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LOCAL EXISTENCE AND STABILITY OF PERIODIC TRAVELING WAVES OF LATTICE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor J. Kato on the occasion of his sixtieth birthday

ABSTRACT. Motivated by a neural network model and a population dynamics model, we consider a discrete Nagumo system with time delay and nonlinear nonlocal interactions. The existence of periodic traveling wave solutions is established by using the symmetric Hopf bifurcation theory recently developed, and the stability of periodic traveling wave solutions under small spatially periodic perturbations is investigated.

1. Introduction. In this paper we consider the spatio-temporal patterns and global dynamics of the following functional differential equations defined on a lattice on the real line

$$\begin{aligned}\frac{du_n(t)}{dt} &= d[u_{n-1}(t) - 2u_n(t) + u_{n+1}(t)] \\ &\quad + f(u_n(t), u_{n-1}(t-\tau), u_n(t-\tau), u_{n+1}(t-\tau)), \\ &\qquad n \in \mathbf{Z}\end{aligned} \tag{1.1}$$

where $d > 0$, $\tau \geq 0$ and $f : R^4 \to R$ is a sufficiently smooth function.

A prototype of system (1.1) in the absence of delay, $\tau = 0$, is the infinite system of coupled nonlinear differential equations

$$\frac{du_n(t)}{dt} = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbf{Z}. \tag{1.2}$$

Such a system can be viewed as a discretization of the well-known Nagumo equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u),$$

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see McKean [**23**], and arises in various application fields. For example, it occurs in the study of population genetics where spatially discretely distributed populations of diploid individuals are considered, see Aronson and Weinberger [**2**] and Fisher [**11**]. System (1.2) was also proposed as a model for propagation of nerve pulses in myelinated axons where the membrane is excitable only at spatially discrete sites, see Bell [**3**], Bell and Cosner [**4**], Chi, Bell and Hassard [**6**], Keener [**17, 18**], Zinner [**28–30**] and references therein.

In system (1.1), the smooth function f accounts for the nonlinear interaction between adjacent components, and we assume that the system's response to such nonlinear interaction is delayed. An example is a network of infinitely many neurons distributed in a line and connected by nearest neighborhood coupling. In certain situations, such a network (after normalization) can be described by

$$\frac{du_i(t)}{dt} = -u_i(t) + g(\alpha u_{i-1}(t-\tau) + \beta u_{i+1}(t-\tau)), \quad i \in \mathbf{Z} \tag{1.3}$$

where $\alpha, \beta \geq 0$, and $\tau \geq 0$ is used to account for the delay in the response and communication between neurons. See Cohen and Grossberg [**9**], Hopfield [**16**], Marcus and Westervelt [**22**] and Pineda [**24**]. Another example is the following system

$$\begin{aligned} \frac{du_i(t)}{dt} &= d[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] \\ &\quad + ru_i(t)\Bigg[1 + \alpha u_i(t) \\ &\quad - (1+\alpha) \sum_{|j-i|\leq 1} \beta_{|j-i|} u_j(t-\tau)\Bigg], \quad i \in \mathbf{Z}, \end{aligned} \tag{1.4}$$

which was employed to describe the growth of a single species population distributed over a patchy environment consisting of infinite number of patches on the real line. Due to accumulation of the population's waste products, the growth rate decreases as the total population increases. See Madras, Wu and Zou [**21**] for details and Britton [**5**] for a spatially continuous analog.

The focus of the present paper is on the local existence and stability of periodic traveling wave solutions of system (1.1). As will be shown, periodic traveling wave solutions of (1.1) subject to a spatially m-periodic initial condition, i.e., $u_{i+m}(t) = u_i(t)$, for $i \in \mathbf{Z}$ and $t \in \mathbf{R}$, is completely described by phase-locked oscillations of the following finite system on a periodic lattice

$$\begin{aligned}\frac{du_n(t)}{dt} &= d[u_{n-1}(t) - 2u_n(t) + u_{n+1}(t)] \\ &\quad + f(u_n(t), u_{n-1}(t-\tau), u_n(t-\tau), u_{n+1}(t-\tau)), \\ &\qquad n = 0, 1, \dots, m-1 \pmod m.\end{aligned} \tag{1.5}$$

Here a phase-locked oscillation of (1.5) is a periodic solution satisfying $u_n(t) = u_{n-1}(t + p/m)$ and $u_n(t) = u_n(t+p)$ for $t \in \mathbf{R}$ and n (mod m). This observation enables us to apply the recently developed symmetric Hopf bifurcation theory of functional differential equations to establish the existence of periodic traveling wave solutions of system (1.1). In Section 3 it is shown that, near several critical values of the delay τ, system (1.1) has periodic traveling wave solutions bifurcating from a spatially homogeneous equilibrium. The smallest critical value is of great importance, for only those bifurcating periodic traveling wave solutions near this smallest value can be stable under small perturbations.

The stability of bifurcating periodic traveling wave solutions is a very difficult problem, even for those solutions near the aforementioned smallest critical value. This is partially due to the multiplicity of associated eigenvalues of the linearization of system (1.5) and partially due to the lack of general stability theory suitable for system (1.1) that consists of infinitely many components. In Section 4 we consider the stability of (spatially) 2-periodic traveling wave solutions under small spatially periodic (not necessarily of period 2) perturbations. The restriction on the periodicity of perturbations eventually reduces the stability problem to one for a finite system of functional differential equations and so the well-developed stability theory of Hopf bifurcation, see Chow and Mallet-Paret [**7**], Claeyssen [**8**], Hale [**14**] and Hassard, Kazarinoff and Wan [**15**], can be applied. It is shown that such a periodic traveling wave solution is asymptotically stable, if f is the usual response function in neural networks and its third order derivative satisfies a certain sign condition.

