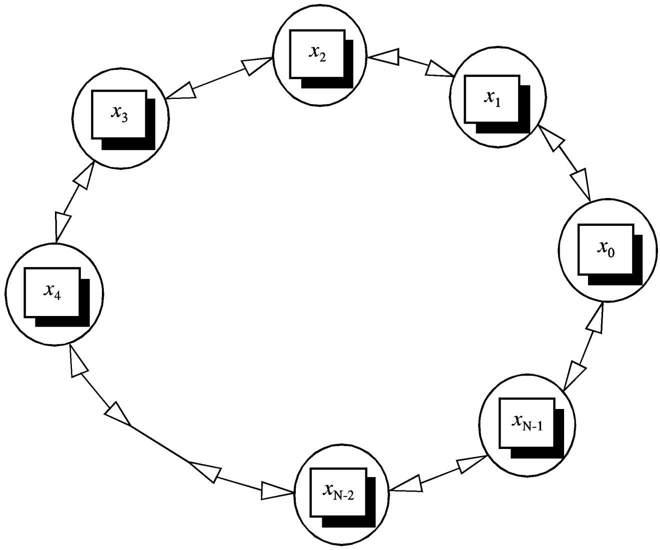

Fig. 1. Schematic picture of a ring of N coupled identical cells.

the seminar paper by [34] on morphogenesis and provides models for many situations in biology, chemistry and electrical engineering. The Hopf bifurcation of this Turing ring has been extensively studied in the literature. We refer to [2,3,13,18,20,30,32,36] and references therein for the current state of the literature.

Our primary interest is to illustrate how the temporal delay in the kinetics and in the coupling of cells affects the type of oscillation that may be observed in the system. In particular, we will show that the temporal delay in the coupling between adjacent cells may cause oscillations in the case where each cell is described by only one state variable, though it has been shown (cf. [16]) that such oscillations cannot occur if the temporal delay is neglected.

Our secondary interest is to illustrate how to apply our global bifurcation theorem to obtain the existence of large-amplitude periodic solutions with prescribed symmetries when the parameter is far away from a bifurcation value in systems of functional differential equations.

We emphasize the importance of temporal delays in the coupling between cells, since in many chemical and biological oscillators (cells coupled via membrane transport of ions), the time needed for transport or processing of chemical components or signals may be of considerable length. While such delay equations in mathematical biology have been studied extensively in the literature (see cf. [6,11,28]), the qualitative study of the effect of temporal delays on oscillations of coupled oscillators has not been found, to the best of our knowledge.

We consider a ring of N identical cells which are coupled symmetrically by diffusion along the sides of an N-gon, as in Fig. 1. Each cell will be regarded as a chemical system with m distinct chemical species. The concentrations $u^{j,i}(t)$ of the ith species in the jth cell is assumed to obey the kinetic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u^{j,i}(t)=f_i(u_t^{j,1},\dots,u_t^{j,m},\alpha),\quad 1\le i\le m,\ 1\le j\le N \tag{4.1}$$

which can be reformulated in the vector from

$$\frac{\mathrm{d}}{\mathrm{d}t}u^{j}(t)=f(u_t^{j},\alpha),\quad 1\le j\le N, \tag{4.2}$$

where $t\in\mathbb{R}$ denotes the time, $\alpha\in\mathbb{R}$ is parameter, $u^j(t)=(u^{j,1}(t),\dots,u^{j,m}(t))^{\mathrm{T}}$, $1\le j\le N$, and $f: C([-\tau,0];\mathbb{R}^m)\to\mathbb{R}^m$ is continuously differentiable. The kinetic equation (4.2) is usually determined by some physical, chemical or biological laws.

