ON A HYPERLOGISTIC DELAY EQUATION

by JIANSHE YU,† JIANHONG WU* and XINGFU ZOU

(Received 25 January, 1995)

1. Introduction. Consider the following hyperlogistic equation

$$\frac{d}{dt}N(t) = rN(t)\prod_{j=1}^{m}\left[1 - \frac{N(t-\tau_j)}{K}\right]^{\alpha_j}, \qquad t \geq 0, \tag{1.1}$$

where $r, K, \tau_j \in (0, \infty)$, and $\alpha_j = p_j/q_j$ are rational numbers with q_j odd, p_j and q_j are co-prime, $1 \leq j \leq m$, and $\prod_{j=1}^{m}(-1)^{\alpha_j} = -1$.

When $m = 1$ and $\alpha_1 = 1$, Eq. (1.1) reduces to the well-known delay logistic equation

$$\frac{d}{dt}N(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{1.2}$$

which has been extensively investigated by many authors. See for example [**3**, **5**, **6**, **7**, **10**, **13**, **16**]. Other related work includes [**1**, **2**, **12**] (in the case $m = 1$ and $\alpha_1 \neq 1$) and [**4**] (in the case $\alpha_1 = \ldots = \alpha_m = 1$). Allowing $m \neq 1$, we wish to discuss the effect of different delayed terms on the oscillatory and asymptotic behaviors of solutions.

By making a change of variables

$$x(t) = \frac{N(t)}{K} - 1,$$

one can write (1.1) as

$$\frac{d}{dt}x(t) + r[1 + x(t)]\prod_{j=1}^{m}x^{\alpha_j}(t-\tau_j) = 0. \tag{1.3}$$

We are interested in those solutions $x(t)$ of (1.3) satisfying $x(t) \geq -1$ which correspond to solutions $N(t)$ of (1.1) satisfying $N(t) \geq 0$. Thus, the initial condition

$$\begin{cases} x(t) = \phi(t) \geq -1, & t \in [t_0 - \tau, t_0], \\ \phi \in C([t_0 - \tau, t_0], [-1, \infty)) & \text{and} \quad \phi(t_0) > -1 \end{cases} \tag{1.4}$$

should be specified, where $\tau = \max\{\tau_1, \ldots, \tau_m\}$. It can be easily shown that for any t_0 and any ϕ satisfying (1.4) Eq. (1.3)–(1.4) has a unique solution $x(t; t_0, \phi)$ on $[t_0 - \tau, \infty)$ and $x(t) > -1$ for $t \geq t_0$.

Of major concern in this paper is the oscillatory property of equation (1.3). We will show that all solutions of (1.3)–(1.4) are oscillatory when $\sum_{j=1}^{m}\alpha_j < 1$, but at least one non-oscillatory solution exists when $\sum_{j=1}^{m}\alpha_j > 1$. For the case where $\sum_{j=1}^{m}\alpha_j = 1$, we will

† Research partially supported by NNSF-P. R. China.
* Research partially supported by NSERC-Canada.

establish an equivalence, as far as oscillation is concerned, between (1.3) and its so-called quasilinearized equation

$$\frac{d}{dt}y(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) = 0, \tag{1.5}$$

whose oscillation has been thoroughly studied in [**8**, **9**, **14**, **15**]. Consequently, some existing results can be applied to give necessary and sufficient conditions for the oscillation of Eq. (1.3) when $\sum_{j=1}^{m} \alpha_j = 1$.

