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Self-Sustained Oscillations in a Ring Array of Coupled Lossless Transmission Lines

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In this paper, we derive a neutral difference-differential system with diffusion which arises from a ring array of coupled lossless transmission lines. We investigate the problem of self-sustained oscillations of the considered transmission lines and apply a global Hopf bifurcation theorem to establish the existence of multiple large amplitude phase-locked periodic solutions in the corresponding neutral system.

1. INTRODUCTION

In 1961, in their enlightening studies on distributed (nonlumped) transmission line theory, Nagumo and Shimura [49] obtained a difference-differential equation of neutral type and discussed the self-oscillation phenomena in the transmission line. This work was later extended by Shimura [55] to a lossless transmission line terminated with a tunnel diode and a lumped parallel capacitor. Shortly after Nagumo and Shimura, on basis of the studies on nonlinear mixed initial-boundary problems arising from distributed transmission line theory [10, 48], a more general difference-differential equation of neutral type was derived independently by Brayton [8, 9] and the self-sustained periodic solutions of small amplitude were also proved to exist in transmission lines. The main idea used to obtain such difference-differential equations is the reduction of classical telegrapher's partial differential equation which describes the voltage and

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current changes in a transmission line by introducing the d'Alembert solution of the wave equation and using the boundary condition at terminals. This idea goes back to at least as far as Abolinia and Mishkis [1, 2], where they demonstrated the existence and uniqueness of solutions to a mixed problem for hyperbolic systems by converting them to integral-functional equations, with integral along characteristics.

Since then, the above derived neutral equation, we call it the *LLTL equation* in the sequel, has been extensively investigated and many new methods and new results have been developed. We refer to [16, 20, 28, 29, 35–37, 44, 57] for stability and instability considerations in LLTL equation and to [8, 9, 39, 43, 49, 55, 65] for discussions of the existence of periodic solutions. On the other hand, several generalizations of LLTL equation were also presented in [13, 14, 43, 46]. In particular, Cooke and Krumme [13] gave a systematic procedure for reducing transimission line problems, which are described by linear partial differential equations subject to certain nonlinear initial-boundary conditions, to initial value problems for differential-difference or integral differential-difference equations. It should be emphasized that it is the LLTL equation that has motivated the theory, which was first systematically discussed by Hale [28], for the *D*-type neutral functional differential equations (NFDEs).

Strictly speaking, a single transmission line as considered above is less usuall than multiconductor line in applications. As an electric circuit, a self-contained single transmission line is assumed to be removed far enough from other lines so that it is not affected by any electrical changes occuring in the latter. As soon as a second transmission line is placed close to the first one the fields of the first line induce a voltage and a current on the second. *Capacitive coupling* is then produced by the electric field and *inductive coupling* results from the magnetic field. The classical applications of telephone (or telegraph) line and high-voltage power transmission line are often examples of coupling. The coupling phenomenon is also utilized in practice to realize directional couplers and interdigital filters. Moreover, in the modern high-speed integrated circuit (IC) technology, coupling among a group of physically close transmission lines are very common and interconnects in high-density IC are usually treated as transmission lines. We refer to [12, 23, 45, 52, 56, 60] and the references therein for the detailed discussions on coupled electric circuits and transmission lines.

Motivated by Endo and Mori [19] and Winnerl et al. [64] we consider in this paper a *ring array* of mutually coupled lossless transmission lines. For simplicity, we assume the transmission lines are *resistively coupled* and the capactive and inductive coupling among the system are neglected. We also assume that each linked transmission line is identical and terminates at each end by a lumped linear or nonlinear circuit element. By employing telegrapher's equation at each line together with a coupling term in the initial-boundary condition, we derive a *symmetric difference-differential system of neutral type*, which is equivalent to the original partial differential equations governing the coupled lines. We believe this is the first time that a diffusive system of neutral functional differential equations is derived. We use a modified version of the local and *global* bifurcation theory developed in [39] and [66] to study self-sustained oscillations and to prove the existence and multiplicity of phase-locked and synchronous periodic solutions. Due to the global nature of the bifurcation theorem, comparing our results with those of Shimura [55] and Brayton [8, 9] (for single transmission line equations) the results on the existence of periodic solutions we shall present are global in the sense that the parameter can be *far away* from the local bifurcation value.