We assume that the coupling is "nearest-neighbor" and symmetric in the sense that the interaction between any neighboring pair of cells takes the same form. For simplicity, we also assume that the coupling between adjacent cells is linear. In practice, there are a number of mechanisms and transport processes, whereby the concentrations of a chemical species in one cell could affect the concentration of the same species in the adjacent cells and such an effect takes place after a certain amount of time. So we have a system of retarded functional differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u^{j}(t)=f(u_t^{j},\alpha)-K(\alpha)(2u_t^j-u_t^{j-1}-u_t^{j+1}),\quad 1\le j\le N, \tag{4.3}$$

where $K(\alpha): C([-\tau,0];\mathbb{R}^m)\to\mathbb{R}^m$ is a bounded linear operator and the mapping $\alpha\in\mathbb{R}\to K(\alpha)\in\mathscr{L}(C([-\tau,0];\mathbb{R}^m;\mathbb{R}^m)$ is continuously differentiable. $K(\alpha)$ represents *coupling strength* and the additional term

$$K(\alpha)(u_t^{j-1}-u_t^j)+K(\alpha)(u_t^{j+1}-u_t^j)$$

in Eq. (4.3) is usually supported by the ordinary law of diffusion, i.e. each chemical substance moves from region of greater to region of less concentration, at a rate proportional to the gradient of the concentration. For details, we refer to [13, 16, 34] and references therein.

Suppose that

$$f(0,\alpha)=0. \tag{4.4}$$

Then $(0,\dots,0,\alpha)$ is a stationary solution of Eq. (4.3) and the linearization of Eq. (4.3) at $(0,\dots,0,\alpha)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}x^{j}(t)=\mathrm{D}f(0,\alpha)x_t^j-K(\alpha)[2x_t^j-x_t^{j-1}-x_t^{j+1}],\quad 1\le j\le N. \tag{4.5}$$

Consequently, $\lambda\in\mathbb{C}$ is an eigenvalue of $(0,\dots,0,\alpha)$ iff there exists a nonzero vector $(z^1,\dots,z^N)\in\mathbb{C}^{Nm}$ such that

$$\mathrm{diag}(\lambda\cdot\mathrm{Id}-Df(0,\alpha)(\mathrm{e}^{\lambda\cdot}\mathrm{Id}))z=\delta(\alpha,\lambda)z, \tag{4.6}$$

where $\mathrm{diag}(\lambda\cdot\mathrm{Id}-Df(0,\alpha)(\mathrm{e}^{\lambda\cdot}\mathrm{Id}))$ denotes the block-diagonal $mN\times mN$ matrix and $\delta(\alpha,\lambda):\mathbb{C}^{mN}\to\mathbb{C}^{nN}$ is defined by

$$(\delta(\alpha,\lambda)z)_j=K(\alpha)[\mathrm{e}^{\lambda\cdot}(z^{j-1}+z^{j+1}-2z^j)],\quad 1\le j\le N.$$

Therefore, we have

$$\Delta_\alpha(\lambda) = \operatorname{diag}(\lambda\,\mathrm{Id} - \mathrm{D}f(0,\alpha)(\mathrm{e}^{\lambda\cdot}\mathrm{Id})) - \delta(\alpha,\lambda) \tag{4.7}$$

for $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

To prevent complicating the argument with generalities, we consider the case where each chemical system has only one chemical species, i.e. $m = 1$. In this case, Hopf bifurcation cannot occur if the temporal delay is neglected (see, cf. [16, 18]). However, as we will show, the temporal delay in the coupling cells provides an important resource for various types of oscillations that may be observed.

In the case where $m = 1$, we have

$$\Delta_\alpha(\lambda) = \operatorname{diag}(\lambda - \mathrm{D}f(0,\alpha)(\mathrm{e}^{\lambda\cdot})) - K(\alpha)\mathrm{e}^{\lambda\cdot}\delta,$$

where $\delta : \mathbb{R}^N \to \mathbb{R}^N$ is the discretized Laplacian defined by

$$(\delta x)_i = x_{i+1} - 2x_i + x_{i-1}, \quad 1 \le i \le N.$$

Let $\xi = \mathrm{e}^{\mathrm{i}\,2\pi/N}$ be a primitive Nth root of the unity in $\mathbb{C}$. Then

$$\xi^{-j} = \overline{\xi}^{j} = \xi^{N-j}, \quad 0 \le j \le N-1.$$

Put

$$\mathbb{C}_r^N := \{(1, \xi^r, \xi^{2r}, \ldots, \xi^{(N-1)r})^{\mathrm{T}} x;\; x \in \mathbb{C}\}, \quad 0 \le r \le N-1.$$