2. Preliminaries. Consider the system

$$\frac{dx_n(t)}{dt} = d(\Delta x(t))_n + f(x_n(t), x_{n-1}(t-\tau), x_n(t-\tau), x_{n+1}(t-\tau)), \quad n \in \mathbf{Z}, \tag{2.1}$$

where $d > 0$ is a given constant, $f : \mathbf{R}^4 \to \mathbf{R}$ is a C^4 function and Δ is the discrete Laplacian operator defined by

$$(\Delta x)_n = x_{n-1} - 2x_n + x_{n+1},$$
$$x = \{x_n\}_{-\infty}^{\infty} \in l^{\infty},$$

where

$$l^{\infty} \triangleq \{x = \{x_n\}_{-\infty}^{\infty}; x_n \in \mathbf{R} \quad \text{for } n \in \mathbf{Z} \quad \text{and } \sup_{n\in\mathbf{Z}} |x_n| < \infty\}.$$

A *traveling wave* of (2.1) is a solution $x(t) = \{x_n(t)\}_{-\infty}^{\infty}$ of (2.1) given by

$$x_n(t) = \varphi(n + ct) \quad \text{for } n \in \mathbf{Z}, \quad t \in \mathbf{R} \tag{2.2}$$

for some positive constant c and for a continuously differentiable map $\phi : \mathbf{R} \to \mathbf{R}$.

The objective of this paper is to investigate spatially periodic traveling waves, that is, traveling waves satisfying (2.2) and

$$x_n(t) = x_{n+m}(t), \quad t \in \mathbf{R},\ n \in \mathbf{Z} \tag{2.3}$$

for some positive integer m. Clearly m is a period of ϕ and the periodic traveling wave is also temporally periodic with a period m/c. Moreover, a periodic traveling wave of (2.1) satisfying (2.2) and (2.3) gives a solution $y(t) = \{y_n(t)\}_{n=1}^{m}$, $y_n(t) = x_n(t)$, $1 \le n \le m$, of the finite system of delay differential equations

$$\frac{dy_n(t)}{dt} = d(\Delta y(t))_n + f(y_n(t), y_{n-1}(t-\tau), y_n(t-\tau), y_{n+1}(t-\tau)), \quad n \pmod{m} \tag{2.4}$$

satisfying

$$y_n(t) = y_{n-1}\left(t + \frac{1}{c}\right), \quad n \pmod m, \quad t \in \mathbf{R}, \tag{2.5}$$

$$y_n(t) = y_n\left(t + \frac{m}{c}\right), \quad n \pmod m, \quad t \in \mathbf{R}. \tag{2.6}$$

Conversely, any solution $y(t) = \{y_n(t)\}_{n=1}^{m}$ of (2.4) satisfying (2.5) and (2.6) gives a periodic traveling wave $x(t) = \{x_n(t)\}_{n\in\mathbf{Z}}$, defined by $x_n(t) = y_l(t)$ if $n = km + l$ for some integer k and some $0 \le l \le m-1$, of (2.1) satisfying (2.3) and (2.2) with $\phi(t) = y_m(t/c)$, $t \in \mathbf{R}$.

Following Alexander and Auchumuty [**1**], and Wu and Krawcewicz [**26**], we call a solution of (2.4) satisfying (2.5) and (2.6) a *phase-locked oscillation*. The above argument claims that there exists a one-to-one correspondence between a phase-locked oscillation of a finite system (2.4) and a periodic traveling wave of the infinite system (2.1).

We will use a symmetric Hopf bifurcation theorem developed in [**12, 19, 20**] to establish the local existence of periodic traveling waves. To this end, we first introduce this theorem.

Let N be a given positive integer and C_τ denote the Banach space of all continuous functions from $[-\tau, 0]$ into $\mathbf{R}^N$ with the supremum norm. Consider the following one parameter family of retarded equations

$$\frac{dx}{dt} = f(x_t, \mu), \tag{2.7}$$

where $x \in \mathbf{R}^N$, $\mu \in \mathbf{R}$, $f : C_\tau \times \mathbf{R} \to \mathbf{R}^N$ is a continuously differentiable compact mapping satisfying the following conditions.

(P1) There exists an orthogonal representation $\rho : \mathbf{Z}_n \to GL(\mathbf{R}^N)$ of $\mathbf{Z}_n$ on $\mathbf{R}^N$ such that

$$f(\rho(r)\phi, \mu) = \rho(r)f(\phi, \mu), \quad \phi \in C_\tau, \quad \mu \in \mathbf{R}, \quad r \in \mathbf{Z}_n,$$

where $\rho(r)\phi \in C_\tau$ is defined as $(\rho(r)\phi)(\theta) = \rho(r)\phi(\theta)$ for $\theta \in [-\tau, 0]$.

(P2) $f(0, \mu) = 0$ for all $\mu \in \mathbf{R}$ and $D\bar{f}(0,0) : \mathbf{R}^N \to \mathbf{R}^N$ is an isomorphism, where $\bar{f}$ denotes the restriction of f to $\mathbf{R}^N \times \mathbf{R}$, and $D\bar{f}(0,0)$ denotes the derivative of $\bar{f}$ with respect to the first variable x, evaluated at $(0,0)$.

Let $\mathbf{C}^N := \mathbf{R}^N + i\mathbf{R}^N$ and $\{\varepsilon_1, \dots, \varepsilon_N\}$ denote the standard basis of $\mathbf{R}^N$. For any $\lambda \in \mathbf{C}$ and $1 \le j \le N$, define $e^{\lambda \cdot}\varepsilon_j : [-\tau, 0] \to \mathbf{C}^N$ by $e^{\lambda \cdot}\varepsilon_j(\theta) = e^{\lambda\theta}\varepsilon_j$, $\theta \in [-\tau, 0]$. Let $\Delta_\mu(\lambda) : \mathbf{C}^N \to \mathbf{C}^N$ be defined by $\Delta_\mu(\lambda) = \lambda I - Df(0,\mu)(e^{\lambda \cdot}I)$ where

$$Df(0,\mu)(e^{\lambda \cdot}I) = (Df(0,\mu)(e^{\lambda \cdot}\varepsilon_1), \dots, Df(0,\mu)(e^{\lambda \cdot}\varepsilon_N)).$$

Denote by

$$\mathbf{C}^N = \mathbf{C}_0^N \oplus \mathbf{C}_1^N \oplus \cdots \oplus \mathbf{C}_{n-1}^N$$

the isotypical decomposition of the $\mathbf{Z}_n$-action on $\mathbf{C}^N$, where $\mathbf{C}_l^N$, $0 \le l \le n-1$, is the direct sum of all one-dimensional $\mathbf{Z}_n$-irreducible subspaces V of $\mathbf{C}^N$ such that the restricted action of $\mathbf{Z}_n$ on V is isomorphic to the $\mathbf{Z}_n$-action on $\mathbf{C}$ defined by

$$\rho_l(\varepsilon^{i(2\pi j/n)})z = e^{i(2\pi l j/n)}z, \quad z \in \mathbf{C},\ 0 \le j \le n-1.$$

