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Asymptotic and Periodic Boundary Value Problems of Mixed FDEs and Wave Solutions of Lattice Differential Equations[1]

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We discuss the existence and approximation of solutions of asymptotic or periodic boundary value problems of mixed functional differential equations. Our approach is via monotone iteration and non-standard ordering in the profile set for asymptotic boundary value problems and via S^1-degree and equivariant bifurcation theory for periodic boundary value problems. Applications will be given to wave fronts and to slowly oscillatory spatially periodic traveling waves of lattice delay differential equations arising from population genetics, population dynamics, and neural networks. © 1997 Academic Press

1. INTRODUCTION

The purpose of this paper is to establish the existence and approximation schemes of solutions to periodic or asymptotic boundary value problems of the following functional differential equation

$$\frac{d}{dt}x(t)=f(x(t), x(t-\tau)) + \sum_{j=1}^{m} a_j[g(x(t+r_j-\varepsilon\tau))+g(x(t-r_j-\varepsilon\tau))-2g(x(t-\varepsilon\tau))] \quad (1.1)$$

where $x\in\mathbb{R}$, $f\colon \mathbb{R}^2\to\mathbb{R}$ and $g\colon \mathbb{R}\to\mathbb{R}$ are continuous mappings, $\tau\geqslant 0$ is a given constant, $\varepsilon=0$ or 1, a_j and r_j, $1\leqslant j\leqslant m$, are given constants.

This equation has the important feature that the history and the future status of the system both affect its change rate at the present time. It is called a *mixed functional differential equation* and arises from various application fields. For example, in optimal control problems with delays

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the Euler equation determining the optimal solution often involves systems of functional differential equations with both advanced and delayed terms, see Pontryagin, Gamkreledze and Michenko [50]. A specific mixed functional differential equation arising in such way from a competitive economy was investigated by Rustichini [51, 52]. Physical justification of (1.1) was also discussed by Schulman [55, 56] in the field of time symmetric electrodynamics and absorber theory of Weeler and Feynman [61]. The qualitative analysis of mixed functional differential equations is quite complicated, and even the basic existence-uniqueness theory has not been established.

In this paper, we will investigate those mixed functional differential equations arising from the study of traveling waves of the following infinitely coupled system of delay differential equations defined in a linear lattice on the real line

$$\frac{d}{dt}u_n(t)=f(u_n(t),u_n(t-\tau)) + \sum_{j=1}^{m} a_j[g(u_{n-j}(t-\varepsilon\tau))+g(u_{n+j}(t-\varepsilon\tau))-2g(u_n(t-\varepsilon\tau))] \tag{1.2}$$

where $n\in\mathbb{Z}$ (the lattice of all integers). Prototypes of (1.2), which will be used to motivate our assumptions and to illustrate the general results, include

$$\frac{d}{dt}u_n(t)=u_n(t-\varepsilon\tau)[1-u_n(t)]+\sum_{j=1}^{m} a_j[u_{n-j}(t))+u_{n+j}(t)-2u_n(t)], \tag{1.3}$$

$$\frac{d}{dt}u_n(t)=u_n(t)[1-u_n(t-\tau)]+\sum_{j=1}^{m} a_j[u_{n-j}(t)+u_{n+j}(t)-2u_n(t)], \tag{1.4}$$

$$\frac{d}{dt}u_n(t)=-\alpha u_n(t)+a_0\,g(u_n(t-\varepsilon\tau)) + \sum_{j=1}^{m} a_j[g(u_{n-j}(t-\varepsilon\tau))+g(u_{n+j}(t-\varepsilon\tau))]. \tag{1.5}$$

arising from population genetics (Aronson and Weinberger [3], Fisher [20] and Mckean [44]), population growth (Hastings [24], Krawcewicz and Wu [31], Levin [39–41], Madras, Wu and Zou [42], Murray [45]), and neural networks (Chua and Yang [12, 13], Cohen and Grossberg [17], Hopefield [26, 27] and Pineda [49]), respectively. Other examples include a model for propagation of nerve pulses in myelinated axons where the membrane is excitable only at spatially discrete sites (Bell [7], Bell and Cosner [8], Britton [9], Chi *et al.* [10], Keener [28, 29] and Zinner

[64–66]). See also Cahn *et al.* [10], Chow and Shen [16], and the excellent survey of Chow and Mallet-Paret [15].

A *traveling wave* of (1.2) is a solution $u_n(t) = x(t - nc)$, where c is a given constant and $x: \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying the following mixed functional differential equation

$$\frac{d}{dt} x(t) = f(x(t), x(t-\tau)) + \sum_{j=1}^{m} a_j [g(x(t+jc-\varepsilon\tau)) + g(x(t-jc-\varepsilon\tau)) - 2g(x(t-\varepsilon\tau))]. \quad (1.6)$$

Of our main concern in this paper is the existence of periodic traveling waves and traveling wave fronts of (1.2). These are special wave solutions of (1.2) which are periodic with respect to the spatial and temporal variables or are convergent as $t \to \pm\infty$. The above discussion naturally leads us to the consideration of periodic boundary value problems or asymptotic boundary value problems of the mixed functional differential equation (1.6) or its general form (1.1).

Our first objective is to establish the existence of the asymptotic boundary value problem (1.1) related to the wave fronts of the lattice delay differential equation (1.2) in the case where $f(0,0) = f(K,K) = 0$ for a constant $K > 0$, by using the technique of monotone iteration. It will be shown that an iterative scheme, using an upper solution as initial iteration, monotonically converges to a solution of (1.1), and that the existence of a lower solution will guarantee that the limit of the iterative scheme satisfies the asymptotic boundary condition. We will also show that an ordered pair of upper and lower solutions can be constructed from a careful analysis of the related *characteristic equation* of (1.1) at a trivial solution. Since the monotone iterative scheme involves only linear scalar nonhomogeneous ordinary differential equations, our method can be utilized for computing the wave fronts numerically. The monotone iteration technique to be developed can be applied when f satisfies the *quasi-monotonicity* condition that $f(x, y)$ is monotonically increasing with respect to the second argument. Examples of such a functional include the delayed Fisher nonlinearity $f = u(t-\tau)[1 - u(t)]$ arising from population genetics; and the delayed positive nonlinear feedback $f = -\alpha u(t) + a_0 g(u(t-\tau))$ with a nonnegative constant a_0 and a sigmoidal g arising from neural networks.

In the case where the functional f does not satisfy the quasi-monotonicity condition, the above iterative scheme is not necessarily monotone with respect to the pointwise ordering of the space of all possible solutions, called *profiles*, of (1.1) subject to the asymptotic boundary value condition. Consequently, we are unable to establish the convergence of the scheme. In this case, we will further restrict the profile set and its ordering, motivated by the work of Smith and Thieme [59, 60] in a different content. Under additional conditions such as small delay, we will show that every function in the iterative scheme starting from an upper solution in the restricted profile set will remain in the profile set and the iteration scheme is monotone with respect to the new ordering of the profile set. This, coupled with the existence of a lower solution, will again lead to the existence of a solution of (1.1) subject to the asymptotic boundary value condition. The asymptotic boundary value problem (1.1) related to the population model (1.4) with delayed logistic nonlinearity will be carefully analyzed. It will be shown that with certain restrictions on d, if τ is small, then (1.4) has a wave front.

Our second objective is to obtain the existence of periodic waves for (1.2), and in particular, for (1.5) under general conditions on the coefficients $(a_0, a_1, ..., a_m)$. The approach is to examine the existence and global continuation of periodic solutions of the related mixed functional differential equation (1.6) by applying the S^1-degree and the equivariant bifurcation theory developed in Erbe *et al.* [19]. This involves: (i) establishing a-priori bounds for possible periodic solutions which are related to the disipativeness of system (1.6); (ii) locating local Hopf bifurcation points and evaluating the S^1-degree from the information of the distribution of zeros of a characteristic equation; and (iii) ruling out periodic waves of certain large fixed periods by first using the idea of Chow and Mallet-Paret [14], Nussbaum [46], and Nussbaum and Potter [48] that a periodic wave of large periods can be associated to a certain cyclic system of ordinary differential equations and then by using a Liapunov function and the Nussabaum's spectral theory of circulant matrices (Nussbaum [46]).

The rest of this paper is organized as follows. In Section 2, we will present our main results and their applications to those lattice delay differential equations arising from genetics, population dynamics and neural networks. Sections 3, 4, and 5 will be devoted to the detailed discussions and proofs of the main results for wave fronts in the case of quasimonotone nonlinearity, for wave fronts in the case of non-quasimonotone nonlinearity and for periodic traveling waves, respectively.

2. MAIN RESULTS AND APPLICATIONS

We start with the mixed functional differential equation

$$\frac{d}{dt}x(t)=f(x_t)+\sum_{j=1}^{m} a_j[g(x(t+r_j))+g(x(t-r_j))-2g(x(t))], \quad (2.1)$$

where a_j, $1 \leqslant j \leqslant m$, are real numbers, r_j, $1 \leqslant j \leqslant m$, positive constants, $g: \mathbb{R} \to \mathbb{R}$ and $f: X \to \mathbb{R}$ given mappings to be specified later. Moreover, for each $x: \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, x_t stands for the element of X defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, here $\tau \geqslant 0$ is a given constant and X is the Banach space of continuous functions defined on $[-\tau, 0]$, equipped with the super-norm.

In the following, we will use $F1, F2, \ldots$ and $P1, P2, \ldots$ to denote conditions related to wave fronts and periodic traveling waves, respectively. We first consider the existence of asymptotic boundary value problem of (2.1) related to wave fronts of lattice delay differential equations.

We assume

(F1) a_j and r_j are positive constants, $1 \leqslant j \leqslant m$;

(F2) there exists a constant $K > 0$ such that $f(\hat{0}) = f(\hat{K}) = 0$ and $f(\hat{x}) \neq 0$ for $x \in (0, K)$, here $\hat{x}$ denotes the constant mapping from $[-\tau, 0]$ into $\mathbb{R}$ with the constant value x;

(F3) $g \in C^2([0, K]; \mathbb{R})$, $g(0) = 0$, $0 < g'(x) \leqslant g'(0)$ for $x \in [0, K]$;

(F4) there exists a constant $\mu > 0$ such that for any $\phi, \psi \in X$ with $0 \leqslant \phi(s) \leqslant \psi(s) \leqslant K$ for $s \in [-\tau, 0]$, one has $f(\psi) - f(\phi) + \mu[\psi(0) - \phi(0)] \geqslant 2A[g(\psi(0)) - g(\phi(0))]$ with $A = \sum_{j=1}^{m} a_j$;

(F5) $f \in C^2(X; \mathbb{R})$, $\sup\{|f''(\phi)|; \phi \in X_K\} < \infty$ and $f'(\phi)\, e^{\lambda \cdot} - 2Ag'(\phi(0)) \leqslant f'(0)\, e^{\lambda \cdot} - 2Ag'(0)$ for any real number λ and for any $\phi \in X_K$, where $X_K := \{\phi \in X;\ \phi(s) \in [0, K] \text{ for } s \in [-\tau, 0]\}$, and $e^{\lambda \cdot} \in X$ is defined by $e^{\lambda \cdot}(\theta) = e^{\lambda\theta}$ for $\theta \in [-\tau, 0]$;

(F6) There exist $0 < \lambda_1 < \lambda_2$ such that $\Delta(\lambda_1) = \Delta(\lambda_2) = 0$ and $\Delta(\lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$, where

$$\Delta(\lambda) = \lambda - f'(0)\, e^{\lambda \cdot} - \sum_{j=1}^{m} a_j g'(0)[e^{\lambda r_j} + e^{-\lambda r_j} - 2].$$

Condition (F3) imposed on the function g is motivated by the linear or nonlinear diffusivity in the population growth model and the neural network model to be described later. (F4) and (F5) are related to the quasimonotonicity condition widely used in the literature of monotone dynamical systems generated by functional differential equations (Ahmad and Vatsala [1], Kerscher and Nagel [30], Kunish and Schappacher [34], Ladas and Lakshmikantham [35], Lakshmikantham and Leela [36], Leela and Moaura [38], and Smith [57, 58]). These conditions will play an important role in ensuring the monotonicity of the iteration employed in the proof of the existence of the solution to the asymptotic boundary value problem of (2.1) subject to the asymptotic boundary condition

$$\lim_{t \to \infty} x(t) = K, \qquad \lim_{t \to -\infty} x(t) = 0. \tag{2.2}$$

Finally, the transcendental equation

$$\Delta(\lambda) = 0 \tag{2.3}$$

will be called the *characteristic equation* of (2.1) at the trivial solution. As will be seen in next section, an upper solution of (2.1)–(2.2) can be constructed from some solutions (called *characteristic values*) of (2.3).

Our first general result is as follows:

THEOREM 2.1. *Assume that* (F1)–(F6) *hold. Then the asymptotic boundary value problem* (2.1)–(2.2) *has a solution.*

EXAMPLE 2.1. Consider the following system of lattice differential equations

$$C\frac{du_n(t)}{dt} = -\frac{1}{R}u_n(t) + A_0 g(u_n(t)) + \sum_{j=1}^{m} A_j[g(u_{n-j}(t)) + g(u_{n-j}(t))] + I$$

as a model, suggested by the work of Chua and Yang [11, 12], Cohen and Grossberg [17], and Hopfield [26, 27], for a network of infinitely many cells located in a linear lattice on the real line. Here, it is assumed that each cell is made of a linear capacitor, a nonlinear voltage-controlled current source and a few resistive linear circuit elements, and that cells communicate with each other directly only through its nearest m-neighbors. In the equation, C and R are positive constants denoting the capacitance and the resistance of each cell, the transfer (input-output or activation) function $g: \mathbb{R} \to \mathbb{R}$ is a sigmoidal (that is, a smooth and nondecreasing function with graph asymptotic to two horizontal lines), $(A_0, A_1, ..., A_m)$ are the interactive parameters, and I denotes the input control effect. In what follows, we will assume that $I = 0$. This can always be achieved by some translation of coordinates. Note that $A_j, j = 0, ..., m$, can be either positive or negative, corresponding to the excitatory or inhibitory interaction of cells. The above equation can be rewritten as

$$\frac{du_n(t)}{dt} = -\alpha u_n(t) + a_0 g(u_n(t)) + \sum_{j=1}^{m} a_j[g(u_{n-j}(t)) + g(u_{n-j}(t))], \tag{2.4}$$

where $\alpha = 1/RC$ and $a_j = A_j/C$ for all $j = 0, ..., m$. Applying Theorem 2.1, we get

COROLLARY 2.1. *Assume*

(i) a_j, $0 \leqslant j \leqslant m$ *are positive constants*;

(ii) $g \in C^2(\mathbb{R}; \mathbb{R})$, $g(0)=0$, $\lim_{x \to \pm\infty} g(x) = \pm 1$, $g'(x) > 0$ *and* $xg''(x) < 0$ *for* $x \neq 0$;

(iii) $(a_0 + 2\sum_{j=1}^m a_j)\, g'(0) > \alpha$.