Then

$$\mathbb{C}^N = \mathbb{C}_0^N \oplus \mathbb{C}_1^N \oplus \cdots \oplus \mathbb{C}_{N-1}^N, \tag{4.8}$$

and for every $x \in \mathbb{C}$, $r \in \{0, \ldots, N-1\}$ and $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned}
&(\Delta_\alpha(\lambda)(1, \xi^r, \xi^{2r}, \ldots, \xi^{(N-1)r})^T x)_i \\
&\quad = [\lambda\xi^{(i-1)r} - \mathrm{D}f(0,\alpha)(\mathrm{e}^{\lambda\cdot})\xi^{(i-1)r} - K(\alpha)\mathrm{e}^{\lambda\cdot}(\xi^{ir} - 2\xi^{(i-1)r} + \xi^{(i-2)r})]x \\
&\quad = [\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} - K(\alpha)\mathrm{e}^{\lambda\cdot}(\xi^r - 2 + \xi^{-r})]\xi^{(i-1)r}x \\
&\quad = [\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} - K(\alpha)\mathrm{e}^{\lambda\cdot}(2\mathrm{Re}\,\xi^r - 2)]\xi^{(i-1)r}x \\
&\quad = \left[\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} - 2\left(\cos\frac{2\pi r}{N} - 1\right)K(\alpha)\mathrm{e}^{\lambda\cdot}\right]\xi^{(i-1)r}x \\
&\quad = \left[\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} + 4\sin^2\frac{\pi r}{N}K(\alpha)\mathrm{e}^{\lambda\cdot}\right]\xi^{(i-1)r}x.
\end{aligned} \tag{4.9}$$

Consequently, we have

Proposition 4.1. *In the case where $m = 1$, we have*

$$\det \Delta_\alpha(\lambda) = \prod_{r=0}^{N-1}\left[\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} + 4\sin^2\frac{\pi r}{N}K(\alpha)\mathrm{e}^{\lambda\cdot}\right].$$

So $\lambda \in \mathbb{C}$ *is an eigenvalue of* $(0,\alpha)$ *iff there exists an* $r \in \{0,\dots,N-1\}$ *such that*

$$p_r(\alpha,\lambda) := \lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} + 4\sin^2\frac{\pi r}{N}K(\alpha)\mathrm{e}^{\lambda\cdot} = 0. \tag{4.10}$$

Remark 4.1. *Note that* $\sin^2 \pi r/N = \sin^2 \pi(N-r)/N$ *for each* $r \in \{0,\dots,N-1\}$. *So the zero of* $\det \Delta_\alpha(\lambda) = 0$ *has always an even algebraic multiplicity, except possibly the zero of*

$$\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} = 0$$

which corresponds to Eq. (4.10) *with* $r=0$, *and the zero of*

$$\lambda - \mathrm{D}f(0,\alpha)\mathrm{e}^{\lambda\cdot} + 4K(\alpha)\mathrm{e}^{\lambda\cdot} = 0 \quad \textit{if } N \textit{ is even}$$

which corresponds to Eq. (4.10) *with* $r = N/2$. *It is a common feature that the presence of a symmetry forces purely imaginary eigenvalues to be multiple. One may apply the Hopf bifurcation theorem from a multiple eigenvalue in* [9]. *But our priminary interest here is the existence of periodic solutions with prescribed symmetries.*

It is easy to check that system (4.3) is equivariant with respect to the action of the dihedral group D_N, where the subgroup $\mathbb{Z}_N$ permutes the variable cyclically, sending cell j to $j-1 \pmod N$), whereas the flip interchanges, sending cell j to $-j \pmod N$). More previsely, the dihedral group D_N is generated by a rotation θ and a reflection κ such that

$$\langle\theta\rangle \cong \mathbb{Z}_N, \qquad \langle\kappa\rangle \cong \mathbb{Z}_2, \qquad \kappa\theta\kappa^{-1} = \theta^{-1}$$

and system (4.3) is equivariant with respect to the representation $\rho : D_N \to GL(\mathbb{R}^N)$ defined by

$$(\rho(\theta)x)_j = x_{j-1}, \qquad (\rho(\kappa)x)_j = x_{-j}, \ (\mathrm{mod}\, N), \quad x \in \mathbb{R}^N.$$