Clearly, $\Delta_\mu(\lambda)\mathbf{C}_l^N \subset \mathbf{C}_l^N$ for $0 \le l \le n-1$. So we can define $\Delta_{\mu,l}(\lambda) := \Delta_\mu(\lambda)|_{\mathbf{C}_l^N}$, $0 \le l \le n-1$. We further assume

(P3) There exist ε_0, δ_0 and $\omega_0 > 0$ such that

(i) $\det \Delta_0(u+iv) = 0$ with $(u,v) \in \partial\Omega$ if and only if $u = 0$ and $v = \omega_0$, where $\Omega = (0,\varepsilon_0) \times (\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0)$;

(ii) $\det \Delta_\mu(i\omega) = 0$ with $(\mu,\omega) \in [-\delta_0, \delta_0] \times [\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0]$ if and only if $\mu = 0$ and $\omega = \omega_0$;

(iii) $\det \Delta_{\pm\delta_0}(\lambda) \neq 0$ for $\lambda \in \partial\Omega$.

The following local symmetric Hopf bifurcation theorem is taken from Krawcewicz and Wu [**20**] and represents an analog for differential delay equations of the corresponding results in Fiedler [**10**] and Golubitsky, Schaeffer and Steward [**13**] for ordinary and partial differential equations.

Theorem 2.1. *If* (P1)–(P3) *are satisfied and*

$$\deg_B(\det \Delta_{-\delta_0,r}(\cdot), \Omega) \neq \deg_B(\det \Delta_{\delta_0,r}(\cdot), \Omega)$$

for some $r \in \{0, 1, \dots, n-1\}$, *where* $\deg_B$ *denotes the Brouwer degree, then there exists a sequence of triples* $\{(x^{(k)}, \mu^{(k)}, \omega^{(k)})\}_{k=1}^\infty$ *such that*

(i) $\mu^{(k)} \to \mu_0$, $\omega^{(k)} \to \omega_0$, $x^{(k)}(t) \to 0$ *uniformly for* $t \in \mathbf{R}$ *as* $k \to \infty$;

(ii) $x^{(k)}$ *is a* $2\pi/\omega^{(k)}$*-periodic solution of* (2.7) *with* $\mu = \mu_k$ *for* $k = 1, 2, \ldots$;

(iii) $\rho(e^{i(2\pi/n)})x^{(k)}(t) = x^{(k)}(t - (2\pi/\omega^{(k)})(r/n))$ *for* $t \in \mathbf{R}$ *and* $k = 1, 2, \ldots$.

In later applications of Theorem 2.1, we have the situation where $n = N$ and $\mathbf{Z}_n$ acts on $\mathbf{R}^n$ by permuting the variables cyclically, i.e., $[\rho(e^{i(2\pi/n)})x]_j = x_{j-1}$, $j \pmod n$, $x \in \mathbf{R}^n$. In this situation, we have $\mathbf{C}_r^N = \{(1, \xi^r, \ldots, \xi^{(n-1)r})^T x; x \in \mathbf{R}\}$, $0 \le r \le n-1$ with $\xi = e^{i(2\pi/n)}$ and (iii) becomes $x_{j-1}^{(k)}(t) = x_j^{(k)}(t - (2\pi/\omega^{(k)})(r/n))$, $t \in \mathbf{R}$, $k = 1, 2, \ldots$.

3. Existence of periodic traveling waves. Throughout this section we fix $m \ge 1$, and assume that there exists a constant $k \in \mathbf{R}$ such that $f(k,k,k,k) = 0$. Then $K = (k, \ldots, k)$ is a steady state of system (2.1). Let

$$\alpha_0 = f_1(k,k,k,k) \qquad \beta_0 = f_3(k,k,k,k),$$
$$\beta_2 = f_2(k,k,k,k) \qquad \beta_4 = f_4(k,k,k,k),$$

where

$$f_j = \frac{\partial f(u_1,u_2,u_3,u_4)}{\partial u_j}, \quad j = 1,2,3,4.$$

For technical reasons, we always assume

$$(\mathrm{A}_0) \qquad \beta_2 = \beta_4 \triangleq \beta.$$

The linearization of (2.1) at K is, writing $n \pmod m$, given by

$$(3.1) \qquad \frac{dx_n(t)}{dt} = d(\Delta x(t))_n + \alpha_0 x_n(t) + \beta_0 x_n(t-\tau) + \beta[x_{n-1}(t-\tau) + x_{n+1}(t-\tau)]$$

or, using matrix expressions

$$(3.2) \qquad \frac{dx(t)}{dt} = Mx(t) + Nx(t-\tau),$$

where

$$M = \begin{pmatrix} \alpha_0 - 2d & d & 0 & \cdots & 0 & d \\ d & \alpha_0 - 2d & d & \cdots & 0 & 0 \\ 0 & d & \alpha_0 - 2d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \cdots & d & \alpha_0 - 2d \end{pmatrix}$$

and

$$N = \begin{pmatrix} \beta_0 & \beta & 0 & 0 & \cdots & 0 & \beta \\ \beta & \beta_0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & \beta & \beta_0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta & 0 & 0 & 0 & \cdots & \beta & \beta_0 \end{pmatrix}.$$

In the case $\tau > 0$, we normalize the time variable by $y_n(t) = x_n(\tau t)$. Then (2.1) and (3.2) become, respectively,

$$(2.1^*) \qquad \frac{dy_n(t)}{dt} = \tau d(\Delta y(t))_n + \tau f(y_n(t), y_{n-1}(t-1), y_n(t-1), y_{n+1}(t-1)), \quad n \pmod m$$

and

$$(3.2^*) \qquad \frac{dy(t)}{dt} = \tau[My(t) + Ny(t-1)].$$

Let $\Lambda_\tau(\lambda) = \lambda I - \tau M - \tau N e^{-\lambda}$, where I is the $m \times m$ identity matrix. Then the characteristic equation of (3.2*) is

$$(3.3) \qquad \det \Lambda_\tau(\lambda) = 0.$$

Lemma 3.1. *Under* (A_0), *we have*

$$\det \Lambda_\tau(\lambda) = \prod_{j=0}^{m-1} p_j(\tau, \lambda)$$

where

$$p_j(\tau,\lambda) = \lambda - \tau\alpha_0 + 4\tau\, d\sin^2\frac{j\pi}{m} - \tau\left(\beta_0 + 2\beta - 4\beta\sin^2\frac{j\pi}{m}\right)e^{-\lambda},$$
$$0 \le j \le m-1.$$

