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Hopf Bifurcations of Functional Differential Equations with Dihedral Symmetries

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We discuss the joint impact of temporal delay and spatial dihedral symmetries on the occurence and multiplicity of Hopf bifurcations for a system of FDEs. By applying the equivariant degree theory we establish a result on the existence of multiple branches of nonconstant periodic solutions and classify their symmetries. General results are illustrated by a ring of identical oscillators with identical coupling between adjacent cells. © 1998 Academic Press

1. INTRODUCTION

In this paper we develop a Hopf bifurcation theory for functional differential equations (FDEs) with dihedral symmetries via the equivariant degree theory and present an orbit type classification of possible Hopf bifurcations. We also discuss the joint impact of the time delay and the spatial dihedral symmetry on the multiplicity of bifurcating solutions in coupled cells.

The advantage of using the equivariant degree lies in the fact that it provides a local bifurcation invariant containing the full topological information on the "essential" types of bifurcated solutions and gives more control over the global bifurcation phenomena.

In this paper we use only a simplified, but easier to compute, version of the equivariant degree, which is called the *primary equivariant degree*.

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However, this degree already gives us enough information to classify effectively the types of possible Hopf bifurcations with dihedral symmetry. Due to the topological nature of the equivariant degree method, we can avoid several technical difficulties, such as symmetric generic approximations, encountered in Fiedler's approach (cf. [4]), which is based on a reduction of a system to a fixed point space with a cyclic group action. Nevertheless, his method may lead to the same results as ours for a system of coupled oscillators.

There are several advantages of using the equivariant degree method. It can be applied, without additional technicalities, to the study of Hopf bifurcation problems for FDEs exactly in the same way as for ODEs, and it also provides a local bifurcation invariant reflecting the true topological nature of the bifurcating branches of solutions (cf. [13, 19]). More precisely, a nonzero component of this local invariant indicates that "generically" there is a branch of solutions corresponding to the associated orbit type (possibly submaximal). The global Hopf bifurcation results are based on the fact that for a bounded branch of solutions the sum of local invariants must be zero. In our case the local bifurcation invariants take into account also submaximal orbit types and therefore the obtained relations may give better characterizations of bounded branches. In this paper we present the full classification of the local bifurcation invariants, including the submaximal orbit types, which can also be used for the description of the global bifurcations.

We should also mention that this method can be applied in a standard way to several Hopf bifurcation problems with more complicated symmetries like $SU(2)$ and $O(3)$ ($SU(n)$ or $O(n)$, in general). However, the equivariant degree method is just one of many ways to study Hopf bifurcation problems and we do not claim that similar results can not be obtained through another approach (for example, by using the equivariant singularity theory, cf. [8–10]). Still, we believe that equivariant degree method is a standard and relatively simple technique and can be effectively used to study various symmetric bifurcation problems. We refer to the monographs [4, 8, 10, 12] for a detailed account of the subject for ordinary differential equations and partial differential equations. As for FDEs, an analytic (local) Hopf bifurcation theorem was obtained in [31] as an analogy of the Golubitsky–Stewart theorem [9]. Moreover, a topological Hopf bifurcation theory was developed in [18] for FDEs in the case where the spatial symmetry group is the abelian group $\mathbb{Z}_N$ or $\mathbb{Z}_\infty := S^1$. While the problem of looking for local bifurcations of periodic solutions with prescribed symmetries can always be reduced to one where the spatial symmetry group is $\mathbb{Z}_N$ or $\mathbb{Z}_\infty$ (see [4, 9]), examining the global interaction of all bifurcated periodic solutions requires the consideration of the full symmetry group of the equation. Our results, especially the presented application to coupled cells arising from neural networks with memory [29, 30], illustrate that a nonabelian action, due to the fact that its irreducible representations may contain many different orbit types, can cause spontaneous bifurcations of multiple branches of periodic solutions with various symmetry properties. For example, if a coupled oscillators consist of N cells with N being a prime number, then at certain critical values of the parameter (usually the delay) the system possess at least $2(N+1)$ distinct branches of nonconstant periodic solutions with certain spatiotemporal patterns.

The rest of the paper is organized as follows: Section 2 summarizes some results about the equivariant degree theory, Section 3 contains the general results on Hopf bifurcations of FDEs with dihedral group symmetry, and Section 4 presents some applications of the general results to coupled cells.

2. G-EQUIVARIANT DEGREE

Our main technical tool is the equivariant degree which was introduced by Ize *et al.* (cf. [13–15]). We use the standard notation following [25] and [16]. Let G be a compact Lie group, V a finite dimensional orthogonal representation of G, and $\Omega \subset V \oplus \mathbb{R}^n$ an open bounded invariant set. We consider an equivariant map $f\colon V \oplus \mathbb{R}^n \to V$ such that $f(x) \neq 0$ for $x \in \partial\Omega$. The equivariant degree $\deg_G(f, \Omega)$ of f in Ω is defined as an element of the "stabilized" equivariant homotopy group of the representation sphere $S^V := S(\mathbb{R} \oplus V)$

$$\Pi^G := \lim_{\to k} \Pi^G_{S^{\mathbb{R}^k \oplus V \oplus \mathbb{R}^n}}(S^{\mathbb{R}^k \oplus V}),$$

in the following way: Let $\eta\colon V \oplus \mathbb{R}^n \to [0, 1]$ be an invariant function such that $\eta^{-1}(0) = \overline{\Omega}$ and $\eta^{-1}(0, 1) \cap f^{-1}(0) = \emptyset$. We put $f_\eta\colon [-1, 1] \times V \oplus \mathbb{R}^n \to \mathbb{R} \oplus V$, $f_\eta(t, x) = (t + 2\eta(x), f(x))$ for $t \in [-1, 1]$ and $x \in V \oplus \mathbb{R}^n$. The equivariant homotopy class $[f_\eta] \in \Pi^G_{S^{V \oplus \mathbb{R}^n}}(S^V)$ defines an element $\deg_G(f, \Omega)$ in Π^G called the *equivariant degree* of f in Ω. The equivariant degree has all the standard properties like the *existence*, *additivity*, *homotopy*, *suspension*, and *excision*.

Let $\Phi_n(G)$ denote the set of all the orbit types (H) in V such that the Weyl group $W(H) = N(H)/H$ of H is *bi-orientable*, i.e., $W(H)$ admits an invariant (fixed) orientation with respect to left and right translations, and $\dim W(H) = n$. Then the free $\mathbb{Z}$-module $A_n(G) := \mathbb{Z}[\Phi_n(G)]$ is a subgroup of Π^G (see [22] or [15]). We denote by G-Deg(f, Ω) the image of $\deg_G(f, \Omega)$ under the natural projection $\Pi^G \to A_n(G)$. We call G-Deg(f, Ω) the *primary degree*[1] of f in Ω. We will write $G\text{-Deg}(f, \Omega) = \sum_{(H)} n_H(H)$, where the

[1] The primary degree is related to the primary orbit types (H) in V, i.e., satisfying $\dim W(H) = n$.

summation is taken over all the primary orbit types in V. The primary degree was introduced independently of the work of Ize *et al.* in [5] (see also [21, 22]).

The primary degree of $f: \bar{\Omega} \to V$ can be expressed by an analytic formula: Approximate f by a *regular normal map* (i.e., a map satisfying certain normality and transversality conditions, see [21] for more details) $g: \bar{\Omega} \to V$ such that $\sup_{x \in \Omega} \|f(x) - g(x)\| < \eta$ with $2\eta := \inf_{x \in \partial\Omega} \|f(x)\|$. In particular, for every orbit type (H) in Ω the map $g_H := g_{|\Omega_H}: \Omega_H \to V^H$ has zero as a regular value. Then

$$G\text{-Deg}(f, \Omega) = \sum_{(H) \in \Phi_n(G)} n_H(H), \tag{2.1}$$

and

$$n_H = \sum_{W(H)\, x \subset g_H^{-1}(0)} \operatorname{sign} \det Dg_H(x)_{|S_x},$$

where S_x denotes the linear slice to the orbit $W(H)x$ in the space $V^H \oplus \mathbb{R}^n$ at x, i.e., the subspace of $V^H \oplus \mathbb{R}^n$ which is orthogonal to $W(H)x$ at x. We choose bases in V^H and S_x in such a way that the orientation of the tangent space $T_x W(H)x$, induced by the chosen invariant orientation of $W(H)$, followed by the orientation of S_x gives the orientation of V^H followed by the standard orientation of $\mathbb{R}^n$.

In the case where $n = 1$ the computation of the primary degree can be be reduced to the calculations of a related S^1-degree (see [2] or [17] for more details, technical formulae and comprehensive examples).