Note that the isotypical decomposition of $\mathbb{C}^N$ with respect to the subrepresentation of $\mathbb{Z}_N$ is given by Eq. (4.8). By Eq. (4.9) we have

$$\Delta_{\alpha,r}(\lambda) = \Delta_\alpha(\lambda)|_{\mathbb{C}^N_r} = p_r(\alpha,\lambda).$$

Consequently, applying Theorem 3.2 we obtain

Theorem 4.1. *Suppose that there exist two positive numbers* ε *and* β_0 *and an integer* $r \in \{0,\dots,N-1\}$ *such that*

(i) $\mathrm{D}\bar{f}(0,0) - 4\sin^2(\pi j/N)\overline{K}(0) \neq 0$ *for* $j \in \{0,\dots,N-1\}$;

(ii) $p_j(\alpha, \mathrm{i}\beta) = 0$ *for some* $j \in \{0,\dots,N-1\}$ *and* $(\alpha,\beta) \in [\alpha_0-\varepsilon, \alpha_0+\varepsilon] \times [\beta_0-\delta, \beta_0+\delta]$ *iff* $\alpha = 0$, $\beta = \beta_0$ *and* $j = r \pmod N$);

(iii) $p_j(\alpha_0, u+\mathrm{i}v) = 0$ *for some* $j \in \{0,\dots,N-1\}$ *and* $(u,v) \in \partial\Omega$ *with* $\Omega := (0,\varepsilon) \times (\beta_0 - \delta, \beta_0+\delta)$ *iff* $j = r \pmod N$), $u=0$ *and* $v = \beta_0$;

(iv) $deg_B(p_r(\alpha_0-\varepsilon,\cdot),\Omega) \neq deg_B(p_r(\alpha_0+\varepsilon,\cdot),\Omega)$.

Then there exists a sequence of triples $\{(x^k,\alpha^k,\beta^k)\}_{k=1}^\infty$ *such that*

(a) $\alpha^k \to \alpha_0$, $\beta^k \to \beta_0$, $x^k(t) \to 0$ *uniformly for* $t \in \mathbb{R}$ *as* $k \to \infty$.

(b) $x^k(t)$ *is a* $2\pi/\beta^k$*-periodic solution of Eq.* (4.3) *with* $\alpha=\alpha^k$ *for* $k=1,2,\dots$.
(c) $x_{i-1}^{k-1}(t)=x_i^k(t-(2\pi/\beta_k)\cdot(r/N))$ *for* $t\in\mathbb{R}$ *and* $k=1,2,\dots,i\,(\mathrm{mod}\,N)$.
Moreover, if $p_j(\alpha_0,\mathrm{i}\beta)=0$ *implies* $j=r\,(\mathrm{mod}\,N)$ *and* $\beta=\beta_0$, *then* $2\pi/\beta^k$ *is the minimal period of* $x^k(t)$.

The oscillation obtained in the above result is called a *synchronous oscillation* in the case of $r=0$ in which oscillators are in phase, a *phase-locked oscillation* in the case of $r\neq 0$ in which each cell oscillates just like the others except not necessarily in phase with each other. We refer to [3] for a detailed discussion in the case where the temporal delay is neglected.

Finally, we demonstrate how to apply our global bifurcation theorem to obtain large-amplitude periodic solutions with prescribed symmetries when the parameter is far away from bifurcation values.

Example 4.1. Consider the following system of retarded functional differential equations

$$\dot{x}_i(t)=-\alpha x_i(t)+\alpha h(x_i(t))[2g(x_i(t-1))-g(x_{i-1}(t-1))-g(x_{i+1}(t-1))], \tag{4.11}$$

where $i\,(\mathrm{mod}\,N), \alpha\geq 0,\ h,g:\mathbb{R}\to\mathbb{R}$ are continuously differentiable, h does not vanish and $g(0)=0,\ g'(0)>0$. Such a system can be obtained from

$$\dot{y}_i(t)=-f(y_i(t))+d[y_{i+1}(t-\alpha)+y_{i-1}(t-\alpha)-2y_i(t-\alpha)],$$

by rescaling the time and making a certain change of variables.