Let

$$c^* = \inf\left\{c > 0;\ \lambda < -\alpha + g'(0)\left[a_0 + \sum_{j=1}^m a_j(e^{\lambda jc} + e^{-\lambda jc})\ for\ \lambda > 0\right\}.$$

Then for every $c < c^*$, *equation* (2.4) *has a traveling wave front* $u_n(t) = x(t - nc)$ *such that* $\lim_{t \to -\infty} x(t) = 0$ *and* $\lim_{t \to \infty} x(t) = K$, *where* $K > 0$ *is the unique positive solution of the algebraic equation*

$$\alpha K = \left(a_0 + 2\sum_{j=1}^m a_j\right) g(K).$$

Note that in the above result $c^* > 0$ always holds. In fact, with the change of variable $\lambda c = \theta$, the inequality defining c^* becomes

$$\frac{\theta}{c} < -\alpha + g'(0)\left[\left[a_0 + \sum_{j=1}^m a_j(e^{j\theta} + e^{-j\theta})\right],\qquad \theta > 0.$$

The right hand is a smooth function which is concave up and with a positive value (under the condition (iii)) at $\theta = 0$. c^* is the value of c where the graph of the right hand side function of the above inequality has a double coincidence point with the straight line θ/c. See Fig. 2.1.

EXAMPLE 2.2. We consider the following lattice delay differential equations

$$\frac{d}{dt} u_n(t) = h(u_n(t), u_n(t-\tau)) + \sum_{j=1}^m a_j[u_{n-j}(t) + u_{n+j}(t) - 2u_n(t)]. \qquad (2.5)$$

This can be regarded as the model for the growth of a single species population distributed over a patchy environment consisting of infinitely many patches, where $u_n(t)$ stands for the density of the population in the nth patch, $h(u_n(t), u_n(t-\tau))$ is the intrinsic growth rate, and $\sum_{j=1}^m a_j[u_{n-j}(t) + u_{n+j}(t) - 2u_n(t)]$ represents the effect of the spatial dispersal, where $a_j > 0$, $1 \leqslant j \leqslant m$. It can also be regarded as the discrete analog of the well known Fisher equation arising from population genetics and bifurcation process, and the existence of traveling wave fronts in the

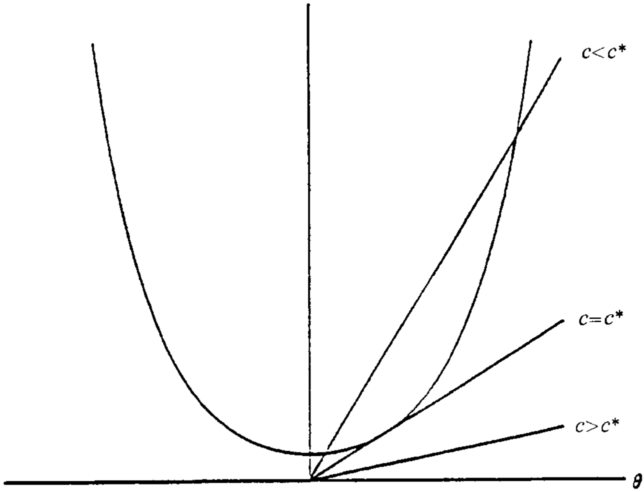

FIG. 2.1. The functions defining c^*.

continuous model with delay was established by Schaaf [53]. An immediate application of Theorem 2.1 leads to the following.

COROLLARY 2.2. *Assume that*

(i) $h \in C^2(\mathbb{R}^2; \mathbb{R})$ *and there exists a constant* $K > 0$ *such that* $h(0, 0) = h(K, K) = 0$ *and* $h(x, x) > 0$ *for all* $x \in (0, K)$;

(ii) $(\partial/\partial y)\, h(x, y) \geqslant 0$ *for all* $x, y \in [0, K]$;

(iii) $(\partial/\partial z)\, h(x, y) \leqslant (\partial/\partial z)\, h(0, 0)$ *for* $x, y \in [0, K]$, *where* $z = x$ *or* y;

(iv) $(\partial/\partial x)\, h(0, 0) + (\partial/\partial y)\, h(0, 0) > 0$.

Let

$$c^* := \inf \left\{ c;\ \lambda - \frac{\partial}{\partial x} h(0, 0) + 2 \sum_{j=1}^{m} a_j < \frac{\partial}{\partial y} h(0, 0)\, e^{-\lambda\tau} \right.$$
$$\left. + \sum_{j=1}^{m} a_j (e^{\lambda j c} + e^{-\lambda j c}) \text{ for } \lambda > 0 \right\}.$$

Then for every $c < c^*$, *equation* (2.5) *has a traveling wave front* $u_n(t) = x(t - nc)$ *such that* $\lim_{t \to \infty} x(t) = K$ *and* $\lim_{t \to -\infty} x(t) = 0$.

We should mention that in the absence of delay, the existence of traveling wave fronts of (2.5) for $m = 1$ was established by Zinner *et al.* [66] via continuation and comparison methods.

One of the crucial conditions in Theorem 2.1 is the quasimonotonicity (F4) which is not satisfied by the delayed logistic map $h(x, y) = [K - y]\, x$. In the next result, we will replace (F4) by the following:

(F4*) There exists a constant $\mu > 0$ such that for any $\phi, \psi \in X$ with $0 \leqslant \phi(s) \leqslant \psi(s) \leqslant K$ and $[\psi(s) - \phi(s)]\, e^{\mu s}$ nondecreasing in $s \in [-\tau, 0]$, one has $f(\psi) - f(\phi) + \mu[\psi(0) - \phi(0)] \geqslant 2A[g(\psi(0)) - g(\phi(0))]$, where $A = \sum_{j=1}^{m} a_j$.

This condition is in the spirit of the non-standard ordering of the phase-space introduced by Smith and Thieme [59, 60] in order to obtain the (strong) order-preserving property of solution semiflows defined by non-cooperative functional differential equations, and in order to apply the powerful theory of monotone dynamical systems. See also Krisztin and Wu [32, 33].

Our idea for establishing the existence of the solution of the asymptotic boundary value problem (2.1)–(2.1) under condition (F4*) is to seek solutions in the following *profile* set

$$\Gamma^* = \left\{ \rho: \mathbb{R} \to [0, K]; \quad \begin{array}{l} \rho \text{ is continuous and nondecreasing;} \\ \lim_{t \to -\infty} \rho(t) = 0 \text{ and } \lim_{t \to \infty} \rho(t) = K; \text{ and} \\ [\rho(t+s) - \rho(t)]\, e^{\mu t} \text{ is nondecreasing for any fixed } s > 0. \end{array} \right\}$$

and to use an iteration starting from an upper solution in Γ^*, where a continuous function $\rho: \mathbb{R} \to [0, K]$ is called an *upper solution* of (2.1) if it is differentiable almost everywhere and satisfies

$$\frac{d}{dt} x(t) \geqslant f(x_t) + \sum_{j=1}^{m} a_j [g(x(t + r_j)) + g(x(t - r_j)) - 2g(x(t))] \qquad (2.6)$$

a.e. on $\mathbb{R}$. Lower solutions can be similarly defined by reversing the inequality in (2.6).

THEOREM 2.2. *Assume* (F1)–(F3) *and* (F4*) *are satisfied. Suppose also that* (2.1) *has an upper solution* ρ^+ *in* Γ^* *and a lower solution* $\rho^-: \mathbb{R} \to [0, K]$ *such that* $\rho^- \not\equiv 0$, $0 \leqslant \rho^-(t) \leqslant \rho^+(t) \leqslant K$ *and* $[\rho^+(t) - \rho^-(t)]\, e^{\mu t}$ *is nondecreasing in* $t \in \mathbb{R}$. *Then* (2.1)–(2.2) *has a solution in* Γ^*.

Of practical importance is, of course, to construct the pair of upper and lower solutions satisfying the conditions in Theorem 2.2. At this time, we are unable to construct such a pair for general form of (2.1). But the next example indicates that in some special situations, it is still possible to construct this pair from the careful analysis of the related characteristic equation.

EXAMPLE 2.3. Consider the following lattice delay differential equations with delayed logistic nonlinearility

$$\frac{d}{dt}u_n(t)=u_n(t)[1-u_n(t-\tau)]+d[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)], \quad n\in\mathbb{Z}. \tag{2.7}$$

We can show that if d is sufficiently large, then

$$\rho^+(t)=\frac{1}{1+\alpha e^{-\lambda_1 t}}$$

with a positive α is indeed an upper solution in Γ^*, where λ_1 is the minimal positive real solution of the characteristic equation $c\lambda-1-d[e^{\lambda}+e^{-\lambda}-2]=0$. A lower solution, similar to the one given by Atkinson and Reuter [4] and Schumacher [54] in the continuous diffusion case, can be constructed. Therefore, we obtain

COROLLARY 2.3. *For every* $d\geqslant(e/2(e-1))$ *there exist* $\tau^*=\tau^*(d)>0$ *and* $0<c_1(d)<c_2(d)$ *so that if* $0\leqslant\tau<\tau^*$ *then for every* $c\in(c_1(d), c_2(d))$, *Eq.* (2.7) *has a traveling wave front* $u_n(t)=x(t-nc)$ *satisfying* $\lim_{t\to-\infty}x(t)=0$ *and* $\lim_{t\to\infty}x(t)=1$.

As already mentioned, the main idea in the proof of Theorem 2.1 or 2.2 is the monotone iteration

$$\frac{d}{dt}x_n(t)=-\mu x_n(t)+f((x_{n-1})_t)+\mu x_{n-1}(t)$$
$$+\sum_{j=1}^{m}a_j[g(x_{n-1}(t+r_j))+g(x_{n-1}(t-r_j))-2g(x_{n-1}(t))],$$
$$n=1, 2, \ldots$$

starting from an upper solution. Since only linear scalar nonhomogeneous ordinary differential equations are involved, the monotone iteration provides a very simple and effective method for computing the wave fronts numerically.

We now consider how time delay induces nonlinear periodic oscillations in the mixed functional differential equation

$$\frac{d}{dt}x(t)=f(x_t)+\sum_{j=1}^{m}a_j[g(x(t+r_j-\tau))+g(x(t-r_j-\tau))-2g(x(t-\tau))]. \tag{2.8}$$

We will regard the delay τ as a parameter and look for periodic solutions of (2.8) from the Hopf bifurcation point of view. A general result about the existence and global continuation of periodic solutions of mixed functional differential equations was obtained by Erbe *et al.* [19], we here state a special form of their results in our setting and present an application to the existence of periodic traveling waves of a lattice delay differential equation motivated by the neural network model.

We assume that

(P1) a_j and r_j are given constants and r_j are positive, $1 \leqslant j \leqslant m$;

(P2) $f \in C^2(X;\mathbb{R})$ is completely continuous, $f(\hat{0}) = 0$ and $f(\hat{x}) \neq 0$ for $x \neq 0$;

(P3) $g \in C^2(\mathbb{R};\mathbb{R})$ and $g(0) = 0$;

(P4) There exists a sequence of increasing positive real numbers $\{\tau_j\}$ and a sequence of positive real numbers $\{\omega_j\}$ so that $\Delta(\tau, i\omega) = 0$ for real $\tau, \omega \geqslant 0$ if and only if $\tau = \tau_j$ and $\omega = \omega_j$ for some $j \geqslant 1$, where

$$\Delta(\tau, \lambda) = \lambda - f'(0)\, e^{\lambda \cdot} - \sum_{j=1}^{m} a_j g'(0)[e^{\lambda r_j} + e^{-\lambda r_j} - 2]\, e^{-\lambda \tau};$$

(P5) $\gamma_j = -1$, where

$$\gamma_j = \deg_B(H_j^-, \Omega_j) - \deg_B(H_j^+, \Omega_j)$$

$$\Omega_j = \{(u, v) \in \mathbb{R}^2;\, 0 < u < \varepsilon,\, \omega_j - \varepsilon < v < \omega_j + \varepsilon\}$$

$$H_j^{\pm}(u, v) = \Delta(\tau_j \pm \delta, u + iv),\ (u, v) \in \mathbb{R}^2$$

here ε and δ are small positive real numbers and $\deg_B$ is the Brouwer degree.

Condition (P4) relates to the existence of purely imaginary zeros of the related characteristic equation and (P5) is the topological analog of the usual transversality condition. Condition $\gamma_j \neq 0$ implies that τ_j is a critical value where a branch of Hopf bifurcation takes place. Let Σ_j be the nonempty connected component through $(0, \tau_j, 2\pi/\omega_j)$ in the set

$$\Gamma_j = \text{closure}\{(x, \tau, T);\, x(t) \text{ is a non-constant T-periodic solution of (2.8)}\},$$

where the closure is taken in the space $Y \times \mathbb{R}^2$, Y is the Banach space of all bounded continuous functions defined on $\mathbb{R}$ equipped with the super-norm. Then we have the following global Hopf bifurcation theorem:

THEOREM 2.3. *Assume that* (P1)–(P5) *hold. Then* Σ_j *is unbounded.*

EXAMPLE 2.4. We return to the neural network model (2.4). It was observed by Hopfield [26, 27] and Marcus and Westervelt [43] that cells do not communicate and response instantaneously and sustained oscillations can arise from large relative size of the delay (relative to the relaxation time of the system) in the communication and response among cells (See also Belair [5], Belair *et al.* [6], Gopalsamy and He [22], Herz *et al.* [25], and Wu and Zou [63] for related work). This naturally leads to the following infinite system of delay differential equations

$$\frac{du_n(t)}{dt} = -\alpha u_n(t) + a_0 g(u_n(t-\tau)) + \sum_{j=1}^{m} a_j [g(u_{n-j}(t-\tau)) + g(u_{n-j}(t-\tau))]. \tag{2.9}$$

Note that if $u_n(t) = x(t-nc)$, $c>0$, is a traveling wave of (2.9), then x satisfies

$$\frac{d}{dt} x(t) = -\alpha x(t) + a_0 g(x(t-\tau)) + \sum_{j=1}^{m} a_j [g(x(t-\tau+jc)) + g(x(t-\tau-jc))]. \tag{2.10}$$

A traveling wave $u_n(t) = x(t-nc)$ is said to be *spatially p-periodic* if p is a positive integer and $u_n(t) = u_{n+p}(t)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Clearly, x is a pc-periodic function and thus, a spatially p-periodic wave is also periodic with respect to the time variable.

To detect the bifurcation of spatially periodic traveling waves, we define

$$\gamma = g'(0) \tag{2.11}$$

and

$$\beta_p = a_0 + 2 \sum_{j=1}^{m} a_j \cos\left(\frac{2\pi}{p} j\right). \tag{2.12}$$

Also, in order to rule out periodic traveling waves with prescribed periods, we need the following real numbers, corresponding to each given integer q:

$$b_i = \sum \{a_j;\, 1 \leqslant j \leqslant m, jq \text{ or } -jq = i-2 (\text{mod } pq)\},\ 1 \leqslant i \leqslant pq \tag{2.13}$$

and

$$\gamma_{p,q} = \max\left\{ Re\left(\sum_{j=2}^{pq} b_j e^{i(2\pi/pq)(j-1)n} \right);\, n = 0, ..., pq-1 \right\}. \tag{2.14}$$

Then we have the following existence result for periodic traveling waves:

THEOREM 2.4. *Assume that*

(i) $g \in C^2(\mathbb{R}; \mathbb{R})$, $g(0) = 0$, $\lim_{x \to \pm\infty} g(x) = \pm 1$, $g'(x) > 0$ *and* $xg''(x) < 0$ *for* $x \neq 0$;

(ii) $a_0 + 2\sum_{j=1}^{m} a_j < \alpha/\gamma$;

(iii) *there exists a positive integer p such that* $\beta_p < -\alpha/\gamma$;

(iv) *there exists a positive integer q such that* $pq \geqslant 4$ *is an even integer and* $\gamma_{p,q} < \alpha/\gamma$.

