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Asymptotic Periodicity, Monotonicity, and Oscillation of Solutions of Scalar Neutral Functional Differential Equations

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We consider the periodic scalar neutral functional differential equation $(d/dt)[x(t) - c(t)x(t-\tau)] = -h(t, x(t)) + h(t-\sigma, x(t-\sigma))$, where c is continuously differentiable, h is increasing in its second argument, and both c and h are 1-periodic in the t-variable. The two time-lags τ and σ are not required to be the same. It is shown that, under certain conditions, (i) the set of 1-periodic solutions is an ordered arc and each solution is convergent to a periodic solution; (ii) the asymptotic and oscillatory behaviors of each solution are completely classified in terms of the value of the first integral at the initial condition. © 1996 Academic Press, Inc.

1. INTRODUCTION

Consider the following periodic scalar neutral functional differential equation

$$\frac{d}{dt}\left[x(t) - c(t)x(t-\tau)\right] = g(t, x(t), x(t-\sigma)), \tag{1.1}$$

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where τ and σ are nonnegative constants, and

(A) $c \in C^1(R; R)$, $|c(t)| < 1$, and $c(t+1) = c(t)$ for all $t \in R$;

(B) $g \in C(R^3; R)$ is locally Lipschitz continuous in its second and third arguments, strictly increasing in its third argument, and $g(t+1, x, y) = g(t, x, y)$ for $(t, x, y) \in R^3$.

Let $r = \max\{\tau, \sigma\}$ and let $C = C([-r, 0]; R)$ be the Banach space of continuous functions from $[-r, 0]$ into R with the norm $|\phi|_C = \sup_{-r \leq s \leq 0} |\phi(s)|$, $\phi \in C$. Then for each given $\phi \in C$, there exists $\gamma(\phi) > 0$ and a unique continuous mapping $x = x(\phi)$: $[-r, \gamma(\phi)) \to R$ such that $x(s) = \phi(s)$ for $s \in [-r, 0]$, $x(t) - c(t)x(t-\tau)$ is continuously differentiable and satisfies (1.1) on $[0, \gamma(\phi))$. Such a function is called the solution of (1.1) through ϕ an we refer to Hale [14] or Hale and Verduyn Lunel [15] for a detailed account of the fundamental existence−uniqueness theory.

To motivate the problem we are going to investigate and to provide a brief account of the history of the subject, let us start with the autonomous case where c is a constant and g is independent of t:

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = g(x(t), x(t-\sigma)). \tag{1.2}$$

We assume the following condition is satisfied:

(C) $g(x, x) = 0$ for all $x \in R$.

Prototypes of Eq. (1.2) satisfying condition (C) are

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -\sinh[x(t) - x(t-\sigma)] \tag{1.3}$$

and

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -h(x(t)) + h(x(t-\sigma)), \tag{1.4}$$

for a locally Lipschitz continuous and increasing function h. In the case where $c = 0$, Eq. (1.3) arises from the study of the motion of a classically radiating electron (see Kaplan, Sorg, and Yorke [17]), and Eq. (1.4) has been used as a model for some population growth, the spread of epidemics, and the dynamics of capital stocks. We refer to Cooke and Kaplan [5], Cooke and Yorke [6], and Győri [7] for more details. The additional term $cx(t-\tau)$ may be regarded as a certain feedback control mechanism which adjusts the change of the system according to its past growth rate, or may be viewed as the relapse of the infectious disease considered in the Cooke−Kaplan model of epidemics. In Győri and Wu [8], Eq. (1.4) was used to model the transmission dynamics of material in an active compartmental system with one compartment and one pipe coming out of and returning into the compartment, where the delay τ is the length of time required in the process in which some material is produced, while the delay σ represents the transit time for the material flow to pass through the pipe.

Note that while Eq. (1.3) and Eq. (1.4) both satisfy condition (C), only Eq. (1.4) has a first integral, that is,

$$x(t) - cx(t-\tau) + \int_{t-\sigma}^{t} h(x(s))\,ds = x(0) - cx(-\tau) + \int_{-\sigma}^{0} h(x(s))\,ds.$$

Furthermore, an immediate consequence of the assumption (C) is that *every constant function is a solution of* (1.2). This implies that the set of equilibria contains at least an ordered arc parametrized by the real line. This simple observation naturally motivates the following conjecture:

Conjecture. Every solution of (1.2) converges to a constant as $t \to \infty$.

In the case where $c = 0$, Eq. (1.2) reduces to a retarded functional differential equation for which the conjecture has been proven to be true. We refer to Haddock [9] and references therein for the proof and various important and useful technical tools developed in order to prove the conjecture. On the other hand, if c is nonzero, then a solution of (1.2) is not necessarily differentiable. This lack of differentiability is one of the sources for the difficulties in the qualitative analysis of neutral equations.

The above conjecture was first formulated by Haddock [9] in the case $\tau = \sigma$ in 1987 after the successful development inspired by the similar problem for retarded equations. In the case where $0 \le c < 1$ and $\tau = \sigma$, this conjecture was proved by Wu [28], using some elementary comparison method and differential inequalities technique. It was further confirmed by Haddock, Krisztin, Terjéki, and Wu [11], using an extension of the LaSalle's invariance principle of Razumikhin type, and by Haddock, Nkashama, and Wu [13], using a general convergence theorem for a semiflow defined on a function space where each constant function is an equilibrium and the semiflow preserves the ordering between a constant equilibrium and an arbitrary initial point (a slightly weaker order-preserving assumption than that frequently used in the literature of the theory of monotone dynamical systems). Also, in Wu and Freedman [31], the following nonstandard ordering was introduced for the phase space

$$\phi \le \psi \qquad \text{in } C \text{ if and only if } \phi(0) - c\phi(-\tau) \le \psi(0) - c\psi(-\tau),$$
$$\phi(s) \le \psi(s),\ s \in [-r, 0],$$

and it was shown that the solution semiflow defined by (1.2) is eventually strongly monotone with respect to this ordering. This enabled them to apply some general results due to Hirsch [16] to verify the conjecture.

The assumption $\tau = \sigma$ seems crucial in the aforementioned approaches. This is not surprising as the conjecture is no longer true if this assumption is not satisfied. For example, in Krisztin [18], it was observed that if $c = \sqrt{2} - 1$, $\tau = \pi/4$, $\sigma = 7\pi/4$, $h(x) = x$ then Eq. (1.4) does have a periodic solution $x = \sin t$. Some sufficient conditions guaranteeing the validity of the conjecture in the case where $\tau \neq \sigma$ were given in Krisztin and Wu [19], where we introduced a dense strongly ordered subspace X of C, defined by

$$X = \left\{\phi \in C^1([-r,0]; R) : \dot{\phi}(0) - c\dot{\phi}(-r) = -h(\phi(0)) + h(\phi(-\sigma))\right\},$$

endowed with the usual C^1-topology. For each $\mu \geq 0$, we followed Smith and Thieme [21] to define an order $\leq_\mu$ on X by

$$\phi \leq_\mu \psi \qquad \text{in } X \text{ iff } \phi(s) \leq \psi(s),\ \dot{\phi}(s) + \mu\phi(s) \leq \dot{\psi}(s) + \mu\psi(s), \quad s \in [-r,0].$$

We then showed that, under certain technical conditions on τ, σ, c, and h, the solutions of Eq. (1.2) define a strongly order-preserving semiflow (with respect to the order $\leq_\mu$) on X so that the powerful convergence and stability theory of monotone dynamical systems first developed by Hirsch [16] and Matano [20] and later improved by Smith and Thieme [22–24] can be applied to conclude that the solution semiflow is convergent (to constants). The drawback of this approach is that the convergence was only established for the solutions starting from a point in X. One could conclude the convergence of solutions starting from a point in C if the stability of each constant function (equilibrium) in the phase space C is known. Unfortunately, even the local stability of each constant function for system (1.2) is a quite complicated problem, as the set of equilibria (constant functions) is not a discrete set.