Clearly, $(0,\dots,0,\alpha)$ is a stationary solution of Eq. (4.11) and the linearization of this solution is

$$\dot{z}_i(t)=-\alpha z_i(t)-\alpha\mu[2z_i(t-1)-z_{i-1}(t-1)-z_{i+1}(t-1)],$$

where $\mu=h(0)g'(0)$. Therefore,

$$\Delta_{\alpha,r}(\lambda)=\lambda+\alpha+4\sin^2\frac{\pi r}{N}\alpha\mu e^{-\lambda}. \tag{4.12}$$

Letting $\lambda=\mathrm{i}\beta$ in Eq. (4.12), we obtain

$$\begin{cases}\alpha+4\alpha\mu\sin^2\dfrac{\pi r}{N}\cos\beta=0\\ \beta-4\alpha\mu\sin^2\dfrac{\pi r}{N}\sin\beta=0.\end{cases} \tag{4.13}$$

Assume that there exists $r\in\{0,\dots,N-1\}$ such that

$$\mu>\frac{1}{4\sin^2(\pi r/N)}. \tag{4.14}$$

Let $\beta_{0,r}\in(\frac{\pi}{2},\pi)$ denote the unique solution of

$$\cos\beta_{0,r}=-\frac{1}{4\mu\sin^2(\pi r/N)}$$

and define

$$\alpha_{0,r} = -\beta_{0,r}\cot\beta_{0,r}.$$

Then $i\beta_{0,r}$ is an eigenvalue of the stationary solution $(0,\dots,0,\alpha_{0,r})$. Moreover, assume $u(\alpha)+iv(\alpha)$ satisfies Eq. (4.12) with $u(\alpha_{0,r})+iv(\alpha_{0,r})=i\beta_{0,r}$. Then

$$u+\alpha+4\alpha\mu\sin^2\frac{\pi r}{N}e^{-u}\cos v=0, \qquad v-4\alpha\mu\sin^2\frac{\pi r}{N}e^{-u}\sin v=0.$$

Differentiating both sides of Eq. (4.12) with respect to α, we get

$$(1+\alpha_{0,r})u'(\alpha_{0,r}-(\beta_{0,r}+\alpha_{0,r})v'(\alpha_{0,r})=0,$$

$$(\beta_{0,r}+\alpha_{0,r})u'(\alpha_{0,r})+(1+\alpha_{0,r})v'(\alpha_{0,r})=\beta_{0,r}$$

from which it follows that

$$u'(\alpha_{0,r})=\frac{(\beta_{0,r}+\alpha_{0,r})\beta_{0,r}}{(\beta_{0,r}+\alpha_{0,r})^2+(1+\alpha_{0,r})^2}>0. \tag{4.15}$$

Consequently,

$$c_{r,\alpha_{0,r},\beta_{0,r}}=deg_B(\Delta_{\alpha_{0,r}-\varepsilon,r}(\cdot),\Omega)-deg_B(\Delta_{\alpha_{0,r}+\varepsilon,r}(\cdot),\Omega)<0 \tag{4.16}$$

for sufficiently small $\delta,\varepsilon>0$, where $\Omega=(0,\varepsilon)\times(\beta_{0,r}-\varepsilon,\beta_{0,r}+\varepsilon)$. Consequently, By Theorem 3.2 we have

Proposition 4.2. *If there exists an* $r\in\{0,\dots,N-1\}$ *such that Eq.* (4.14) *is satisfied, then there exists a sequence of triples* $\{(x^k,\alpha^k,\beta^k)\}_{k=1}^{\infty}$ *such that*
(i) $\alpha^k\to\alpha_{0,r}$, $\beta^k\to\beta_{0,r}$, $x^k(t)\to 0$ *uniformly for* $t\in\mathbb{R}$ *as* $k\to\infty$.
(ii) $x^k(t)$ *is a* $2\pi/\beta^k$*-periodic solution of Eq.* (4.11) *with* $\alpha=\alpha^k$, $k=1,2,\dots$.
(iii) $x^k_{i-1}(t)=x^k_i(t-(2\pi/\beta^k)\cdot(r/N))$ *for* $t\in\mathbb{R}$ *and* $k=1,2,\dots,i\,(\mathrm{mod}\,N)$.