2. The case $\sum_{j=1}^{m} \alpha_j < 1$.

THEOREM 2.1. *If* $\alpha = \sum_{j=1}^{m} \alpha_j < 1$, *then every solution of Eq.* (1.3)–(1.4) *oscillates.*

Proof. Assume, by way of contradiction, that Eq. (1.3)–(1.4) has a non-oscillatory solution $x(t)$. We first suppose that $x(t)$ is eventually positive. Then, by (1.3), we eventually have

$$\frac{d}{dt}x(t) = -r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) < 0,$$

which implies that $x(t)$ is eventually decreasing. thus

$$x(t-\tau_j) \geq x(t) \quad \text{eventually,} \quad \text{for } j = 1, \ldots, m.$$

and hence

$$\frac{d}{dt}x(t) + r(1+x(t))x^{\alpha}(t) \leq \frac{d}{dt}x(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0.$$

Thus

$$\frac{d}{dt}x^{1-\alpha}(t) \leq -(1-\alpha)r[1+x(t)] \leq -(1-\alpha)r,$$

which implies that $x^{1-\alpha}(t) \to -\infty$, as $t \to \infty$. This is impossible since $x(t) > 0$ eventually and $1-\alpha > 0$.

We next suppose that $x(t)$ is eventually negative. Noting that $x(t) > -1$ for $t \geq 0$, we have eventually

$$\begin{aligned} \frac{d}{dt}x(t) &= -r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \\ &= r(1+x(t))\prod_{j=1}^{m} [-x(t-\tau_j)]^{\alpha_j} > 0, \end{aligned}$$

which implies that $x(t)$ is eventually increasing. Hence, there exists $T_1 > 0$ such that $x(t-\tau_j) \leq x(t) < 0$ and $1 + x(t) > 1 + x(T_1) > 0$, for all $t > T_1$ and $j = 1, \ldots, m$. Therefore

$$\frac{d}{dt}x(t) + r(1+x(t))x^{\alpha}(t) \geq \frac{d}{dt}x(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \qquad t > T_1,$$

and hence

$$\begin{aligned}\frac{d}{dt}x^{1-\alpha}(t) &\leq -r(1-\alpha)(1+x(t)) \\ &< -r(1-\alpha)(1+x(T_1)) < 0, \qquad t \geq T_1.\end{aligned}$$

Integrating the above inequality from T_1 to $t > 0$ and letting $t \to \infty$, we would get $x^{1-\alpha}(t) \to -\infty$, as $t \to \infty$. This is a contradiction to the fact that $x(t) > -1$ for $t \geq 0$, and completes the proof.

3. The case $\sum_{j=1}^{m} \alpha_j > 1$.

THEOREM 3.1. *If* $\alpha = \sum_{j=1}^{m} \alpha_j > 1$, *then Eq.* (1.3) *has a non-oscillatory solution.*

In order to complete the proof of Theorem 3.1, we will need the following Lemma from [**15**].

LEMMA 3.2. *Every solution of Eq.* (1.5) *with* $\sum_{j=1}^{m} \alpha_j = 1$ *oscillates if and only if*

$$r\sum_{j=1}^{m} \alpha_j \tau_j > \frac{1}{e}.$$

Moreover, the above inequality holds if and only if

$$\begin{cases}\dfrac{d}{dt}y(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \leq 0 \text{ has no eventually positive solution,} \\ \dfrac{d}{dt}y(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \geq 0 \text{ has no eventually negative solution.}\end{cases}$$

Proof of Theorem 3.1. Choose rational numbers $\beta_j = r_j/s_j \in [0, \infty)$ with s_j odd, $1 \leq j \leq m$, such that

$$\beta_j \leq \alpha_j, \quad \text{for} \quad j = 1, \ldots, m, \qquad \sum_{j=1}^{m} \beta_j = 1, \qquad \prod_{j=1}^{m} (-1)^{\beta_j} = -1.$$

Let $\varepsilon > 0$ satisfy

$$r\varepsilon \sum_{j=1}^{m} \beta_j \tau_j \leq \frac{1}{e}.$$

Then, by Lemma 3.2, the following equation

$$\frac{d}{dt}x(t) + r\varepsilon \prod_{j=1}^{m} x^{\beta_j}(t-\tau_j) = 0 \ . \tag{3.1}$$

has a positive solution $x(t)$ defined on $[t_0, \infty)$ for some $t_0 \geq 0$. It is clear that $x(t) \to 0$, as $t \to \infty$. Since $\beta_j \leq \alpha_j$ and $\sum_{j=1}^{m} \beta_j < \sum_{j=1}^{m} \alpha_j$, we have

$$\lim_{t\to\infty}(1+x(t))\frac{\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j)}{\prod_{j=1}^{m} x^{\beta_j}(t-\tau_j)} = 0.$$