Proof. Let $\xi = e^{i(2\pi/m)}$ and $W_k = (1, \xi^k, \dots, \xi^{(m-1)k})^T$, $0 \le k \le m-1$. Then $\{W_0, W_1, \dots, W_{m-1}\}$ spans the space $\mathbf{C}^m$. Noting that $\xi^{-k} = \overline{\xi^k}$, we have, for every $j, k \in \{0, 1, \dots, m-1\}$ that

$$\begin{aligned}(\Lambda_\tau(\lambda)W_k)_j &= (\lambda - \tau(\alpha_0 - 2d) - \tau\beta_0 e^{-\lambda})\xi^{(j-1)k} \\ &\quad - \tau(d + \beta e^{-\lambda})(\xi^{(j-2)k} + \xi^{jk}) \\ &= [\lambda - \tau(\alpha_0 - 2d) - \tau\beta_0 e^{-\lambda} \\ &\quad - \tau(d + \beta e^{-\lambda})(\xi^k + \xi^{-k})]\xi^{(j-1)k} \\ &= [\lambda - \tau\alpha_0 + \tau d(2 - 2\mathrm{Re}\,\xi^k) - \tau e^{-\lambda}(\beta_0 + 2\beta\mathrm{Re}\,\xi^k)]\xi^{(j-1)k} \\ &= \left[\lambda - \tau\alpha_0 + \tau d\left(2 - 2\cos\frac{2k\pi}{m}\right)\right. \\ &\quad \left. - \tau e^{-\lambda}\left(\beta_0 + 2\beta\cos\frac{2k\pi}{m}\right)\right]\xi^{(j-1)k} \\ &= \left[\lambda - \tau\alpha_0 + 4\tau\, d\sin^2\frac{k\pi}{m}\right. \\ &\quad \left. - \tau e^{-\lambda}\left(\beta_0 + 2\beta - 4\beta\sin^2\frac{k\pi}{m}\right)\right]\xi^{(j-1)k}.\end{aligned}$$

Thus $\Lambda_\tau(\lambda)W_k = p_k(\tau,\lambda)W_k$ and $\det\Lambda_\tau(\lambda) = \prod_{k=0}^{m-1} p_k(\tau,\lambda)$. □

Lemma 3.2. *The following statements hold.*

(i) *The equation*

$$p_j(\tau,\lambda) = 0 \tag{3.4$_j$}$$

has purely imaginary roots λ for some $\tau > 0$ if and only if

$$\left|\alpha_0 - 4d\sin^2\frac{j\pi}{m}\right| < \left|\beta_0 + 2\beta - 4\beta\sin^2\frac{j\pi}{m}\right| \tag{H$_j$}$$

is satisfied.

(ii) *For each $j \in \{0, 1, \ldots, m-1\}$ such that* (H_j) *holds, the least positive τ for $(3.4)_j$ to have purely imaginary roots is*

$$(3.5)_j \qquad \tau_j = \frac{-1}{(\beta_0 + 2\beta - 4\beta \sin^2(j\pi/m))} \left(\frac{\omega_j}{\sin \omega_j}\right)$$

and the corresponding pair $\pm i\omega_j$ of the purely imaginary roots of $p_j(\tau, \lambda) = 0$ are given by

$(3.6)_j$

$$\omega_j = \begin{cases} \arccos(4d\sin^2(j\pi/m) - \alpha_0)/(\beta_0 + 2\beta - 4\beta \sin^2(j\pi/m)) \\ \qquad \text{if } \beta_0 + 2\beta - 4\beta\sin^2(j\pi/m) < 0, \\ 2\pi - \arccos(4d\sin^2(j\pi/m) - \alpha_0)/(\beta_0 + 2\beta - 4\beta\sin^2(j\pi/m)) \\ \qquad \text{if } \beta_0 + 2\beta - 4\beta\sin^2(j\pi/m) > 0. \end{cases}$$

Proof. Substituting $\lambda = i\omega$, $\omega > 0$, into $(3.4)_j$ yields

$$i\omega - \tau\left(\alpha_0 - 4d\sin^2\frac{j\pi}{m}\right) - \tau\left(\beta_0 + 2\beta - 4\beta\sin^2\frac{j\pi}{m}\right)e^{-i\omega} = 0,$$

which is equivalent to

$$\begin{cases} \omega = -\tau(\beta_0 + 2\beta - 4\beta\sin^2(j\pi/m))\sin\omega, \\ \alpha_0 - 4d\sin^2(j\pi/m) = -(\beta_0 + 2\beta - 4\beta\sin^2(j\pi/m))\cos\omega. \end{cases}$$

Straightforward calculation and the monotonicity of $x/\sin x$ on $(0, \pi)$ then leads to the conclusion. □

Let us now explore the symmetry of system (2.1^*) in order to apply Theorem 2.1. Define $\mathbf{Z}_m = \{e^{i(2\pi/m)j}; 0 \le j \le m-1\}$. Then $\mathbf{Z}_m$ is a group with the usual operation

$$e^{i(2\pi/m)j_1} \cdot e^{i(2\pi/m)j_2} = e^{i(2\pi/m)(j_1+j_2)}, \quad j_1, j_2 \pmod m.$$

Define the orthogonal representation $\rho : \mathbf{Z}_m \to GL(\mathbf{R}^m)$ of the Z_m cyclic permutation on $\mathbf{R}^m$ by

$$(\rho(e^{i(2\pi/m)k})x)_j = x_{j-k}, \quad \text{for } x = (x_1, x_2, \ldots, x_m)^T \in \mathbf{R}^m \\ \text{and } j, k \pmod m.$$

Then we have the following result, the proof of which is straightforward and is omitted.