Since the self-sustained oscillation occurs in the lossless transmission line, we may regard it as an *electric oscillator*. It should be noted that there recently has been great interest in the study of coupled nonlinear oscillators. For example, Alexander and Auchmuty [3] have considered the global bifurcation of phase-locked oscillations in the coupled brusselators and van del Pol oscillators. In their series of papers, Endo and Mori [17–19] have discussed the mode analysis of one-dimensional and two-dimensional multimode oscillators. As a mathematical model for slow-wave electrical activity of the gastro-intestinal tract of humans and animals, Allian and Linkens [4] have proposed a tubular structure which comprises one-dimensional rings and two-dimensional arrays of interconnected nonlinear oscillators with third-power conductance characteristics. Similar mathematical models for the electrical activity in humans and animals are also postulated by Linkens et al. [42] and Sarna et al. [54], where a series of simulated relaxation oscillators are resistively coupled as a chain. Other problems related to coupled electric oscillators are addressed by Gollab et al. [26] on periodicity and chaos and are systematically reviewed by Grasman [25] on various applications.

This paper is now organized as follows. In Section 2, we use the standard reduction procedure developed in [9, 13, 57] to derive the governing diffusive neutral equations for resistively coupled lossless transmission lines of a ring structure. To investigate the global bifurcation of the neutral equations, three lemmas concerning the periods and upper and lower bounds are prepared in Section 3. Section 4 is devoted to the global Hopf bifurcation analysis and the existence of self-sustained phase-locked and synchronous periodic solutions of large amplitudes is proved. In Section 5, we draw some conclusions and discuss briefly some of the implications of the lossless transmission line problem. Finally, in the Appendix, we sketch a local and global Hopf bifurcation theory for neutral equations of mixed type (with both delayed and advanced arguments).

2. Coupled LLTL Diffusive Neutral Equations

Let N be a positive integer. We consider a ring of N mutually coupled lossless transmission line (LLTL) networks which are interconnected by a common resistor R. We assume all coupled LLTL networks are identical, each of which is a uniformly distributed lossless transmission line with series inductance L_s and parallel capacitance C_s per unit length of the line.