Let $\theta_p \in (\pi/2, \pi)$ *be given so that* $\cos\theta_p = \alpha/\gamma\beta_p$, *and define* $\tau_p = -(\theta_p/\alpha)\cot\theta_p$. *Then for each* $\tau > \tau_p$ *there exists a constant* $c > 0$ *such that* (2.9) *has a spatially p-periodic traveling wave* $u_n(t) = x(t - nc)$ *and the period of x is between* 2τ *and* $pq\tau$.

The above result shows that τ_p is the critical value of delay where a branch of spatially p-periodic waves bifurcates from the trivial solution. The *profile x* is of a period larger than 2τ and thus will be called *slowly oscillatory waves*, borrowing a terminology from the study of scalar functional differential equations (see, for example, Nussbaum and Mallet-Paret [47]). As will be made clear in Section 5 where Theorem 2.4 will be proved, there exists a sequence of critical values $\tau_p < \tau_{p,1} < \tau_{p,2} \cdots$ such that at each $\tau_{p,k}$, $k \geqslant 1$, a branch of spatially p-periodic rapidly oscillatory waves (those waves whose profiles have periods less than 2τ) occurs as well, and thus for large τ, we will have the coexistence of at least one slowly oscillatory and multiple rapidly oscillatory waves.

A similar result holds when $\beta_p > \alpha/\gamma$. We refer the reader to Section 5 for more details.

In the case where $p = q = 2$, we have the following

COROLLARY 2.4. *Assume that* (i) *of Theorem* 2.4 *holds, and let*

$$b_2 = a_0 + \sum \{a_j;\ 1 \leqslant j \leqslant m, j \text{ is even}\};$$

$$b_4 = \sum \{a_j;\ 1 \leqslant j \leqslant m, j \text{ is odd}\}.$$

Moreover, assume that

$$\begin{cases} b_2 - b_4 < -\dfrac{\alpha}{\gamma} \\[2ex] |b_2 + b_4| < \dfrac{\alpha}{\gamma}, \end{cases}$$

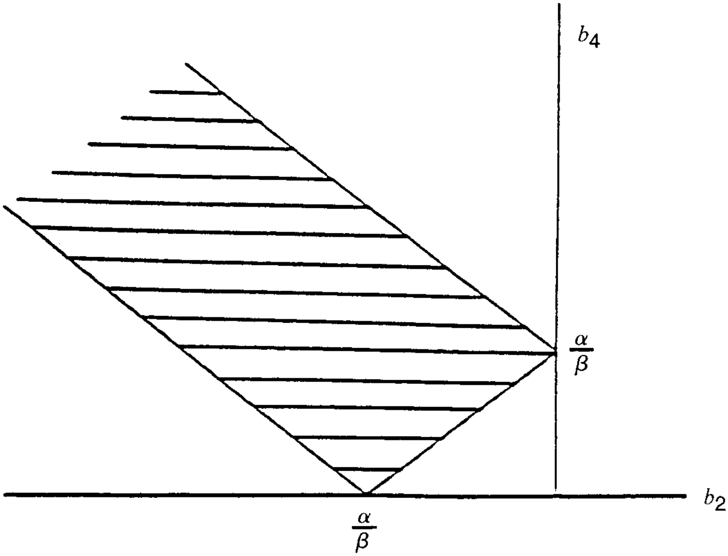

FIG. 2.2. The region of (b_2, b_4) when (2.9) has a slowly oscillatory spatially 2-periodic traveling wave.

that is, (b_2, b_4) *belongs to the shaded region in Fig.* 2.2. *Then for each* $\tau > \tau_2 := -(\theta_2/\alpha) \cot \theta_2$ *there exists a constant* $c > 0$ *such that* (2.9) *has a spatially* 2-*periodic traveling wave* $u_n(t) = x(t - nc)$ *and the period of* x *is between* 2τ *and* 4τ, *where*

$$\cos \theta_2 = \frac{\alpha}{\gamma(b_2 - b_4)}, \quad \theta_2 \in \left(\frac{\pi}{2}, \pi\right).$$

In the case where $p = 3$ and $q = 2$, we have the following

COROLLARY 2.5. *Assume that* (i) *of Theorem* 2.4 *holds, and let*

$$b_2 = a_0 + \sum \{a_j;\ 1 \leqslant j \leqslant m, j = 0 (\bmod 3);\}$$

$$b_4 = \sum \{a_j;\ 1 \leqslant j \leqslant m, j = 1 (\bmod 3)\};$$

$$b_6 = \sum \{a_j;\ 1 \leqslant j \leqslant m, j = 2 (\bmod 3)\}.$$

Moreover, assume that

$$\begin{cases} |b_2+b_4+b_6|<\dfrac{\alpha}{\gamma} \\ \left| b_4-\dfrac{b_2+b_6|}{2} \right| <\dfrac{\alpha}{\gamma} \\ b_2-\dfrac{(b_4+b_6)}{2}<-\dfrac{\alpha}{\gamma}. \end{cases}$$

Then for each $\tau>\tau_3:=-(\theta_3/\alpha)\cot\theta_3$ *there exists a constant* $c>0$ *such that* (2.9) *has a spatially* 3*-periodic traveling wave* $u_n(t)=x(t-nc)$ *and the period of* x *is between* 2τ *and* 6τ, *where*

$$\cos\theta_3=\frac{\alpha}{\gamma[b_2-1/2(b_4+b_6)]}.$$

3. ASYMPTOTIC BOUNDARY VALUE PROBLEMS OF MIXED FDEs: QUASIMONOTONE NONLINEARITIES

Consider the following asymptotic boundary value problem of mixed functional differential equation

$$\frac{d}{dt}x(t)=f(x_t)+\sum_{j=1}^{m}a_j[g(x(t+r_j))+g(x(t-r_j))-2g(x(t))], \quad (3.1)$$

$$\lim_{t\to-\infty}x(t)=0, \quad \lim_{t\to\infty}x(t)=K, \quad (3.2)$$

where

(H1) a_j and r_j are positive constants, $1\leqslant j\leqslant m$;

(H2) K and τ are given positive constants, $f\colon X_K\to\mathbb{R}$ is continuous, $f(\hat{0})=f(\hat{K})=0$ and $f(\hat{x})\neq 0$ for $x\in(0,K)$;

(H3) $g\colon[0,K]\to\mathbb{R}$ is continuous and monotonically increasing and $g(0)=0$;

(H4) there exists a constant $\mu>0$ such that for any $\phi,\psi\in X$ with $0\leqslant\phi(s)\leqslant\psi(s)\leqslant K$ for $s\in[-\tau,0]$, one has $f(\psi)-f(\phi)+\mu[\psi(0)-\phi(0)]\geqslant 2A[g(\psi(0))-g(\phi(0))]$ with $A=\sum_{j=1}^{m}a_j$.

Define the set of profiles by

$$\Gamma=\left\{\rho\colon\mathbb{R}\to[0,K];\ \begin{array}{l}\rho \text{ is continuous and nondecreasing;}\\ \lim_{t\to-\infty}\rho(t)=0 \text{ and } \lim_{t\to\infty}\rho(t)=K\end{array}\right\}$$

and define $H: C(\mathbb{R}; \mathbb{R}) \to C(\mathbb{R}; \mathbb{R})$ by

$$H(\rho)(t) = f(\rho_t) + \mu\rho(t) + \sum_{j=1}^{m} a_j[g(\rho(t+r_j)) + g(\rho(t-r_j)) - 2g(\rho(t))], \quad t \in \mathbb{R}.$$

PROPOSITION 3.1. *Assume* (H1)–(H4) *are satisfied. Let $\rho \in \Gamma$ and $\hat{\rho} \in C(\mathbb{R}; \mathbb{R})$ with $\rho(t) \leqslant \hat{\rho}(t)$ for $t \in \mathbb{R}$. Then*

(i) $0 \leqslant H(\rho)(t) \leqslant f(\hat{K}) + \mu K$ *for* $t \in \mathbb{R}$;

(ii) $H(\rho)(t)$ *is nondecreasing. Moreover, if ρ is strictly increasing on $(-\infty, a]$ with some $a \in \mathbb{R}$, then so is $H(\rho)$ on $(-\infty, a]$;*

(iii) $H(\rho)(t) \leqslant H(\hat{\rho})(t)$ *for* $t \in \mathbb{R}$.

Proof. (i) and (iii) are immediate consequences of (H3) and (H4). To verify (ii), we fix $t \in \mathbb{R}$ and $s > 0$. Using (H3) and (H4), we get

$$\begin{aligned} &H(\rho)(t+s) - H(\rho)(t) \\ &\quad = f(\rho_{t+s}) - f(\rho_t) + \mu[\rho(t+s) - \rho(t)] - 2A[g(\rho(t+s)) - g(\rho(t))] \\ &\qquad + \sum_{j=1}^{m} a_j[g(\rho(t+s+r_j)) - g(\rho(t+r_j))] \\ &\qquad + \sum_{j=1}^{m} a_j[g(\rho(t+s-r_j)) - g(\rho(t-r_j))] \\ &\quad \geqslant 0. \end{aligned}$$

Moreover, if ρ is strictly increasing on $(-\infty, a]$ for some $a \in \mathbb{R}$, then $g(\rho(t+s-r_j)) > g(\rho(t-r_j))$ for $t \leqslant a$ provided that $t+s \leqslant a$. Consequently, $H(\rho)(t+s) > H(\rho)(t)$ for $t+s \leqslant a$. This completes the proof.

We now rewrite (3.1) as

$$\frac{d}{dt}x(t) = -\mu x(t) + H(x)(t). \tag{3.3}$$

It is easy to verify that $x: \mathbb{R} \to [0, K]$ is a solution of (3.3) with $\lim_{t \to -\infty} x(t) = 0$ if and only if it solves the following integral equation

$$x(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(x)(s)\, ds. \tag{3.4}$$

DEFINITION 3.1. A continuous function $\rho: \mathbb{R} \to [0, K]$ is called an *upper solution* of (3.1) if it is differentiable almost everywhere, $\lim_{t \to -\infty} \rho(t) = 0$ and satisfies

$$\frac{d}{dt}\rho(t) \geqslant f(\rho_t) + \sum_{j=1}^{m} a_j[g(\rho(t+r_j)) + g(\rho(t-r_j)) - 2g(\rho(t))] \quad (3.5)$$

a.e. on $\mathbb{R}$. *Lower solutions* of (3.1) can be similarly defined by reversing the inequality in (3.5).

Our goal is (i) to construct an ordered pair of upper and lower solutions $0 \leqslant \rho^-(t) \leqslant \rho^+(t) \leqslant K$ for $t \in \mathbb{R}$; and (ii) to construct a monotone sequence of functions starting from ρ^+ and approaching to a solution of the asymptotic boundary value problem (3.1)–(3.2). To start the iteration, let us first assume that there exists an upper solution $\rho^+(t)$ and a lower solution $\rho^-(t)$ of (3.1) with $0 \leqslant \rho^-(t) \leqslant \rho^+(t) \leqslant K$ for $t \in \mathbb{R}$. We assume ρ^- is a nontrivial lower solution (that is, $\rho^- \not\equiv 0$ on $\mathbb{R}$) and $\rho^+(t)$ is nondecreasing for $t \in \mathbb{R}$ and $\lim_{t \to \infty} \rho^+(t) = K$. It is easy to verify that $x_1: \mathbb{R} \to \mathbb{R}$ given by

$$x_1(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\rho^+)(s)\, ds, \quad t \in \mathbb{R} \quad (3.6)$$

is a well defined C^1-function. Some of the important properties of x_1 are formulated as follows:

PROPOSITION 3.2. *The function x_1 defined by* (3.6) *satisfies*

(i) $(d/dt)\, x_1(t) > 0$ *for* $t \in \mathbb{R}$;

(ii) $\rho^-(t) \leqslant x_1(t) \leqslant \rho^+(t)$ *for* $t \in \mathbb{R}$;

(iii) $\lim_{t \to -\infty} x_1(t) = 0$ *and* $\lim_{t \to +\infty} x_1(t) = K$.

Proof. Using the monotonicity of ρ^+ and (ii) of Proposition 2.1, we get

$$\begin{aligned}\frac{d}{dt} x_1(t) &= -\mu e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\rho^+)(s)\, ds + H(\rho^+)(t)\\ &= -\mu e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\rho^+)(s)\, ds + \mu e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\rho^+)(t)\, ds\\ &= \mu e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} [H(\rho^+)(t) - H(\rho^+)(s)]\, ds > 0.\end{aligned}$$

Applying the L' Hospital's rule, we get

$$\lim_{t\to-\infty} x_1(t) = \lim_{t\to-\infty} \frac{e^{\mu t} H(\rho^+)(t)}{\mu e^{\mu t}} = \lim_{t\to-\infty} \frac{1}{\mu} H(\rho^+)(t) = 0;$$

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} \frac{e^{\mu t} H(\rho^+)(t)}{\mu e^{\mu t}} = \lim_{t\to+\infty} \frac{1}{\mu} H(\rho^+)(t) = K.$$

The inequality $\rho^-(t) \leqslant x_1(t) \leqslant \rho^+(t)$ for $t \in \mathbb{R}$ follows from the definition of x_1, the upper solution and the monotonicity $H(\rho^+)(t) \geqslant H(\rho^-)(t)$ for $t \in \mathbb{R}$. This completes the proof.

Note that by (iii) of Proposition 3.1, we have

$$\begin{aligned}\frac{d}{dt} x_1(t) &= -\mu x_1(t) + H(\rho^+)(t)\\ &\geqslant -\mu x_1(t) + H(x_1)(t)\\ &= f((x_1)_t) + \sum_{j=1}^{m} a_j [g(x_1(t+r_j)) + g(x_1(t-r_j)) - 2g(x_1(t))], \quad t \in \mathbb{R}.\end{aligned}$$

Therefore, x_1 is an upper solution of (3.1) and we can repeat the above process for the pair (x_1, ρ^-) to obtain another upper solution

$$x_2(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(x_1)(s)\, ds, \quad t \in \mathbb{R}. \tag{3.7}$$

Inductively, we can define

$$x_n(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(x_{n-1})(s)\, ds, \quad t \in \mathbb{R}, n \geqslant 2 \tag{3.8}$$

and obtain:

PROPOSITION 3.3. *The above sequence is well-defined and satisfies*

(i) $(d/dt)\, x_n(t) > 0$ *for* $t \in \mathbb{R}$;

(ii) $\lim_{t\to-\infty} x_n(t) = 0$, $\lim_{t\to+\infty} x_n(t) = K$;

(iii) $\rho^-(t) \leqslant x_n(t) \leqslant x_{n-1}(t) \leqslant \rho^+(t)$ *for* $t \in \mathbb{R}$ *and* $n \geqslant 2$.