Lemma 3.3. *System* (2.1^*) *is equivariant with respect to the above* $\mathbf{Z}_m$*-action.*

Now we can state and prove our local existence result.

Theorem 3.4. *Assume that there exists a* $j \in \{0, 1, \ldots, [m/2]\}$ *such that* (H_j) *holds. Assume also that*

$$(\mathrm{A}_1) \qquad \begin{gathered} \alpha_0 + \beta_0 + 2\beta - 4(d+\beta)\sin^2\frac{k\pi}{m} \neq 0 \\ \text{for } k = 0, 1, \ldots, [m/2]. \end{gathered}$$

Then $\tau = \tau_j$ *determined by* $(3.5)_j$ *and* $(3.6)_j$ *is a Hopf bifurcation value of synchronous (if* $j = 0$*) or phase-locked (if* $j \neq 0$*) oscillations of* (2.1^*)*. More precisely, there exists a sequence of triples* $\{(y^{(l)}, \tau^{(l)}, \omega^{(l)})\}$ *such that*

(i) $\tau^{(l)} \to \tau_j$, $\omega^{(l)} \to \omega_j$ *and* $y^{(l)}(t) \to K$ *uniformly for* $t \in \mathbf{R}$ *as* $l \to \infty$;

(ii) $y^{(l)}(t)$ *is a* $2\pi/\omega^{(l)}$*-periodic solution of* (2.1^*) *with* $\tau = \tau^{(l)}$ *for* $l = 1, 2, \ldots$.

(iii) $y_k^{(l)}(t) = y_{k-1}^{(l)}(t + (2\pi/\omega^{(l)})(j/m))$ *for* $t \in \mathbf{R}$, $l = 1, 2, \ldots$, *and* $k \pmod m$.

Proof. It is obvious that $p_j(\tau, \lambda)$ is analytic both in λ and τ. Now since

$$\begin{aligned} \frac{\partial p_j(\tau,\lambda)}{\partial \lambda}\bigg|_{\tau=\tau_j,\lambda=i\omega_j} &= 1 + \tau_j\left(\beta_0 + 2\beta\cos\frac{2j\pi}{m}\right)e^{-i\omega_j} \\ &= 1 + i\omega_j - \tau_j\left(\alpha_0 - 4d\sin^2\frac{j\pi}{m}\right) \\ &\neq 0 \quad (\text{since } \omega_j > 0), \end{aligned}$$

the implicit function theorem implies that there exist $\delta_j > 0$ and a unique continuously differentiable $\lambda : (\tau_j - \delta_j, \tau_j + \delta_j) \to \mathbf{C}$ such that

$\lambda(\tau_j) = i\omega_j$ and $p_j(\tau, \lambda(\tau)) = 0$ for $\tau \in (\tau_j - \delta_j, \tau_j + \delta_j)$. It can be easily calculated that

$$\frac{d}{d\tau}(\operatorname{Re}\lambda(\tau))\Big|_{\tau=\tau_j} = \frac{\omega_j^2}{\tau_j(1 - \tau_j\alpha_0 + \tau_j\, d\sin^2(j\pi/m))^2 + \omega_j^2} > 0.$$

Thus, $\deg_B(p_j(\tau_j - \varepsilon, \cdot), \Omega) \neq \deg_B(p_j(\tau_j + \varepsilon, \cdot), \Omega)$ for some sufficiently small $\varepsilon > 0$, where $\Omega = (0, \varepsilon) \times (\omega_j - \varepsilon, \omega_j + \varepsilon)$ and $\deg_B$ is the Brouwer degree. Noting that $\det \Lambda_\tau(\lambda)$ is also analytic in τ and λ, the conditions (i)–(iii) in (P3) of Theorem 2.1 can then be easily verified. Clearly (P1) and (P2) in Theorem 2.1 are guaranteed by Lemma 3.3 and (A_1) respectively. Therefore, the conclusions of this theorem follow immediately from Theorem 2.1. □

Corollary 3.5. *Under the assumptions of Theorem* 3.4, *near* $\tau = \tau_j$ *and the equilibrium* K, *the system* (2.1*) *has a synchronous or phase-locked oscillation* $x^{(l)}(t) = (x_0^{(l)}(t), \ldots, x_{m-1}^{(l)}(t))$ *satisfying*

$$x_n^{(l)}(t) = x_{n-1}^{(l)}\left(t + \frac{2\pi}{\omega^{(l)}}\frac{j\tau^{(l)}}{m}\right),$$
$$x_n^{(l)}(t) = x_n^{(l)}\left(t + \frac{2\pi\tau^{(l)}}{\omega^{(l)}}\right).$$

Hence system (2.1) *has periodic traveling solution(s) satisfying* (i)–(ii) *with* $T^{(l)} = (2\pi\tau^{(l)}/\omega^{(l)})$ *and*

$$\varphi^{(l)}(t) = \begin{cases} x_0^{(l)}((2\pi/\omega^{(l)})(j\tau^{(l)}/m)t) & \text{if } j \neq 0, \\ x_0^{(l)}((2\pi\tau^{(l)}/\omega^{(l)})t), & \text{if } j = 0, \end{cases}$$

where $\omega^{(l)}$, $\tau^{(l)}$ *and* j *are as in Theorem* 3.4.

In the remainder of this section, we consider some special values of m. First let us take $m = 3$. Then (H_1) becomes

$$|\alpha_0 - 3d| < |\beta_0 - \beta| \tag{3.7}$$

and (A_1) becomes

$$\alpha_0 + \beta_0 - \beta - 3d \neq 0 \tag{3.8}$$

and

$$\alpha_0 + \beta_0 + 2\beta \neq 0. \tag{3.9}$$

Note that (3.8) is implied by (3.7). Thus we have the following explicit criterion.

Corollary 3.6. *Suppose* (3.7) *and* (3.9) *hold. Then system* (2.1) *has periodic traveling wave solution(s) near the equilibrium* K.