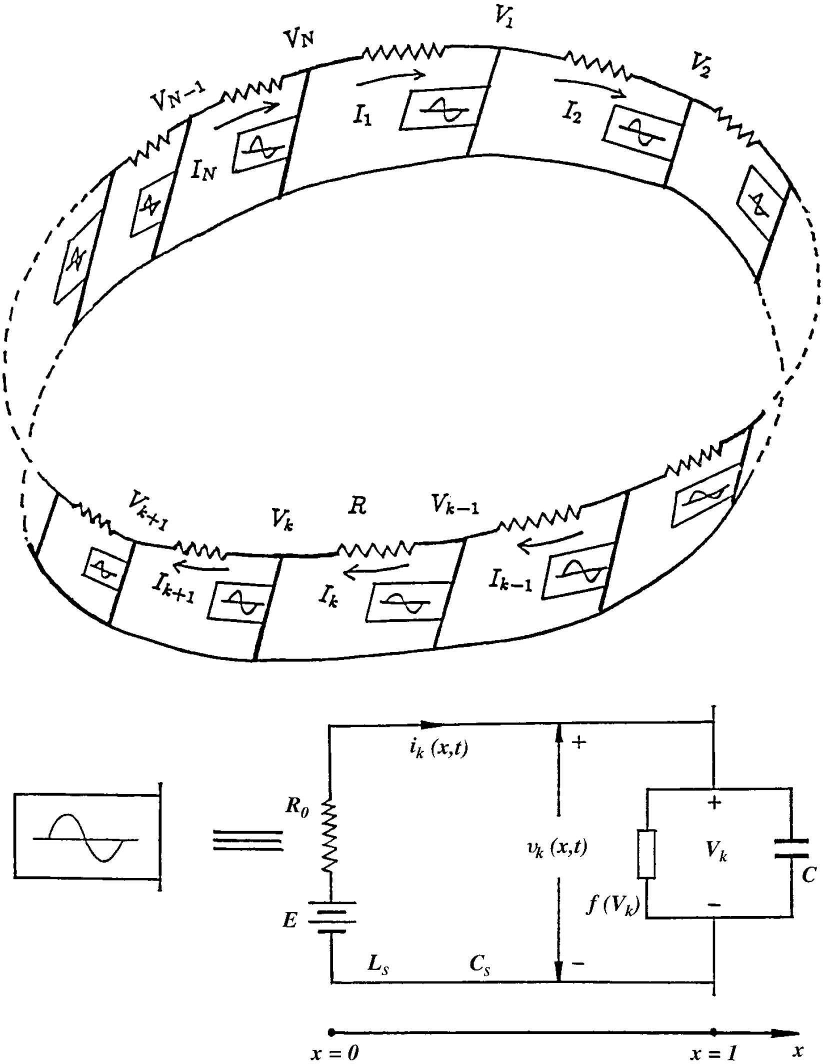

Figure 2.1

To derive the network equations, let us take an x-axis in the direction of the line, with two ends of the normalized line at $x=0$ and $x=1$. See Fig. 2.1.

Let $i_k(x, t)$ denote the current flowing in the kth line at time t and distance x down the line and $v_k(x, t)$ denote the voltage across the line at t and x. It is well-known [38, 45, 59] that the functions i_k and v_k obey the following partial differential equations (*Telegrapher's equation*)

$$\begin{aligned} L_s \frac{\partial i_k}{\partial t} &= -\frac{\partial v_k}{\partial x} \\ C_s \frac{\partial v_k}{\partial t} &= -\frac{\partial i_k}{\partial x}, \qquad k = 1, 2, ..., N. \end{aligned} \tag{2.1}$$

When these N networks are interconnected resistively in the way as shown in Fig. 2.1, the middle lines have coupling terms from the preceeding and succeeding lines, and at two ends $x=0$ and $x=1$, the line gives rise to the boundary conditions

$$\begin{aligned} 0 &= E - v_k(0, t) - R_0 i_k(0, t) \\ -C \frac{d}{dt} v_k(1, t) &= -i_k(1, t) + f(v_k(1, t)) - (I_{k-1} - I_k) \\ v_k(1, t) - v_{k+1}(1, t) &= RI_k(t) \end{aligned} \tag{2.2}$$

where E is the constant dc dias voltage, $f(v_k(1, t))$ is the current ($V-I$ characteristic) through the nonlinear resistor in the direction shown in Fig. 2.1 and I_k is the network current coupling term.

Under equilibrium conditions, $\partial i_k / \partial x = \partial v_k / \partial x = 0$. We have $i_k(0, t) = i_k(1, t)$ and $v_k(0, t) = v_k(1, t)$. Thus, Eq. (2.1) and (2.2) have the following equilibrium equations

$$\begin{aligned} E - v_k - R_0 i_k &= 0 \\ i_k &= f(v_k) - \frac{1}{R}(v_{k+1} - 2v_k + v_{k-1}). \end{aligned} \tag{2.3}$$

We assume that (2.3) has a unique homogeneous solution $(v_k, i_k) = (v^*, i^*)$, for all $1 \leqslant k \leqslant N$. By changing variables, the equilibrium can be shifted from (v^*, i^*) to $(0, 0)$ and Eq. (2.1) and (2.2) reduce to

$$\begin{aligned} L_s \frac{\partial i_k}{\partial t} &= -\frac{\partial v_k}{\partial x} \\ C_s \frac{\partial v_k}{\partial t} &= -\frac{\partial i_k}{\partial x} \end{aligned}$$

$$0 = v_k(0, t) + R_0 i_k(0, t)$$

$$-C \frac{d}{dt} v_k(1, t) = -i_k(1, t) + \tilde{g}(v_k(1, t)) - \frac{1}{R}(v_{k+1} - 2v_k + v_{k-1})(1, t) \tag{2.4}$$

where $\tilde{g}(v_k) = f(v_k + v^*) - f(v^*)$.