To obtain a large-amplitude periodic solution of Eq. (4.11) when α is far away from $\alpha_{0,r}$, we need the following.

Proposition 4.3. *If* $xg(x)/h(x)>0$ *for all* $x\neq 0$ *and* $\lim_{x\to\infty}xg(x)/h(x)=\infty$, *then Eq.* (4.11) *has no non-trivial* 4*-periodic solution with* $x_{j-1}(t)=x_j(t-2)$ *for* $j\,(\mathrm{mod}\,N)$, $t\in\mathbb{R}$.

Proof. By way of contradiction, if Eq. (4.11) has a non-trivial 4-periodic solution with $x_{j-1}(t)=x_j(t-2)$ for $t\in\mathbb{R}$, then $x_j(t)$ is a nontrivial 4-periodic solution of the following scalar equation

$$\begin{aligned}\dot{x}_1(t)&=-\alpha x_1(t)+\alpha h(x_1(t))[g(x_1(t-3))+g(x_1(t+1))-2g(x_1(t-1))]\\&=-\alpha x_1(t)+\alpha h(x_1(t))[g(x_1(t-3))+g(x_1(t-3))-2g(x_1(t-1))]\\&=-\alpha x_1(t)+2\alpha h(x_1(t))[g(x_1(t-3))-g(x_1(t-1))].\end{aligned}$$

Consequently, $(x_1,x_2,x_3,x_4)^{\mathrm{T}}$, with $x_i(t)=x_1(t-i+1)$, $i=1,2,3,4$, satisfies the following system of ordinary differential equations

$$\begin{aligned}
\dot{x}_1 &= -\alpha x_1 + 2\alpha h(x_1)[g(x_4)-g(x_2)],\\
\dot{x}_2 &= -\alpha x_2 + 2\alpha h(x_2)[g(x_1)-g(x_3)],\\
\dot{x}_3 &= -\alpha x_3 + 2\alpha h(x_3)[g(x_2)-g(x_4)],\\
\dot{x}_4 &= -\alpha x_4 + 2\alpha h(x_4)[g(x_3)-g(x_1)].
\end{aligned} \tag{4.17}$$

Let

$$V(x_1,x_2,x_3,x_4)=\sum_{i=1}^{4}\int_0^{x_i}\frac{g(s)}{h(s)}\,\mathrm{d}s.$$

Then $V(x_1,x_2,x_3,x_4)$ is positive definite and the derivative of V along solutions of Eq. (4.17) is negative definite. Therefore, a standard Liapunov stability theorem implies that the zero solution of Eq. (4.17) is globally asymptotically stable. Consequently, Eq. (4.17) has no non-trivial periodic solution, a contradiction. This completes the proof. □

Theorem 4.2. *Assume that n is even, $\mu>\frac{1}{4}$, $\lim_{x\to\infty}(xg(x)/h(x))=\infty$, $xg(x)/h(x)>0$ for $x\neq 0$ and h,g are both bounded. Then for each $\alpha>\alpha_{0,n/2}=-\beta_{0,n/2}\cot\beta_{0,n/2}$, where $\cos\beta_{0,n/2}=-1/4\mu$ and $\beta_{0,n/2}\in(\pi/2,\pi)$, system (4.11) has a non-trivial periodic solution satisfying $x_j(t)=x_j(t+p)$ for some $p\in(2,4)$ and $x_{j-1}(t)=x_j(t-\frac{p}{2})$, $t\in\mathbb{R}$, $j\,(\mathrm{mod}\,N)$.*