Next we let $m = 4$. Then (H_1) and (H_2) become respectively

$$|\alpha_0 - 2d| < |\beta_0|, \tag{3.10}$$

and

$$|\alpha_0 - 4d| < |\beta_0 - 2\beta|. \tag{3.11}$$

Furthermore, (A_1) becomes

$$\alpha_0 + \beta_0 + 2\beta \neq 0, \tag{3.12}$$

$$\alpha_0 + \beta_0 - 2d \neq 0, \tag{3.13}$$

$$\alpha_0 + \beta_0 - 2\beta - 4d \neq 0. \tag{3.14}$$

Hence, we have

Corollary 3.7. *Assume either*

(i) (3.10), (3.12) *and* (3.14), *or*

(ii) (3.11), (3.12) *and* (3.13) *hold.*

Then system (2.1) *has periodic traveling wave solution*(*s*) *near the equilibrium* K.

Remark 3.8. Corollary 3.6 and Corollary 3.7 show that, for fixed α_0, β and d, large $|\beta_0|$ can induce periodic traveling waves. Also, for fixed α_0, β_0 and d, large $|\beta|$ causes periodic traveling waves. In other words, periodic traveling waves can arise if the delayed effect is significant and if the delay is sufficiently large.

4. Stability of traveling waves. Theorem 3.4 shows that, for each fixed $m \geq 1$, if (H_j) holds for some integer $0 \leq j \leq m-1$, then system (2.1) has a bifurcation of periodic traveling waves near a certain critical value $\tau_j > 0$. In the case where (H_j) holds for several integers $j \in \{0, 1, \dots, m-1\}$, the smallest critical value $\tau_m^* = \min\{\tau_j; (H_j)$ holds$\}$ is of most importance. This is because, when τ is near $\tau_j > \tau_m^*$ the characteristic equation (3.3) always has zeros of positive real parts and hence the bifurcated periodic solutions of (2.1*) are unstable under small perturbations (or equivalently, the bifurcated periodic traveling wave solutions of (2.1) are unstable under spatially m-periodic small perturbations). The only bifurcating periodic traveling waves that are possibly asymptotically stable under small perturbations (at least under small spatially periodic perturbations) are those bifurcated waves when τ is near τ_m^*. Note that τ_m^* may depend on m as well.

In the sequel, we make the following assumptions

(C1) $\alpha_0 < 0$, $\beta_0 < 0$, $\beta > 0$.

(C2) $\beta > d$, $\beta_0 + 2\beta < 0$.

(C3) $\beta_0 + 2\beta - \alpha_0 < 4(\beta - d)$.

(C4) $-\beta_0 d < \beta(2d - \alpha_0)$.

Lemma 4.1. *Under* (C_1)–(C_4), *we have the following conclusions*:

(i) *there exists a constant* $\tau^* > 0$ *such that* $\tau_{2k}^* = \tau^*$ *for every positive integer* k;

(ii) τ^* *is given by* $\tau_{m/2}$ *when* m *is even and* $\tau_{m/2}$ *is defined by* $(3.5)_{m/2}$;

(iii) $\tau_{2k+1}^* > \tau^*$ *for every positive integer* k, *so* $\tau^* = \inf\{\tau_k^*; k \in \mathbf{Z}^+\}$.

Proof. Assume m is an even integer. By (C_1) and (C_3), we have

$$|\alpha_0 - 4d| = 4d - \alpha_0 < 2\beta - \beta_0 = |2\beta - \beta_0|$$

which means that $(H_{m/2})$ holds. Suppose (H_j) holds with $j \in \{0, 1, \dots, m/2-1\}$. Note that $(4dx-\alpha_0)/(\beta_0+2\beta-4\beta x)$ is an increasing function of x under (C_4). So $\sin^2(j\pi/m) < \sin^2((m/2)\pi/m)$ implies

$$\frac{4d\sin^2(j\pi/m) - \alpha_0}{\beta_0 + 2\beta - 4\beta\sin^2(j\pi/m)} < \frac{4d\sin^2((m\pi/2)/m) - \alpha_0}{\beta_0 + 2\beta - 4\beta\sin^2((m\pi/2)/m)},$$

which yields $\omega_j > \omega_{m/2}$. Now, in terms of the expressions of τ_j and $\tau_{m/2}$ in Lemma 3.2, we can get $\tau_j > \tau_{m/2}$. Thus $\tau^* = \tau_m^* = \tau_{m/2}$. Note that $\tau_{m/2}$ is in fact independent of the choice of the even integer m. This justifies (i) and (ii). (iii) can be verified in a similar manner. □

It can be easily calculated that, when $\tau = 0$, all the eigenvalues of (3.1) are real and are given by

$$\lambda_j = \alpha_0 + \beta_0 + 2\beta - 4(\beta + d)\sin^2 \frac{j\pi}{m}, \quad 0 \le j \le m-1.$$

So, if (C_1) and (C_2) hold, then $\lambda_j < 0$ for $j \in \{0, 1, \dots, m\}$.

In the remainder of this section, we always assume that (C_1)–(C_4) hold so that (2.1*) has phase-locked oscillation(s) occurring near $\tau = \tau^* = \tau_{m/2}$, where

$$\tau^* = \tau_{m/2} = \frac{1}{2\beta - \beta_0}\frac{\omega^*}{\sin\omega^*},$$
$$\omega^* = \omega_{m/2} = \arccos\frac{4d - \alpha_0}{\beta_0 - 2\beta}.$$

Moreover, it is obvious that, for $\tau \in [0, \tau^*]$ all the eigenvalues have negative real parts, except for the pair of purely imaginary ones $\pm i\omega^* = \pm i\omega_{m/2}$ corresponding to $(3.4)_{m/2}$ at $\tau = \tau^* = \tau_{m/2}$. Thus, the algorithm for stability of Hopf bifurcations developed in Hassard, Kazarinoff and Wan [**15**] is applicable to system (2.1*).