We now solve the partial differential equation (2.4). It is known [57, 59] that there exists a unique solution (*d'Alembert solution*) $i_k(x, t)$ and $v_k(x, t)$ which are of the form

$$\begin{aligned} v_k(x, t) &= \tfrac{1}{2}[\phi_k(x - \sigma t) + \psi_k(x + \sigma t)] \\ i_k(x, t) &= \frac{1}{2Z}[\phi_k(x - \sigma t) - \psi_k(x + \sigma t)] \end{aligned} \tag{2.5}$$

where

$$\sigma = \frac{1}{\sqrt{L_s C_s}}, \qquad Z = \sqrt{\frac{L_s}{C_s}} \tag{2.6}$$

are respectively the propagation velocity of waves and the characteristic impedance of the line, and

$$\phi_k \in C^1(-\infty, 1], \qquad \psi_k \in C^1[0, \infty).$$

Let

$$\begin{aligned} \phi_{k_1}(t) &= \phi_k(1 - \sigma t), & \phi_{k_0}(t) &= \phi_k(-\sigma t) \\ \psi_{k_1}(t) &= \psi_k(1 + \sigma t), & \psi_{k_0}(t) &= \psi_k(\sigma t) \end{aligned}$$

and $V_k(t) = v_k(1, t)$. We have from (2.5) that

$$\begin{aligned} \phi_{k_1}(t) &= V_k(t) + Z i_k(1, t), & \phi_{k_0}(t) &= v_k(0, t) + Z i_k(0, t) \\ \psi_{k_1}(t) &= V_k(t) - Z i_k(1, t), & \psi_{k_0}(t) &= v_k(0, t) - Z i_k(0, t). \end{aligned} \tag{2.7}$$

Note that $\phi_{k_1}(t) = \phi_{k_0}(t - 1/\sigma)$ and $\psi_{k_1}(t) = \psi_{k_0}(t + 1/\sigma)$. By (2.7) and the first boundary condition in (2.4), we get

$$\begin{aligned} V_k(t) + Z i_k(1, t) &= -q \psi_{k_1}(t - r) \\ V_k(t) - Z i_k(1, t) &= \psi_{k_1}(t) \end{aligned} \tag{2.8}$$

where

$$r = \frac{2}{\sigma} \qquad \text{and} \qquad q = \frac{Z - R_0}{Z + R_0}. \tag{2.9}$$

Now, the second boundary condition in (2.4) gives

$$i_k(1, t) = CV_k'(t) + \tilde{g}(V_k(t)) - \frac{1}{R}(V_{k+1}(t) - 2V_k(t) + V_{k-1}(t)). \quad (2.10)$$

Substituting (2.10) into (2.8) and eliminating $\psi_{k_1}(t-r)$ lead to

$$\begin{aligned} V_k(t) + Z\left[CV_k' + \tilde{g}(V_k) - \frac{1}{R}(V_{k+1}(t) - 2V_k(t) + V_{k-1}(t))\right] \\ = -qV_k(t-r) + qZ[CV_k'(t-r) + \tilde{g}(V_k(t-r))] \\ - \frac{qZ}{R}[V_{k+1}(t-r) - 2V_k(t-r) + V_{k-1}(t-r)]. \end{aligned}$$