THEOREM 2.1. (Ulrich Type Formula for G-Degree, cf. [16].) *Let G be a compact Lie group, V be an orthogonal representation of G, and $f: V \oplus \mathbb{R} \to V$ an equivariant map such that $f(x) \neq 0$ for all $x \in \partial\Omega$, where $\Omega \subset V \oplus \mathbb{R}$ is an open, invariant, and bounded subset. Then*

$$n_H = [I_{S^1}(F^H)_{\mathbb{Z}_1} - I_{S^1}(F^{[H]})_{\mathbb{Z}_1}] \Big/ \left| \frac{W(H)}{S^1} \right|,$$

where n_H are the coefficients in (2.1), $S^1 \subset W(H)$, I_{S^1} *denotes the S^1-fixed point index corresponding to the primary S^1-degree* (*see* [16] *for more details*), *and $f := Id - F$: $V \oplus \mathbb{R} \to V$.*

In what follows we will denote by D_N the dihedral group of order $2N$. In the case $G = D_N \times S^1$, the primary G-degree has an additional important property, called *multiplicativity property* (which is also true in more general case, cf. [16, Theorem 3.4]):

PROPOSITION 2.2. *Assume that V is an orthogonal $G = D_N \times S^1$-representation and U is an orthogonal D_N-representation. Let $f: V \oplus \mathbb{R} \to V$ (resp. $g: U \to U$) be an equivariant map such that $f(x) \neq 0$ for $x \in \partial\Omega$ (resp. $g(x) \neq 0$ for $x \in \partial\mathcal{U}$), where $\Omega \subset V \oplus \mathbb{R}$ (resp. $\mathcal{U} \subset U$) is an invariant open bounded subset. Then*

$$G\text{-Deg}(g \times f, \mathcal{U} \times \Omega) = D_N\text{-deg}(g, \mathcal{U}) \cdot G\text{-Deg}(f, \Omega),$$

where $A(D_N) = A_0(D_N)$ denotes the Burnside ring of D_N and $A_1(G)$ has a natural structure of an $A(D_N)$-module.

To derive the computational formulae for the primary $G = D_N \times S^1$-degree we will need a description of the subgroup structure of D_N and $D_N \times S^1$ as well as the multiplication tables for $A_1(D_N \times S^1)$. First, we classify the conjugacy classes of D_N as follows: If N is an odd number, then

TABLE I

Multiplication Table for $A(D_N)$, Where $l = gcd(m, k)$, $m \mid N$ and $k \mid N$

	$(\mathbf{D}_m)$ $2m \nmid n$	$(\mathbf{D}_m)$ $2m \mid n$	$(\tilde{\mathbf{D}}_m)$ $2m \nmid n$	$(\tilde{\mathbf{D}}_m)$ $2m \mid n$	$(\mathbb{Z}_m)$
$(\mathbf{D}_k)$ $2k \nmid N$	$(D_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$(D_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\mathbf{D}_k)$ $2k \mid N$	$(D_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$2(D_l) + \frac{Nl - 2mk}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k \nmid N$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k \mid N$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$\frac{lN}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l) + \frac{Nl - mk}{2mk}(\mathbb{Z}_l)$	$2(\tilde{D}_l) + \frac{Nl - 2mk}{2mk}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$
$(\mathbb{Z}_k)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{Nl}{km}(\mathbb{Z}_l)$	$\frac{2Nl}{km}(\mathbb{Z}_l)$

$\Phi_0(D_N)=\{(D_k), (\mathbb{Z}_k); k \mid N\}$; and if N is even then $\Phi_0(D_N)=\{(D_k), (\tilde{D}_k), (\mathbb{Z}_k); k \mid N\}$, where

$$\tilde{D}_k=\mathbb{Z}_k \cup \kappa\xi_N\mathbb{Z}_k, \qquad \xi_N=e^{2i\pi/N}, \qquad \text{and} \qquad \kappa=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have Table I (see [16]) for the Burnside ring $A(D_N):=A_0(D_N)$.

All the generators of $A_1(D_N\times S^1)$ are the *m-folded θ-twisted* subgroups $K^{(\theta,m)}:=\{(\gamma,z)\in K\times S^1; \theta(\gamma)=z^m\}$, where K is a subgroup of D_N and $\theta: K\to S^1$ a homomorphism (cf. [8]).

We next proceed with the classification (up to conjugacy classes) of the nontrivial l-folded θ-twisted subgroups of $D_N\times S^1$: We have the subgroups $D_k^{(c,l)}$ and $\tilde{D}_k^{(c,l)}$, where $c: D_k\to\mathbb{Z}_2$ is a homomorphism such that $\ker c=\mathbb{Z}_k$, and the subgroup $D_k^{(d,l)}$ (when k is even), where $d: D_k\to\mathbb{Z}_2$ is a homomorphism such that $\ker d=D_{k/2}$. Moreover, if k is divisible by 4 then there exists one more conjugacy class of the subgroup $D_k^{(\hat{d},l)}$, where $\ker\hat{d}=\hat{D}_{k/2}:=\mathbb{Z}_{k/2}\cup\kappa\hat{\xi}_k\mathbb{Z}_{k/2}$ with $\hat{\xi}_k=e^{2i\pi/k}$. In the case of subgroups

TABLE II

Multiplication Table for $A(D_n)$, Where $l=gcd(m,k)$, $m \mid n$ and $k \mid n$

	$(\mathbf{D}_m)$ $2m\nmid n$	$(\mathbf{D}_m)$ $2m \mid n$	$(\tilde{\mathbf{D}}_m)$ $2m\nmid n$	$(\tilde{\mathbf{D}}_m)$ $2m \mid n$	$(\mathbb{Z}_m)$
$(\mathbf{D}_k)$ $2k\nmid n$	$(D_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$(D_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\mathbf{D}_k)$ $2k \mid n$	$(D_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$2(D_l)+\frac{nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k\nmid n$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k \mid n$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$\frac{ln}{2mk}(\mathbb{Z}_l)$	$(\tilde{D}_l)+\frac{nl-mk}{2mk}(\mathbb{Z}_l)$	$2(\tilde{D}_l)+\frac{nl-2mk}{2mk}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$
$(\mathbb{Z}_k)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{nl}{km}(\mathbb{Z}_l)$	$\frac{2nl}{km}(\mathbb{Z}_l)$

Table III

Multiplication Table, Where We Assume $m = gcd(k, r)$ Is Such That $2m \mid N$

	$(\mathbf{D}_k^{(d,l)})$, $2 \mid k$ $2k \nmid N$	$(\mathbf{D}_k^{(d,l)})$, $2 \mid k$ $2k \mid N$
$(\mathbf{D}_r)$, $2 \mid r$ $2r \nmid N$	Excluded	$2(D_m^{(d,l)}) + \frac{lm-2kr}{2kr}(\mathbb{Z}_m^{(d,l)})$
$(\mathbf{D}_r)$, $2 \mid r$ $2r \mid N$	$2(D_m^{(d,l)}) + \frac{ml-2kr}{2kr}(\mathbb{Z}_m^{(d,l)})$	$4(D_m^{(d,l)}) + \frac{ml-4kr}{2kr}(\mathbb{Z}_m^{(d,l)})$

$\mathbb{Z}_k^{(\varphi_v, l)}$, the homomorphism φ_v is given by $\varphi_v(z) = z^v$, where v is an integer and $z \in \mathbb{Z}_k \subset S^1 \subset \mathbb{C}$. In the case where k is an even number, we have the homomorphism $d: \mathbb{Z}_k \to \mathbb{Z}_2$ such that $\ker d = \mathbb{Z}_{k/2}$, for which we have the l-folded d-twisted subgroup $\mathbb{Z}_k^{(d,l)}$.

We have Tables II and III for $A(D_N) \times A_1(D_N \times S^1) \to A_1(D_N \times S^1)$ (cf. [16]).

3. HOPF D_N-SYMMETRIC BIFURCATION THEOREMS

Let $\tau \geqslant 0$ be a given constant, n a positive integer and $C_{n,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbb{R}^n$ equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|, \qquad \varphi \in C_{n,\tau}.$$

In what follows, if $x : [-\tau, A] \to \mathbb{R}^n$ is a continuous function with $A > 0$ and if $t \in [0, A]$, then $x_t \in C_{n,\tau}$ is defined by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-\tau, 0].$$

Also, for any $x \in \mathbb{R}^n$ we will use $\bar{x}$ to denote the constant mapping from $[-\tau, 0]$ into $\mathbb{R}^n$ with the value $x \in \mathbb{R}^n$.

Consider the following one parameter family of retarded functional differential equations

$$\dot{x} = f(x_t, \alpha), \tag{3.1}$$

where $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $f: C_{n,\tau} \times \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable and completely continuous mapping. Assume there is an orthogonal representation $\Theta : \Gamma \to O(n)$ of $\Gamma := D_N$, $N > 2$, on $\mathbb{R}^n$, which naturally induces an isometric Banach representation of Γ on the space $C_{n,\tau}$ with the action $\cdot : \Gamma \times C_{n,\tau} \to C_{n,\tau}$ given by:

$$(\gamma\varphi)(\theta) := \Theta(\gamma)(\varphi(\theta)), \qquad \gamma \in \Gamma, \quad \theta \in [-\tau, 0].$$

We make the following assumptions

(A1) The mapping f is Γ-equivariant, i.e.,

$$f(\gamma\varphi, \alpha) = \gamma f(\varphi, \alpha), \qquad \varphi \in C_{n,\tau}, \quad \alpha \in \mathbb{R}, \quad \gamma \in \Gamma.$$

(A2) $f(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, i.e., $(0, \alpha)$ is a *stationary solution* of (3.1) for every $\alpha \in \mathbb{R}$.