When both c and g depend on t periodically and when an analog of condition (C) is satisfied, one naturally ask if every solution of (1.1) converges to a periodic function. In Wu [29], an asymptotic periodicity theorem was established for the periodic neutral equation (1.1) under the assumption that $g(t, x, x) = 0$, $\tau = \sigma = 1$. More precisely, it was shown that for every $\phi \in C$, there exists a constant $k(\phi)$ such that $\lim_{t \to \infty}[x(\phi)(t) - k(\phi)/(1 - c(t))] = 0$. The asymptotic behaviors were investigated in a series of papers by Arino and his collaborators (see, for example, Arino and Bourad [2], Arino and Haourgui [3], and Arino and Seguier [4]) for the following scalar neutral equation

$$\frac{d}{dt}[x(t) - cx(t-\tau)] = -h(t, x(t)) + h(t-\sigma, x(t-\sigma)), \quad (1.5)$$

under the assumption that $\tau = \sigma$ and the orbits of h under the translations in time, $h_s(t, x) = h(t + s, x)$, $s \in R$, are precompact in the space of bounded continuous functions on R, for each fixed x. Their approach was based on the monotonicity of solutions and the existence of the first integral of (1.5) given by

$$J(t, \phi) = \phi(0) - c(t)\phi(-\tau) + \int_{-\sigma}^{0} h(t + s, \phi(s))\, ds.$$

They proved that the asymptotic behaviors of solutions of (1.5) are completely determined by the values of the first integral at the corresponding initial functions. In particular, they proved that if $J(0, \phi) = J(0, \psi)$ and if $x(0, \phi)(t)$ and $x(0, \psi)(t)$ are all bounded solutions on $[-r, \infty)$ then $\lim_{t \to \infty}[x(0, \psi)(t) - x(0, \phi)(t)] = 0$. While Wu's result applies to the general neutral equation (1.1) without first integrals, the approach of Arino and his collaborators could be applied to Eq. (1.5) (with a first integral) with almost-periodic coefficients. Again, both approaches depend heavily on the assumption that $\tau = \sigma$.

The purpose of this paper is to discuss the asymptotic periodicity of solutions of Eq. (1.5) when $\tau \neq \sigma$ and $h(t, x)$ is 1-periodic in t, increasing in x and $h(t, 0) = 0$. Our approach is based on a combination of the aforementioned methods and results and requires the monotone and eventual strong monotone property with respect to a nonstandard order $\leq_\mu$, the existence of a first integral, and the equivalence of (1.5) to an infinite delay differential equation. Since $\tau = \sigma$ is not assumed, there is no monotonicity with respect to the standard ordering of C and the ordering used by Wu and Freedman [31]. Following Smith and Thieme [21], we consider a dense subspace X (the space of Lipschitz continuous functions) of C and a cone with nonempty interior in X. Then the monotonicity and eventual monotonicity are shown in X with respect to the order relations $\leq_\mu$ and $\ll_\mu$ induced by the cone and its interior, respectively.

The main result is that if there exists $\mu > 0$ such that

$$0 \leq e^{\mu\tau}c(t) < 1$$

and

$$\mu + \min\{0, \dot{c}(t) - \mu c(t)\}e^{\mu\tau} > \sup_{(t, x)\in R^2} \frac{\partial}{\partial x}h(t, x),$$

then the following very detailed information about the structure of the solution set of (1.5) can be established:

(i) *Boundedness in* X. If $\phi \in X$ then $x(0, \phi)$ is bounded on $[-r, \infty)$.

(ii) *Structure of* 1-*Periodic Solutions*. For any $k \in R$ there exists a unique 1-periodic solution p^k such that $J(t, p_t^k) = k$ for all $t \in R$. Moreover, $k_1 < k_2$ implies $p_t^{k_1} \ll_\mu p_t^{k_2}$ for all $t \in R$.

(iii) *Asymptotic Periodicity in* C. If $\phi \in C$, $J(0, \phi) = k$, and $x(0, \phi)$ is bounded on $[-r, \infty)$, then $\lim_{t\to\infty}[x(0, \phi)(t) - p^k(t)] = 0$.

(iv) *Oscillation in* X. If $\phi \in X$ and $J(0, \phi) = k$, then $x(0, \phi)$ is oscillatory around p^k with respect to the ordering $\ll_\mu$, that is, neither $x_t(0, \phi) - p_t^k \gg_\mu 0$ nor $x_t(0, \phi) - p_t^k \ll_\mu 0$ holds for all large t.

Let us remark that the general theory of monotone dynamical systems can guarantee asymptotic periodicity only for a dense subset of initial values of the phase space which is X here. If we already know (ii), then a general convergence theorem of Takáč [25] or Wu [29] can give asymptotic periodicity in the whole X. In this paper we show more; namely, asymptotic periodicity holds for all of those initial values from C for which the corresponding solutions are bounded on $[-r, \infty)$.

2. MONOTONICITY IN A NONSTANDARD ORDERING

Let $r \geq \tau > 0$ be given constants, $C = C([-r, 0]; R)$ the Banach space of continuous functions from $[-r, 0]$ into R with the norm $|\phi|_C = \sup_{-r\leq s\leq 0}|\phi(s)|$, $\phi \in C$. For every $x \in C([-r, \infty); R)$ and every $t \geq 0$, define $x_t \in C$ by $x_t(s) = x(t + s)$, $-r \leq s \leq 0$.

Consider the following scalar neutral functional differential equation

$$\frac{d}{dt}[x(t) - c(t)x(t - \tau)] = f(t, x_t), \tag{2.1}$$

where

(i) $c \in C^1(R; R)$, $0 \leq c(t) < 1$ for all $t \in R$;

(ii) $f \in C(R \times C; R)$ and f is locally Lipschitz continuous in its second argument.

It is well known that the Cauchy initial value problem for (2.1) is well posed in the space C. In particular, for each $t_0 \in R$ and $\phi \in C$ there exists $\gamma = \gamma(t_0, \phi) > t_0$ and a unique continuous map $x : [t_0 - r, \gamma(t_0, \phi)) \to R$ such that $x_{t_0} = \phi$, $x(t) - c(t)x(t - \tau)$ is differentiable and (2.1) is satisfied on $[t_0, \gamma(t_0, \phi))$. We call this function the solution of (2.1) through (t_0, ϕ), denoted by $x(t_0, \phi)$, and call $[t_0 - r, \gamma(t_0, \phi))$ the maximal interval of existence of $x(t_0, \phi)$. It is known that if $\gamma(t_0, \phi) < \infty$, then $\lim_{t\to\gamma(t_0,\phi)-0}|x_{t_0}|_C = \infty$. We refer to Hale [14] for more details.

In order to describe a certain order-preserving property of the solutions of (2.1), we follow Smith and Thieme [21] and introduce a dense subspace of C. Let

$$X = \{\phi \in C : \phi \text{ is Lipschitz continuous on } [-r, 0]\}$$

and define

$$|\phi|_X = \max\left\{|\phi|_C, \sup\left\{\left|\frac{\phi(t) - \phi(s)}{t - s}\right| : -r \le s < t \le 0\right\}\right\}.$$

Then X is a Banach space. We also use

$$C_L[a,b] = \{x \in C([a,b]; R) : x \text{ is Lipschitz continuous on } [a,b]\},$$

$$|x|_{C_L[a,b]} = \max\left\{|x|_{C[a,b]}, \sup\left\{\left|\frac{x(t) - x(s)}{t - s}\right| : a \le s < t \le b\right\}\right\},$$

$$|x|_{C[a,b]} = \sup\{|x(s)| : a \le s \le b\}.$$

Let $\mu \ge 0$ be a given constant and define

$$K_\mu = \{\phi \in X : \phi(s) \ge 0 \text{ for } s \in [-r,0], \dot{\phi}(s) + \mu\phi(s) \ge 0 \text{ a.e. in } [-r,0]\}.$$

K_μ is a closed cone in X. Its interior can be given by

$$\operatorname{Int} K_\mu = \left\{\phi \in X : \phi(s) > 0 \text{ for } s \in [-r,0], \operatorname{ess\,inf}_{[-r,0]}(\dot{\phi} + \mu\phi) > 0\right\}.$$

Using the cone K_μ, we can define an order relation $\le_\mu$ in X such that $\phi \le_\mu \psi$ if $\psi - \phi \in K_\mu$. We will write $\phi \ll_\mu \psi$ if $\psi - \phi \in \operatorname{Int} K_\mu$. The relation $\phi <_\mu \psi$ will mean that $\phi \le_\mu \psi$ and $\phi \ne \psi$. An equivalent definition for $\phi \le_\mu \psi$ is that $\phi(s) \le \psi(s)$ and $e^{\mu s}[\psi(s) - \phi(s)]$ is nondecreasing for $s \in [-r, 0]$. We refer to Smith and Thieme [21] for detailed discussions of the space X and its ordering induced by K_μ.