This simplifies to

$$\begin{aligned} \frac{d}{dt}[V_k(t) - qV_k(t-r)] = & -\frac{1}{ZC}V_k(t) - \frac{q}{ZC}V_k(t-r) \\ & - \bar{g}(V_k) + q\bar{g}(V_k(t-r)) \\ & + \frac{1}{RC}[V_{k+1}(t) - qV_{k+1}(t-r) - 2(V_k(t) \\ & - qV_k(t-r)) + V_{k-1}(t) - qV_{k-1}(t-r)] \end{aligned}$$

where $\bar{g}(V_k) = 1/C\,\tilde{g}(V_k)$. Define for each $\alpha \in \mathbb{R}$ the operator $D(\alpha)$: $C([-r, 0]; \mathbb{R}) \to \mathbb{R}$ by

$$D(\alpha)\varphi = \varphi(0) - \alpha\varphi(-r), \quad \varphi \in C([-r, 0]; \mathbb{R}). \quad (2.12)$$

Following [8, 9], we assume

$$\bar{g}(v) = -\gamma v + g(v), \quad v \in \mathbb{R}, \quad \gamma > 0 \quad (2.13)$$

where g is a continuous function. Using (2.12) and (2.13), we obtain from (2.11) the following LLTL-network coupling equations

$$\begin{aligned} \frac{d}{dt}D(q)V_t^k = & -\left(\frac{1}{ZC} - \gamma\right)V^k(t) - q\left(\frac{1}{ZC} + \gamma\right)V^k(t-r) - g(V^k) \\ & + qg(V^k(t-r)) + \frac{1}{RC}D(q)(V_t^{k+1} - 2V_t^k + V_t^{k-1}) \\ & k = 1, 2, ..., N, \quad (\text{mod } N) \end{aligned} \quad (2.14)$$

where for each $1 \leqslant k \leqslant N$, $t \in \mathbb{R}$, $V_t^k \in C([-r, 0]; \mathbb{R})$ is defined by $V_t^k(\theta) = V_k(t+\theta)$ for all $\theta \in [-r, 0]$.

Note that Eq. (2.14) is a functional differential equation of neutral type with one time delay $r > 0$ (see [28, 29]). If there is no coupling between these N networks, then (2.14) reduces to a single LLTL-network equation

$$\frac{d}{dt} D(q) V_t^k = -\left(\frac{1}{ZC} - \gamma\right) V^k(t) - q\left(\frac{1}{ZC} + \gamma\right) V^k(t-r)$$
$$- g(V^k) + qg(V^k(t-r)). \tag{2.15}$$

This equation was first obtained by Naguma and Shimura [49], Shimura [55] (for $R_0 = 0$) and Brayton [9] (for any $R_0 \geqslant 0$) and was extensively investigated. See [8, 9, 16, 20, 28, 36, 37, 39, 43, 44, 49, 55, 57, 65] and the references therein.

Remark 2.1. Equation (2.14) can be viewed as a neutral system with discrete diffusion and therefore may be considered as a special example of the Rashevsky-Turing theory [51, 61]. Consequently, as in the Turning ring case, Eq. (2.14) bears a symmetry of the cyclic group $\mathbb{Z}_N$ and bifurcation of discrete waves may occur. We will discuss the existence of discrete waves in the remaining part of this paper. For more details about Turing rings and the symmetry of equations we refer to [21, 24, 27] and the references cited there.

3. Periods and a Priori Bounds

In this section, we prove three lemmas which will be needed in the study of global Hopf bifurcation of discrete waves and phase-locked oscillations. The first two lemmas concern the periods of periodic solutions of Eq. (2.14). In the third lemma we give a priori bounds on the amplitude of possible periodic solutions of Eq. (2.14).