Since $\mathbb{R}^n$ is an orthogonal representation of the group D_N, we have the following unique isotypical decomposition of $\mathbb{R}^n$ with respect to the action of D_N

$$V := \mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \tag{3.2}$$

where $k = (N+1)/2$ if N is odd, or $k = (N+4)/2$ if N is even, and

(i) $V_0 := V^\Gamma = \{v \in V; \forall \gamma \in \Gamma \ \gamma v = v\}$;

(ii) Each one of the subrepresentations V_j $(j = 1, ..., k)$, called *isotypical components*, is a direct sum of all subrepresentations of V equivalent to a fixed irreducible orthogonal representation of D_N described as follows:

(a1) For every integer number $1 \leqslant j < [\![N/2]\!]$ there is an orthogonal representation ρ_j (of real type) of D_N on $\mathbb{C}$ given by:

$$\gamma z := \gamma^j \cdot z, \qquad \gamma \in \mathbb{Z}_N, \quad z \in \mathbb{C};$$
$$\kappa z := \bar{z},$$

where $\gamma^j \cdot z$ denotes the usual complex multiplication;

(a2) There is a representation $c : D_N \to \mathbb{Z}_2 \subset O(1)$, such that $\ker c = \mathbb{Z}_N$;

(a3) For N even, there is an irreducible representation $d : D_N \to \mathbb{Z}_2 \subset O(1)$ such that $\ker d = D_{N/2}$, and

(a4) For N divisible by 4, there is an irreducible representation $\hat{d} : D_N \to \mathbb{Z}_2 \subset O(1)$ such that $\ker \hat{d} = \hat{D}_{N/2}$.

We will denote by $U := \mathbb{C}^n$ the complexification of $V = \mathbb{R}^n$. It is not difficult to see that the isotypical decomposition (3.2) induces the following isotypical decomposition of the complex representation U:

$$U = U_0 \oplus U_1 \oplus \cdots \oplus U_k, \tag{3.3}$$

where $U_0 := U^{\Gamma}$ and each of the isotypical components U_j is characterized by complex representation of the following types:

(b1) For $1 < j \leqslant [\![N/2]\!]$ the representation η_j on $\mathbb{C} \oplus \mathbb{C}$ is given by

$$\gamma(z_1, z_2) := (\gamma^j \cdot z_1, \gamma^{-j} \cdot z_2), \qquad \gamma \in \mathbb{Z}_N, \quad z_1, z_2 \in \mathbb{C},$$
$$\kappa(z_1, z_2) := (z_2, z_1);$$

(b2) The representation $c : D_N \to \mathbb{Z}_2 \subset U(1)$, such that $\ker c = \mathbb{Z}_N$;

(b3) In the case when N is even, the representation $d : D_N \to \mathbb{Z}_2 \subset U(1)$, such that $\ker d = D_{N/2}$; and

(b4) In the case when N is even, the representation $\hat{d} : D_N \to \mathbb{Z}_2 \subset U(1)$ such that $\ker \hat{d} = \hat{D}_{N/2}$.

An element $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ is called a stationary solution of (3.1) if $f(\bar{x}, \alpha) = 0$. A complex number $\lambda \in \mathbb{C}$ is said to be a *characteristic value* of the stationary solution (x, α) if it is a root of the following *characteristic equation*

$$\det_{\mathbb{C}} \Delta_{(x, \alpha)}(\lambda) = 0, \tag{3.4}$$

where

$$\Delta_{(x, \alpha)}(\lambda) := \lambda \,\mathrm{Id} - D_x f(x, \alpha)(e^{\lambda} \cdot \mathrm{Id}).$$

A stationary solution (x_0, α_0) is called *nonsingular* if $\lambda = 0$ is not a characteristic value of (x_0, α_0), and a nonsingular stationary point (x_0, α_0) is called a *center* if it has a purely imaginary characteristic value. We will call (x_0, α_0) an *isolated center* if it is the only center in some neighborhood of (x_0, α_0) in $\mathbb{R}^n \times \mathbb{R}$.

We now make the following assumption:

(A3) There is a stationary solution $(0, \alpha_0)$ which is an isolated center such that $\lambda = i\beta_0$, $\beta_0 > 0$, is a characteristic value of $(0, \alpha_0)$.

Let $\Omega_1 := (0, b) \times (\beta_0 - c, \beta_0 + c) \subset \mathbb{C}$. Under assumption (A3), the constants $b > 0$, $c > 0$ and $\delta > 0$ can be chosen such that the following condition is satisfied:

(*) For every $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$ if there is a characteristic value $u + iv \in \partial\Omega_1$ of $(0, \alpha)$ then $u + iv = i\beta_0$ and $\alpha = \alpha_0$.

Note that $\Delta_{(0, \alpha)}(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$. It follows that $\det_{\mathbb{C}} \Delta_{(0, \alpha_0 \pm \delta)}(\lambda) \neq 0$ for $\lambda \in \partial\Omega_1$.

Since the mapping f is Γ-equivariant, for every $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the operator $\Delta_{(0,\alpha)}(\lambda)\colon \mathbb{C}^n \to \mathbb{C}^n$ is Γ-equivariant and consequently for every isotypical component U_j of $U = \mathbb{C}^n$ we have $\Delta_{(0,\alpha)}(\lambda)(U_j) \subseteq U_j$ for $j = 0, 1, ..., k$. We put

$$\Delta_{\alpha, j}(\lambda) := \Delta_{(0,\alpha)}(\lambda)|_{U_j}\colon U_j \to U_j.$$

Solutions $\lambda \in \mathbb{C}$ of the equation

$$\det{}_{\mathbb{C}} \Delta_{\alpha, j}(\lambda) = 0$$

where $j = 0, 1, ..., k$, will be called the jth *isotypical characteristic values* of $(0, \alpha)$. It is clear that λ is a characteristic value of the solution $(0, \alpha)$ if and only if it is a jth isotypical characteristic value of $(0, \alpha)$ for some $j = 0, 1, ..., k$.

Following the idea of a crossing number in a nonequivariant case (cf. [3, 6, 17, 20]), we define

$$c_{1,j}(\alpha_0, \beta_0) := \deg_B(\det{}_{\mathbb{C}} \Delta_{\alpha_0 - \delta, j}(\cdot), \Omega) - \deg_B(\det{}_{\mathbb{C}} \Delta_{\alpha_0 + \delta, j}(\cdot), \Omega)$$

for $0 \leqslant j \leqslant k$. The number $c_{1,j}(\alpha_0, \beta_0)$ will be called the jth *isotypical crossing number*, for the isolated center $(0, \alpha_0)$ corresponding to the characteristic value $i\beta_0$. The crossing number $c_{1,j}(\alpha_0, \beta_0)$ indicates how many jth characteristic values (counted with algebraic multiplicity) of the stationary points $(0, \alpha)$ "escape" from the region Ω_1 when α crosses the value α_0.

Since an integer multiple of $i\beta_0$ can also be an jth isotypical characteristic value of $(0, \alpha_0)$, we define for $l > 1$

$$c_{l,j}(\alpha_0, \beta_0) := \deg_B(\det{}_{\mathbb{C}} \Delta_{\alpha_0 - \delta, j}(\cdot), \Omega_l) - \deg_B(\det{}_{\mathbb{C}} \Delta_{\alpha_0 + \delta, j}(\cdot), \Omega_l)$$

where $\Omega_l := (0, b) \times (l\beta_0 - c, l\beta_0 + c) \subset \mathbb{C}$ and the constants $b > 0$, $c > 0$ and $\delta > 0$ can be chosen to be sufficiently small so that there are no characteristic values of $(0, \alpha)$ in $\partial\Omega_l$ except perhaps $il\beta_0$ for $\alpha = \alpha_0$. In other words, $c_{l,j}(\alpha_0, \beta_0) = c_{1,j}(\alpha_0, l\beta_0)$. If $il\beta$ is not an jth isotypical characteristic value of $(0, \alpha_0)$ then clearly $c_{l,j}(\alpha_0, \beta_0) = 0$.

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(0, \alpha_0)$, i.e., the existence of Hopf bifurcations at the stationary point $(0, \alpha_0)$, and to associate with $(0, \alpha_0)$ a *local bifurcation invariant*, we will use the standard degree-theoretical approach (cf. [3, 6, 17, 18, 20]). We reformulate the Hopf bifurcation problem for equation (3.1) as an $\Gamma \times S^1$-equivariant bifurcation problem (with two parameters) in an appropriate Hilbert isometric representation of $G = \Gamma \times S^1$. For this purpose we make the following change of variable

$x(t)=z((\beta/2\pi)\,t)$ for $t\in\mathbb{R}$. We obtain the following equation, which is equivalent to (3.1) as

$$\dot{z}(t)=\frac{2\pi}{\beta}\,f(z_{t,\beta},\alpha), \tag{3.5}$$

where $z_{t,\beta}\in C_{n,\tau}$ is defined by

$$z_{t,\beta}(\theta)=z\left(t+\frac{\beta}{2\pi}\,\theta\right), \qquad \theta\in[-\tau,0].$$

Evidently, $z(t)$ is a 1-periodic solution of (3.5) if and only if $x(t)$ is a $(2\pi/\beta)$-periodic solution of (3.1).