The monotonicity (iii) in the above result ensures the existence of

$$x(t) = \lim_{n\to\infty} x_n(t). \tag{3.9}$$

Clearly, the function $x: \mathbb{R} \to \mathbb{R}$ is nondecreasing. We now claim

THEOREM 3.1. *$x\colon \mathbb{R} \to \mathbb{R}$ obtained in* (3.6), (3.7), (3.8), *and* (3.9) *is a solution of the asymptotic boundary value problem* (3.1)–(3.2).

Proof. Applying the Lebesgue's Dominated Convergence Theorem to (3.8), we can establish that

$$x(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(x)(s)\, ds \tag{3.10}$$

from which it follows that x satisfies (3.1). It remains to show that $\lim_{t \to \infty} x(t) = K$.

Note that x is nondecreasing and bounded. So $x^* := \lim_{t \to \infty} x(t) \leqslant K$ exists. Taking the limit $t \to \infty$ in (3.1), we get $f(\hat{x}^*) = 0$. On the other hand, we have $x_n(t) \geqslant \rho^-(t)$ for $n \geqslant 1$ and $t \in \mathbb{R}$. Therefore, $x(t) \geqslant \rho^-(t)$ and hence $x^* \geqslant \sup_{t \in \mathbb{R}} \rho^-(t) > 0$. Consequently, in view of (H2), we must have $x^* = K$. This completes the proof.

It remains to construct a pair of ordered upper and lower solutions.

PROPOSITION 3.4. *Assume, in addition to* (H1)–(H4), *the following holds*:

(H5) *$g\colon [0, K] \to \mathbb{R}$ is continuously differentiable and $0 < g'(x) \leqslant g'(0)$ for $x \in [0, K]$*;

(H6) *$f\colon X \to \mathbb{R}$ is continuously differentiable and $f'(\phi)\, e^{\lambda \cdot} - 2Ag'(\phi(0)) \leqslant f'(0)\, e^{\lambda \cdot} - 2Ag'(0)$ for any real number λ and for any $\phi \in X_K$*;

(H7) *There exists $0 < \lambda_1 < \lambda_2$ such that $\Delta(\lambda_1) = \Delta(\lambda_2) = 0$ and $\Delta(\lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$, where $\Delta(\lambda) = \lambda - f'(0)\, e^{\lambda \cdot} - \sum_{j=1}^{m} a_j g'(0)\, [e^{\lambda r_j} + e^{-\lambda r_j} - 2]$.*

Then $\rho^+(t) = K \min\{e^{\lambda_1 t}, 1\}$ is an upper solution of (3.1).

Proof. Clearly, $0 \leqslant \rho^+(t) \leqslant K$ for $t \in \mathbb{R}$, $\lim_{t \to -\infty} \rho^+(t) = 0$ and $\lim_{t \to +\infty} \rho(t) = K$. So, it suffices to show that $\dot{\rho}^+(t) \geqslant F(\rho^+)(t)$ a.e. on $\mathbb{R}$, where

$$F(\rho^+)(t) = f(\rho_t^+) + \sum_{j=1}^{m} a_j [g(\rho^+(t + r_j)) + g(\rho^+(t - r_j)) - 2g(\rho^+(t))].$$

For $t > 0$, we have from (H3) that

$$\sum_{j=1}^{m} a_j [g(\rho^+(t + r_j)) + g(\rho^+(t - r_j))] \leqslant \sum_{j=1}^{m} a_j [g(K) + g(K)] = 2Ag(K),$$

and from (H4) that

$$f(\hat{K}) - f(\rho_t^+) = f(\hat{K}) - f(\rho_t^+) + \mu[K - \rho^+(t)] \geqslant 2A[g(K) - g(\rho_t^+)].$$

Therefore,

$$\begin{aligned} &F(\rho^+)(t) \\ &\quad = f(\rho_t^+) - 2Ag(\rho^+(t)) + \sum_{j=1}^{m} a_j [g(\rho^+(t+r_j)) + g(\rho^+(t-r_j))] \\ &\quad \leqslant f(\hat{K}) - 2Ag(K) + 2Ag(K) = f(\hat{K}) = 0 = \dot{\rho}^+(t). \end{aligned}$$

For $t \leqslant 0$, we have from (H5) that

$$\begin{aligned} g(\rho^+(t+r_j)) &+ g(\rho^+(t-r_j)) \\ &\leqslant g'(0)[\rho^+(t+r_j) + \rho^+(t-r_j)] \\ &\leqslant Kg'(0)[e^{\lambda_1(t+r_j)} + e^{\lambda_1(t-r_j)}], \end{aligned}$$

and from (H6) that

$$f(\rho_t^+) - 2Ag(\rho^+(t)) \leqslant K[f'(0)\, e^{\lambda_1(t+\cdot)} - 2Ag'(0)\, e^{\lambda_1 t}].$$

Consequently,

$$\begin{aligned} F(\rho^+)(t) &\leqslant K\left[f'(0)\, e^{\lambda_1 \cdot} + \sum_{j=1}^{m} a_j g'(0)[e^{\lambda_1 r_j} + e^{-\lambda_1 r_j} - 2] \right] e^{\lambda_1 t} \\ &= K\lambda_1 e^{\lambda_1 t} = K\dot{\rho}^+(t). \end{aligned}$$

Therefore, ρ^+ is an upper solution of (3.1). This completes the proof.

PROPOSITION 3.5. *Assume, in addition to* (H1)–(H4) *and* (H7), *that*

(H8) $g\colon [0, K] \to \mathbb{R}$ *and* $f\colon X \to \mathbb{R}$ *are twice continuously differentiable and that there exists* $N > 0$ *such that* $|f''(\phi)| \leqslant N$ *for* $\phi \in X_K$.

Then, for sufficiently small $\varepsilon > 0$ *and sufficiently large* $M > 0$, $\rho^-(t) = K \max\{0, (1 - Me^{\varepsilon t})\, e^{\lambda_1 t}\}$ *is a lower solution of* (3.1).

Proof. Assume $M > 0$ is chosen so that at least $M > 1$. Let $s_0 = s_0(M, \varepsilon) < 0$ be such that $Me^{\varepsilon s_0} = 1$. For $t > s_0$, we first use (H4) to get

$$f(\rho_t^-) \geqslant 2Ag(\rho^-(t)).$$

Therefore,

$$
\begin{aligned}
F(\rho^-)(t) :&= f(\rho_t^-) + \sum_{j=1}^{m} a_j [g(\rho^-(t+r_j)) + g(\rho^-(t-r_j)) - 2g(\rho^-(t))] \\
&\geqslant 2Ag(\rho^-(t)) + \sum_{j=1}^{m} a_j [g(0) + g(0) - 2g(\rho^-(t))] \\
&= 0 = \dot{\rho}^-(t).
\end{aligned}
$$

For $t < s_0$, we have

$$K^- \dot{\rho}^-(t) = [\lambda_1 - M(\lambda_1 + \varepsilon)\, e^{\varepsilon t}]\, e^{\lambda_1 t}.$$

Let

$$
\begin{aligned}
f(\phi) &= f'(0)\, \phi - R(\phi), \qquad \phi \in X_K; \\
g(x) &= g'(0)\, x - Q(x), \qquad x \in [0, K].
\end{aligned}
$$

Using assumption (H8), we have

$$
\begin{aligned}
&|R(\phi)| \leqslant N[\sup_{\theta \in [-\tau, 0]} |\phi(\theta)|]^2, \qquad \phi \in X_K; \\
&|Q(x)| \leqslant Gx^2, \qquad \text{for all} \quad x \in [0, K] \text{ and for some } G > 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&K^{-1} F(\rho^-)(t) \\
&= K^{-1} f'(0)\, \rho_t^- + K^{-1} \sum_{j=1}^{m} a_j g'(0) [\rho^-(t+r_j) + \rho^-(t-r_j) - 2\rho^-(t)] \\
&\quad - K^{-1} R(\rho_t^-) - K^{-1} \sum_{j=1}^{m} a_j [Q(\rho^-(t+r_j)) + Q(\rho^-(t-r_j)) - Q(\rho^-(t))] \\
&\geqslant f'(0)\, e^{\lambda_1 (t+\cdot)} + \sum_{j=1}^{m} a_j g'(0) [e^{\lambda_1 (t+r_j)} + e^{\lambda_1 (t-r_j)} - 2e^{\lambda_1 t}] \\
&\quad - Me^{(\varepsilon+\lambda_1)\, t} \left\{ f'(0)\, e^{(\varepsilon+\lambda_1)\cdot} + \sum_{j=1}^{m} a_j g'(0) [e^{(\lambda_1+\varepsilon)\, r_j} + e^{-(\lambda_1+\varepsilon)\, r_j} - 2] \right\} \\
&\quad - K^{-1} R(\rho_t^-) - K^{-1} \sum_{j=1}^{m} a_j [Q(\rho^-(t+r_j)) + Q(\rho^-(t-r_j)) - 2Q(\rho^-(t))].
\end{aligned}
$$

Thus

$$K^{-1} F(\rho^-)(t) \geqslant K^- \dot{\rho}^-(t) = [\lambda_1 - M(\lambda_1 + \varepsilon)\, e^{\varepsilon t}]\, e^{\lambda_1 t} \tag{3.11}$$

provided that

$$\begin{aligned}
&\Delta(\lambda_1+\varepsilon)\, Me^{(\lambda_1+\varepsilon)\,t}\\
&\quad \geqslant \Delta(\lambda_1)\, e^{\lambda_1 t} + K^{-1} R(\rho_t^-)\\
&\qquad + K^{-1} \sum_{j=1}^{m} a_j [\, Q(\rho^-(t+r_j)) + Q(\rho^-(t-r_j)) - 2Q(\rho^-(t))\,].
\end{aligned}$$

So, if we choose $\varepsilon > 0$ sufficiently small so that $\lambda_1 + \varepsilon < \lambda_2$, then (3.11) holds if

$$\begin{aligned}
&\Delta(\lambda_1+\varepsilon)\, M\\
&\quad \geqslant K^{-1}\, e^{-(\lambda_1+\varepsilon)\,t} \left\{ R(\rho_t^-) + \sum_{j=1}^{m} a_j [\, Q(\rho^-(t+r_j)) \right.\\
&\qquad \left. + Q(\rho^-(t-r_j)) - 2Q(\rho^-(t))\,] \right\}.
\end{aligned}$$

Note that there exists a constant $P > 0$ so that

$$\begin{aligned}
&K^{-1}\, |\rho^-(t+\theta)| = |(1 - Me^{\varepsilon(t+\theta)})\, e^{\lambda_1(t+\theta)}| \leqslant Pe^{\lambda_1 t}, \quad t \leqslant 0, \quad \theta \in [-\tau, 0];\\
&K^{-1}\, |\rho^-(t \pm r_j)| \leqslant Pe^{\lambda_1 t}, \quad t \leqslant 0, \quad 1 \leqslant j \leqslant m.
\end{aligned}$$

Therefore, for $t \leqslant s_0 < 0$, if $\varepsilon \leqslant \lambda_1$ then we have

$$\begin{aligned}
&\left| K^{-1} e^{-(\lambda_1+\varepsilon)\,t} \left\{ R(\rho_t^-) + \sum_{j=1}^{m} a_j [\, Q(\rho^-(t+r_j)) \right.\right.\\
&\qquad \left.\left. + Q(\rho^-(t-r_j)) - 2Q(\rho^-(t))\,] \right\} \right|\\
&\quad \leqslant K^{-1} e^{-(\lambda_1+\varepsilon)\,t} [\, NK^2P^2 e^{2\lambda_1 t} + 4K^2 AGP^2 e^{2\lambda_1 t}\,]\\
&\quad = K[\, NP^2 + 4AGP^2\,]\, e^{(\lambda_1-\varepsilon)\,t}\\
&\quad \leqslant KP^2[\, N + 4AG\,].
\end{aligned}$$

Consequently, if we choose $\varepsilon > 0$ sufficiently small and $M > 0$ sufficiently large so that

$$\begin{gathered}
0 < \varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2\\
\Delta(\lambda_1+\varepsilon)\, M > KP^2[\, N + 4AG\,],
\end{gathered}$$

then ρ^- is a lower solution of (3.1). This completes the proof.

Combining Theorem 3.1 and Propositions 3.4–3.5, we obtain:

THEOREM 3.2. *Assume that* (H1)–(H8) *hold. Then the asymptotic boundary value problem* (3.1)–(3.2) *has a solution.*

We can now prove Theorem 2.1 and Corollaries 2.1 and 2.2. First of all, Theorem 2.1 is an immediate consequence of Theorem 3.2, noting the following (F1) $\Leftrightarrow$ (H1), (F2) $\Leftrightarrow$ (H2), (F3) $\Rightarrow$ (H3) + (H5), (F4) $\Leftrightarrow$ (H4), (F5) $\Rightarrow$ (H6) + (H8), and (F6) $\Leftrightarrow$ (H7).

To prove Corollary 2.1, we observe that $u_n(t) = x(t - nc)$ is a traveling wave of (2.4) if $x(t)$ satisfies (3.1) with $f(\phi) = -\alpha\phi(0) + (a_0 + 2A)\, g(\phi(0))$ and $r_j = jc$. Therefore, (i) implies (F1). On the other hand, (ii) and (iii) implies that $-\alpha x + (a_0 + 2A)\, g(x) = 0$ has a unique positive solution. Thus, (F2) holds. (F3) follows from (ii). To obtain (F4), we note that

$$\begin{aligned}
&f(\psi) - f(\phi) - 2A[\, g(\psi(0)) - g(\phi(0))] \\
&\quad = -\alpha[\psi(0) - \phi(0)] + (a_0 + 2A)[\, g(\psi(0)) - g(\phi(0))] \\
&\qquad - 2A[\, g(\psi(0)) - g(\phi(0))] \\
&\quad = (a_0 - \alpha)[\psi(0) - \phi(0)].
\end{aligned}$$

So (F4) holds with any $\mu > \alpha - a_0$. For (F5), we note that

$$\begin{aligned}
&f'(\phi)\, e^{\lambda \cdot} - 2Ag'(\phi(0)) \\
&\quad = -\alpha + (a_0 + 2A)\, g'(\phi(0)) - 2Ag'(\phi(0)) \\
&\quad = -\alpha + a_0\, g'(\phi(0)) \\
&\quad \leqslant -\alpha + a_0\, g'(0) \\
&\quad = f'(0)\, e^{\lambda \cdot} - 2Ag'(0).
\end{aligned}$$

Finally, for Eq. (2.4), the related characteristic equation is

$$\begin{aligned}
\Delta(\lambda) &= \lambda - [-\alpha + (a_0 + 2A)\, g'(0)] - \sum_{j=1}^{m} a_j g'(0)[e^{\lambda jc} + e^{-\lambda jc} - 2] \\
&= \lambda - \left[-\alpha + a_0 g'(0) + \sum_{j=1}^{m} a_j g'(0)(e^{\lambda jc} + e^{-\lambda jc}) \right]
\end{aligned}$$

As $(a_0 + 2A)\, g'(0) > \alpha$, we can easily see that

$$c^* := \inf\left\{ c;\, \lambda < -\alpha + a_0 g'(0) + \sum_{j=1}^{m} a_j g'(0)(e^{\lambda jc} + e^{-\lambda jc}) \text{ for } \lambda > 0 \right\}$$

is a finite number and if $c < c^*$, then (F6) holds. Therefore, Corollary 2.1 follows from Theorem 2.1.