LEMMA 2.1. *Let $T > 0$ be a given constant. If $\phi \in X$, $y \in C([-r,T]; R)$, and $h \in C_L[0,T]$ are given so that*

$$\begin{cases} y(t) - c(t)y(t-\tau) = h(t), & 0 \le t \le T, \\ y_0 = \phi, \end{cases}$$

then $y \in C_L[-r,T]$ and

$$|y|_{C_L[-r,T]} \le \left(\frac{1}{1-c_0} + \frac{c_1}{(1-c_0)^2}\right)\max\{|\phi|_X, |h|_{C_L[0,T]}\},$$

where $c_0 = sup\{c(t) : 0 \le t \le T\}$, $c_1 = sup\{|\dot{c}(t)| : 0 \le s \le T\}$.

Proof. The conclusion that $y \in C_L[-r,T]$ follows from the equality $y(t) = c(t)y(t-\tau) + h(t)$, $t \in [0,T]$, and the method of steps. The above equality also implies

$$|y(t)| \le c(t)|y(t-\tau)| + |h(t)|$$

and

$$\left|\frac{y(t)-y(s)}{t-s}\right| \le c(t)\left|\frac{y(t-\tau)-y(s-\tau)}{t-s}\right| + \left|\frac{c(t)-c(s)}{t-s}\right| |y(s-\tau)| + \left|\frac{h(t)-h(s)}{t-s}\right|$$

for $0 \le s < t \le T$. From the first inequality it follows that

$$|y|_{C[-r,T]} \le \max\left\{|\phi|_C, \frac{1}{1-c_0}|h|_{C[0,T]}\right\},$$

while the second implies

$$|y|_{Lip[-r,T]} \le \max\left\{|\phi|_{Lip[-r,T]}, \frac{c_1}{1-c_0}|y|_{C[-r,T]} + \frac{1}{1-c_0}|h|_{Lip[0,T]}\right\},$$

where $|x|_{Lip[a,b]}$ means $\sup\{|(x(t)-x(s))/(t-s)| : a \le s < t \le b\}$. The stated inequality for $|y|_{C_L[-r,T]}$ can be easily obtained from the estimation for $|y|_{C[-r,T]}$ and $|y|_{Lip[-r,T]}$. This completes the proof.

LEMMA 2.2. *Assume that* $(t_0, \phi) \in R \times X$ *is given and* $t \in [t_0, \gamma(t_0, \phi))$ *is fixed. Then* $x_t(t_0, \phi) \in X$ *and the mapping* $\phi \in X \mapsto x_t(t_0, \phi) \in X$ *is continuous.*

Proof. Let $x(t) = x(t_0, \phi)(t)$ for $t \in [t_0, \gamma(t_0, \phi))$. Integrating (2.1), we obtain

$$x(t) - c(t)x(t-\tau) = \phi(0) - c(t_0)\phi(-\tau) + \int_{t_0}^{t} f(s, x_s)\, ds.$$

Since the right-hand side of the above equality is differentiable, Lemma 2.1 implies that $x_t = x_t(t_0, \phi) \in X$. In order to show the continuity of $x_t(t_0, \phi)$ in $\phi \in X$, we let $\phi, \psi \in X$ be given and define $\gamma^* = \min\{\gamma(t_0, \phi), \gamma(t_0, \psi)\}$. Choose $T \in (t_0, \gamma^*)$ arbitrarily and let $u = x(t_0, \phi)$, $v = x(t_0, \psi)$, $z = u - v$. From the local Lipschitz continuity of f in its second argument and from the compactness of the set $\bigcup_{t\in[t_0,T]}\{u_t, v_t\}$ it follows that

$$|f(t,u_t) - f(t,v_t)| \le M|z_t|_C, \qquad t_0 \le t \le T$$

for some constant $M > 0$. Applying Lemma 2.1 with $y(t) = z(t)$, $h(t) = \phi(0) - c(t_0)\phi(-r) - \psi(0) + c(t_0)\psi(-\tau) + \int_{t_0}^{t}[f(s,u_s) - f(s,v_s)]\,ds$, we obtain

$$\begin{aligned} |z_T|_X &\le |z(\cdot)|_{C_L[-r,T]} \\ &\le \left(\frac{1}{1-c_0} + \frac{c_1}{(1-c_0)^2}\right)\max\Big\{|\phi - \psi)|_X, (1+c_0)|\phi-\psi|_C \\ &\qquad + M(T - t_0 + 1)\sup_{t_0\le t\le T}|z_t|_C\Big\}. \end{aligned}$$

Therefore, our conclusion follows from the continuity of $x_t(t_0,\phi)$ with respect to ϕ in the space C. This completes the proof.

Remark 2.1. H. Smith and T. Zhang pointed out to us that the mapping $t \mapsto x_t(t_0,\phi)$ is not necessarily continuous from R to X. So, even in the case where both c and f are independent of t, the scalar equation does not define a semiflow in the space X.

THEOREM 2.1. *Let* $D(t,\phi) = \phi(0) - c(t)\phi(-\tau)$ *for* $(t,\phi) \in R \times C$. *Assume* $e^{\mu\tau}c_0 < 1$ *and that the following condition is satisfied*:

(M) $\phi \le_\mu \psi$ *implies* $\mu[D(t,\psi) - D(t,\phi)] + f(t,\psi) - f(t,\phi) + \dot{c}(t)[\psi(-\tau) - \phi(-\tau)] \ge 0$.

Then we have

$$\phi \le_\mu \psi \qquad \textit{implies}\; x_t(t_0,\phi) \le_\mu x_t(t_0,\psi)\; \textit{for}\; t \ge t_0.$$

Proof. For any $\epsilon > 0$ define $f_\epsilon(t,\phi) = \epsilon D(t,\phi) + f(t,\phi)$ and denote by $x(t_0,\phi,\epsilon)(t)$ the unique solution of

$$\begin{cases} \dfrac{d}{dt}D(t,x_t) = f_\epsilon(t,\phi), \\ x_{t_0} = \phi. \end{cases}$$

By the well-known continuous dependence on initial data and the right-hand functionals of solutions to neutral equations on the space C, for every fixed positive $T < \min\{\gamma(t_0, \phi), \gamma(t_0, \psi)\}$, there exists $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ then $x(t_0, \phi, \epsilon)(t)$ and $x(t_0, \psi, \epsilon)(t)$ exist on $[0, T]$.

Let $t_1 \in [0, T]$ be the maximal time such that $x(t) \le y(t)$ on $[0, t_1]$ and $e^{\mu t}[y(t) - x(t)]$ is nondecreasing on $[-r, t_1]$, where $x(t) = x(t_0, \phi, \epsilon)(t)$, $y(t) = x(t_0, \psi, \epsilon)(t)$. We want to show that $t_1 = T$.

By way of contradiction, we assume $t_1 < T$. If $x(t_1) = y(t_1)$, then

$$0 = e^{\mu t_1}[y(t_1) - x(t_1)] \ge e^{\mu t}[y(t) - x(t)] \ge 0$$

for all $t \in [-r, t_1]$. Therefore, $\phi = \psi$. Hence, by uniqueness, t_1 is not the maximal time satisfying the stated properties. So, $y(t_1) > x(t_1)$. Let $z(t) = y(t) - x(t)$. By condition (M), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\mu t}D(t, z_t)\right)\Big|_{t=t_1} + e^{\mu t_1}\dot{c}(t_1)z(t_1 - \tau) \\
&\quad = e^{\mu t_1}\left[\mu D(t_1, z_{t_1}) + f(t_1, y_{t_1}) - f(t_1, x_{t_1}) + \dot{c}(t_1)z(t_1 - \tau)\right] \\
&\qquad + \epsilon e^{\mu t_1}D(t_1, z_1) \\
&\quad \ge \epsilon e^{\mu t_1}D(t_1, z_{t_1}) \\
&\quad = \epsilon e^{\mu t_1}\left[z(t_1) - c(t_1)e^{\mu\tau}e^{-\mu\tau}z(t_1 - \tau)\right] \\
&\quad \ge \epsilon e^{\mu t_1}z(t_1)\left[1 - c(t_1)e^{\mu\tau}\right] > 0.
\end{aligned}$$

Therefore, since $(d/dt)(e^{\mu t}D(t, z_t)) + e^{\mu t}\dot{c}(t)z(t - \tau)$ is continuous at $t = t_1$ (continuous from the right if $t_1 = 0$), there exists $\alpha > 0$ and $\delta > 0$ such that $\delta < \tau$ and

$$\frac{d}{dt}\left(e^{\mu t}D(t, z_t)\right) + e^{\mu t}\dot{c}(t)z(t - \tau) \ge \alpha, \qquad t_1 \le t \le t_1 + \delta.$$