Let $S^1=\mathbb{R}^1/\mathbb{Z}$, $W=L^2(S^1;\mathbb{R}^n)$ and define

$$L: H^1(S^1;\mathbb{R}^n)\to W, \quad Lz(t)=\dot{z}(t), \qquad z\in H^1(S^1;\mathbb{R}^n), \quad t\in S^1;$$

$$K: H^1(S^1;\mathbb{R}^n)\to W, \quad Kz(t)=\int_0^1 z(s)\,ds, \quad z\in H^1(S^1;\mathbb{R}^n), \quad t\in S^1.$$

It can easily be shown that $(L+K)^{-1}: W\to H^1(S^1;\mathbb{R}^N)$ exists and the map $F: W\times(\alpha_0-\delta,\ \alpha_0+\delta)\times(\beta_0-c,\ \beta_0+c)\to W$ defined by

$$F(z,\alpha,\beta)=(L+K)^{-1}\left[Kz+\frac{2\pi}{\beta}\,N_f(z,\alpha,\beta)\right]$$

is completely continuous, where $N_f: W\times(\alpha_0-\delta,\ \alpha_0+\delta)\times(\beta_0-c,\ \beta_0+c)\to W$ is defined by

$$N_f(z,\alpha,\beta)(t)=f(z_{t,\beta},\alpha), \qquad t\in S^1,$$
$$(z,\alpha,\beta)\in W\times(\alpha_0-\delta,\alpha_0+\delta)\times(\beta_0-c,\beta_0+c).$$

Moreover, (z,α,β) is a 1-periodic solution of (3.5) if and only if $z=F(z,\alpha,\beta)$.

The space W is an isometric Hilbert representation of the group $G=D_N\times S^1$ with the action being given by

$$(\gamma,\theta)\,x(t)=\gamma(x(t+\theta)), \qquad \theta, t\in S^1, \quad \gamma\in D_N, \quad x\in W.$$

The nonlinear operator F is clearly G-equivariant.

With respect to the restricted S^1-action on W, we have the following isotypical decomposition of the space W

$$W=\overline{\bigoplus_{l=0}^{\infty} W_l},$$

where W_0 is the space of all constant mappings from S^1 into $\mathbb{R}^n$, and W_l with $l>0$ is the vector space of all mappings of the form $x \sin 2l\pi \cdot + y \cos 2l\pi \cdot$, $x, y \in \mathbb{R}^n$. For $l>0$, the subspace W_l can be endowed with a complex structure by

$$i \cdot (x \sin 2l\pi \cdot + y \cos 2l\pi \cdot) = x \cos 2l\pi \cdot - y \sin 2l\pi \cdot, \qquad x, y \in \mathbb{R}^n.$$

Since the above multiplication by i induces an operator $J: W_l \to W_l$ such that $J^2 = -\mathrm{Id}$, it follows that $\mathrm{Id} + J$ is an Γ-equivariant isomorphism and every function in W_l can be uniquely represented as $e^{i2l\pi \cdot}(x+iy)$, $x, y \in \mathbb{R}^n$. In particular, we notice that the above defined complex structure on W_l coincides with the complex structure given by $x+iy \in \mathbb{C}^n$. In addition, the complex isomorphism A_l: $W_l \to U := \mathbb{C}^n$ given by $A_l(e^{i2l\pi \cdot}(x+iy)) = x+iy$, $x+iy \in \mathbb{C}^n$ is Γ-equivariant. Thus, as a complex Γ-representation, W_l is equivalent to U. Consequently, the isotypical Γ-decomposition (3.3) of U induces the following isotypical Γ-decomposition of W_l

$$W_l := W_{0,l} \oplus W_{1,l} \oplus \cdots \oplus W_{k,l},$$

where the isotypical components $W_{j,l}$, $l>0$ can be described exactly by the same conditions (b1)–(b4). On the other hand the component W_0 is exactly the representation $V = \mathbb{R}^n$, which admits the isotypical decomposition (3.2). To unify the notations, we denote this isotypical decomposition by $W_0 = W_{0,0} \oplus \cdots \oplus W_{k,0}$, where for every j we have $W_{j,0} := V_j$. As the complex structure on W_j, with $j>0$ was defined using the S^1-action and as all the subspaces $W_{j,l}$, with $l>0$ are complex Γ-invariant subspaces, $W_{j,l}$ with $l>0$ are also S^1-invariant. Therefore, $W_{j,l}$ are the isotypical G-components of the representation W.

For every j and l we define

$$a_{j,l}(\alpha, \beta) := \mathrm{Id} - (L+K)^{-1} \left[K + \frac{2\pi}{\beta} D_z N_f(0, \alpha, \beta) \right] \bigg|_{W_{j,l}},$$

where $(\alpha, \beta) \in (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$.

We observe that

$$(L+K)^{-1} (e^{i2l\pi \cdot}(x+iy)) = \frac{1}{i2l\pi} e^{i2l\pi \cdot}(x+iy) \tag{3.6}$$

for every $x, y \in \mathbb{R}^n$, and since

$$a_{j,l}(\alpha, \beta) = (L+K)^{-1} \left[L - \frac{2\pi}{\beta} D_z N_f(0, \alpha, \beta) \right] \bigg|_{W_{j,l}},$$

we obtain

$$a_{j,l}(\alpha,\beta)\, e^{i2l\pi\cdot} z = (L+K)^{-1}\left[i2l\pi\, e^{i2l\pi\cdot} z - \frac{2\pi}{\beta}\, e^{i2l\pi\cdot} D_x f(0,\alpha)(e^{il\beta\cdot})\, z\right]$$

$$= e^{i2l\pi\cdot} \frac{1}{il\beta}\, \Delta_{(0,\alpha)}(il\beta)(z).$$

Consequently,

$$a_{j,l}(\alpha,\beta) = \frac{1}{il\beta}\, \Delta_{\alpha,j}(il\beta).$$

The idea of using the topological degrees to study the existence of Hopf bifurcations, and the various symmetry properties of the solutions, is based on the notion of a complementing function. More precisely, let $\lambda = \alpha + i\beta = (\alpha, \beta) \in \mathbb{R}^2 = \mathbb{C}$ and $\lambda_0 = \alpha_0 + \beta_0$. We define a special neighborhood $U(r, \rho)$ of the solution $(0, \lambda_0) \in W \times \mathbb{R}^2$ by

$$U(r,\rho) := \{(z,\lambda) \in W \times \mathbb{C};\ \|z\| < r, \text{ and } |\lambda - \lambda_0| < \rho\}.$$

By taking sufficiently small $r > 0$ and $\rho > 0$, we may assume that the equation

$$F(z,\lambda) = 0, \qquad z \in W, \quad \lambda \in \mathbb{C} = \mathbb{R}^2, \tag{3.7}$$

has no solution (z, λ) such that $(z, \lambda) \in \partial U(r, \rho)$, $z \neq 0$ and $|\lambda - \lambda_0| = \rho$. A G-invariant function $\xi : \overline{U(r, \rho)} \to \mathbb{R}$ defined by

$$\xi(z,\lambda) := |\lambda - \lambda_0|\, (\|z\| - r) + \|z\|$$

is called a *complementing function* with respect to $U(r, \rho)$. Define the mapping $F_\xi : \overline{U(r, \rho)} \to W \times \mathbb{R}$ by $F_\xi(z, \lambda) := (F(z, \lambda), \xi(z, \lambda))$, where $(z, \lambda) \in \overline{U(r, \rho)}$. The mapping F_ξ is a compact G-equivariant field. It is well known that the G-equivariant degree $G\text{-Deg}(F_\xi, U(r, \rho))$ does not depend on the numbers $r > 0$ and $\rho > 0$ (if r and ρ are sufficiently small), thus the standard properties of G-degree imply that if $G\text{-Deg}(F_\xi, U(r, \rho)) \neq 0$ then $(0, \lambda_0)$ is a bifurcation point of (3.7), i.e., there exists a continuum $\mathscr{C} \subset U(r, \rho)$ of nonconstant periodic solutions of (3.7) such that $(0, \lambda_0) \in \overline{\mathscr{C}}$. We can regard the G-degree $G\text{-Deg}(F_\xi, U(r, \rho))$ as a local bifurcation invariant.

The computations in [16] provide us with a complete information needed to evaluate the exact value of $G\text{-Deg}(F_\xi, U(r, \rho))$. To illustrate this point, we need a more detailed description of the G-isotypical components $W_{j,l}$.