Similarly, we can verify that (F1)–(F6) hold for the equation satisfied by the profile of the traveling wave of (2.5) under conditions (i)–(iv) of Corollary 2.2, noting that the related characteristic equation in this case is

$$\Delta(\lambda)=\lambda-\frac{\partial}{\partial x}h(0,0)-\frac{\partial}{\partial y}h(0,0)\,e^{-\lambda\tau}-\sum_{j=1}^{m}a_j(e^{jc\lambda}+e^{-jc\lambda}-2).$$

4. ASYMPTOTIC BOUNDARY VALUE PROBLEMS OF MIXED FDEs: NON-QUASIMONOTONE NONLINEARITIES

The purpose of this section is to establish the existence of solutions to the asymptotic boundary value problem (3.1)–(3.2), replacing the quasi-monotonicity condition (H4) by the following

(H4*) there exists a constant $\mu>0$ such that for any $\phi,\psi\in X$ with $0\leqslant\phi(s)\leqslant\psi(s)\leqslant K$ and $[\psi(s)-\phi(s)]\,e^{\mu s}$ nondecreasing in $s\in[-\tau,0]$, one has $f(\psi)-f(\phi)+\mu[\psi(0)-\phi(0)]\geqslant 2A[g(\psi(0))-g(\phi(0))]$ where $A=\sum_{j=1}^{m}a_j$.

This condition is in the spirit of the non-standard ordering of the phase space introduced by Smith and Thieme [59, 60] in order to obtain the (strong) order-preserving property of solution semiflows defined by non-cooperative functional differential equations.

We will seek wave fronts in the following profile set

$$\Gamma^*=\left\{\rho:\mathbb{R}\to[0,K];\ \begin{array}{l}\rho\text{ is continuous and nondecreasing;}\\ \lim_{t\to-\infty}\rho(t)=0\text{ and }\lim_{t\to\infty}\rho(t)=K;\text{ and}\\ [\rho(t+s)-\rho(t)]\,e^{\mu t}\text{ is nondecreasing for any fixed }s>0.\end{array}\right\}.$$

Again, we define

$$H(\rho)(t)=f(\rho_t)+\mu\rho(t)+\sum_{j=1}^{m}a_j[g(\rho(t+r_j))+g(\rho(t-r_j))-2g(\rho(t))],\quad t\in\mathbb{R}$$

for $\rho\in\Gamma^*$. We can easily show that $H(\rho)(t)\geqslant 0$ for all $t\in\mathbb{R}$. Moreover, for any $\rho\in\Gamma^*$, $t\in\mathbb{R}$, $s>0$ and $\theta\in[-\tau,0]$, we have

$$[\rho_{t+s}(\theta)-\rho_t(\theta)]\,e^{\mu\theta}=[\rho(t+s+\theta)-\rho(t+\theta)]\,e^{\mu(t+\theta)}e^{-\mu t}.$$

Thus, $[\rho_{t+s}(\theta)-\rho_t(\theta)]\,e^{\mu\theta}$ is nondecreasing in $\theta\in[-\tau,0]$. This implies, using (H4*) and (H3), the following monotonicity of $H(\rho)$:

$$
\begin{aligned}
&H(\rho)(t+s)-H(\rho)(t)\\
&\quad = f(\rho_{t+s})-f(\rho_t)+\mu[\rho(t+s)-\rho(t)]-2A[g(\rho(t+s)-g(\rho(t))]\\
&\qquad +\sum_{j=1}^{m} a_j[g(\rho(t+s+r_j))-g(\rho(t+r_j))\\
&\qquad +g(\rho(t+s-r_j))-g(\rho(t-r_j))]\\
&\quad \geqslant 0.
\end{aligned}
$$

Now, we assume that (3.1)–(3.2) has an upper solution $\rho^+ : \mathbb{R} \to [0, K]$ in the profile set Γ^*. Define

$$
x_1(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\rho^+)(s)\, ds, \quad t \in \mathbb{R}. \tag{4.1}
$$

We can apply the same argument as that of Proposition 3.2 to show that $(d/dt)\, x_1(t) \geqslant 0$ and $0 \leqslant x_1(t) \leqslant \rho^+(t)$ for $t \in \mathbb{R}$, $\lim_{t\to-\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_1(t) = K$. Moreover, for fixed $t \in \mathbb{R}$ and $s > 0$, we have

$$
\begin{aligned}
&[x_1(t+s)-x_1(t)]\, e^{\mu t}\\
&\quad = e^{-\mu s} x_1(t+s)\, e^{\mu(t+s)} - x_1(t)\, e^{\mu t}\\
&\quad = e^{-\mu s} \int_{-\infty}^{t+s} e^{\mu\theta} H(\rho^+)(\theta)\, d\theta - \int_{-\infty}^{t} e^{\mu\theta} H(\rho^+)(\theta)\, d\theta\\
&\quad = \int_{-\infty}^{t+s} e^{\mu(\theta-s)} H(\rho^+)(\theta)\, d\theta - \int_{-\infty}^{t} e^{\mu\theta} H(\rho^+)(\theta)\, d\theta\\
&\quad = \int_{-\infty}^{t} e^{\mu\theta} H(\rho^+)(\theta+s)\, d\theta - \int_{-\infty}^{t} e^{\mu\theta} H(\rho^+)(\theta)\, d\theta\\
&\quad = \int_{-\infty}^{t} e^{\mu\theta} [H(\rho^+)(\theta+s) - H(\rho^+)(\theta)]\, d\theta.
\end{aligned}
$$

This implies that

$$
\frac{d}{dt} \{[x_1(t+s)-x_1(t)]\, e^{\mu t}\} = e^{\mu t}[H(\rho^+)(t+s) - H(\rho^+)(t)] \geqslant 0.
$$

Therefore, $x_1 \in \Gamma^*$.

Also, note that

$$
\frac{d}{dt} \rho^+(t) \geqslant -\mu \rho^+(t) + H(\rho^+)(t), \quad \text{a.e. on } \mathbb{R}
$$

and

$$\frac{d}{dt}x_1(t) = -\mu x_1(t) + H(\rho^+)(t), \quad t \in \mathbb{R}.$$

Therefore,

$$\frac{d}{dt}[\rho^+(t) - x_1(t)] \geqslant -\mu[\rho^+(t) - x_1(t)] \quad \text{a.e. on } \mathbb{R}.$$

Consequently,

$$\frac{d}{dt}[\rho^+(t) - x_1(t)]\, e^{\mu t} \geqslant 0 \quad \text{a.e. on } \mathbb{R}$$

from which it follows that $[\rho^+(t) - x_1(t)]\, e^{\mu t}$ is nondecreasing in $t \in \mathbb{R}$. This also implies that $H(\rho^+)(t) \geqslant H(x_1)(t)$ by using (H4*). So x_1 is an upper solution as well.

We now assume that there exists a lower solution ρ^- of (3.1)–(3.2) such that $\rho^- \not\equiv 0$, $0 \leqslant \rho^-(t) \leqslant \rho^+(t)$ and $[\rho^+(t) - \rho^-(t)]\, e^{\mu t}$ is nondecreasing in $t \in \mathbb{R}$. Then $H(\rho^-)(t) \leqslant H(\rho^+)(t)$ by using (H4*), and hence $x_1(t) \geqslant \rho^-(t)$ for $t \in \mathbb{R}$. Moreover,

$$\begin{aligned}
&\frac{d}{dt}\{[x_1(t) - \rho^-(t)]\, e^{\mu t}\} \\
&\quad = \left[\frac{d}{dt}x_1(t) - \frac{d}{dt}\rho^-(t)\right] e^{\mu t} + \mu[x_1(t) - \rho^-(t)]\, e^{\mu t} \\
&\quad \geqslant \{[-\mu x_1(t) + H(\rho^+)(t)] - [-\mu\rho^-(t) + H(\rho^-)(t)] \\
&\qquad + \mu[x_1(t) - \rho^-(t)]\}\, e^{\mu t} \\
&\quad \geqslant 0 \quad \text{a.e. on } \mathbb{R}.
\end{aligned}$$

That is, $[x_1(t) - \rho^-(t)]\, e^{\mu t}$ is nondecreasing in $t \in \mathbb{R}$.

We now summarize the above discussion in the following:

Proposition 4.1. *Assume that* (3.1)–(3.2) *has an upper solution* ρ^+ *in* Γ^* *and a lower solution* $\rho^-: \mathbb{R} \to [0, K]$ *such that* $\rho^- \not\equiv 0$, $0 \leqslant \rho^-(t) \leqslant \rho^+(t)$ *and* $[\rho^+(t) - \rho^-(t)]\, e^{\mu t}$ *is nondecreasing in* $t \in \mathbb{R}$. *Then* x_1 *defined by* (4.1) *is an upper solution in* Γ^* *of* (3.1)–(3.2) *such that* $0 \leqslant \rho^-(t) \leqslant x_1(t) \leqslant \rho^+(t)$, $[x_1(t) - \rho^-(t)]\, e^{\mu t}$ *and* $[\rho^+(t) - x_1(t)]\, e^{\mu t}$ *are nondecreasing in* $t \in \mathbb{R}$.

In general, we can define

$$x_n(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(x_{n-1})(s)\, ds, \quad n \geqslant 2 \tag{4.2}$$

and repeat the above argument, using x_{n-1} as the upper solution of (3.1)–(3.2) in Γ^*, to obtain that

(i) $x_n \in \Gamma^*$;

(ii) $0 \leqslant \rho^-(t) \leqslant x_n(t) \leqslant x_{n-1}(t) \leqslant \cdots \leqslant \rho^+(t)$, $t \in \mathbb{R}$;

(iii) $[x_n(t) - \rho^-(t)]\, e^{\mu t}$ and $[x_{n-1}(t) - x_n(t)]\, e^{\mu t}$ are nondecreasing in $t \in \mathbb{R}$.

We can now use the same argument as that of Theorem 3.1 to establish the following existence result (Theorem 2.2):

THEOREM 4.1. *Assume* (H1)–(H3) *and* (H4*) *are satisfied. Suppose also that* (3.1)–(3.2) *has an upper solution* ρ^+ *in* Γ^* *and a lower solution* $\rho^-: \mathbb{R} \to [0, K]$ *such that* $\rho \not\equiv 0$, $0 \leqslant \rho^-(t) \leqslant \rho^+(t) \leqslant K$ *and* $[\rho^+(t) - \rho^-(t)]\, e^{\mu t}$ *is nondecreasing in* $t \in \mathbb{R}$. *Then* (3.1)–(3.2) *has a solution* x *in* Γ^*, *which can be obtained by* $\lim_{n \to \infty} x_n(t)$.

To demonstrate this general result, we consider the lattice delay differential equation with delayed logistic nonlinearity (2.7). We look for the traveling wave front $u_n(t) = y(t - n\tilde{c})$ with y monotonically increasing, $\lim_{s \to -\infty} y(s) = 0$ and $\lim_{s \to +\infty} y(s) = 1$. Then y must satisfy

$$\frac{d}{dt} y(t) = d[y(t+\tilde{c}) + y(t-\tilde{c}) - 2y(t)] + y(t)[1 - y(t-\tau)], \quad t \in \mathbb{R}; \tag{4.3}$$

$$\lim_{t \to -\infty} y(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} y(t) = 1. \tag{4.4}$$

In what follows, we will write $x(t) = y(\tilde{c}t)$ and let $c = \tilde{c}^{-1}$. The (4.3)–(4.4) is equivalent to

$$c \frac{d}{dt} x(t) = d[x(t+1) + x(t-1) - 2x(t)] + x(t)[1 - x(t - c\tau)], \quad t \in \mathbb{R}; \tag{4.5}$$

$$\lim_{t \to -\infty} x(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} x(t) = 1. \tag{4.6}$$

Clearly, this is a special case of (3.1)–(3.2) with $m = 1$, $r_1 = 1$, $a_1 = d/c$, $g(x) = x$ and $f(\phi) = (1/c)\, \phi(0)[1 - \phi(-c\tau)]$. It is easy to see that (H1)–(H3) are satisfied with $K = 1$.

PROPOSITION 4.2. *For any* $d>0$, *let* $x^*=x^*(d)>1$ *be given so that* $x^*(\ln x^*-1)=2d$. *Then for each* $0\leqslant\tau<(x^*)^{-1}$, *there exists* $\mu>0$ *such that* $f(\phi)=(1/c)\,\phi(0)[1-\phi(-c\tau)]$ *satisfies* (H4*).

Proof. Let ϕ and ψ be in $X=C([-c\tau,0];\mathbb{R}]$ with $0\leqslant\phi(s)\leqslant\psi(s)\leqslant 1$ and $[\psi(s)-\phi(s)]\,e^{\mu s}$ nondecreasing in $s\in[-c\tau,0]$. Then

$$\begin{aligned}
&c[f(\psi)-f(\phi)]\\
&\quad=[\psi(0)-\phi(0)]-[\psi(0)\,\psi(-c\tau)-\phi(0)\,\phi(-c\tau)]\\
&\quad=[\psi(0)-\phi(0)][1-\psi(-c\tau)]-\phi(0)[\psi(-c\tau)-\phi(-c\tau)]\\
&\quad\geqslant[\psi(0)-\phi(0)][1-\psi(-c\tau)]-\phi(0)[\psi(0)-\phi(0)]\,e^{\mu c\tau}\\
&\quad\geqslant -e^{\mu c\tau}[\psi(0)-\phi(0)].
\end{aligned}$$

Thus,

$$\begin{aligned}
&f(\psi)-f(\phi)+\left(\mu-\frac{2d}{c}\right)[\psi(0)-\phi(0)]\\
&\qquad\geqslant\frac{1}{c}\,(\mu c-e^{\mu c\tau}-2d)[\psi(0)-\phi(0)].
\end{aligned}$$

Therefore, if $\tau<(x^*)^{-1}$ and $\mu=(1/c\tau)\ln(1/\tau)$ then

$$\mu c-e^{\mu c\tau}-2d=\frac{1}{\tau}\left[\ln\frac{1}{\tau}-1\right]-2d\geqslant x^*(\ln x^*-1)-2d\geqslant 0.$$

This completes the proof.

In the remainder of this section, we will construct a pair of ordered upper and lower solutions of (4.5)–(4.6) satisfying the conditions in Theorem 4.1. First, we formulate the following observation which can be easily verified by elementary calculus:

PROPOSITION 4.3. *Let*

$$\Delta_{c,d}(\lambda)=c\lambda-1-d[e^{\lambda}+e^{-\lambda}-2],\quad \lambda\in\mathbb{R},$$

where $d>0$. *Then there exists* $c^*=c^*(d)>0$ *such that*

(i) *if* $c<c^*$, $\Delta_{c,d}(\lambda)$ *has no real zeros*;

(ii) *if* $c=c^*$, $\Delta_{c,d}(\lambda)$ *has precisely one double zero* λ^*;

(iii) *if* $c>c^*$, $\Delta_{c,d}(\lambda)$ *have exactly two real zeros* $0<\lambda_1<\lambda_2$, *and* $\Delta_{c,d}(\lambda)>0$ *for all* $\lambda\in(\lambda_1,\lambda_2)$.