Thus, integrating from s to t, we get

$$\begin{aligned}
&e^{\mu t}D(t, z_t) - e^{\mu s}D(s, z_s) + e^{\mu t}c(t)z(t - \tau) - e^{\mu s}c(s)z(s - \tau) \\
&\qquad - \int_s^t c(u)\frac{d}{du}(e^{\mu u}z(u - \tau))\,du \\
&\quad \ge \alpha(t - s), \qquad t_1 \le s \le t \le t_1 + \delta.
\end{aligned}$$

That is,

$$e^{\mu t}z(t) - e^{\mu s}z(s) \ge \alpha(t - s) + \int_s^t c(u)e^{\mu\tau}\frac{d}{du}\left[e^{\mu(u-\tau)}z(u - \tau)\right]du$$

for $t_1 \leq s \leq t \leq t_1 + \delta$. We know that $e^{\mu t}z(t)$ is nondecreasing on $[-r, t_1]$ and $t_1 + \delta - \tau < t_1$. Therefore, the integral is nonnegative and $e^{\mu t}z(t) - e^{\mu s}z(s) \geq \alpha(t - s)$, $t_1 \leq s \leq t \leq t_1 + \delta$ follows. This shows that $e^{\mu t}z(t)$ is increasing on $[t_1, t_1 + \delta]$, contradicting the maximality of t_1.

So, $x_t \leq_\mu y_t$ on $[t_0, T]$. Letting $\epsilon \to 0$, we obtain the monotonicity and complete the proof.

THEOREM 2.2. *Assume* $e^{\mu\tau}c_0 < 1$ *and that the following condition is satisfied*:

(*SM*) $\phi <_\mu \psi$ *implies* $\mu[D(t, \psi) - D(t, \phi)] + f(t, \psi) - f(t, \phi) + \dot{c}(t)[\psi(-\tau) - \phi(-\tau)] > 0$.

Then we have

$$\phi <_\mu \psi \qquad \textit{implies}\; x_t(t_0, \psi) \ll_\mu x_t(t_0, \phi) \;\textit{for}\; t \geq t_0 + r.$$

Proof. Let $x(t) = x(t_0, \phi)(t)$, $y(t) = x(t_0, \psi)(t)$, $z(t) = y(t) - x(t)$. By Theorem 2.1, if $\phi <_\mu \psi$ then $x_t \leq_\mu y_t$ for $t \geq t_0$. It also follows that $x(t) < y(t)$ for $t \geq t_0$. Thus $x_t <_\mu y_t$ for $t \geq t_0$. Let $T \geq t_0 + r$ be fixed. By condition (SM) and the compactness of $\bigcup_{t_0 \leq t \leq T}\{x_t, y_t\}$ in C, we have

$$\beta_0 = \min_{t_0 \leq t \leq T}\{\mu D(t, z_t) + f(t, y_t) - f(t, x_t) + \dot{c}(t)z(t - \tau)\} > 0.$$

So

$$\frac{d}{dt}D(t, z_t) + \mu D(t, z_t) + \dot{c}(t)z(t - \tau) \geq \beta_0 > 0, \qquad t_0 \leq t \leq T.$$

That is,

$$\dot{z}(t) + \mu z(t) - c(t)[\dot{z}(t - \tau) + \mu z(t - \tau)] \geq \beta_0 > 0$$

a.e. on $[t_0, T]$. As $\dot{z}(t - \tau) + \mu z(t - \tau) \geq 0$ a.e. on $[t_0, T]$, we obtain $\dot{z}(t) + \mu z(t) \geq \beta_0 > 0$ a.e. on $[t_0, T]$, that is, $x_t \ll_\mu y_t$ for $t \in [t_0 + r, T]$. The proof is complete.

EXAMPLE 2.1. For illustrative purpose, let us now consider the following scalar neutral functional differential equation

$$\frac{d}{dt}[x(t) - c(t)x(t - \tau)] = g(t, x(t), x(t - \sigma)), \qquad (2.2)$$

where $0 \leq c(t) < 1$ for $t \in R$, $\tau > 0$, $\sigma \geq 0$, $g : R^3 \to R$ is locally Lipschitz continuous in its second and third arguments, and

$$L_1 = \inf_{(t,x,y)\in R^3} \frac{\partial g(t, x, y)}{\partial x} > -\infty, \qquad L_2 = \inf_{(t,x,y)\in R^3} \frac{\partial g(t, x, y)}{\partial y} > -\infty.$$

Then g satisfies the following one-sided global Lipschitz condition in x and y:

(L) If $x_1 \leq x_2$ and $y_1 \leq y_2$, then $g(t, x_2, y_2) - g(t, x_1, y_1) \geq L_1(x_2 - x_1) + L_2(y_2 - y_1)$.

For the above equation, $D(t, \phi) = \phi(0) - c(t)\phi(-\tau)$, $f(t, \phi) = g(t, \phi(0), \phi(-\sigma))$ for $\phi \in C([-r, 0]; R)$, where $r = \max\{\tau, \sigma\}$. We now fix $\mu \geq 0$. Then for any $\psi \geq_\mu \phi$, we have

$$\begin{aligned}
\mu[D(t,\psi) - D(t,\phi)] &+ f(\psi) - f(\phi) + \dot{c}(t)[\psi(-\tau) - \phi(-\tau)] \\
&\geq \mu[\psi(0) - \phi(0)] - c(t)\mu[\psi(-\tau) - \phi(-\tau)] \\
&\quad + \dot{c}(t)[\psi(-\tau) - \phi(-\tau)] \\
&\quad + L_1[\psi(0) - \phi(0)] + L_2[\psi(-\sigma) - \phi(-\sigma)] \\
&= (\mu + L_1)[\psi(0) - \phi(0)] + [\dot{c}(t) - c(t)\mu][\psi(-\tau) - \phi(-\tau)] \\
&\quad + L_2[\psi(-\sigma) - \phi(-\sigma)].
\end{aligned}$$

Note that $\psi \geq_\mu \phi$ implies that

$$\psi(0) - \phi(0) \geq [\psi(-\theta) - \phi(-\theta)]e^{-\mu\theta}, \qquad \theta = \tau, \sigma. \tag{2.3}$$

Let $\alpha^- = -\min\{\alpha, 0\}$ for a real number α. Then for $\phi \leq_\mu \psi$, we have

$$\begin{aligned}
\mu[D(t,\psi) - D(t,\phi)] &+ f(t,\psi) - f(t,\phi) + \dot{c}(t)[\psi(-\tau) - \phi(-\tau)] \\
&\geq (\mu + L_1)[\psi(0) - \phi(0)] + [\dot{c}(t) - c(t)\mu] \\
&\quad \times [\psi(-\tau) - \phi(-\tau)] - L_2^-[\psi(-\sigma) - \phi(-\sigma)] \\
&\geq \left[\mu + L_1 - [\dot{c}(t) - c(t)\mu]^- e^{\mu\tau} - L_2^- e^{\mu\sigma}\right][\psi(0) - \phi(0)].
\end{aligned}$$

Therefore, condition (M) holds if

$$\mu + L_1 - [\dot{c}(t) - c(t)\mu]^- e^{\mu\tau} - L_2^- e^{\mu\sigma} \geq 0. \tag{2.4}$$

Notice also that if $\psi >_\mu \phi$ then $\psi(0) > \phi(0)$ and hence (SM) is satisfied if

$$\mu + L_1 - [\dot{c}(t) - c(t)\mu]^- e^{\mu\tau} - L_2^- e^{\mu\sigma} > 0. \tag{2.5}$$

Other sets of sufficient conditions to guarantee (M) or (SM) can be given, using different combinations of (2.3) in the estimations of $(\mu + L_1)[\psi(0) - \phi(0)] + [\dot{c}(t) - c(t)\mu][\psi(-\tau) - \phi(-\tau)] + L_2[\psi(-\sigma) - \phi(-\sigma)]$.