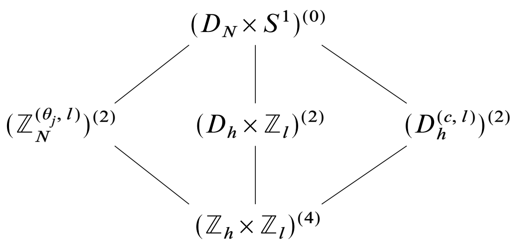
For every isotypical component $W_{j,l}$, we denote by $Y_{j,l}$ the corresponding irreducible representation of G, (i.e., $Y_{j,l}$ is equivalent to every irreducible subrepresentation of $W_{j,l}$). We describe the action of the group $G = D_N \times S^1$, $N \geqslant 3$, on $Y_{j,l}$ with $j > 0$ as follows:

The first type of $W_{j,l}$ corresponds to the irreducible 4-dimensional representations $Y_{j,l}$ of G (described as complex D_N-representations in (b1)), where the action of $G = D_N \times S^1$ on the space $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{C} \oplus \mathbb{C}$ is given by:

$$(\gamma, \tau)(z_1, z_2) := (\gamma^j \tau^l z_1, \gamma^{-j} \tau^l z_2) \qquad \text{for} \quad (\gamma, \tau) \in \mathbb{Z}_N \times S^1;$$
$$(\kappa\gamma, \tau)(z_1, z_2) := (\gamma^{-j} \tau^l z_2, \gamma^j \tau^l z_1) \qquad \text{for} \quad (\kappa\gamma, \tau) \in \kappa\mathbb{Z}_N \times S^1,$$

where $(z_1, z_2) \in \mathbb{C} \oplus \mathbb{C}$, $l = 1, 2, 3, \ldots$, $1 \leqslant j < [\![N/2]\!]$. We put $h = gcd(j, N)$.

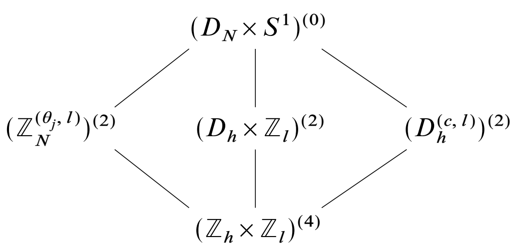
(i1) *$m = N/h$ is odd.* In this case, we have the following lattice of the isotropy groups in $Y_{j,l}$:



where $\theta_j : \mathbb{Z}_N \to S^1$ is given by $\theta_j(\gamma) = \gamma^{-j}$, $\gamma \in \mathbb{Z}_N$, and the numbers in brackets denote the dimension of the corresponding fixed-point space. We define the following element of $A_1(D_N \times S^1)$

$$\deg_{j,l} := (\mathbb{Z}_N^{(\theta_j, l)}) + (D_h \times \mathbb{Z}_l) + (D_h^{(c, l)}) - (\mathbb{Z}_h \times \mathbb{Z}_l).$$

(i2) $m = N/h \equiv 2 \pmod 4$. In this case, we have the following lattice of subgroups in $Y_{j,l}$:



where $\theta_j : \mathbb{Z}_N \to S^1$ is given by $\theta_j(\gamma) = \gamma^{-j}$, $\gamma \in \mathbb{Z}_N$, and the numbers in brackets denote the dimension of the corresponding fixed-point spaces. We define the following element of $A_1(D_N \times S^1)$:

$$\deg_{j,l} := (\mathbb{Z}_N^{(\theta_j, l)}) + (D_h \times \mathbb{Z}_l) + (D_h^{(c, l)}) - (\mathbb{Z}_h \times \mathbb{Z}_l).$$

(i2) $m = N/h \equiv 2 \pmod 4$. In this case, we have the following lattice of isotropy subgroups in $Y_{j,l}$:

$$\begin{array}{ccc} & (D_N \times S^1)^{(0)} & \\ (\mathbb{Z}_N^{(\theta_j,\, l)})^{(2)} & (D_{2h}^{(d,\, l)})^{(2)} & (D_{2h}^{(\hat{d},\, l)})^{(2)} \\ & \mathbb{Z}_{2h}^{(d,\, l)})^{(4)} & \end{array}$$

and we define

$$\deg_{j,l} := (\mathbb{Z}_N^{(\theta_j,\, l)}) + (D_{2h}^{(d,\, l)}) + (D_{2h}^{(\hat{d},\, l)}) - (\mathbb{Z}_{2h}^{(d,\, l)}).$$

(i3) $m = N/h \equiv 4 \pmod 4$. In this case, we have the following lattice of isotropy subgroups in $Y_{j,l}$

$$\begin{array}{ccc} & (D_N \times S^1)^{(0)} & \\ (\mathbb{Z}_N^{(\theta_j,\, l)})^{(2)} & (D_{2h}^{(d,\, l)})^{(2)} & (\tilde{D}_{2h}^{(d,\, l)})^{(2)} \\ & \mathbb{Z}_{2h}^{(d,\, l)})^{(4)} & \end{array}$$

and we put

$$\deg_{j,l} := (\mathbb{Z}_N^{(\theta_s,\, l)}) + (D_{2h}^{(d,\, l)}) + (\tilde{D}_{2h}^{(d,\, l)}) - (\mathbb{Z}_{2h}^{(d,\, l)}).$$

(i4) For an isotypical component $W_{j,l}$ corresponding to irreducible two-dimensional representation $Y_{j,l}$ on $\mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ which is given by

$$(\gamma, \tau)\, z = \tau^l z, \qquad (\gamma, \tau) \in \mathbb{Z}_N \times S^1;$$

$$(\kappa\gamma, \tau)\, z = -\tau^l z, \qquad (\kappa\gamma, \tau) \in \kappa\mathbb{Z}_N \times S^1,$$

where $l = 1, 2, 3, \ldots$ and we have the following lattice of the isotropy groups in $Y_{j,l}$

$$\begin{array}{c} (D_N \times S^1)^{(0)} \\ | \\ (\tilde{D}_N^{(c,\, l)})^{(2)}. \end{array}$$

We define

$$\deg_{j,l} := (\tilde{D}_N^{(c,l)}).$$

(i5) If N is even then there is a two-dimensional irreducible representation on $Y_{j,l} = \mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ given by

$$(g, \tau)\, z = \tau^l z, \qquad \text{if} \quad (g, \tau) \in D_{N/2} \times S^1;$$

$$(g, \tau)\, z = -\tau^l z, \qquad \text{if} \quad (g, \tau) \in (D_N \backslash D_{N/2}) \times S^1.$$

We have the following lattice of the isotropy subgroups in $Y_{j,l}$

$$\begin{array}{c} (D_N \times S^1)^{(0)} \\ | \\ (\tilde{D}_N^{(d,l)})^{(2)}. \end{array}$$

In this case we put

$$\deg_{j,l} := (\tilde{D}_N^{(d,l)}).$$

(i6) Finally, for N even and $j = N/2$, there may also be an isotypical component $W_{N/2,l}$ corresponding to the two dimensional representation on $Y_{N/2,l} := \mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ given by

$$(\gamma, \tau)\, z = \gamma^{N/2} \tau^l z, \qquad \text{where} \quad (\gamma, \tau) \in \mathbb{Z}_N \times S^1;$$

$$(\kappa\gamma, \tau)\, z = -\gamma^{N/2} \tau^l z, \qquad \text{where} \quad (\kappa\gamma, \tau) \in \kappa\mathbb{Z}_N \times S^1.$$

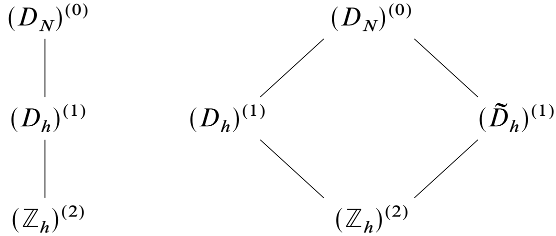
In this case, we have the following lattice of the isotropy groups in $Y_{N/2,l}$

$$\begin{array}{c} (D_N \times S^1)^{(0)} \\ | \\ (D_N^{(\hat{d},l)})^{(2)} \end{array}$$

and we define

$$\deg_{j,l} := (D_N^{(\hat{d},l)}).$$

(j1) For the isotypical component corresponding to the type (a1) of the irreducible representations of D_N, i.e., $W_{j,0} := V_j$, where $1 \leqslant j < [\![N/2]\!]$, we have the following lattice of isotropy groups of $Y_{j,0}$



For m odd   For m even

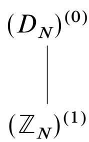
where $h = gcd(j, N)$ and $m := N/h$. If m is odd, we put

$$\deg_j := (D_h) + (\mathbb{Z}_h)$$

and if m is even, we put

$$\deg_j := (D_h) + (\tilde{D}_h) - (\mathbb{Z}_h).$$

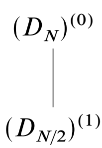
(j2) For the isotypical component $W_{j,l}$ corresponding to the irreducible representation $Y_{j,0}$ of type (a2), we have the following lattice of isotropy groups in $Y_{j,l}$



and we put

$$\deg_j := (\mathbb{Z}_N).$$

(j3) For $W_{j,l}$ corresponding to the irreducible representation $Y_{j,0}$ of type (a3), we have the following lattice of isotropy groups in $Y_{j,l}$



and we put

$$\deg_j := (D_{N/2}).$$

(j4) In the case $j=N/2$, $W_{j,0}=V_j$ corresponds to a one-dimensional irreducible representation $Y_{j,0}$ of type (a4), having the following lattice of the isotropy groups

$$\begin{array}{c}(D_N)^{(0)}\\ |\\ (\hat{D}_{N/2})^{(1)}\end{array}$$

and we put

$$\deg_j := (\hat{D}_{N/2}).$$

We also define, for every $j=0, 1, ..., k$, the number

$$\nu_j(\alpha_0, \beta_0)=\begin{cases}1 & \text{if sign det } a_{j,0}(\alpha_0, \beta_0)=-1,\\ 0 & \text{if sign det } a_{j,0}(\alpha_0, \beta_0)=1.\end{cases}$$