Let $\tau = \mu + \tau^*$ and write (2.1*) as

$$\dot{y}(t) = L_\mu y_t + F_\mu y_t, \tag{4.1}$$

where $L_\mu : C([-1,0]; \mathbf{R}^m) \to \mathbf{R}^m$ is defined by

$$L_\mu\phi = (\mu + \tau^*)M\phi(0) + (\mu + \tau^*)N\phi(-1),$$

and $F_\mu : C([-1,0]; \mathbf{R}^m) \to \mathbf{R}^m$ is given by

$$\begin{aligned}(F_\mu\phi)_n = (\mu + \tau^*)[&f(\phi_n(0), \phi_{n-1}(-1), \phi_n(-1), \phi_{n+1}(-1)) \\ &- \alpha_0\phi_n(0) - \beta_0\phi_n(-1) - \beta(\phi_{n-1}(-1) \\ &- \phi_{n+1}(-1))], \quad n \pmod{m}.\end{aligned}$$

L_μ can be expressed as

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta),$$

where

$$\eta(\theta, \mu) = (\mu + \tau^*)[\delta(\theta)M + \delta(\theta + 1)N],$$

and δ is the Dirac function. Let

$$\mathrm{Dom}\,(A(\mu)) = \left\{ \phi \in C^1([-1, 0]; \mathbf{R}^m); \dot{\phi}(0_-) = \int_{-1}^{0} d\eta(s, \mu)\phi(s) \right\},$$

$$A(\mu)\phi(\theta) = \begin{cases} \dot{\phi}(\theta) & \text{if } -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s) = L_\mu \phi & \text{if } \theta = 0. \end{cases}$$

Then (4.1) can be formally written as

$$\dot{y}_t = A(\mu)y_t + R(\mu)y_t, \tag{4.2}$$

where

$$(R(\mu)\phi)(\theta) = \begin{cases} 0 & \text{if } -1 \le \theta < 0, \\ F_\mu \phi & \text{if } \theta = 0. \end{cases}$$

For any $\psi \in C([0, 1]; \mathbf{C}^m)$ and $\phi \in C([-1, 0], \mathbf{C}^m)$, define a sort of "inner product" by

$$\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-1}^{0} \int_{s=0}^{\theta} \bar{\psi}^T(s - \theta)\, d\eta(\theta, 0)\phi(s)\, ds.$$

Let $A^*(0)$ be the adjoint operator of $A(0)$ relative to the above inner product. Then

$$\mathrm{Dom}\,(A^*(0)) = \left\{ \psi \in C^1([0, 1]; \mathbf{C}^m); \dot{\psi}(0^+) = -\int_{-1}^{0} d\eta^T(s, 0)\psi(-s) \right\},$$

$$A^*(0)\psi(s) = \begin{cases} -\dot{\psi}(s) & \text{if } 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(\theta, 0)\psi(-\theta) & \text{if } s = 0. \end{cases}$$

It is straightforward to show that $\pm i\omega^*$ are eigenvalues of $A(0)$ and $A^*(0)$. Moreover, $q(\theta) = e^{i\omega^* \theta}V$, $\theta \in [-1, 0]$, is an eigenvector of $A(0)$

corresponding to $i\omega^*$, and $q^*(\theta) = De^{i\omega^*\theta}V$, $\theta \in [0,1]$, is an eigenvector of $A^*(0)$ corresponding to $-i\omega^*$ such that

$$\langle q^*, q\rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q}\rangle = 0,$$

where $V = (1, -1, \dots, 1, -1)^T$ and

$$D = \frac{1}{m[1 + (4d - \alpha_0)\tau^* - i\omega^*]}.$$

In what follows, we follow the notation and the algorithm developed in Hassard, Kazarinoff and Wan [**15**] for determining the stability of Hopf bifurcation of periodic solutions for general functional differential equations.

Let

$$z(t) = \langle q^*, y_t\rangle,$$
$$w(z,\bar{z})(\theta) = y_t(\theta) - 2\mathrm{Re}\,(zq(\theta)).$$

Then, at $\mu = 0$, (4.2) is reduced to an ordinary differential equation for a single complex variable

$$\dot{z}(t) = i\omega^* z(t) + \bar{q}^*(0)\hat{F}(z,\bar{z}), \tag{4.3}$$

where

$$\hat{F}(z,\bar{z}) = F_0(w(z,\bar{z}) + 2\mathrm{Re}\,(zq)) = F_0(y_t).$$

Rewrite (4.3) as

$$\dot{z}(t) = i\omega^* z(t) + g(z,\bar{z}) \tag{4.4}$$

with

$$w(z,\bar{z}) = w_{20}\frac{z^2}{2} + w_{11}z\bar{z} + w_{02}\frac{\bar{z}^2}{2} + \cdots \tag{4.5}$$

and

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{4.6}$$

By Hassard, Kazarinoff and Wan [**15**], the stability and direction of the Hopf bifurcation are determined by the coefficients g_{20}, g_{11}, g_{02} and g_{21}. So, in the next step, we calculate these.

Note that

$$y_t(\theta) = w(z,\bar{z})(\theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta).$$

Hence the nth component of $y_t(0)$ and $y_t(-1)$ for $n = 1, 2, \dots, m$ are

$$\text{(4.7)} \qquad \begin{cases} y_n(t) & = y_t^{(n)}(0) = w^{(n)}(z,\bar{z})(0) + (-1)^{n+1}(z+\bar{z}), \\ y_n(t-1) & = y_t^{(n)}(-1) = w^{(n)}(z,\bar{z})(-1) \\ & \qquad\qquad +(-1)^{n+1}(ze^{i\omega^*} + \bar{z}e^{-i\omega^*}). \end{cases}$$

We now specify the function f for the sake of simplicity. We assume

$$\text{(4.8)} \qquad f(v_1, v_2, v_3, v_4) = g\left(\sum_{j=1}^{4} a_j v_j\right)$$

with

$$\text{(4.9)} \qquad \begin{cases} a_2 = a_4 \\ g : \mathbf{R} \to \mathbf{R} \text{ is } C^4, \qquad g(0) = 0, \\ g'(0) = \sup_{x \in \mathbf{R}} g'(x) > 0, \\ \quad \text{and } g''(0) = 0. \end{cases}$$

Such a function is frequently used as a response function (input-output function, or activative function) in the study of neural networks. See Hopfield [**16**], Marcus and Westervelt [**22**], Wu [**25**] and Wu and Zou [**27**]. With the above assumption, we have

$$\alpha_0 = a_1 g'(0), \qquad \beta_0 = a_3 g'(0),$$
$$\beta = a_2 g'(0) = a_4, \qquad g'(0) = a g'(0).$$

Hence, in accordance with (C_1)–(C_4), we assume

$$\text{(4.10)} \qquad \begin{cases} a_1 < 0, \quad a_3 < 0 \quad \text{and} \quad a \triangleq a_2 = a_4 > 0; \\ ag'(0) > d, \qquad a_3 + 2a < 0; \\ (a_3 + 2a - a_1)g'(0) \le 4[ag'(0) - d]; \\ -a_3 d \le a[2d - a_1 g'(0)]. \end{cases}$$