3. OSCILLATION AND ASYMPTOTIC PERIODICITY

We now consider the following scalar neutral functional differential equation

$$\frac{d}{dt}[x(t) - c(t)x(t-\tau)] = -h(t, x(t)) + h(t-\sigma, x(t-\sigma)), \quad (3.1)$$

where

(C1) $\tau > 0$, $\sigma \geq 0$;

(C2) $c \in C^1(R; R)$, $0 \leq c(t) < 1$, $c(t+1) = c(t)$ for $t \in R$;

(C3) $h \in C(R^2; R)$, $h(t, x) = h(t+1, x)$, $h(t, 0) = 0$, $(\partial/\partial x)h(t, x)$ is continuous, and $h(t, x)$ is increasing in x for all $(t, x) \in R^2$.

Comparing with Eq. (2.2) in Example 2.1, we have

$$g(t, x, y) = -h(t, x) + h(t-\sigma, y).$$

So

$$L_1 = -\sup_{(t, x) \in R^2} \frac{\partial}{\partial x} h(t, x),$$

$$L_2 = \inf_{(t, x) \in R^2} \frac{\partial}{\partial x} h(t, x).$$

We assume

(C4) $\sup_{(t, x) \in R^2} (\partial/\partial x) h(t, x) < \infty$;

(C5) there exists $\mu > 0$ such that $\mu - [\dot{c}(t) - c(t)\mu]^- e^{\mu\tau} > \sup_{(t, x) \in R^2} (\partial/\partial x) h(t, x)$.

Clearly, (C5) holds if $\dot{c}(t)$ and $c(t)$ are sufficiently small. Note that (C5) is exactly the condition (2.5). So if we further assume

(C6) $e^{\mu\tau} c_0 < 1$, $c_0 = \sup\{c(t); 0 \leq t \leq 1\}$;

then if $\phi <_\mu \psi$ and $t_0 \in R$ are given, then $x_t(t_0, \phi) \leq_\mu x_t(t_0, \psi)$ for $t \geq t_0$ and $x_t(t_0, \phi) \ll_\mu x_t(t_0, \psi)$ for $t \geq t_0 + r$. Throughout the remainder of this section, we assume (C1)–(C6) are satisfied.

The main result of this paper is the next theorem.

THEOREM 3.1. *Assume that conditions* (C1)–(C6) *are satisfied. Then the solutions of* (3.1) *have the following properties*:

(i) *If* $\phi \in X$ *then* $x(0, \phi)$ *exists on* $[-r, \infty)$ *and the set* $\{x_t(0, \phi) : t \geq 0\}$ *is bounded in* X.

(ii) *For any* $k \in R$ *there is a unique* 1-*periodic solution* p^k *of* (3.1) *such that* $J(t, p_t^k) = k$.

(iii) *If* $\phi \in C$, $J(0, \phi) = k$, $x(0, \phi)$ *is a solution of* (3.1) *on* $[-r, \infty)$ *and the set* $\{x_t(0, \phi) : t \geq 0\}$ *is bounded in* C, *then* $\lim_{t \to \infty}[x(0, \phi)(t) - p^k(t)] = 0$.

(iv) *If* $\phi \in X$ *and* $J(0, \phi) = k$, *then* $x(0, \phi)$ *is oscillatory around* p^k *with respect to the ordering* $\ll_\mu$, *i.e., neither* $x_t(0, \phi) - p_t^k \gg_\mu 0$ *nor* $x_t(0, \phi) - p_t^k \ll_\mu 0$ *holds for all large* t.

In the remaining part of this section we prove Theorem 3.1. The proof is contained in several lemmas. The monotonicity, which is one of the main ingredients of the proof, was considered in Section 2. The other basic tool is that Eq. (3.1) has a first integral, that is, for any solution $x = x(0, \phi)$ of (3.1) defined on $[-r, \infty)$, we have

$$J(t, x_t) := x(t) - c(t)x(t - \tau) + \int_{-\sigma}^{0} h(t + s, x(t + s))\, ds$$
$$= J(0, \phi), \qquad t \geq 0. \tag{3.2}$$

It is easy to show that if $\phi <_\mu \psi$ then $J(t, \phi) < J(t, \psi)$ for all $t \in R$. In fact, $\phi <_\mu \psi$ implies $\phi(s) \leq \psi(s)$ and $e^{\mu s}[\psi(s) - \phi(s)] \leq \psi(0) - \phi(0)$ for $s \in [-r, 0]$, moreover $\phi(0) < \psi(0)$. Therefore, the monotonicity of h in its second argument implies that

$$J(t, \psi) - J(t, \phi) \geq \psi(0) - \phi(0) - c_0[\psi(-\tau) - \phi(-\tau)]$$
$$\geq [1 - c_0 e^{\mu\tau}][\psi(0) - \phi(0)] > 0.$$

Now we can show the boundedness statement of Theorem 3.1.

LEMMA 3.1. *If* $\phi \in X$ *then* $x(0, \phi)$ *exists on* $[-r, \infty)$ *and the set* $\{x_t(0, \phi) : t \geq 0\}$ *is bounded in* X.

Proof. The zero function is a solution of (3.1) since $h(t, 0) = 0$. If $\phi >_\mu 0$, then $x(0, \phi)(t) \geq 0$ for all $t \in [0, \gamma(\phi))$ by Theorem 2.1. So from (3.2) and the monotonicity of h in its second argument it follows that $\sup_{-r \leq t < \gamma(\phi)} |x(t)| \leq |J(0, \phi)|/(1 - c_0)$. Thus, $\gamma(\phi) = \infty$ and $x(0, \phi)(t)$ exists on $[-r, \infty)$ and is bounded. A similar result holds if $\phi <_\mu 0$. For arbitrary $\phi \in X$, we can choose constant functions m, M in X such that $m <_\mu 0 <_\mu M$ and $m <_\mu \phi <_\mu M$. So, $x(0, m)$ and $x(0, M)$ are bounded functions from $[-r, \infty)$ into R. The inequality

$$x(0, m)(t) \leq x(0, \phi)(t) \leq x(0, M)(t)$$

also holds for all $t \geq -r$ such that $x(0, \phi)(t)$ exists. Therefore, we obtain the global existence of $x(0, \phi)$ and the boundedness of $\{x_t(0, \phi) : t \geq 0\}$

in C. Boundedness of $\{x_t(0,\phi): t \geq 0\}$ in X comes from Lemma 2.1 with $y(t) = x(0,\phi)(t)$, $h(t) = y(0) - c(0)y(-\tau) - \int_0^t h(s, y(s))\,ds + \int_0^t h(s-\sigma, y(s-\sigma))\,ds$ and the fact that $h \in C_L[0,\infty)$.

We now derive an important integral equation for bounded solutions of (3.1). First, note that if x is a solution of (3.1) on $[-r,\infty)$, then (3.2) holds. If $t \geq \tau$, then (3.2) implies

$$c(t)x(t-\tau) - c(t)c(t-\tau)x(t-2\tau) + c(t)\int_{t-\sigma-\tau}^{t-\tau} h(s,x(s))\,ds$$
$$= J(0,\phi)c(t).$$

If $t \geq 2\tau$, then

$$c(t)c(t-\tau)x(t-2\tau) - c(t)c(t-\tau)c(t-2\tau)x(t-3\tau)$$
$$+ c(t)c(t-\tau)\int_{t-\sigma-2\tau}^{t-2\tau} h(s,x(s))\,ds = J(0,\phi)c(t)c(t-\tau).$$

Repeating the same argument, we obtain for any nonnegative integer $j \leq [t/\tau]$ the following

$$\left(\prod_{i=0}^{j-1} c(t-i\tau)\right)x(t-j\tau) - \left(\prod_{i=0}^{j} c(t-i\tau)\right)x(t-(j+1)\tau)$$
$$+ \left(\prod_{i=0}^{j-1} c(t-i\tau)\right)\int_{t-\sigma-j\tau}^{t-j\tau} h(s,x(s))\,ds = J(0,\phi)\prod_{i=0}^{j-1} c(t-i\tau),$$

where $\prod_{i=0}^{-1} c(t-i\tau) = 1$. Summarizing up in the above equation as j goes from 0 to $[t/\tau]$, we get

$$x(t) + \sum_{j=0}^{[t/\tau]}\left(\prod_{i=0}^{j-1} c(t-i\tau)\right)\int_{t-\sigma-j\tau}^{t-j\tau} h(s,x(s))\,ds$$
$$-\left(\prod_{i=0}^{[t/\tau]} c(t-i\tau)\right)x(t-([t/\tau]+1)\tau)$$
$$= J(0,\phi)\sum_{j=0}^{[t/\tau]}\prod_{i=0}^{j-1} c(t-i\tau). \tag{3.3}$$

If x is a bounded solution on $(-\infty,\infty)$, then similar arguments lead to

$$x(t) + \sum_{j=0}^{\infty}\left(\prod_{i=0}^{j-1} c(t-i\tau)\right)\int_{t-\sigma-j\tau}^{t-j\tau} h(s,x(s))\,ds$$
$$= J(0,\phi)\sum_{j=0}^{\infty}\prod_{i=0}^{j-1} c(t-i\tau). \tag{3.4}$$

Let us define

$$\mathscr{N} = \left\{ \phi \in X : \text{either } x_t(0,\phi) \in \operatorname{Int} K_\mu \text{ or } -x_t(0,\phi) \in \operatorname{Int} K_\mu \text{ for sufficiently large } t \right\}$$

and $\mathscr{O} = X \setminus \mathscr{N}$. Clearly, by Lemma 2.2 and Theorem 2.2, $\mathscr{N}$ is open and $\mathscr{O}$ is closed. The solution $x_t(0,\phi)$ with $\phi \in \mathscr{N} (\phi \in \mathscr{O})$ will be called *nonoscillatory* (*oscillatory*).