We have the following result:

THEOREM 3.1. *Under the above assumptions,*

$$G\text{-Deg}(F_\xi, U(r, \rho)) = \nu_0\left(\prod_{j=1}^{k}((D_N)-\nu_j(\alpha_0, \beta_0)\deg_j)\right)\left(\sum_{j,l;\,l>0} c_{l,j}(\alpha_0, \beta_0)\deg_{j,l}\right),$$

where $\nu_0 := \nu_0(\alpha_0, \beta_0)$ *and the products are given by the multiplication in the Burnside ring* $A(D_N)$ *and by the multiplication* $A(D_N)\times A_1(D_N\times S^1)\to A_1(D_N\times S^1)$, *respectively.*

Proof. Let Ω_j denote the unit ball in the isotypical component $W_{j,0}$. We donote by $Y_{j,l}$ the irreducible representation (except for these (j, l) such that $j=N/2$) corresponding to the isotypical component $W_{j,l}$, where $l>0$. We also denote by

$$\Omega_{j,l} := \{(v, z)\in Y_{j,l}\oplus\mathbb{C};\ \|v\|<1,\ \tfrac{1}{2}<|z|<2\}.$$

By Theorem 4.5 in [16], we have

$$G\text{-Deg}(F_\xi, U(r, \rho)) = \nu_0\left(\prod_{j=1}^{k} D_N\text{-deg}(a_{j,0}(\alpha_0, \beta_0), \Omega_j)\right)\left(\sum_{j,l;\,l>0} c_{l,j}(\alpha_0, \beta_0)\, G\text{-Deg}(f_{j,l}, \Omega_{j,l})\right),$$

where $f_{j,l}\colon Y_{j,l} \oplus \mathbb{C} \to Y_{j,l} \oplus \mathbb{R}$ is defined by

$$f_{j,l}(v, z) = (z \cdot v, |z|(\|v\| - 1) + \|v\| + 1).$$

The computations of G-Deg$(f_{j,l}, \Omega_{j,l})$ were essentially done in Example 4.6 in [16], where the Ulrich Type Formula (Theorem 2.2) was applied to show that for every (j, l) such that $l > 0$ we have G-Deg$(f_{j,l}, \Omega_{j,l}) = \deg_{j,l}$. In order to compute D_N-deg$(a_{j,0}(\alpha_0, \beta_0), \Omega_j)$, we can use the properties of the Ulrich equivariant degree (the case $n = 0$, see [27] or [17]) and the standard computations based mostly on the evaluation of appropriate fixed point indices). In particular we can verify that D_N-Deg$(a_{j,0}(\alpha_0, \beta_0)) = (D_N) - \nu(\alpha_0, \beta_0) \deg_j$. ∎

THEOREM 3.2. *Under the above assumptions, for every nonzero crossing number* $c_{l,j}(\alpha_0, \beta_0)$ *there exist, bifurcating from* $(0, \alpha_0, \beta_0)$, *branches of non-constant periodic solutions of* (3.5) *such that*:

(i1) *if the corresponding to the index* (j, l) *element* $\deg_{j,l}$ *is* $(\mathbb{Z}_N^{(\theta_j, l)}) + (D_h \times \mathbb{Z}_l) + (D_h^{(c,l)}) - (\mathbb{Z}_h \times \mathbb{Z}_l)$, *i.e.*, $m \equiv 1 \pmod 2$, *then there are* 2 *branches of periodic solutions with the orbit type* $(\mathbb{Z}_N^{(\theta_j, l)})$, $m = N/h$ *branches with the orbit type* $(D_h \times \mathbb{Z}_l)$, *and* $m = N/h$ *branches with the orbit type* $(D_h^{(c,l)})$;

(i2) *if* $\deg_{j,l} = (\mathbb{Z}_N^{(\theta_j, l)}) + (D_{2h}^{(d,l)}) + (D_{2h}^{(\hat{d},l)}) - (\mathbb{Z}_{2h}^{(d,l)})$ (*i.e.*, $m \equiv 2 \pmod 4$), *then there are* 2 *branches of periodic solutions with orbit type* $(\mathbb{Z}_N^{(\theta_j, l)})$, $N/2h$ *branches with the orbit type* $(D_{2h}^{(d,l)})$, *and* $N/2h$ *branches with the orbit type* $(D_{2h}^{(\hat{d},l)})$;

(i3) *if* $\deg_{j,l} = (\mathbb{Z}_N^{(\theta_s, l)}) + (D_{2h}^{(d,l)}) + (\tilde{D}_{2h}^{(d,l)}) - (\mathbb{Z}_{2h}^{(d,l)})$ (*i.e.*, $m \equiv 0 \pmod 4$), *then there are* 2 *branches of periodic solutions of type* $(\mathbb{Z}_N^{(\theta_j, l)})$, $N/2h$ *branches of orbit type* $(D_{2h}^{(d,l)})$, *and* $N/2h$ *branches of the orbit type* $(\tilde{D}_{2h}^{(d,l)})$;

(i4) *if* $\deg_{j,l} = (\tilde{D}_N^{(c,l)})$, *then there is one branch of periodic solutions of the orbit type* $(\tilde{D}_N^{(c,l)})$;

(i5) *if* $\deg_{j,l} = (D_N^{(d,l)})$, *then there exists one branch of periodic solutions of the orbit type* $(D_N^{(d,l)})$;

(i6) *if* $\deg_{j,l} = (D_N^{(\hat{d},l)})$, *then there exists one branch of periodic solutions of the orbit type* $(D_N^{(\hat{d},l)})$.

Proof. Using the fact that all the orbit types mentioned in Theorem 3.2 are maximal, it follows from Theorem 3.1 that if the crossing number $c_{l,j}(\alpha_0, \beta_0)$ is nonzero, then there is a nonzero component $c_{l,j}(\alpha_0, \beta_0) \deg_{j,l}$ of the degree G-Deg$(F_\xi, U(r, \rho))$. Consequently, from the existence property of the G-degree, it follows that to every maximal orbit type (H) contained in $\deg_{j,l}$ corresponds to at least $|G/H|$ branches bifurcating from $(0, \alpha_0, \beta_0)$ nonconstant periodic solutions of the orbit type exactly equal to (H). ∎

Remark 3.3. Note that in Theorem 3.2, for a sequence of nonconstant periodic solutions $x(t)$ of (3.1) corresponding to the 1-periodic solutions $(z_k(t), \alpha_k, \beta_k)$ of (3.5) such that $(z_k(t), \alpha_k, \beta_k) \to (0, \alpha_0, \beta_0)$ in $W \oplus \mathbb{R}^2$ as $k \to \infty$, $2\pi/\beta_k$ is not necessary the minimal period of $x_k(t)$. However, by applying the same idea as in [18], one can show that if p_k is a minimal period of $x_k(t)$ such that $\lim_{k \to \infty} p_k = p_0$ then there exists an integer r such that $2\pi/\beta_0 = rp_0$ and $ir\beta_0$ is a characteristic value of $(0, \alpha)$. In particular, if other purely imaginary characteristic values of $(0, \alpha_0)$ are not integer multiples of $\pm i\beta_0$, then $2\pi/\beta_k$ is the minimal period of $x_k(t)$.

Remark 3.4. We should emphasize that the computational formula for $G\text{-Deg}(F_\xi, U(r, \rho))$ gives more information than what was used in the proof of Theorem 3.2. For example, we did not refer to the factor

$$v_0 \left(\prod_{j=1}^{k} ((D_N) - v_j(\alpha_0, \beta_0) \deg_j) \right) \in A(D_N)$$

which provides additional information about the type of symmetries involved in this Hopf bifurcation. At this stage, we are unable to predict the existence of branches of periodic solutions corresponding to submaximal orbit types, but the computational formula for $G\text{-Deg}(F_\xi, U(r, \rho))$ indicates that there is a potential for this type of branches. It was shown in [19] that in the case of a finite dimensional symmetric bifurcation, a nonzero component of the equivariant degree corresponding to an orbit type (H) (possibly submaximal), implies that the existence of a branch of solutions with the orbit type (H) can be achieved using arbitrarily small perturbations of the original mapping.