Thus, there exists $t_1 > t_0$ such that

$$(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) < \varepsilon \prod_{j=1}^{m} x^{\beta_j}(t-\tau_j), \quad \text{for} \quad t \geq t_1,$$

and hence

$$\frac{d}{dt}x(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) < \frac{d}{dt}x(t) + r\varepsilon \prod_{j=1}^{m} x^{\beta_j}(t-\tau_j) = 0, \quad \text{for} \quad t \geq t_1. \tag{3.2}$$

Set $y(t) = \ln(1 + x(t))$. Then, from (3.2) we have

$$\frac{d}{dt}y(t) + r\prod_{j=1}^{m} [e^{y(t-\tau_j)} - 1]^{\alpha_j} < 0, \quad \text{for} \quad t \geq t_1,$$

which yields

$$y(t) > r\int_t^\infty \prod_{j=1}^{m} [e^{y(s-\tau_j)} - 1]^{\alpha_j}\, ds, \quad \text{for} \quad t \geq t_1. \tag{3.3}$$

Define $\mathbf{X}$ to be the set of piecewise continuous functions $z : [t_1, -\tau, \infty) \to [0, 1]$ and endow $\mathbf{X}$ with the usual pointwise ordering $\leq$, that is

$$z_1 \leq z_2 \Leftrightarrow z_1(t) \leq z_2(t), \quad \text{for all} \quad t \geq t_1 - \tau.$$

Then $(\mathbf{X}; \leq)$ becomes an ordered set. It is obvious that for any nonempty subset $\mathbf{M}$ of $\mathbf{X}$, $\inf(M)$ and $\sup(M)$ exist. So $(\mathbf{X}; \leq)$ is actually a complete lattice. Define a mapping Ψ on $\mathbf{X}$ as follows:

$$(\Psi z)(t) = \begin{cases} \dfrac{r}{y(t)}\displaystyle\int_t^\infty \prod_{j=1}^{m} [e^{y(s-\tau_j)z(s-\tau_j)} - 1]^{\alpha_j}\, ds, & t \geq t_1, \\ \dfrac{t}{t_1}(\Psi z)(t_1) + \left(1 - \dfrac{t}{t_1}\right), & t_1 - \tau \leq t \leq t_1. \end{cases}$$

For each $z \in \mathbf{X}$, we can show that

$$0 \leq (\Psi z)(t) \leq \frac{r}{y(t)}\int_t^\infty \prod_{j=1}^{m} [e^{y(s-\tau_j)} - 1]\, ds < 1, \quad \text{for} \quad t \geq t_1,$$

and

$$0 \le (\Psi z)(t) \le 1, \quad \text{for} \quad t \in [t_1 - \tau, t_1].$$

This shows that $\Psi\mathbf{X} \subseteq \mathbf{X}$. Moreover, it can be easily verified that Ψ is a monotone increasing mapping. Therefore, by the Knaster-Tarski fixed-point theorem (see [**11**]), we know that there exists a $z \in \mathbf{X}$ such that $\Psi z = z$, that is

$$z(t) = \begin{cases} \dfrac{r}{y(t)} \displaystyle\int_t^\infty \prod_{j=1}^m [e^{y(s-\tau_j)z(s-\tau_j)} - 1]^{\alpha_j}\, ds, & \text{for} \quad t \ge t_1, \\ \dfrac{t}{t_1}(\Psi z)(t_1) + \left(1 - \dfrac{t}{t_1}\right), & t_1 - \tau \le t \le t_1. \end{cases} \tag{3.4}$$