For the sake of presentation, let us introduce the following three functions:

$$g_{1,d}(\lambda) = e^{\lambda} - d[e^{\lambda} + e^{-\lambda} - 2], \quad \lambda \in \mathbb{R};$$

$$g_2(\lambda) = \frac{1 + e^{\lambda}}{e^{\lambda} + e^{-\lambda}}, \quad \lambda \in \mathbb{R};$$

$$g_{3,d}(\lambda) = 1 + d[e^{\lambda} + e^{-\lambda} - 2], \quad \lambda \in \mathbb{R}.$$

PROPOSITION 4.4. *Define*

$$h_1(s) = \frac{s}{2(s-1)}, \quad s > 1;$$

$$h_2(s) = \frac{s}{(s-1)[(s+1)\ln s - (s-1)]}, \quad s > 1.$$

Then

(i) *h_1 and h_2 are decreasing on $(1, \infty)$;*

(ii) *$h_2(s) > h_1(s)$ for $s \in (1, e)$, $h_1(e) = h_2(e) = (e/2(e-1))$, $h_1(1^+) = h_2(1^+) = \infty$;*

(iii) *for each $d > (e/2(e-1))$ there exist $1 < s_1 := s_1(d) < s_2 := s_2(d) < e$ such that $h_1(s_1) = d$, $h_2(s_2) = d$ and $h_1(s) < d < h_2(s)$ for $s \in (s_1, s_2)$;*

(iv) *$c(s) := (g_{3,d}(\ln s)/\ln s)$ is decreasing in $s \in (s_1, s_2)$;*

(v) *for each $d > (e/2(e-1))$ and for every $c \in (c(s_2), c(s_1))$ there exists $s \in (s_1, s_2)$ such that $c = c(s)$ and if $\lambda_1 = \ln s$, then $g_2(\lambda_1) \leqslant g_{3,d}(\lambda_1)$, $g_{1,d}(\lambda_1) \leqslant g_{3,d}(\lambda_1)$, $\Delta_{c,d}(\lambda_1) = 0$ and $(d/d\lambda)\,\Delta_{c,d}(\lambda_1) > 0$.*

Proof. We suggest the reader to consult Fig. 4.1 to understand the proof. First of all, (i) and (iii) can be verified by elementary calculations. To prove (iv), we note that

$$\frac{d}{ds}\,c(s) = \frac{d(s-1)[(s+1)\ln s - (s-1)] - s}{(s \ln s)^2} < 0$$

since $d < h_2(s)$ for $s \in (s_1, s_2)$.

Also, elementary calculation leads to $g_{1,d}(\lambda_1) \leqslant g_{3,d}(\lambda_1)$ and $\Delta_{c,d}(\lambda_1) = 0$ for $\lambda_1 = \ln s$ if $d > (e/2(e-1))$, $s \in (s_1, s_2)$ and $c = c(s)$. By Proposition 4.3, $\Delta_{c(s),d}(\lambda)$ has two real zeros. But since $(d/d\lambda)\,\delta_{c(s),d}(\lambda_1) > 0$, we can conclude that $\lambda_1 = \ln s$ is the smaller positive real zero of $\Delta_{c(s),d}(\lambda) = 0$. This completes the proof.

Now we are ready to get an upper solution of (4.5)–(4.6).

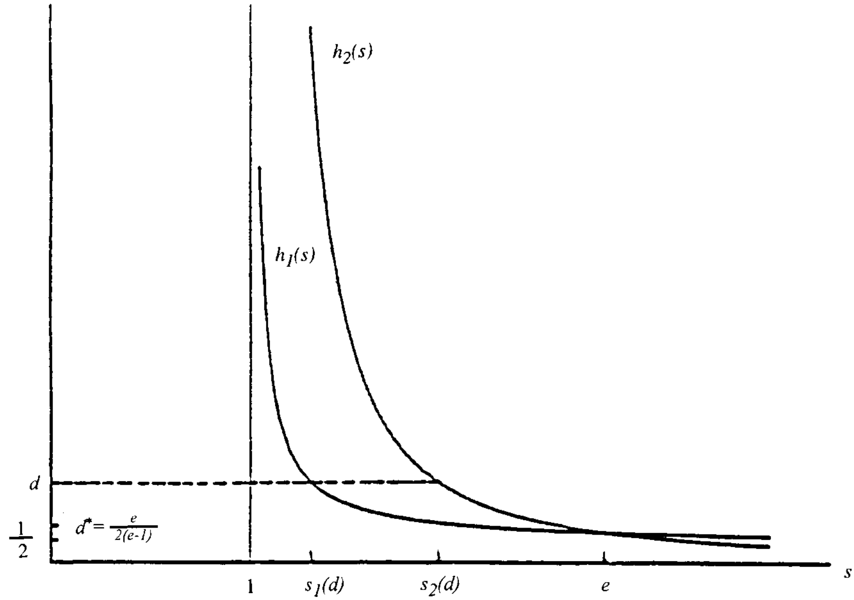

FIG. 4.1. The functions of h_1 and h_2.

PROPOSITION 4.5. *For every $d \geqslant (e/2(e-1))$ and $\tau \leqslant 1/c(s_1(d))$, define*

$$\rho_\alpha^+(t) = \frac{1}{1+\alpha e^{-\lambda_1 t}}, \quad t \in \mathbb{R}$$

where $\lambda_1 = \ln s$, $s \in (s_1, s_2)$ is a given number. Then

(i) *ϕ_α^+ is an upper solution to (4.5) for every $\alpha > 0$;*

(ii) *For any $\alpha > 0$, $\rho_\alpha^+(t)$ is nondecreasing in $t \in \mathbb{R}$ and $\lim_{t\to-\infty} \rho_\alpha^+(t) = 0$, $\lim_{t\to\infty} \rho_\alpha^+(t) = 1$. Moreover, $[\rho_\alpha^+(t+s) - \rho_\alpha^+(t)]\, e^{\mu t}$ is nondecreasing in $t \in \mathbb{R}$ for any fixed $s > 0$, provided $\mu > \lambda_1$.*

Proof. By direct calculation, we get

$$\dot{\rho}_\alpha^+(t) = \frac{\alpha\lambda_1 e^{-\lambda_1 t}}{(1+\alpha e^{-\lambda_1 t})^2}$$

and

$$\rho_\alpha^+(t+1) + \rho_\alpha^+(t-1) - 2\rho_\alpha^+(t)$$
$$= \frac{\alpha e^{-\lambda_1 t}(e^{\lambda_1} + e^{-\lambda_1} - 2)(\alpha e^{-\lambda_1 t} - 1)}{(1+\alpha e^{-\lambda_1 t})(1+\alpha e^{\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1} e^{-\lambda_1 t})}.$$

Moreover, since $c\tau \leqslant 1$, we have

$$\begin{aligned}
&\rho_\alpha^+(t)[1-\rho_\alpha^+(t-c\tau)] \\
&\quad = \frac{\alpha e^{\lambda_1 c\tau} e^{-\lambda_1 t}}{(1+\alpha e^{-\lambda_1 t})(1+\alpha e^{\lambda_1 c\tau} e^{-\lambda_1 t})} \\
&\quad \leqslant \frac{\alpha e^{\lambda_1} e^{-\lambda_1 t}}{(1+\alpha e^{\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1 t})}.
\end{aligned}$$

So, in order for ρ_α^+ to be an upper solution of (4.5), we only need

$$\frac{e^{\lambda_1}}{1+\alpha e^{\lambda_1} e^{-\lambda_1 t}} + \frac{d(e^{\lambda_1}+e^{-\lambda_1}-2)(\alpha e^{-\lambda_1 t}-1)}{(1+\alpha e^{\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1} e^{-\lambda_1 t})} \leqslant \frac{c\lambda_1}{1+\alpha e^{-\lambda_1 t}}. \tag{4.7}$$

This is true, because by Proposition 4.4, we have

$$\begin{aligned}
&\frac{e^{\lambda_1}}{1+\alpha e^{\lambda_1} e^{-\lambda_1 t}} + \frac{d(e^{\lambda_1}+e^{-\lambda_1}-2)(\alpha e^{-\lambda_1 t}-1)}{(1+\alpha e^{\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1} e^{-\lambda_1 t})} - \frac{c\lambda_1}{1+\alpha e^{-\lambda_1 t}} \\
&\quad = \frac{[g_{1,d}(\lambda_1)-c\lambda_1] + \alpha e^{-\lambda_1 t}[e^{\lambda_1}+e^{-\lambda_1}][g_2(\lambda_1)-c\lambda_1] - \alpha e^{-2\lambda_1 t}\Delta_{c,d}(\lambda_1)}{(1+\alpha e^{\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1} e^{-\lambda_1 t})(1+\alpha e^{-\lambda_1 t})} \\
&\quad \leqslant 0.
\end{aligned}$$

This shows (i), (ii) can be verified directly. ∎

PROPOSITION 4.6. *Let $d \geqslant (e/2(e-1))$ and $c \in (c(s_2), c(s_1))$ be fixed.*

(i) *Let $\varepsilon > 0$ be such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, and $M > 0$ be such that $M \geqslant (e^{-\varepsilon c\tau}/\Delta_{c,d}(\lambda_1+\varepsilon))$. Then, $\rho^-(t) = \max\{0, (1-Me^{\varepsilon t})\, e^{\lambda_1 t}\}$ is a lower solution of* (4.5);

(ii) *$\rho^-(t) \leqslant \rho_\alpha^+(t)$ for $t \in \mathbb{R}$ if $\alpha > 0$ is sufficiently small*;

(iii) *Assume $\mu \geqslant \lambda_1$ and $0 < \alpha < (\mu/2(\lambda_1+\mu))$. Let ε be such that $0 < \varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, and $M > 0$ be sufficiently large such that $(\sqrt{2}-1) \leqslant \alpha M < M-1$ and $M \geqslant (e^{-\varepsilon c\tau}/\Delta_{c,d}(\lambda_1+\varepsilon))$. Then, $[\rho_\alpha^+(t) - \rho^-(t)]\, e^{\mu t}$ is nondecreasing in $t \in \mathbb{R}$.*

Proof. The argument of (i) is similar to that of Proposition 3.5. Verification of (ii) and (iii) is straightforward but tedious, and thus is omitted.

Now, combining Lemmas 4.2–4.6 and applying Theorem 4.1, we finally have the following result from which Corollary 2.3 follows.

COROLLARY 4.1. *Assume that* $d \geqslant (e/2(e-1))$ *and* $\tau \leqslant \min\{(x^*)^{-1}, (c(s_1(d))^{-1}\}$. *Then for every* $c \in (c(s_2), c(s_1))$, *system* (2.7) *has a traveling wave front* $u_n(t) = x(t - nc^{-1})$, $n \in \mathbb{Z}$, *where* $(d/dt)\, x(t) > 0$ *for* $t \in \mathbb{R}$, $\lim_{t \to -\infty} x(t) = 0$ *and* $\lim_{t \to \infty} x(t) = 1$. *Moreover, we have the following estimate*

$$\rho^-(t) \leqslant \phi(t) \leqslant \rho_\alpha^+(t), \qquad t \in \mathbb{R}$$

where ρ^- *and* ρ_α^+ *are given in Propositions* 4.5 *and* 4.6.

5. PERIODIC TRAVELING WAVES

As mentioned in Section 2, Theorem 2.3 can be established by an argument similar to that of Erbe *et al.* [19], using the S^1-degree and bifurcation theory of Dylawerski *et al.* [18] and Geba and Marzantowicz [21]. In this section, we will take (2.9)–(2.10) as an example to demonstrate the main steps of the proofs of Theorems 2.3 and 2.4.

We consider (2.9) with $g \in C^2(\mathbb{R}; \mathbb{R})$ and $g(0) = 0$. Recall that if $u_n(t) = x(t - nc)$ is a spatially p-periodic traveling wave of (2.9), then $x(t + pc) = x(t)$ and $x\colon \mathbb{R} \to \mathbb{R}$ satisfies the mixed functional differential equation (2.10). We now normalize the period of x by

$$y(t) = x\left(\frac{pc}{2\pi}\, t\right), \tag{5.1}$$

then y is 2π-periodic and (2.10) is equivalent to

$$\frac{2\pi}{pc}\,\dot{y}(t) = -\alpha y(t) + a_0\, g\left(y\left(t - \frac{2\pi}{pc}\,\tau\right)\right) + \sum_{j=1}^{m} a_j \left[g\left(y\left(t - \frac{2\pi}{pc}\,\tau - j\,\frac{2\pi}{p}\right)\right) + g\left(y\left(t - \frac{2\pi}{pc}\,\tau + j\,\frac{2\pi}{p}\right)\right)\right]. \tag{5.2}$$

The Hopf bifurcation problem of (5.2) was discussed in Alexander and Auchmuty [2]. From now on, we will fix the positive integer p. Then, for given constants c and τ and for a given 2π-periodic mapping $y\colon \mathbb{R} \to \mathbb{R}$, we can define

$$F(y, \tau, c)(t) = \frac{pc}{2\pi} \left\{ -\alpha y(t) + a_0\, g\left(y\left(t - \frac{2\pi}{pc}\,\tau\right)\right) + \sum_{j=1}^{m} a_j \left[g\left(y\left(t - \frac{2\pi}{pc}\,\tau - j\,\frac{2\pi}{p}\right)\right) + g\left(y\left(t - \frac{2\pi}{pc}\,\tau + j\,\frac{2\pi}{p}\right)\right)\right]\right\}.$$

Therefore, (5.2) can be written as

$$\dot{y}(t) = F(y, \tau, c)(t). \tag{5.3}$$

Restricting to the subspace of all constant mappings, F induces a mapping $\hat{F}\colon \mathbb{R} \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by

$$\hat{F}(z, \tau, c) = \frac{pc}{2\pi}\left[-\alpha z + \left(a_0 + 2 \sum_{j=1}^{m} a_j \right) g(z) \right], \quad (z, \tau, c) \in \mathbb{R} \times \mathbb{R} \times (0, \infty). \tag{5.4}$$

A point $(z, \tau, c) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ is called a *stationary point* if $\hat{F}(z, \tau, c) = 0$. Clearly, if $z = 0$ is the only fixed point of $(1/\alpha)[a_0 + 2\sum_{j=1}^{m} a_j]\, g(x)$, then $(0, \tau, c)$ is the only stationary point corresponding to $(\tau, c) \in \mathbb{R} \times (0, \infty)$.

To apply the S^1-bifurcation theory developed by Erbe, Geba, Krawcewicz and Wu [19] for parameterized mixed functional differential equations, we need to verify that $D_z \hat{F}(z, \tau, c)$, the derivative of $\hat{F}$ with respect to the first argument, is an isomorphism at given $(0, \tau, c)$. Note that

$$D_z \hat{F}(0, \tau, c) = \frac{pc}{2\pi}\left[-\alpha + \left(a_0 + 2 \sum_{j=1}^{m} a_j \right) \gamma \right], \quad \gamma = g'(0). \tag{5.5}$$

Therefore, if

$$\gamma \left(a_0 + 2 \sum_{j=1}^{m} a_j \right) \neq \alpha, \tag{5.6}$$

then $D_z \hat{F}(0, \tau, c)$ is indeed an isomorphism.