Remark 3.5. The degree $G\text{-Deg}(F_\xi, U(r, \rho))$ is a local bifurcation invariant characterizing the Hopf bifurcation from $(0, \alpha_0, \beta_0)$ which can be used to describe the global behavior of bifurcated branches of solutions. More precisely, assume that (3.1) has only isolated centers $(0, \alpha)$ and let $\mathscr{S}$ be a bounded in $H^1(S^1; \mathbb{R}^n) \times \mathbb{R}^2$ branch of nontrivial solutions of (3.1). Then the set of centers $(0, \alpha)$ belonging to $\mathscr{S}$ is finite, i.e., $\mathscr{S} \cap \{0\} \times \mathbb{R}^2 = \{(0, \alpha - 1), ..., (0, \alpha_M)\}$ and the corresponding sum of local bifurcation invariants is zero, i.e.,

$$\sum_{s=1}^{M} G\text{-Deg}(F_{\xi_s}, U(r_s, \rho_s)) = 0, \tag{3.1}$$

where $U(r_s, \rho_s)$ is a special neighborhood of $(0, \alpha_s, \beta_s)$ and ξ_s is a corresponding complementing function near $(0, \alpha_s, \beta_s)$. As the G-degree is fully computable for the D_N-symmetric Hopf bifurcation and the relations (3.8) take into account not only the maximal orbit types, but also the interaction between all the orbit types (including submaximal orbit types) of the bifurcated solutions, this type of global bifurcation result provides a set of relations which can be used to gain more information about the existence of large amplitude periodic solutions or even to prove the existence of multiple solutions of (3.1). We refer to papers [17] and [18] for more details and examples.

4. HOPF BIFURCATIONS IN A RING OF IDENTICAL OSCILLATORS

In this section we consider a ring of identical oscillators with identical coupling between adjacent cells. Such a ring was modeled by Turing (cf. [26]) and provides models for various situations in biology, chemistry, and electrical engineering. The local Hopf bifurcation of this Turing ring has been extensively studied in the literature, see [1, 7, 11, 18, 24, 30] and references therein.

There are many reasons to emphasize the importance of temporal delays in coupling between cells, for example, in many chemical or biological oscillators the time needed for transport or processing of chemical components or signals may be of considerable length (see [18]).

We will study how the temporal delay in the kinetics and in the coupling of cells together with the dihedral symmetries of the system may cause various types of oscillations in the case where each cell is described by only one state variable. It has been shown in [8] that such oscillations can not occur if the temporary delay is neglected.

We consider a ring of N identical cells coupled symmetrically by diffusion along the sides of an N-gon (see Fig. 1). Each cell may be regarded as a chemical system with m distinct chemical species. In what follows we will assume, for the sake of simplicity, that $m=1$. However, our method, based on the use of the G-equivariant degree, can also be effectively applied to more complex systems, in particular the case where $m>1$. We denote by $u^j(t)$ the concentration of the chemical species in the jth cell, $0\leqslant j\leqslant N-1$. We assume that the coupling is "nearest-neighbor" and symmetric in the sense that the interaction between any neighboring pair of cells takes the same form. For simplicity, we also assume that the coupling between adjacent cells is linear. Thus, we have the following system of retarded functional differential equations

$$\frac{d}{dt}u^j(t)=f(u_t^j,\alpha)+K(\alpha)(u_t^{j-1}-2u_t^j+u_t^{j+1}),\qquad 0\leqslant j\leqslant N-1,\qquad (4.1)$$

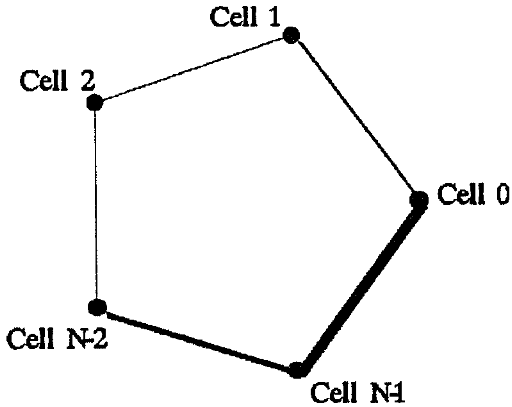


FIGURE 1

where $t \in \mathbb{R}$ denotes the time, $\alpha \in \mathbb{R}$ is a parameter, $u_t^j(\theta) = u^j(t+\theta)$, $0 \leqslant j \leqslant N-1$, $f\colon C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is continuously differentiable, and $K(\alpha) : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is a bounded linear operator and the mapping $K\colon \mathbb{R} \to L(C([-\tau, 0]; \mathbb{R}), \mathbb{R})$ is continuously differentiable. In (4.1) we assume that the integer $j+1$ has taken modulo N.

The function f describes the kinetic law obeyed by the concentrations u^j in every cell, and $K(\alpha)$ represents coupling strength, where the additional term

$$K(\alpha)(u_t^{j-1} - u_t^j) + K(\alpha)(u_t^{j+1} - u_t^j), \qquad 0 \leqslant j \leqslant N-1$$

in (4.1) is usually supported by the ordinary law of diffusion, i.e., the chemical substance moves from a region of greater concentration to a region of less concentration, at a rate proportional to the gradient of the concentration. We refer to [1, 26] for more details.

We assume that

$$f(0, \alpha) = 0. \tag{4.2}$$

Then $(0, 0, ..., 0, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ is a stationary solution of (4.1) and the linearization of (4.1) at $(0, 0, ..., 0, \alpha)$ is

$$\frac{d}{dt} x^j(t) = D_x f(0, \alpha)\, x_t^j + K(\alpha)[x_t^{j-1} - 2x_t^j + x_t^{j+1}], \quad 0 \leqslant j \leqslant N-1. \tag{4.3}$$

Therefore, a number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution $(0, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ if there exists a nonzero vector $z = (z_0, ..., z_{N-1}) \in \mathbb{C}^N$ such that

$$\operatorname{diag}(\lambda \operatorname{Id} - D_x f(0, \alpha)(e^{\lambda \cdot} \operatorname{Id}))\, z + r(\alpha, \lambda)\, z = 0,$$

where $\mathrm{diag}(\lambda\,\mathrm{Id} - D_x f(0,\alpha)(e^{\lambda\cdot}\,\mathrm{Id}))$ denotes the diagonal $N \times N$ matrix and $r(\alpha, \lambda)\colon \mathbb{C}^N \to \mathbb{C}^N$ is defined by

$$\{r(\alpha,\lambda)\,z\}_j = K(\alpha)[e^{\lambda\cdot}(z_{j-1} - 2z_j + z_{j+1})]; \qquad 0 \leqslant j \leqslant N-1.$$

We put $\{\delta z\}_j = z_{j-1} - 2z_j + z_{j+1}$, $0 \leqslant j \leqslant N-1$. The operator δ is the discretized Laplacian. Therefore, a number λ is a characteristic value if and only if the matrix

$$\Delta_\alpha(\lambda) = \mathrm{diag}(\lambda\,\mathrm{Id} - D_x f(0,\alpha)(e^{\lambda\cdot}\,\mathrm{Id})) - r(\alpha,\lambda)$$

is singular, i.e., the following characteristic equation is satisfied:

$$\det\nolimits_{\mathbb{C}} \Delta_\alpha(\lambda) = 0.$$

We have the following

Proposition 4.1. *A number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution if and only if*

$$\det\nolimits_{\mathbb{C}} \Delta_\alpha(\lambda) = \prod_{r=0}^{N-1} \left[\lambda - D_x f(0,\alpha)\, e^{\lambda\cdot} + 4 \sin^2 \frac{\pi r}{N} K(\alpha)\, e^{\lambda\cdot} \right] = 0.$$

Proof. For every $z \in \mathbb{C}$ and $r \in \{0, 1, ..., N-1\}$, we have

$$\begin{aligned}
&(\Delta_\alpha(\lambda)(1, \xi^r, ..., \xi^{(N-1)r})\, z)_{j+1} \\
&\quad = [\lambda \xi^{jr} - D_x f(0,\alpha)(e^{\lambda\cdot})\, \xi^{jr} - K(\alpha)\, e^{\lambda\cdot}(\xi^{(j+1)r} - 2\xi^{jr} + \xi^{(j-1)r}]\, z \\
&\quad = [\lambda - D_x f(0,\alpha)\, e^{\lambda\cdot} - K(\alpha)\, e^{\lambda\cdot}(\xi^r - 2 + \xi^{-r})]\, \xi^{jr} z \\
&\quad = [\lambda - D_x f(0,\alpha)\, e^{\lambda\cdot} - K(\alpha)\, e^{\lambda\cdot}(2\,\mathrm{Re}\,\xi^r - 2)]\, \xi^{jr} z \\
&\quad = \left[\lambda - D_x f(0,\alpha)\, e^{\lambda\cdot} - 2\left(\cos\frac{2\pi r}{N} - 1\right) K(\alpha)\, e^{\lambda\cdot} \right] \xi^{jr} z \\
&\quad = \left[\lambda - D_x f(0,\alpha)\, e^{\lambda\cdot} + 4\sin^2\frac{\pi r}{N} K(\alpha)\, e^{\lambda\cdot} \right] \xi^{jr} z. \quad \blacksquare
\end{aligned}$$

It is well-known (see [8]) that a Hopf bifurcation from a stationary solution $(0, \alpha)$ cannot occur if Eq. (4.1) has no temporal delay. However, the temporal delay in the coupling cells may cause various types of oscillations in the system (4.1), as will be illustrated in the following.