Proof. Let

$$\begin{aligned}
\mathscr{S}=cl\{(z,\alpha,p);\ & x(t)=z(t/p) \text{ is a } p\text{-periodic solution of Eq. (4.11) with}\\
& x_{j-1}(t)=x_j(t-p/2),\ t\in\mathbb{R},\ j\,(\mathrm{mod}\,n)\},\\
& \subset C^1(S^1;\mathbb{R}^N)\times\mathbb{R}^2
\end{aligned}$$

and $\mathscr{C}$ denote the connected component of $\mathscr{S}$ containing $(0,\alpha_{0,n/2},2\pi/\beta_{0,n/2})$. By Proposition 4.2, $\mathscr{C}$ is nonempty. Moreover, by Theorem 3.3 and Eq. (4.16), $\mathscr{C}$ must be unbounded. On the other hand, since $\beta_{0,n/2}\in(\pi/2,\pi)$, if $(z,\alpha,p)\in\mathscr{C}$ is close to $(0,\alpha_{0,n/2},2\pi/\beta_{0,n/2})$, then $p\in(2,4)$. Using a similar argument to that in [7], we can easily show that system (4.11) has no 2-periodic solution satisfying $x_{j-1}(t)=x_j(t)$. Therefore, by Proposition 4.3, we have

$$\{p;(z,\alpha,p)\in\mathscr{C}\}\subset[2,4].$$

On the other hand, let $Q=\sup_{x\in\mathbb{R}}|h(x)|+\sup_{x\in\mathbb{R}}|g(x)|$, then we can easily show that the absolute value of each component of every periodic solution of Eq. (4.11) is bounded by $4Q^2$. Therefore, the set $\{\alpha; (z,\alpha,p)\in\mathscr{C}\}$ must be unbounded. But Eq. (4.11) with $\alpha=0$ clearly has no non-trivial periodic solution. Consequently,

$$(\alpha_{0,n/2},\infty)\subset\{\alpha;(z,\alpha,p)\in\mathscr{C}\}.$$

That is, for every $\alpha>\alpha_{0,n/2}$ there exists a p-periodic solution $x(t)$ of Eq. (4.11) such that $p\in(2,4), x_{j-1}(t)=x_j(t-p/2)$. This completes the proof. □

Remark 4.3. *Similar arguments can also be applied to examine the global continua of the local bifurcation from $\alpha=\alpha_{0,r}$ from arbitrary $r=0,1,\ldots,n-1$ and for arbitrary n.*

References

[1] J.C. Alexander, Patterns at priminary Hopf bifurcations of a plexus of identical oscillators, SIAM J. Appl. Math. 46 (1986) 199–221.
[2] J.C. Alexander, Bifurcation of zeros of parametrized functions, J. Funct. Anal. 29 (1978) 37–53.
[3] J.C. Alexander, G. Auchmuty, Global bifurcations of phase-locked oscillators, Arch. Rational Mech. Anal. 93 (1986) 253–270.
[4] J.C. Alexander, P.M. Fitzpatrick, The homotopy of a certain spaces of nonlinear operators, and its relation to global bifurcation of the fixed points of parametrized condensing operators, J. Funct. Anal. 34 (1979) 87–106.
[5] J.C. Alexander, J.A. Yorke, Global bifurcation of periodic orbits, Amer. J. Math. 100 (1978) 263–292.
[6] U. an der Heiden, Delays in physiological systems, J. Math. Biol. 8 (1978) 345–364.
[7] S.N. Chow, J. Mallet-Paret, The Fuller index and global Hopf bifurcations, J. Differential Equations 29 (1978) 263–292.
[8] S.N. Chow, J. Mallet-Paret, J.A. Yorke, A periodic orbit index which is a bifurcation invariant, in: J. Palis (Ed.), Lecture Notes in Math., vol., 1007, Springer, New York, 1983, pp. 109–131.
[9] S.N. Chow, J. Mallet-Paret, J.A. Yorke, Global Hopf bifurcation from a multiple eigenvalue, Nonlinear Anal. TMA 2 (1978) 753–763.
[10] M. Crandall, P. Rabinowitz, Bifurcation from simple eigenvalues, J. Func. Anal. 8 (1971) 321–340.
[11] J.M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, Lecture Notes in Biomath., vol. 20, Springer, New York, 1977.
[12] L.H. Erbe, K. Gęba, W. Krawcewicz, J. Wu, S^1-degree and global Hopf bifurcation theory of functional differential equations, J. Differential Equations 98 (1992) 277–298.
[13] B. Fiedler, Global Bifurcation of Periodic Solutions with Symmetry, Lecture Notes in Math., vol. 1309, Springer, New York, 1988.
[14] F.B. Fuller, An index of fixed point type for periodic orbits, Amer. J. Math. 89 (1967) 133–148.
[15] K. Gęba, W. Krawcewicz, J. Wu, An equivariant degree with applications to symmetric bifurcation problems, I: construction of the degree, Proc. London Math. Soc. 69 (1994) 377–398.
[16] M. Golubitsky, D.G. Schaeffer, I.N. Stewart, Singularities and Groups in Bifurcation Theory, vol. II, Springer, New York, 1988.
[17] M. Golubitsky, I.N. Stewart, Hopf bifurcation in the presence of symmetry, Arch. Rational Mech. Anal. 87 (1985) 107–165.
[18] M. Golubitsky, I.N. Stewart, Hopf bifurcation with dihedral group symmetry: coupled nonlinear oscillators, in: M. Golubitsky, J. Guckenheimer (Eds.), Multiparameter bifurcation theory, Contemporary Math. 56 (1986) 131–137.
[19] J.K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[20] L.N. Howard, Nonlinear oscillations, in: F.R. Hoppensteadt (Ed.), Oscillations in Biology, AMS Lecture Notes in Math. 17 (1979) 1–69.
[21] J. Ize, Bifurcation theory for Fredholm operators, Mem. Amer. Math. Soc., vol. 184, American Mathematical Society, Providence, 1976.