The linearization of (5.2) at the stationary point $(0, \tau, c)$ has the form

$$\begin{aligned} \frac{2\pi}{pc}\, \dot{y}(t) = {} & -\alpha y(t) + a_0 \gamma y \left(t - \frac{2\pi}{pc} \tau \right) \\ & + \sum_{j=1}^{m} a_j \gamma \left[y\left(t - \frac{2\pi}{pc} \tau - j \frac{2\pi}{p} \right) + y\left(t - \frac{2\pi}{pc} \tau + j \frac{2\pi}{p} \right) \right]. \end{aligned} \tag{5.7}$$

The *characteristic values* of the stationary point $(0, \tau, c)$ are complex numbers λ satisfying the *characteristic equation*

$$\begin{aligned} \lambda = {} & -\frac{pc}{2\pi} \alpha + \frac{pc}{2\pi} a_0 \gamma e^{-\lambda (2\pi/pc)\tau} \\ & + \frac{pc}{2\pi} \sum_{j=1}^{m} a_j \gamma e^{-\lambda (2\pi/pc)\tau} [e^{-\lambda (2\pi/p) j} + e^{\lambda (2\pi/p) j}]. \end{aligned} \tag{5.8}$$

A stationary point $(0, \tau, c)$ is a *center* if the exists an integer $k \geqslant 1$ such that ik is a characteristic value. Substituting $\lambda = ik$ to (5.8), we get

$$\begin{aligned} ik\frac{2\pi}{pc} &= -\alpha + \gamma\left[a_0 + \sum_{j=1}^{m} a_j(e^{-ik(2\pi/p)\,j} + e^{ik(2\pi/p)\,j})\right] e^{-ik(2\pi/pc)\,\tau} \\ &= -\alpha + \gamma\left[a_0 + 2\sum_{j=1}^{m} a_j \cos\left(k\frac{2\pi}{p} j\right)\right] e^{-ik(2\pi/pc)\,\tau} \\ &= -\alpha + \gamma\beta_{p,k} e^{-ik(2\pi/pc)\,\tau}, \end{aligned} \tag{5.9}$$

where

$$\beta_{p,k} := a_0 + 2\sum_{j=1}^{m} a_j \cos\left(k\frac{2\pi}{p} j\right).$$

In particular, if $k = 1$ then

$$\beta_{p,1} = \beta_p := a_0 + 2\sum_{j=1}^{m} a_j \cos\left(\frac{2\pi}{p} j\right).$$

Writing (5.9) in terms of its real and imaginary parts, we get

$$\begin{cases} \alpha = \beta_{p,k}\gamma \cos\left(k\dfrac{2\pi}{pc}\tau\right) \\ -k\dfrac{2\pi}{pc} = \beta_{p,k}\gamma \sin\left(k\dfrac{2\pi}{pc}\tau\right). \end{cases}$$

That is,

$$\begin{cases} \cos\left(k\dfrac{2\pi}{pc}\tau\right) = \dfrac{\alpha}{\gamma\beta_{p,k}} \\ \tan\left(k\dfrac{2\pi}{pc}\tau\right) = -k\dfrac{2\pi}{pc\alpha}. \end{cases} \tag{5.10}$$

Therefore, if $|\beta_{p,k}| > \alpha/\gamma$ then we can solve the first equation of (5.10). Substituting the result into the second equation of (5.10) we can determine the real τ and c. In summary, we have established the following:

PROPOSITION 5.1. *Assume that $|\beta_{p,k}| > \alpha/\gamma$ for some fixed positive integers p and k. Let $\theta_{p,k} \in (\pi/2, \pi)$ or $(3\pi/2, 2\pi)$, depending on whether*

$\beta_{p,k}<0$ or $\beta_{p,k}>0$, be given so that $\cos\theta_{p,k}=\alpha/\gamma\beta_{p,k}$. *For each integer $j\geqslant 0$, define*

$$\theta_{p,k,j}=\theta_{p,k}+2j\pi;$$

$$c_{p,k}=-k\frac{2\pi}{p\alpha}\cot\theta_{p,k};$$

$$\tau_{p,k,j}=-\frac{\theta_{p,k,j}}{\alpha}\cot\theta_{p,k}.$$

Then the set of centers of (5.3) *is* $\{(0,\tau_{p,k,j},c_{p,k});\ k\geqslant 1,\ j\geqslant 0\}$ *and thus, is isolated in* $\mathbb{R}\times\mathbb{R}\times(0,\infty)$.

Our next step is to evaluate the so-called *crossing number* of the stationary point $(0,\tau_{p,j},c_p)$, where

$$\tau_{p,j}:=\tau_{p,1,j}:=-\frac{\theta_p+2j\pi}{\alpha}\cot\theta_p;$$

$$c_p:=c_{p,1}=-\frac{2\pi}{p\alpha}\cot\theta_p;$$

$$\theta_p:=\theta_{p,1}.$$

The crossing number is defined by (see Erbe *et al.* [19])

$$\gamma(0,\tau_{p,j},c_p)=deg_B(\varDelta;\Omega),$$

where deg_B is the Brouwer degree and

$$\varDelta(\tau,c)=i\frac{2\pi}{pc}-\left[-\alpha+a_0\gamma e^{-i(2\pi/pc)\tau}+\sum_{j=1}^{m}a_j\gamma e^{-i(2\pi/pc)\tau}(e^{-i(2\pi/p)j}+e^{i(2\pi/p)j})\right]$$
$$=i\frac{2\pi}{pc}-[-\alpha+\beta_p\gamma e^{-i(2\pi/pc)\tau}],$$

and $\Omega=(\tau_{p,j}-\delta,\tau_{p,j}+\delta)\times(c_p-\delta,c_p+\delta)$ for small $\delta>0$ (the function $\varDelta$ is obtained by looking for the action of the equation (5.2) on the function e^{it}, the basis for the first isotypical component of the space of 2π-periodic continuous mappings, where S^1 acts by shifting the argument). Define

$$H(\tau,u,c)=\left(u+i\frac{2\pi}{pc}\right)-[-\alpha+\beta_p\gamma e^{-(u+i(2\pi/pc))\tau}]$$

where $(u,c)\in D:=(0,\varepsilon)\times(c_p-\varepsilon,c_p+\varepsilon)$ for a small $\varepsilon>0$. Then we have the following observations:

(i) $H(\tau, 0, c) = \Delta(\tau, c)$;

(ii) $H(\tau, u, c) \neq 0$ if $|\tau - \tau_{p,j}| \leqslant \varepsilon$ and $(u, c) \in \partial D \backslash \{(0, c); |c - c_p| < \varepsilon\}$;

(iii) $H(\tau_{p,j} \pm \varepsilon, 0, c) \neq 0$ for $|c - c_p| < \varepsilon$.

Therefore, using Lemma 2.5 of Erbe *et al.* [19], we get

$$\gamma(0, \tau_{p,j}, c_p) = deg_B(H(\tau_{p,j} - \varepsilon, \cdot), D) - deg_B(H(\tau_{p,j} + \varepsilon, \cdot), D). \quad (5.11)$$

PROPOSITION 5.2. *The crossing number* $\gamma(0, \tau_{p,j}, c_p)$ *at* $(0, \tau_{p,j}, c_p)$ *is* -1 *for every given integer* $j \geqslant 0$.

Proof. For the sake of simplicity, we let $v = (2\pi/pc)$. Assume $u = u(\tau)$ and $v = v(\tau)$ are the smooth functions of $\tau \in (\tau_{p,j} - \delta, \tau_{p,j} + \delta)$ such that

$$u + iv + \alpha - \beta_p \gamma e^{-(u+iv)\tau} = 0. \quad (5.12)$$

Differentiating both sides of (5.12) with respect to τ and then evaluating at $\tau = \tau_{p,j}$, $c = c_p$, we get

$$\begin{aligned} \frac{d}{d\tau}(u + iv) &= \frac{-(u+iv)\,\beta_p \gamma e^{-(u+iv)\tau}}{1 + \beta_p \gamma \tau e^{-(u+iv)\tau}} \\ &= \frac{-i(2\pi/pc)\,\beta_p \gamma e^{-i(2\pi/pc)\tau}}{1 + \beta_p \gamma \tau e^{-i(2\pi/pc)\tau}} \\ &= \frac{-i(2\pi/pc)(i(2\pi/pc) + \alpha)}{1 + \tau(\alpha + i(2\pi/pc))}. \end{aligned}$$

Therefore,

$$\left.\frac{d}{d\tau} u(\tau)\right|_{\tau = \tau_{p,j}} = \frac{(2\pi/pc_p)^2}{(1 + \alpha\tau_{p,j})^2 + ((2\pi\tau_{p,j})/(pc_p))^2} > 0$$

Consequently, from (5.11), we get $\gamma(0, \tau_{p,j}, c_p) = -1$. This completes the proof.

We can now apply the global bifurcation theorem (Theorem 2.3) to conclude that the connected component S_p through $(0, \tau_p, c_p)$, $\tau_p := \tau_{p,0} = -(\theta_p/\alpha) \cot \theta_p$, in the closure of the subset $\{(y, \tau, c);\ y$ is a non-constant 2π-periodic solution of (5.2), $\tau \in \mathbb{R}$, $c > 0\}$ of the space $X \times \mathbb{R}^2$ must be non-empty and unbounded, where X is the Banach space of 2π-periodic continuous functions equipped with the super-norm. This is equivalent to say that the connected component Σ_p through $(0, \tau_p, c_p)$ in the closure of the subset $\{(x, \tau, c);\ x$ is a non-constant pc-periodic solution of (2.10), $\tau \in \mathbb{R}$, $c \geqslant 0\}$ of the space $Y \times \mathbb{R}^2$ must be nonempty and unbounded, where Y is the Banach space of all bounded and continuous functions equipped with the super-norm.

The following result establishes *a-priori* bounds for periodic solutions of (2.10).

PROPOSITION 5.3. *There exists a constant* $M > 0$, *independent of* τ *and* c, *such that* $\sup_{t \in \mathbb{R}} |x(t)| \leqslant M$ *for every given periodic solution* x *of* (2.10), *under condition* (i) *of Theorem* 2.4.

Proof. Clearly, for $t, s \in \mathbb{R}$, we have

$$x(t) = e^{-\alpha(t-s)} x(s) + \int_s^t e^{-\alpha(t-\theta)} \left\{ a_0 g(x(\theta - \tau)) + \sum_{j=1}^m a_j [g(x(\theta - \tau + jc)) + g(x(\theta - \tau - jc))] \right\} d\theta.$$

Letting $s \to -\infty$ and considering the fact that $|g(x)| \leqslant 1$ for all $x \in \mathbb{R}$, we have

$$|x(t)| \leqslant \frac{1}{\alpha} \left[|a_0| + 2 \sum_{j=1}^m |a_j| \right] := M$$

for $t \in \mathbb{R}$. This completes the proof.

The following result establishes *a priori* bounds for the period of periodic solutions of (2.10).

PROPOSITION 5.4. *For fixed positive integers* p *and* q, *let* $\gamma_{p,q}$ *be defined by* (2.14). *If* $pq \geqslant 4$ *is an even integer and if* $\gamma_{p,q} < \alpha/\gamma$, *then system* (2.10) *has no non-constant* $pq\tau$*-periodic solution. In other words, if* $\gamma_{p,q} < \alpha/\gamma$ *for positive integers* p *and* q *such that* $pq \geqslant 4$ *is even, then there exists no* $(x, \tau, c) \in \Sigma_p$ *so that* $c = q\tau$.

Proof. Let x be a $pq\tau$-periodic solution of (2.10), and define

$$x_i(t) = x(t + \tau - i\tau), \qquad 1 \leqslant i \leqslant pq;$$

$$X(t) = (x_1(t), x_2(t), ..., x_{pq}(t))^T;$$

$$G(X(t)) = (g(x_1(t)), g(x_2(t)), ..., g(x_{pq}(t))^T;$$

$$b_i = \sum \{a_j; 1 \leqslant j \leqslant m, jq \text{ or } -jq = i - 2 (\text{mod } pq), 1 \leqslant i \leqslant pq.$$

Then we get

$$\dot{X}(t) = -\alpha X(t) + BG(X(t)), \tag{5.13}$$

where B is the $pq \times pq$ circulant matrix

$$B = \begin{pmatrix} 0 & b_2 & b_3 & \cdots & b_{pq} \\ b_{pq} & 0 & b_2 & \cdots & b_{pq-1} \\ b_{pq-1} & b_{pq} & 0 & \cdots & b_{pq-2} \\ & \cdots & \cdots & \cdots & \\ b_2 & b_3 & b_4 & \cdots & 0 \end{pmatrix}.$$

Let $V(X) = \sum_{j=1}^{pq} \int_0^{x_j} g(x)\, dx$. Then

$$\dot{V}_{(5.13)}(X(t)) = -\alpha [X(t)]^T G(X(t)) + [G(X(t))]^T BG(X(t)).$$

Using the Nussbaum's spectral theorem for circulant matrices (Nussbaum [46]), we get

$$[G(X(t))]^T BG(X(t)) \leqslant \gamma_{p,q} [G(X(t))]^T G(X(t)).$$

Therefore,

$$\begin{aligned} &\dot{V}_{(5.13)}(X(t)) \\ &\quad \leqslant -\sum_{j=1}^{pq} x_j(t)\, g(x_j(t)) \left[\alpha - \frac{g(x_j(t))}{x_j(t))} \gamma_{p,q} \right] \\ &\quad \leqslant -[\alpha - \gamma_{p,q}\gamma] \sum_{j=1}^{pq} x_j(t)\, g(x_j(t)). \end{aligned}$$

By the LaSalle's Invariance Principle (see LaSalle [37]), we conclude that $X(t)$ is convergent to a constant as $t \to \infty$. This shows that $X(t)$ cannot be a non-constant periodic solution of (5.13). So, x cannot be a non-constant $pq\tau$-periodic solution of (2.10).

We are now in the position to prove Theorem 2.4 and Corollaries 2.4–2.5: First of all, as $\gamma(0, \tau_p, c_p) = -1 \neq 0$, $\sum_p$ must be nonempty. In other words, $(0, \tau_p, c_p)$ is a Hopf bifurcation point. Next, we note that $pc_p/\tau_p = 2\pi/\theta_p \in (2, 4)$ if $\beta_p < -\alpha/\gamma$. Therefore, in the neighborhood of $(0, \tau_p, c_p)$, every element $(\phi, \tau, c) \in \sum_p$ must satisfy $(pc/\tau) \in (2, 4) \subset (2, pq)$. By Propositions 5.3 and 5.4 and since Σ_p is connected, we know that the unbounded component Σ_p must satisfy

$$\Sigma_p \subset \left\{ (x, \tau, c);\ \sup_{t \in \mathbb{R}} |x(t)| \leqslant M, \frac{pc}{\tau} \in (2, pq) \right\}.$$

We now claim that Σ_p does not intersect with the hyperplane $\tau = 0$. If fact, if $(x, 0, c) \in \Sigma_p$ for some $x \in Y$ and $c \geqslant 0$, then there exists a sequence $(x_n, \tau_n, c_n) \in \Sigma_p$ such that $x_n \to x$ in Y, $\tau_n \to 0$ and $c_n \to c$. As $pc_n/\tau_n \in (2, pq)$, we must have $c_n \to 0$ as well. Therefore, x must satisfy the ordinary differential equation

$$\frac{d}{dt} x(t) = -\alpha x(t) + \left(a_0 + 2 \sum_{j=1}^{m} a_j \right) g(x(t)).$$

Now the assumption $(a_0 + 2\sum_{j=1}^{m} a_j)\, \gamma < \alpha$ implies that $x = 0$. This leads to a contradiction to the obvious fact that $(0, 0, 0) \notin \Sigma_p$.