Since all the second derivatives of f in this case are zero at the equilibrium $K = (0,0,\dots,0)$, we know that $g_{20} = g_{11} = g_{02} = 0$, and hence we only need to calculate g_{21}. A straightforward but tedious calculation shows

$$
\begin{aligned}
(F_0 y_t)_n = \tau^* g'''(0)[& a_1^3 y_n^3(t) \\
&+ a_1^2 y_n^2(t)(a_2 y_{n-1}(t-1) + a_3 y_n(t-1) + a_4 y_{n+1}) \\
&+ a_1 a_2 y_n(t) y_{n-1}(t)(a_2 y_{n-1}(t-1) + a_3 y_n(t-1) + a_4 y_{n+1}) \\
&+ a_1 a_3 y_n(t) y_n(t-1)(a_3 y_n(t-1) + a_4 y_{n+1}(t-1)) \\
&+ a_1 a_4 y_n(t) y_{n+1}(t-1) a_4 y_{n+1}(t-1) \\
&+ a_2^2 y_{n-1}^2(t-1)(a_2 y_{n-1}(t-1) + a_3 y_n(t-1) + a_4 y_{n+1}) \\
&+ a_2 a_3 y_{n-1}(t-1) y_n(t-1)(a_3 y_n(t-1) + a_4 y_{n+1}(t-1)) \\
&+ a_2 a_4 y_{n-1}(t-1) y_{n+1}(t-1) a_4 y_{n+1}(t-1) \\
&+ a_3^2 y_n^2(t-1)(a_3 y_n(t-1) + a_4 y_{n+1}(t-1)) \\
&+ a_3 a_4 y_n(t-1) y_{n+1}(t-1) a_4 y_{n+1}(t-1) \\
&+ a_4^3 y_{n+1}(t-1) + h.o.t.], \quad n \pmod{m},
\end{aligned}
$$

and

$$
\begin{aligned}
g(z,\bar z) &= \bar q^*(0)(F_0 y_t) \\
&= \overline{D} V^T (F_0 y_t) \\
&= \overline{D}[(F_0 y_t)_1 + \dots + (F_0 y_t)_{(m/2)-1} \\
&\qquad - (F_0 y_t)_2 - \dots - (F_0 y_t)_{m/2}] \\
&= \frac{m}{2}\tau^* g'''(0)\overline{D}[E z^2 \bar z + \cdots],
\end{aligned}
$$

where

$$
\begin{aligned}
E &= E_2 e^{i2\omega^*} + E_1 e^{i\omega^*} + E_0 + E_{-1} e^{-i\omega^*}, \\
E_2 &= 2a_1[a(2a - a_3) + a_3(a_3 - a)], \\
E_1 &= 6a_3 a^2 - 9a^3 - 3a_1 a^2 \\
&\qquad - 2(2a - a_3)(3a^2 + 2a_1^2) + 6a_3(a_3 - a)^2, \\
E_0 &= 6a_1^3 + 6a_1 a(2a - a_3) + 4a_1 a_3(a_3 - a), \\
E_{-1} &= -2a_1^2(2a - a_3).
\end{aligned}
$$

Thus,

$$
c_1(0) = \frac{g_{21}}{2} = \frac{m}{2}\tau^* g'''(0)\overline{D}E,
$$

where $\overline{D}$ is the conjugate of D appeared in the expression of the eigenfunction $q^*(\theta)$. Therefore,

$$\begin{aligned}\operatorname{Re} c_1(0) = \frac{\tau^* g'''(0)}{2[(1+(4d-\alpha-0)\tau^*)^2+\omega^{*2}]}&[\omega^*(E_2 \sin 2\omega^* \\ &+ (E_1 - E_{-1})\sin\omega^*) \\ + (1+(4d-\alpha_0)\tau^*)(E_2\cos 2\omega^* &+ (E_1+E_{-1})\cos\omega^* + E_0)].\end{aligned}$$

According to the general theory in Hassard, Kazarinoff and Wan [**15**], the stability of the periodic solutions of Hopf bifurcation are determined by the sign of $\operatorname{Re} c_1(0)$, and hence by the sign of $g'''(0)\Gamma$ where

$$\begin{aligned}\Gamma \triangleq \omega^*&[E_2\sin 2\omega^* + (E_1 - E_{-1})\sin\omega^*] \\ &\cdot [1+(4d-\alpha_0)\tau^*] \\ &\cdot [E_2\cos 2\omega^* + (E_1 - E_{-1})\cos\omega^* + E_0].\end{aligned}$$

Therefore, we have the following result.

Theorem 4.2. *Assume f is defined by* (4.8) *and satisfies* (4.9) *and* (4.10). *Assume also that $g'''(0)\Gamma \neq 0$. Then the periodic solutions of* (2.1^*) *bifurcating at $\tau = \tau^* = \tau_{m/2}$ are asymptotically stable* (*respectively unstable*) *if $g'''(0)\Gamma < 0$* (*respectively $g'''(0)\Gamma > 0$*). *Thus, if $g'''(0)\Gamma < 0$, then near $\tau = \tau^*$ system* (2.1) *with f defined by* (4.8)–(4.10) *has periodic traveling wave solutions that are asymptotically stable under small perturbations of spatially m-periodic functions for arbitrary even m; and if $g'''(0)\Gamma > 0$, then near $\tau = \tau^*$ system* (2.1) *with f defined by* (4.8)–(4.10) *has unstable periodic traveling wave solutions.*

Example 4.3. Let $a_1 = -4$, $a = 1$ and $a_3 = -4$. Then $E_2 = -208$, $E - 1 = -1231$, $E_0 = -844$ and $E_{-1} = -192$. It can be easily shown that (4.10) is satisfied if and only if $g'(0) > 2d$. Let $g'(0)/d = r$. Then numerical calculation shows that $\Gamma = \Gamma(r)$ changes sign in $r \in (4.9, 4.95)$. In fact, we have $\Gamma(4.9) = 8.405825$ and $\Gamma(4.95) = -3.9044275$. Therefore, the stability of the periodic traveling wave solutions of the corresponding system may change as r varies.

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