[22] J. Ize, I. Massabó, V. Vignoli, Degree theory for equivariant maps, I. Trans. Amer. Math. Soc. 315 (1980) 433–510.

[23] J. Ize, I. Massabó, V. Vignoli, Degree theory for equivariant maps, II. The general S^1-action, Mem. Amer. Math. Soc., vol. 100, American Mathematical Society, Providence, 1992.

[24] N. Koppel, Forced and coupled oscillators in biological applications, Proc. Int. Congr. Math., Warsaw, 1983.

[25] M.A. Krasnosel'skii, Topological Methods in the Theory of Non-linear Integral Equations, Pergamon Press, New York, 1965.

[26] W. Krawcewicz, P. Vivi, J. Wu, Computation formulae of an equivalent degree with applications to symmetric bifurcations, Nonlinear Studies 4 (1997) 89–120.

[27] H. Lasota, J.A. York, Bounds for periodic solutions of differential equations in Banach spaces, J. Differential Equations 10 (1971) 83–91.

[28] N. MacDonald, Time Lags in Biological Models, Lecture Notes in Biomath., vol. 28, Springer, New York, 1979.

[29] J. Mallet-Paret, J.A. York, Oriented families of periodic orbits, their sources, sinks and continuation, J. Differential Equations 43 (1982) 419–450.

[30] H.G. Othmer, L.E. Scriven, Instability and dynamics pattern in cellular networks, J. Theoret. Biol. 32 (1971) 507–537.

[31] P.H. Rabinowitz, Some global results for non-linear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–573.

[32] D.H. Sattinger, Bifurcation and symmetry breaking in applied mathematics, Bull. Amer. Math. Soc. 3 (1980) 779–819.

[33] S. Smale, A mathematical model of two cells via Turing's equation, in: J.D. Cowan (Ed.), Some Mathematical Questions in Biology V, AMS Lecture Notes on Mathematics in the Life Sciences, vol. 6, American Mathematical Society, Providence, RI, 1974, pp. 15–26.

[34] A. Turing, The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. B 237 (1952) 37–72.

[35] A. Vanderbauwhede, Local Bifurcation and Symmetry, Research Notes in Mathematics, vol. 75, Pitman, Boston, 1982.

[36] S.A. van Gils, T. Valkering, Hopf bifurcation and symmetry: standing and travelling waves in a circular-chain, Japan J. Appl. Math. 3 (1986) 207–222.