By (3.4), $z(t)$ is continuous on $[t_1 - \tau, \infty)$. Moreover, since $z(t) > 0$ for $t \in [t_1 - \tau, t_1)$, we must have $z(t) > 0$, for all $t \ge t_1$. Set $w(t) = y(t)z(t)$. Then $w(t)$ is positive, continuous on $[t_1 - \tau, \infty)$ and satisfies

$$w(t) = r \int_t^\infty \prod_{j=1}^m [e^{w(s-\tau_j)} - 1]^{\alpha_j}\, ds, \quad \text{for} \quad t \ge t_1. \tag{3.5}$$

Differentiating (3.5) yields

$$\frac{d}{dt} w(t) + r \prod_{j=1}^m [e^{w(t-\tau_j)} - 1]^{\alpha_j} = 0, \quad \text{for} \quad t \ge t_1,$$

which shows that $e^{w(t)} - 1$ is a positive solution of (1.3) on $[t_1, \infty)$. This completes the proof.

4. The case $\sum_{j=1}^m \alpha_j = 1$.

The following theorem establishes an equivalence between the oscillation of Eq. (1.3)–(1.4) and the oscillation of Eq. (1.5):

THEOREM 4.1. *When* $\sum_{j=1}^m \alpha_j = 1$, *every solution of Eq.* (1.3)–(1.4) *oscillates if and only if every solution of Eq.* (1.5) *oscillates.*

Proof. $\Rightarrow$: Assume that Eq. (1.5) has a non-oscillatory solution $y(t)$. Since $-y(t)$ is also a solution of Eq. (1.5), we may assume that $y(t)$ is eventually positive. We will prove that Eq. (1.3)–(1.4) has a non-oscillatory solution for some t_0. To this end, we only need to prove that the following equation

$$\frac{d}{dt} z(t) + r \prod_{j=1}^m (1 - e^{-z(t-\tau_j)})^{\alpha_j} = 0 \tag{4.2}$$

has an eventually positive solution. Let t_0 be such that $y(t - \tau) > 0$ for $t \ge t_0$. Using the inequality $1 - e^{-x} \le x$ for $x \ge 0$, we have

$$\frac{d}{dt} y(t) + r \prod_{j=1}^m (1 - e^{-y(t-\tau_j)})^{\alpha_j} \le \frac{d}{dt} y(t) + r \prod_{j=1}^m y^{\alpha_j}(t - \tau_j) = 0, \quad \text{for} \quad t \ge t_0. \tag{4.3}$$

It can be easily shown that $y(t) \to 0$, as $t \to \infty$. Integrating the above inequality from t to ∞, we obtain

$$y(t) \ge r \int_t^\infty \prod_{j=1}^m (1 - e^{-y(s-\tau_j)})^{\alpha_j}, \quad \text{for} \quad t \ge t_0.$$

Now a similar argument to the proof of Theorem 3.1 shows that (4.2) would have an eventually positive solution $z(t)$ on $[t_0, \infty)$ satisfying $z(t) > 0$ for all $t \ge t_0$.

$\Leftarrow$: Assume, for the sake of contradiction, that (1.3)–(1.4) has a non-oscillatory solution $x(t)$ for every t_0. Then $1+x(t)>0$, for $t \ge t_0$. We now distinguish two cases:

Case (*i*): $x(t)$ is eventually positive. Then there exists $T \ge t_0$ such that $x(t)>0$, for $t \ge T$. From (1.3) it follows that

$$\frac{d}{dt}x(t) + r\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \le \frac{d}{dt}x(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0. \tag{4.4}$$

This, together with Lemma 3.2, implies that (1.5) has a non-oscillatory solution, contrary to the assumption that every solution of (1.5) oscillates.