It is clear that the system (4.1) is equivariant with respect to the action of the dihedral group D_N, where the subgroup $\mathbb{Z}_N$ of rotations acts on $\mathbb{R}^N$

in such a way that the generator $\xi := e^{2\pi i/N}$ sends the jth coordinate of the vector $x = (x_0, x_1, ..., x_{N-1}) \in \mathbb{R}^N$ to the $j+1 \pmod N$ coordinate, and the flip κ sends the jth coordinate of x to the $-j \pmod N$ coordinate. We assume that $N > 2$ and denote this representation by $\Theta : D_N \to O(N)$.

First, we consider the action of $\mathbb{Z}_N$ on the complexification $U := \mathbb{C}^N$ of the representation Θ. It is clear that the $\mathbb{Z}_N$-isotypical decomposition of U is given by

$$U = \tilde{U}_0 \oplus \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_{N-1},$$

where $\tilde{U}_r := \{z(1, \xi^r, \xi^{2r}, ..., \xi^{(N-1)r}); z \in \mathbb{C}\}$. Since the flip κ sends $\tilde{U}_r$ onto $\tilde{U}_{-r}$, where $-r$ is taken $(\bmod N)$, thus $U_0 := \tilde{U}_0$ and $U_r := \tilde{U}_r \oplus \tilde{U}_{-r}$ for $0 < r < [\![N/2]\!]$ are the isotypical components of U with respect to the action of D_N. If N is even, there is one more isotypical component $U_{N/2} := \tilde{U}_{N/2}$. It is easy to see that the isotypical components U_r, $0 < r < [\![N/2]\!]$, correspond to the representations of D_N on $\mathbb{C} \oplus \mathbb{C}$ of the type (b1) given by

$$\gamma(z_1, z_2) = (\gamma^r \cdot z_1, \gamma^r \cdot z_2) \qquad \text{if} \quad \gamma \in \mathbb{Z}_N;$$
$$\kappa(z_1, z_2) = (z_2, z_1),$$

where $(z_1, z_2) \in \mathbb{C} \oplus \mathbb{C}$.

If N is an even number, the isotypical component $U_{N/2}$ is equivalent to the D_N-representation on $\mathbb{C} = \mathbb{R}^2$ of the type (b3), where

$$gz = \begin{cases} -z & \text{if} \quad g \in D_N \backslash D_{N/2}, \\ z & \text{if} \quad g \in D_{N/2}. \end{cases}$$

We make the following hypothesis:

(H1) *There exists* $(0, \alpha_0) \in \mathbb{R}^N \times \mathbb{R}$ *such that* $(0, \alpha_0)$ *is an isolated center of* (4.1) *such that* $\det_{\mathbb{C}} \Delta_{(0, \alpha_0)}(i\beta_0) = 0$, $\beta_0 > 0$.

It is straightforward to obtain the next two technical results:

COROLLARY 4.2. *A complex number* $\lambda \in \mathbb{C}$ *is a* j*th isotypical characteristic value of* $(0, \alpha)$, *where* $0 < j < [\![N/2]\!]$, *if and only if*

$$p_{\alpha, j}(\lambda) := \lambda - D_x f(0, \alpha)\, e^{\lambda \cdot} + 4 \sin^2 \frac{\pi j}{N} K(\alpha)\, e^{\lambda \cdot} = 0.$$

COROLLARY 4.3. *Under the assumption* (H1), *the* j*th isotypical crossing number for the isolated center* $(0, \alpha_0)$ *corresponding to the value* $il\beta_0$ *is equal to*

(i) *for* $0 < j < [\![N/2]\!]$,

$$c_{l,j}(\alpha_0, \beta_0) = 2(\deg_B(p_{\alpha_0-\delta,j}(\cdot), \Omega_l) - \deg_B(p_{\alpha_0+\delta,j}(\cdot), \Omega_l));$$

(ii) for $j = 0$ or $j = N/2$ (if N is even),

$$c_{l,j}(\alpha_0, \beta_0) = \deg_B(p_{\alpha_0-\delta,j}(\cdot), \Omega_l) - \deg_B(p_{\alpha_0+\delta,j}(\cdot), \Omega_l),$$

where $\Omega_l := (0, b) \times (l\beta_0 - c, l\beta_0 + c) \subset \mathbb{C}$ *and the constants* $b > 0$, $c > 0$ *and* $\delta > 0$ *are sufficiently small.*

Using Theorem 3.2, we can establish the following

THEOREM 4.4. *Assume the hypothesis* (H1) *is satisfied. If* $c_{1,j}(\alpha_0, \beta_0) \neq 0$, *then the stationary point* $(0, \alpha_0)$ *is a bifurcation point of* (4.1). *Moreover,*

(i) *if* $1 < j < N/2$, $h = gcd(j, N)$ *and* N/h *is odd, then there are at least 2 branches of periodic solutions corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, N/h *branches of periodic solutions corresponding to the orbit type* $(D_h \times \mathbb{Z}_1)$, *and* N/h *branches of periodic solutions corresponding to the orbit type* $(D_h^{(c,1)})$;

(ii) *if* $1 < j < N/2$, $h = gcd(j, N)$ *and* $N/h \equiv 2 \pmod 4$, *then there are at least 2 branches of periodic solution corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(\hat{d},1)})$;

(iii) *if* $1 < j < N/2$, $h = gcd(j, N)$ *and* $N/h \equiv 0 \pmod 4$, *then there are at least 2 branches of periodic solution corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(\tilde{D}_{2h}^{(d,1)})$;

(iv) *if* $j = N/2$, *then there exists at least one branch of periodic solutions corresponding to the orbit type* $(D_N^{(\hat{d},l)})$;

(v) *if* $j = 0$, *then there exists at least one branch of periodic solutions corresponding to the orbit type* $(D_N \times \mathbb{Z}_1)$.

EXAMPLE 4.5. We consider the following system of retarded functional differential equations (cf. [18])

$$\dot{x}_j(t) = -\alpha x_j(t) + \alpha h(x_j(t))[g(x_{j-1}) - 2g(x_j(t-1)) + g(x_{j+1}(t-1))], \tag{4.4}$$

where $0 \leqslant j \leqslant N-1$ and we use the convention that $j+1$ is always taken (mod N), $\alpha > 0$, h, $g: \mathbb{R} \to \mathbb{R}$ are continuously differentiable, h does not vanish and $g(0)=0$, $g'(0)>0$. By using an appropriate change of variables and rescaling the time, Eq. (4.4) can be transformed into an equation of the same type as Eq. (4.1). In addition, we have

$$p_{\alpha,j}(\lambda)=\lambda+\alpha+4\sin^2\frac{\pi r}{N}\alpha\mu e^{-\lambda},$$

where $\mu=h(0)\,g'(0)$. Assume that there exists j, $0<j<N/2$, such that $\mu>1/(4\sin^2(\pi j/N))$, then the number $i\beta_{0,j}$, where $\beta_{0,j}\in(\pi/2,\pi)$, is the unique solution of $\cos\beta_{0,j}=-1/(4\sin^2(\pi j/N))$ is a jth isotypical characteristic value corresponding to the stationary solution $(0,\alpha_{0,j})$, where $\alpha_{0,,j}=-\beta_{0,j}\cot\beta_{0,j}$. It can be computed (see [18]) that $(0,\alpha_{0,j})$ satisfies the assumption (H1) and we have $c_{1,j}(\alpha_{0,j},\beta_{0,j})<0$. Consequently, by Theorem 4.4 we have

PROPOSITION 4.6. *Let* $h=gcd(j,N)$. *If there exists* j, $0<j<N/2$ *such that* $\mu>1/(4\sin^2(\pi j/N))$, *then the stationary solution* $(0,\alpha_0)$ *is a bifurcation point for Eq.* (4.4). *In particular,*

(i) *if* N/h *is odd, then there are at least* 2 *branches of periodic solution corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, N/h *branches of periodic solutions corresponding to the orbit type* $(D_h\times\mathbb{Z}_1)$, *and* N/h *branches of periodic solutions corresponding to the orbit type* $(D_h^{(c,1)})$.

(ii) *if* $N/h\equiv 2 \pmod 4$, *then there are at least* 2 *branches of periodic solution corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(\hat{d},1)})$;

(iii) *if* $N/h\equiv 0 \pmod 4$, *then there are at least* 2 *branches of periodic solution corresponding to the orbit type* $(\mathbb{Z}_N^{(\theta_j,1)})$, $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and* $N/2h$ *branches of periodic solutions corresponding to the orbit type* $(\tilde{D}_{2h}^{(d,1)})$.

COROLLARY 4.7. *Assume that* N *is a prime number. If there exists* j, $0<j<N/2$, *such that* $\mu>1/(4\sin^2(\pi j/N))$, *then there are at least* $2(N+1)$ *different branches of nonconstant periodic solutions of* (4.4) *bifurcating from the stationary solution* $(0,\alpha_0)$.

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