Therefore, the projection of Σ_p onto the τ-space in unbounded and is contained in $[0, \infty)$. This shows that for every $\tau > \tau_p$, there exists a pc-periodic solution of (2.10) with $(pc/\tau) \in (2, pq)$, completing the proof of Theorem 2.4.

Corollaries 2.4 and 2.5 are immediate consequences of Theorem 2.4 after some elementary calculations.

Using a similar argument for $\Sigma_{p,j}$, the connected component through $(0, \tau_{p,j}, c_p)$ in the closure of the subset $\{(x, \tau, c);\ x$ is a non-constant pc-periodic solution of (2.10), $\tau \in \mathbb{R}$, $c \geqslant 0\}$ of the space $Y \times \mathbb{R}^2$, we can get the following existence of one slowly oscillatory wave and multiple rapidly oscillatory waves of (2.9):

THEOREM 5.1. *Assume that*

(i) $g \in C^2(\mathbb{R}; \mathbb{R})$, $g(0) = 0$, $\lim_{x \to \pm\infty} g(x) = \pm 1$, $g'(x) > 0$ *and* $xg''(x) < 0$ *for* $x \neq 0$;

(ii) $a_0 + 2\sum_{j=1}^{m} a_j < \alpha/\gamma$;

(iii) *there exists a positive integer* p *such that* $\beta_p < -\alpha/\gamma$;

(iv) *there exists a positive integer* q *such that* $pq \geqslant 4$ *is an even integer and* $\gamma_{p,q} < \alpha/\gamma$

Let $\theta_p \in (\pi/2, \pi)$ *be given so that* $\cos\theta_p = (\alpha/\gamma\beta_p)$. *Define*

$$c_p = -\frac{2\pi}{p\alpha} \cot\theta_p, \qquad \tau_p = -\frac{\theta_p}{\alpha} \cot\theta_p;$$

$$\theta_{p,j} = \theta_p + 2j\pi;$$

$$\tau_{p,j} = -\frac{\theta_{p,j}}{\alpha} \cot\theta_p,\ j \geqslant 0.$$

Then for each fixed integer $j \geqslant 0$ *and for each* $\tau > \tau_{p,j}$ *there exist constants* $\alpha_1, ..., \alpha_j > 0$ *such that* (2.9) *has* j *spatially p-periodic traveling waves* $u_{n,j}(t) = x_j(t - n\alpha_j)$ *and the period of* x_j *is in*

$$(2\tau, pq\tau), \left(\frac{2}{3}\tau, \frac{4}{5}\tau\right), ..., \left(\frac{4}{4j+2}\tau, \frac{4}{4j+1}\tau\right),$$

respectively.

Similarly, in the case where $\beta_p > (\alpha/\gamma)$, we have

THEOREM 5.2. *Assume that* (i) *and* (ii) *of Theorem* 5.1 *hold and*

(iii) *there exists a positive integer p such that* $\beta_p > \alpha/\gamma$;

(iv) *there exists a positive integer q such that* $pq \geqslant 2$ *and* $\gamma_{p,q} < \alpha/\gamma$.

Let $\theta_p \in ((3\pi/2), 2\pi)$ *be given so that* $\cos \theta_p = \alpha/\gamma\beta_p$. *Define*

$$c_p = -\frac{2\pi}{p\alpha} \cot \theta_p, \qquad \tau_p = -\frac{\theta_p}{\alpha} \cot \theta_p;$$

$$\theta_{p,j} = \theta_p + 2j\pi;$$

$$\tau_{p,j} = -\frac{\theta_{p,j}}{\alpha} \cot \theta_p, \ j \geqslant 0.$$

Then for each fixed integer $j \geqslant 0$ *and for each* $\tau > \tau_{p,j}$ *there exist constants* $\alpha_1, ..., \alpha_j > 0$ *such that* (2.9) *has* j *spatially p-periodic traveling waves* $u_{n,j}(t) = x_j(t - n\alpha_j)$ *and the period of* x_j *is in*

$$(\tau, q\tau), \left(\frac{2}{5}\tau, \frac{1}{2}\tau\right), ..., \left(\frac{1}{j+1.5}\tau, \frac{1}{j+1}\tau\right),$$

respectively.

REFERENCES

1. S. Ahmad and A. S. Vatsala, Comparison results of reaction-diffusion equations with delay in abstract cones, *Rend. Sem. Mat. Univ. Padova* **65** (1981), 19–34.
2. J. C. Alexander and G. Auchumuty, Global bifurcation of phase-locked oscillations, *Arch. Rational Mech. Anal.* **93** (1986), 253–270.
3. D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion and nerve propagation, *in* "Lecture Notes in Mathematics," Vol. 446, pp. 5–49, Springer-Verlag, Berlin, 1975.
4. C. Atkinson and G. E. H. Reuter, Deterministic epidemic waves, *Math. Proc. Cambridge Philos. Soc.* **80** (1976), 315–330.

5. J. Belair, Stability in a model of a delayed neural network, *J. Dynam. Differential Equations* **5** (1993), 607–623.
6. J. Belair, S. A. Campbell, and P. van den Driessche, Frustration, stability, and delay-induced oscillations in a neural network model, *SIAM J. Appl. Math.* **56** (1996), 245–255.
7. J. Bell, Some threshold results for models of myelinated nerves, *Math. Biosci.* **54** (1981), 181–190.
8. J. Bell and C. Cosner, Threshold behaviour and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons, *Quart. Appl. Math.* **42** (1984), 1–14.
9. N. F. Britton, Spatial structures and periodic traveling waves in an integro-differential reaction-diffusion population model, *SIAM J. Appl. Math.* **50** (1990), 1663–1688.
10. J. W. Cahn, S.-N. Chow, and E. S. Van Vleck, Spatially discrete nonlinear diffusion equations, *Rocky Mountain J. Math.* **25** (1995), 87–118.
11. H. Chi, J. Bell, and B. Hassard, Numerical solution of a nonlinear advance-delay differential equation from nerve conduction theory, *J. Math. Biol.* **24** (1986), 583–601.
12. L. O. Chua and L. Yang, Cellular neural networks: theory, *IEEE Trans. Circuits Systems* **35** (1988), 1257–1272.
13. L. O. Chua and L. Yang, Cellular neural networks: applications, *IEEE Trans. Circuits Systems* **35** (1988), 1273–1290.
14. S.-N. Chow and J. Mallet-Paret, The Fuller index and global Hopf bifurcation, *J. Differential Equations* **29** (1978), 66–88.
15. S.-N. Chow and J. Mallet-Paret, Pattern formation and spatial chaos in lattice dynamical systems, I and II, *IEEE Trans. Circuits Systems* **42** (1995), 746–756.
16. S.-N. Chow and W. Shen, Dynamics in a discrete Nagumo equation: Spatial topological chaos, *SIAM J. Appl. Math.* **55** (1995), 1764–1781.
17. M. A. Cohen and S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE Trans. Systems Man Cybernet* **13** (1983), 815–826.
18. G. Dylawerski, K. Geba, J. Jodel, and W. Marzantowicz, S^1-equivariant degree and the Fuller index, *Ann. Polon. Math.* **52** (1991), 243–280.
19. L. H. Erbe, K. Geba, W. Krawcewicz, and J. Wu, S^1-degree and global Hopf bifurcation theory of functional differential equations, *J. Differential Equations* **98** (1992), 277–298.
20. R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937), 353–269.
21. K. Geba and W. Marzantowicz, Global bifurcation of periodic solutions, *Topol. Methods Nonlinear Anal.* **1** (1993), 67–94.
22. K. Gopalsamy and X. He, Stability in asymmetric Hopfield nets with transmission delays, *Physica D* **76** (1994), 344–358.
23. A. Hastings, Dynamics of a single species in a spatially varying environments: The stabilizing role of high dispersal rates, *J. Math. Biology* **16** (1982), 49–55.
24. D. Hankerson and B. Zinner, Wave fronts for a cooperative tridiagonal system of differential equations, *J. Dynam. Differential Equations* **5** (1993), 359–373.
25. A. Herz, B. Salzer, R. Kühn, and J. L. van Hemmen, Hebbian learning reconsidered: Representation of static and dynamic objects in associative neural nets, *Biol. Cybernet.* **60** (1989), 457–467.
26. J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci. USA* **79** (1982), 2554–2558.
27. J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-stage neurons, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3088–3092.
28. J. P. Keener, Propagation and its failure in the discrete Nagumo equation, *in* "Proceedings of Conference on Ordinary and Partial Differential Equations," Dundee, 1986.

29. J. P. Keener, Propagation and its failure to coupled systems of discrete excitable cells, *SIAM J. Appl. Math.* **47** (1987), 556–572.
30. W. Kerscher and R. Nagel, Asymptotic behavior of one-parameter semigroups of positive operators, *Acta Appl. Math.* **2** (1984), 297–309.
31. W. Krawcewicz and J. Wu, Discrete waves and phase-locked oscillations in the growth of a single-species population over a patchy environment, *Open Systems Inform. Dynam.* **1** (1992), 127–147.
32. T. Krisztin and J. Wu, Monotone semiflows generated by neutral equations with different delays in neutral and retarded parts, *Acta Math. Univ. Comenian* **63** (1993), 207–220.
33. T. Krisztin and J. Wu, Asymptotic periodicity, monotonicity and oscillation of solutions of scalar neutral functional differential equations, *J. Math. Anal. Appl.* **199** (1996), 502–525.
34. K. Kunish and W. Schappacher, Order preserving evolution operators of functional differential equations, *Boll. Unione Mate. Italina* **16B** (1979), 480–500.
35. G. Ladas and V. Lakshmikantham, "Differential Equations in Abstract Spaces," Academic Press, New York, 1972.
36. V. Lakshmikantham and S. Leela, "Nonlinear Differential Equations in Abstract Spaces," Pergamon Press, Oxford, 1981.
37. J. LaSalle, "The Stability of Dynamical Systems," Regional Conferences Series in Applied Mathematics, SIAM, Philadephia, 1976.
38. S. Leela and V. Moauro, Existence of solutions of delay differential equations on closed subsets of a Banach space, *Nonlinear Anal.* **2** (1978), 47–58.
39. S. A. Levin, Dispersion and population interaction, *Amer. Natur.* **108** (1974), 207–228.
40. S. A. Levin, Spatial patterning and the structure of ecological communities, *in* "Some Mathematical Questions in Biology," Vol. 8, pp. 1–36, Amer. Math. Soc., Providence, RI, 1976.
41. S. A. Levin, Population models and community structure in heterogeneous environments, *in* "Mathematical Ecology" (T. G. Hallan and S. A. Levin, Eds.), pp. 295–321, Springer-Verlag, New York, 1986.
42. N. Madras, J. Wu, and X. Zou, Local-nonlocal interaction and spatial-temporal patterns in single species population over a patchy environment, *Canad. Appl. Math. Quart.* **4** (1996), 109–134.
43. C. M. Marcus and R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39** (1989), 347–359.
44. H. K. Mckean, Nagumo's equation, *Adv. Math.* **4** (1970), 209–223.
45. J. D. Murray, "Mathematical Biology," Springer-Verlag, New York, 1989.
46. R. D. Nussbaum, Circulant matrices and differential-delay equations, *J. Differential Equations* **60** (1985), 201–217.
47. R. D. Nussbaum and J. Mallet-Paret, Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation, *Ann. Math. Pura Appl.* **145** (1986), 33–128.
48. R. D. Nussbaum and J. B. Potter, Cyclic differential equations and periodic three solutions of differential-delay equations, *J. Differential Equations* **46** (1982), 379–408.
49. F. J. Pineda, Generalization of back-propagation to recurrent neural networks, *Phys. Rev. Lett.* **59** (1987), 2229–2232.
50. L. S. Pontryagin, R. V. Gamkreledze, and E. F. Mischenko, "The Mathematical Theory of Optimal Processes," Interscience, New York, 1962.
51. A. Rustichini, Functional differential equations of mixed type: The linear autonomous case, *J. Dynam. Differential Equations* **1** (1989), 121–143.
52. A. Rustichini, Hopf bifurcation for functional differential equations of mixed type, *J. Dynam. Differential Equations* **1** (1989), 145–177.

53. K. W. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, *Trans. Amer. Math. Soc.* **302** (1987), 587–605.
54. K. Schumacher, Traveling-front solutions for integro-differential equations, *in* "Lecture Notes in Biomath.," Vol. 38, pp. 296–309, Springer-Verlag, Berlin/New York, 1980.
55. L. S. Schulman, Correlating arrows of time, *Phys. Rev. D* **7** (1973), 2868–2874.
56. L. S. Schulman, Some differential difference equations containing both advance and retardation, *J. Math. Phys.* **15** (1974), 195–198.
57. H. L. Smith, Monotone semiflows generated by functional differential equations, *J. Differential Equations* **66** (1987), 420–442.
58. H. L. Smith, "Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems," Amer. Math. Soc., Providence, RI, 1995.
59. H. L. Smith and H. R. Thieme, Monotone semiflows in scalar non-quasimonotone functional differential equations, *J. Math. Anal. Appl.* **150** (1990), 289–306.
60. H. L. Smith and H. R. Thieme, Strongly order preserving semiflows generated by functional differential equations, *J. Differential Equations* **93** (1991), 332–363.
61. J. A. Wheeler and R. P. Feynman, Classical electrodynamics in terms of direct interparticle action, *Rev. Mod. Phys.* **21** (1949), 425–433.
62. J. Wu, Bifurcating waves in coupled cells described by delay-differential equations, *in* "Pattern Formation: Symmetry Methods and Applications (Waterloo, Ontario, 1993)," pp. 347–358, Fields Inst. Commun. 5, Amer. Math. Soc., Providence, RI, 1996.
63. J. Wu and X. Zou, Patterns of sustained oscillation in neural networks with delayed interactions, *J. Appl. Math. Comput.* **73** (1995), 55–75.
64. B. Zinner, Stability of traveling wave fronts for the discrete Nagumo equation, *SIAM J. Math. Anal.* **22** (1991), 1016–1020.
65. B. Zinner, Existence of traveling wave front solutions for the discrete Nagumo equation, *J. Differential Equations* **96** (1992), 1–27.
66. B. Zinner, G. Harris, and W. Hudson, Traveling wavefronts for the discrete Fisher's equation, *J. Differential Equations* **105** (1993), 46–62.