The following result gives a characterization of oscillatory solutions in terms of the first integral.

LEMMA 3.2. *$J(0,\phi) = 0$ for some $\phi \in X$ if and only if $\phi \in \mathscr{O}$.*

Proof. We prove the equivalent statement: $J(0,\phi) \neq 0$ if and only if $\phi \in \mathscr{N}$.

If $\phi \in \mathscr{N}$ then, without loss of generality, we may assume that for sufficiently large t, $x(t) > 0$ and $e^{\mu t} x(t)$ is increasing, where $x = x(0,\phi)$. In particular, $x(t) > e^{-\mu\tau} x(t-\tau)$ follows. If $J(0,\phi) = 0$, then (3.2) and $h(t, x(t)) > 0$ imply that $x(t) < c(t) x(t-\tau)$ for all large t. Thus $e^{-\mu\tau} < c(t)$ if t is large, a contradiction to (C6). Therefore, $\phi \in \mathscr{N}$ implies $J(0,\phi) \neq 0$.

Now we show that $J(0,\phi) > 0$ implies $\phi \in \mathscr{N}$. The case $J(0,\phi) < 0$ is analogous. By way of contradiction, suppose that $J(0,\phi) > 0$ and $\phi \in \mathscr{O}$. Choose a constant function M from X such that $M >_\mu 0$ and $M >_\mu \phi$. We have $M \in \mathscr{N}$ because of $h(t,0) = 0$ and Theorem 2.2. Define a continuous function $H : [0,1] \to X$ by $H(\lambda) = \lambda M + (1-\lambda)\phi$. Then $H(1) = M \in \mathscr{N}$ and $H(0) = \phi \in \mathscr{O}$. It is easy to see that $H(\lambda)$ is monotone with respect to $<_\mu$. As $\mathscr{O}$ is closed, there is a $\lambda_0 \in [0,1)$ such that $H(\lambda_0) \in \mathscr{O}$ and $H(\lambda) \in \mathscr{N}$ if $\lambda \in (\lambda_0, 1]$. Choose a sequence $\{\lambda_n\}_{n=1}^\infty$ such that $\{\lambda_n\}$ is strictly decreasing and $\lambda_n \to \lambda_0$ as $n \to \infty$. Let $\phi_n = H(\lambda_n)$, $x^n = x(0,\phi_n)$, $J_n = J(0,\phi_n)$, $n = 0,1,2,\ldots$. From the monotonicity of $H(\lambda)$ it follows that $\{J_n\}_{n=1}^\infty$ tends to J_0 in a strictly decreasing way. Since $\phi_n \in \mathscr{N}$ for $n = 1,2,\ldots,$ for each $n \geq 1$ there exists T_n such that $x^n(t) > 0$ and $\dot{x}^n(t) + \mu x^n(t) > 0$ a.e. on $[T_n, \infty)$. In particular, $x^n(t_2) > e^{-\mu(t_2 - t_1)} x^n(t_1)$ follows for $T_n \leq t_1 < t_2 < \infty$. From $x^n(t) - c(t) x^n(t-\tau) + \int_{t-\sigma}^t h(s, x^n(s))\, ds = J_n > J_0 > 0$ and from $x^n(t) > 0$ for $t \geq T_n$, it follows that there exists $\beta_0 > 0$ such that $\max_{t-\sigma \leq s \leq t} x^n(s) \geq \beta_0$ for $t \geq T_n + r$. Then we get $x^n(t) > e^{-2\mu\sigma}\beta_0$ for all $t \geq T_n + r$. The constant β_0 can be chosen independently of n. Thus, we have $x^n(t) \geq \beta > \infty$ for all $t \geq T_n + r$, where $\beta = e^{-2\mu\sigma}\beta_0$.

Next we want to show that $\liminf_{t\to\infty} x^0(t) \geq \beta$. From (3.3) we obtain

$$x^n(t) - x^0(t) + \sum_{j=0}^{[t/\tau]} \left(\prod_{i=0}^{j-1} c(t-i\tau) \right) \times \int_{t-\sigma-j\tau}^{t-j\tau} \left[h(s, x^n(s)) - h(s, x^0(s)) \right] ds$$
$$= (J_n - J_0) \sum_{j=0}^{[t/\tau]} \prod_{i=0}^{k-1} c(t-i\tau) + \left(\prod_{i=0}^{[t/\tau]} c(t-i\tau) \right) \left[x^n(t - ([t/\tau]+1)\tau) - x^0(t - ([t/\tau]+1)\tau) \right].$$

For any $\epsilon > 0$ one can find t_0 and n_0 such that for $t \geq t_0$ and $n \geq n_0$ the right-hand side is less than ϵ. We know from monotonicity that all terms of both sides are nonnegative. Consequently, $x^n(t) - x^0(t) < \epsilon$ for all $t \geq t_0$, $n \geq n_0$. If $t \geq T_n + r$ also holds, then $x^0(t) > \beta - \epsilon$ follows. Therefore, $\liminf_{t\to\infty} x^0(t) \geq \beta$ follows because $\epsilon > 0$ was arbitrary.

We cannot have $x_t^0 \in K_\mu$ for any $t \geq 0$ because of $x_t^0 \neq 0$ and Theorem 2.2. We already know that $x^0(t) > 0$ for all large t. Thus one can find sequences $\{t_n\}$ and $\{s_n\}$ such that $t_n \to \infty$, $t_n \geq T_n + n$, $s_n \in [-r, 0]$, and

$$x^0(t_n) < e^{\mu s_n} x^0(t_n + s_n).$$

Lemma 2.1 implies that the sequence of functions $x^0(t + t_n)$ is equicontinuous on each compact subinterval of R. Thus, by the Arzela–Ascoli theorem, there are subsequences of $\{t_n\}$ and $\{s_n\}$ (denoted again by $\{t_n\}$ and $\{s_n\}$) such that $s_n \to s^* \in [-r, 0]$ and

$$x^0(t + t_n) \to y(t), \qquad n \to \infty$$

uniformly in t on each compact subinterval of R. The limit function y satisfies

$$y(t) - c(t+\omega) y(t-\tau) + \int_{t+\omega-\sigma}^{t+\omega} h(s, y(s))\, ds = J_0$$

for all $t \in R$ and for some $\omega \in [0, 1]$, and also

$$y(0) \leq e^{\mu s^*} y(s^*).$$

By using the diagonalization process, a subsequence of $\{t_n\}$ (denoted again by $\{t_n\}$) can be chosen such that

$$x^n(t + t_n) \to z(t), \qquad n \to \infty,$$

uniformly in t on each compact subinterval of R. The function z also satisfies

$$z(t) - c(t + \omega)z(t - \tau) + \int_{t+\omega-\sigma}^{t+\omega} h(s, z(s))\, ds = J_0$$

because $J_n \to J_0$. From $t_n \geq T_n + n$ it also follows that $e^{\mu t}z(t)$ is monotone nondecreasing on R. Then $z_t \in K_\mu$ for all $t \in R$. By Theorem 2.2, $z_t \in \operatorname{Int} K_\mu$ for all $t \in R$. That is, $e^{\mu t}z(t)$ is strictly increasing on R. From $x^n(t) \geq x^0(t)$, $t \geq -r$, we get $z(t) \geq y(t)$ for all $t \in R$. Equation (3.4) holds for both y and z with $t + \omega$ instead of t and $J(0, \phi) = J_0$. Subtracting these two equations, we obtain

$$z(t) - y(t) + \sum_{j=0}^{\infty} \left(\prod_{i=0}^{j-1} c(t + \omega - i\tau) \right)$$

$$\times \int_{t+\omega-\sigma-j\tau}^{t+\omega-j\tau} [h(s, z(s)) - h(s, y(s))]\, ds = 0, \qquad t \in R.$$

This implies $z \equiv y$ since $h(s, \cdot)$ is increasing. Therefore, $e^{\mu t}y(t)$ is also strictly increasing, contradicting $y(0) \leq e^{\mu s^*}y(s^*)$. This completes the proof.