Case (*ii*): $x(t)$ is eventually negative. Since $1+x(t)>0$, for $t \ge t_0$, and $x(t)<0$ for $t \ge T$, for some $T \ge t_0$, we have

$$\frac{d}{dt}x(t) = r(1+x(t))\prod_{j=1}^{m}[-x(t-\tau_j)]^{\alpha_j} > 0, \quad \text{for} \quad t \ge T,$$

from which we can easily see that $x(t)\nearrow 0$ as $t \to \infty$. On the other hand, in view of Lemma 3.2, we can choose $\varepsilon \in (0,1)$ such that

$$r(1-\varepsilon)\sum_{j=1}^{m}\alpha_j\tau_j > \frac{1}{e}. \tag{4.5}$$

Now, let $T_1 > T$ be sufficiently large such that $1 > 1 + x(t) > 1 - \varepsilon$, for $t \ge T$. Then, by (1.3) we have

$$\frac{d}{dt}x(t) + r(1-\varepsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \ge \frac{d}{dt}x(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \quad \text{for} \quad t \ge T+\tau, \tag{4.6}$$

which is also a contradiction since, by Lemma 3.2, (4.5) implies that the inequality

$$\frac{d}{dt}x(t) + r(1-\varepsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \ge 0$$

can not have an eventually negative solution. This completes the proof.

The following corollary is an immediate result of Theorem 4.1 and Lemma 3.2.

COROLLARY 4.2. *If* $\sum_{j=1}^{m}\alpha_j = 1$, *then every solution of* (1.3)–(1.4) *oscillates* (*or every positive solution of* (1.1) *oscillates about the steady state* K) *if and only if*

$$r\sum_{j=1}^{m}\alpha_j\tau_j > \frac{1}{e}.$$

REFERENCES

1. O. W. G. Aliello, The existence of nonoscillatory solutions to a generalized nonautonomous delay logistic equation, *Math. Model.* **149** (1990), 114–123.

2. Ming-Po Chen, J. S. Yu, D. Zeng and J. W. Li, Global attractivity in a generalized nonautonomous delay logistic equation, *Bull. Inst. Math. Acad. Sinica* **22** (1994), 91–99.

3. M. E. Gilpin and F. J. Ayala, Global models of growth and competition, *Proc. Nat. Acad. Sci.* U.S.A. **70** (1973), 3590–3593.

4. K. Gopalsamy and B. S. Lalli, Oscillatory and asymptotic behavior of a multiplicative delay logistic equation, *Dynamics and Stability of Systems* **7** (1992), 35–42.

5. G. E. Hutchinson, Circular causal systems in ecology, *Ann. New York Acad. Sci.* **50** (1948), 221–246.

6. G. S. Jones, On the nonlinear differential difference equations $f(x) = -f(x-1)[1+f(x)]$, *J. Math. Anal. Appl.* **4** (1962), 440–469.

7. S. M. Lenhart and C. C. Travis, Global stability of a biological model with time delay, *Proc. Amer. Math. Soc.* **96** (1986), 75–78.

8. H. Onose, Oscillatory properties of the first order nonlinear advanced and delayed differential inequalities, *Nonlinear Anal.* **8** (1984), 171–180.

9. I. P. Stavroulakis, Nonlinear delay differential inequalities, *Nonlinear Anal.* **6** (1982), 382–396.

10. J. Sugie, On the stability for a population growth equation with time delay. *Proc. Roy. Soc. Edinburgh* **120A** (1992), 179–184.

11. A. Tarski, A lattice theoretical fixed-point theorem and its applications, *Pacific J. Math.* **5** (1955), 285–309.

12. Z. C. Wang, J. S. Yu and L. H. Huang, The nonoscillatory solutions of delay logistic equations, *Chinese J. Math.* **21** (1993), 81–90.

13. E. M. Wright, A nonlinear difference differential equation, *J. Reine Angew. Math.* **194** (1955), 66–87.

14. J. Yan, Oscillation of solutions of first order delay differential equations, *Nonlinear Anal.* **11** (1987), 1279–1287.

15. J. S. Yu, First order nonlinear differential inequalities with deviating arguments, *Acta Math. Sinica* **33** (1990), 152–159.

16. B. G. Zhang and K. Gopalsamy, Global attractivity in the delay logistic equation with variable parameters, *Math. Proc. Camb. Phil. Soc.* **107** (1990), 579–590.

YU
DEPARTMENT OF APPLIED MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA
HUNAN 410082, P.R. CHINA

WU AND ZOU
DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
NORTH YORK
ONTARIO
CANADA M3J 1P3