The following result shows that each oscillatory solution converges to zero.

LEMMA 3.3. *If* $\phi \in X$ *and* $J(0, \phi) = 0$ *then* $x(0, \phi)(t) \to 0$ *as* $t \to \infty$.

Proof. In the same way as in the proof of Lemma 3.2, one can find a sequence $\{\phi_n\}_{n=0}^{\infty}$ in X such that $\phi \leq_\mu \leq \phi_0 <_\mu \cdots <_\mu \phi_n <_\mu \cdots <_\mu \phi_2 <_\mu \phi_1$, $\phi_0 \in \mathscr{O}$, $\phi_1, \phi_2 \cdots \in \mathscr{N}$, $\phi_n \to \phi_0$, $n \to \infty$. By Lemma 3.2, $J(0, \phi_0) = 0$. From $\phi_n \to \phi_0$, $J(0, \phi_n) \to J(0, \phi_0)$ follows. If t is large enough, then $x(0, \phi_n)(t) > 0$ and $e^{\mu t}x(0, \phi_n)(t)$ is increasing. Hence it follows that $J(0, \phi_n) \geq x(0, \phi_n)(t) - c(t)x(0, \phi_n)(t - \tau) \geq x(0, \phi_n)(t)[1 - c(t)e^{\mu\tau}]$ for all large t. By the monotonicity, $x(0, \phi)(t) \leq x(0, \phi_0)(t) \leq x(0, \phi_n)(t)$, $t \geq -r$. Using $J(0, \phi_n) \to 0$, we get $\limsup_{t\to\infty} x(0, \phi)(t) \leq 0$. Analogously, one can show that $\liminf_{t\to\infty} x(0, \phi(t)) \geq 0$. The proof is complete.

We now consider the structure of the set of 1-periodic solutions of (3.1). The next lemma says that this set is an ordered arc parametrized by R.

LEMMA 3.4. *For any* $k \in R$, (3.1) *has one and only one* 1-*periodic solution* $p = p^k$ *with* $J(0, p_0^k) = k$. *Moreover, if* $k_1 < k_2$ *then* $p_t^{k_1} \ll_\mu p_t^{k_2}$ *for all* $t \in R$.

Proof. Denote by P the set of 1-periodic continuous functions endowed with the supremum norm. Consider the operator G defined in P by

$$G(x)(t) = \int_{t-\sigma}^{t} h(s, x(s))\, ds + c(t) \int_{t-\sigma-\tau}^{t-\tau} h(s, x(s))\, ds + c(t)c(t-\tau) \int_{t-\sigma-2\tau}^{t-2\tau} h(s, x(s))\, ds + \cdots.$$

Let $p(t) = 1 + c(t) + c(t)c(t-\tau) + \cdots$. We have $p \in P$. By (3.4), the solutions x of (3.1) which are 1-periodic with $k = J(t, x_t)$ are exactly the points x in P such that $(Id + G)(x) = kp$.

We now show that the derivative $Id + K$ of $Id + G$ at any $\phi \in P$ is an isomorphism, where

$$K\psi(t) = \int_{t-\sigma}^{t} \frac{\partial}{\partial x} h(s, \phi(s))\psi(s)\, ds + c(t) \int_{t-\sigma-\tau}^{t-\tau} \frac{\partial}{\partial x} h(s, \phi(s))\psi(s)\, ds + c(t)c(t-\tau) \int_{t-\sigma-2\tau}^{t-2\tau} \frac{\partial}{\partial x} h(s, \phi(s))\psi(s)\, ds + \cdots.$$

It is easy to see that $K : P \to P$ is compact. Thus, $Id + K$ is an isomorphism if it is injective. It is enough to show that if $(Id + K)y(t) = 0$ for $t \in R$ with a 1-periodic y, that is,

$$y(t) + \int_{t-\sigma}^{t} \frac{\partial}{\partial x} h(s, \phi(s)) y(s)\, ds + c(t) \int_{t-\sigma-\tau}^{t-\tau} \frac{\partial}{\partial x} h(s, \phi(s)) y(s)\, ds + c(t)c(t-\tau) \int_{t-\sigma-2\tau}^{t-2\tau} \frac{\partial}{\partial x} h(s, \phi(s))\psi(s)\, ds + \cdots = 0, \qquad t \in R,$$

then $y(t) = 0$ for $t \in R$. Since y satisfies an equation of the form (3.1), for which there is a corresponding first integral, with $(\partial/\partial x)h(s, \phi(s))y$ instead of $h(s, y)$ and the conditions analogous to (C1)–(C6) are satisfied, Lemma 3.3 implies $y(t) \to 0$ as $t \to \infty$. Then $y(t) = 0$ for all $t \in R$ because of the periodicity. This also shows that for each fixed k, (3.1) has at most one 1-periodic solution x with $J(t, x_t) = k$.

Define $F : P \times R \to R$ by $F(x, k) = (Id + G)x - kp$. Since 0 is a solution of (3.1), we have $F(0, 0) = 0$. Moreover we have seen that the derivative of F with respect to x at $(0, 0)$ is an isomorphism. Consequently, by the implicit function theorem, there is an open interval (a, b) such that

$0 \in (a, b)$ and, for every $k \in (a, b)$, (3.1) has an 1-periodic solution, denoted by p^k, such that $p^0 = 0$ and p^k is continuous as a mapping from (a, b) to P. If $k_1, k_2 \in (a, b)$, $k_1 < k_2$, then $x(t) = p^{k_1}(t) - p^{k_2}(t)$ satisfies Eqs. (3.1) and (3.2) such that $h(t, x)$ is changed to $h^*(t, x)$, where $h^*(t, x) = h(t, p^{k_2}(t) + x) - h(t, p^{k_2}(t))$. Since h^* satisfies (C3)–(C5) and the value of the corresponding first integral for x is $k_2 - k_1 > 0$, Lemma 3.2 can be applied to conclude $x_0 \in \mathscr{N}$. From the periodicity and $k_2 - k_1 > 0$ we obtain $x_t \gg_\mu 0$ for all $t \in R$. This means that $p_t^{k_2} \gg_\mu p_t^{k_1}$, $t \in R$. Hence, using (3.2) and the monotonicity of $h(t, \cdot)$, we can get $p^{k_2}(t) - p^{k_1}(t) \leq (k_2 - k_1)/(1 - c(t)e^{\mu t})$ for all $t \in R$. This shows the continuity of $k \mapsto p^k$ from (a, b) to C. Applying (3.4) and Lemma 2.1 it can be shown that $k \mapsto p^k$ is also continuous as a function from (a, b) to X. The limit $\lim_{k \to b-} p^k(t) = y(t)$ is uniform in t an defines an 1-periodic function y with $J(t, y_t) = b$. Let $p^b = y$. In the same way as above, we can show that there is an open interval containing b such that for any k in this interval, p^k is the unique 1-periodic solution. Continuing this process, we get a unique 1-periodic solution p^k for any real k. The continuity of $k \mapsto p^k$ and the monotonicity with respect to $\ll_\mu$ come from Lemmas 2.1 and 3.2.

The convergence and oscillation statement of Theorem 3.1 for solutions with initial functions from X is an easy corollary of Lemmas 3.2 and 3.3.

LEMMA 3.5. *If $\phi \in X$ and $J(0, \phi) = k$, then* $\lim_{t \to \infty} [x(0, \phi)(t) - p^k(t)] = 0$ *and $x(0, \phi)$ oscillates around p^k.*

Proof. The difference $x(0, \phi)(t) - p^k(t)$ satisfies (3.1) such that h is changed to $h^*(t, x) = h(t, p^k(t) + x) - h(t, p^k(t))$. For the corresponding first integral, $J^*(t, x_t(0, \phi) - p_t^k) = J(0, \phi) - k = 0$ holds. Since h^* has the properties (C3)–(C5), Lemmas 3.2 and 3.3 are also valid for the modified equation. Consequently, $\phi - p_0^k \in \mathscr{O}$, $x(0, \phi)(t) - p^k(t) \to 0$ as $t \to \infty$ and the proof is complete.

Only one statement of Theorem 3.1 is left to show. This is contained in the next lemma.

LEMMA 3.6. *If $\phi \in C$, $J(0, \phi) = k$, $x(0, \phi)$ exists on $[-r, \infty)$ and the set $\{x_t(0, \phi) : t \geq 0\}$ is bounded in C, then* $\lim_{t \to \infty} [x(0, \phi)(t) - p^k(t)] = 0$.

Proof. It is enough to consider the case $J(0, \phi) = 0$. Otherwise take the difference $\phi - p_0^k$ and use the idea of the proof of Lemma 3.5.

Thus, assume $\phi \in C$, $J(0, \phi) = 0$, and $x(0, \phi)(t)$ exists and bounded on $[-r, \infty)$. We have to show that $x(0, \phi)(t) \to 0$ as $t \to \infty$ since $p^0 = 0$.

We will use the Poincaré map $T : C \to C$, $T(\phi) = x_1(0, \phi)$ and the discrete dynamical system $\{T^n : n = 0, 1, \ldots\}$ in C. It is known (see, e.g., Hale [14]) that T is continuous. Define the ω-limit set $\omega_C(\phi) =$

$\bigcap_{n\geq 0}\operatorname{Cl}\bigcup_{j=n}^{\infty}T^j(\phi)$ where the closure is taken in C. The boundedness of $\{x_t(0,\phi): t\geq 0\}$ in C implies its precompactness in C (see, e.g., Hale [14]). Then it follows that $\omega_C(\phi)$ is a nonempty, compact, invariant subset of C and $T^n(\phi)\to\omega_C(\phi)$ as $n\to\infty$. Clearly, it is enough to prove that $\omega_C(\phi)$ contains only the zero function.

Let $\psi\in\omega_C(\phi)$. The invariance property of $\omega_C(\phi)$ means that there is a bounded continuous function on R such that $y_0=\psi$ and y satisfies (3.1) on R. Moreover, $J(t,y_t)=0$ and

$$y(t)+\sum_{j=0}^{\infty}\left(\prod_{i=0}^{j-1}c(t-i\tau)\right)\int_{t-\sigma-j\tau}^{t-j\tau}h(s,y(s))\,ds=0 \tag{3.5}$$

also hold for all $t\in R$.

Claim. $Y=\{y_n : n\in Z\}$ is precompact in X.

In order to show the Claim let $\{n_l\}$ be a sequence of integers. Since Y is a subset of the C-compact, invariant set $\omega_C(\phi)$, there exists a subsequence of $\{n_l\}$ (denoted again by $\{n_l\}$) and a bounded continuous function z on R such that

$$y(t+n_l)\to z(t),\qquad l\to\infty$$

uniformly on compact subsets of R. Function z also satisfies (3.1) and (3.5) for all t. Equation (3.5) and (C2) then imply that y and z are differentiable on R and their derivates can be obtained by differentiating (3.5) term by term

$$\begin{aligned}\dot y(t)=&-\sum_{j=1}^{\infty}\sum_{m=0}^{j-1}\left(\prod_{i=0,\,i\neq m}^{j-1}c(t-i\tau)\right)\dot c(t-m\tau)\int_{t-\sigma-j\tau}^{t-j\tau}h(s,y(s))\,ds\\&-\sum_{j=0}^{\infty}\left(\prod_{i=0}^{j-1}c(t-i\tau)\right)\big[h(t-j\tau,y(t-j\tau))-h(t-\sigma-j\tau,\\&\qquad\qquad y(t-\sigma-j\tau))\big].\end{aligned}$$

It is sufficient to prove that $y_{n_l}\to z_0$ in X as $l\to\infty$. We already know the convergence in C. Then the convergence in X follows from $\max_{-r\leq t\leq 0}|\dot y(n_l+t)-\dot z(t)|\to 0$, $l\to\infty$. From the above series representations of the derivatives of y and z one obtains

$$\begin{aligned}&\dot y(n_l+t)-\dot z(t)\\&\quad=-\sum_{j=1}^{\infty}\sum_{m=0}^{j-1}\left(\prod_{i=0,\,i\neq m}^{j-1}c(t-i\tau)\right)\dot c(t-m\tau)\end{aligned}$$

$$\times \int_{t-\sigma-j\tau}^{t-j\tau} \left[h(s, y(n_l + s)) - h(s, z(s)) \right] ds$$

$$- \sum_{j=0}^{\infty} \left(\prod_{i=0}^{j-1} c(t - i\tau) \right) \left[h(t - j\tau, y(n_l + t - j\tau)) \right.$$

$$- h(t - \sigma - j\tau, y(n_l + t - \sigma - j\tau))$$

$$\left. - h(t - j\tau, z(t - j\tau)) + h(t - \sigma - j\tau, z(t - \sigma - j\tau)) \right].$$

Hence from the uniform convergence $y(n_l + t) \to z(t)$ on compact subsets of R, we can get that $\max_{-r \le t \le 0} |\dot{y}(n_l + t) - \dot{z}(t)| \to 0$, $l \to \infty$. This completes the proof of the Claim.

By the Claim, the set $A = \mathrm{Cl}\{y_n : n \in Z\}$, where the closure is taken in X, is compact in X. We have $J(0, \alpha) = 0$ for all $\alpha \in A$. In the same way as in the proof of Lemma 3.2, for any $\alpha \in \mathscr{O}$ there is $\beta \in \mathscr{N}$ such that $\beta \gg_{\mu} \alpha$ and $J(0, \beta) > 0$ is as small as we want. Then $x(0, \beta)(t) > 0$ and $e^{\mu t} x(0, \beta)(t)$ is increasing for sufficiently large t. Then for large t, one gets $x(0, \beta)(t)(1 - c(t)e^{\mu\tau}) < J(t, x_t(0, \beta))$. Similarly, there is a $\delta \in \mathscr{N}$ such that $\delta \ll_{\mu} \alpha$ and $x(0, \delta)(t)(1 - c(t)e^{\mu\tau}) > J(t, x_t(0, \delta))$ for all large t and $J(0, \delta) < 0$ is as close to zero as we want.

Now fix an $\epsilon > 0$. For any given $\alpha \in A$ there exist $\delta, \beta \in \mathscr{N}$ and $t_{\alpha} \ge 0$ such that $\delta \ll_{\mu} \alpha \ll_{\mu} \beta$ and, applying Theorems 2.1 and 2.2,

$$-\epsilon \le x(0, \delta)(t) < x(0, \alpha)(t) < x(0, \beta)(t) < \epsilon, \qquad t \ge t_{\alpha}.$$

There is an open ball U_{α} centered at ϕ in X such that $\delta \ll_{\mu} U_{\alpha} \ll_{\mu} \beta$ and

$$|x(0, \nu)(t)| < \epsilon, \qquad \nu \in U_{\alpha}, t \ge t_{\alpha}.$$

The set $\{U_{\alpha} : \alpha \in A\}$ gives an open cover of A. A being compact, there is a finite subcover $U_{\alpha_1}, \dots, U_{\alpha_N}$. Let $T = \max\{t_{\alpha_1}, \dots, t_{\alpha_N}\}$. Then

$$|x(0, \alpha)(t)| < \epsilon, \qquad \alpha \in A, t \ge T$$

follows. Applying this for $\alpha = y_{-n}$, $n = 1, 2, \dots$, we obtain that $|y(t)| < \epsilon$ for all $t \in R$. Since $\epsilon > 0$ was arbitrary, $y = 0$ can be obtained. Consequently, $\psi = y_0 = 0$ and this means that $\omega_C(\phi)$ contains only the zero function. The proof is complete.

Remark 3.1. We conclude this paper with the following comments:

(1) More general linear neutral operators $D(t, \phi)$, as discussed in [11], can be allowed in the above results.

(2) The right hand side $f(t, \phi)$ can also be more general than in Eq. (3.1). The only crucial assumption is that the equation has a first integral.

(3) It would be interesting to get extensions of the above results for systems such as those arising from compartmental systems (see, e.g.,

[7, 8, 18]). In this direction, we mention that the monotonicity results of Section 2 have been extended to systems of (non-neutral) functional differential equations by Smith and Thieme in [23].

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