We consider the following NFDEs

$$\frac{d}{dt} D(q) x_t^k = -ax^k(t) - bqx^k(t-r) - g(x^k(t)) + qg(x^k(t-r))$$
$$+ dD(q)[x_t^{k+1} - 2x_t^k + x_t^{k-1}]$$
$$k = 1, 2, ..., N, \quad (\text{mod } N) \tag{3.1}$$

where constants $d \geqslant 0$, $q \in [0, 1)$, $D(q)$: $C([-r, 0]; \mathbb{R}) \to \mathbb{R}$ is defined by

$$D(q)\varphi = \varphi(0) - q\varphi(-r), \qquad \varphi \in C([-r, 0]; \mathbb{R}), \tag{3.2}$$

r, a and b are positive constants, g is a differentiable function with $g(0)=g'(0)=0$. Note that Eq. (3.1) is a condensed form of Eq. (2.14). The parameters r, a, b, d and q are of physical meanings (see (2.6) and (2.9)). Note also that Eq. (3.1) is a special case of the following more general NFDEs

$$\frac{d}{dt} D(q) x_t^k = F(q, x^k(t), x^k(t-r)) + dD(q)[x_t^{k+1} - 2x_t^k + x_t^{k-1}] \tag{3.3}$$

where q, d, r are constants as before and $F\colon \mathbb{R}^3 \to \mathbb{R}$ is locally Lipschitzian.

In analyzing the global branch of phase-locked oscillations, we need the following information on the periods of possible periodic solutions to Eq. (3.1).

Lemma 3.1. *For every integer $m>0$, Eq. (3.3) has no nonconstant $2r/m$-periodic solution $x(t) := \{x^k(t)\}_{k=1}^N$ with $x^{k-1}(t) = x^k(t-r/m)$ for all $t \in \mathbb{R}$ and $k = 1, 2, ..., N$, (mod N).*

Proof. We consider two cases separately.

Case (I). m is odd. Note that if $x(t)$ is a nonconstant $2r/m$-periodic solution with $x^{k-1}(t) = x^k(t-r/m)$, then

$$x^{k-1}(t) = x^k\left(t - \frac{r}{m}\right) = x^k\left(t - r + \frac{m-1}{2}\frac{2r}{m}\right) = x^k(t-r).$$

It suffices to show that the lemma is true for $m=1$.

Suppose to the contrary that Eq. (3.3) has a nonconstant $2r$-periodic solution $x(t)$ with $x^{k-1}(t) = x^k(t-r)$. Let $y^k(t) = x^k(t-r)$. We have $x^{k+1}(t) = x^k(t+r) = x^k(t-r) = y^k(t)$. Similarly, $y^{k+1}(t) = x^k(t)$, $x^{k-1}(t) = y^k(t)$ and $y^{k-1}(t) = x^k(t)$. Therefore, by (3.2)

$$\begin{aligned} D(q) x_t^{k+1} &= x^{k+1}(t) - q x^{k+1}(t-r) \\ &= y^k(t) - q y^{k+1}(t-r) \\ &= y^k(t) - q x^k(t) = D(q) x_t^{k-1} \\ &\quad k = 1, 2, 3, ..., N, \quad (\text{mod } N) \end{aligned}$$

and $(x^k(t), y^k(t))$ satisfies the following ordinary differential equations

$$\begin{aligned} \frac{d}{dt}[x^k(t) - q y^k(t)] &= F(q, x^k(t), y^k(t)) + 2d(1+q)(y^k(t) - x^k(t)) \\ \frac{d}{dt}[y^k(t) - q x^k(t)] &= F(q, y^k(t), x^k(t)) + 2d(1+q)(x^k(t) - y^k(t)). \end{aligned} \tag{3.4}$$

Put

$$u(t) = x^k(t) - qy^k(t)$$
$$v(t) = y^k(t) - qx^k(t). \tag{3.5}$$

Then

$$x^k(t) = \frac{u(t) + qv(t)}{1 - q^2}$$
$$y^k(t) = \frac{qu(t) + v(t)}{1 - q^2}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we see that $(u(t), v(t))$ is a solution to the following system of ordinary differential equations

$$u'(t) = F\left(q, \frac{u + qv}{1 - q^2}, \frac{qu + v}{1 - q^2}\right) + 2d(v - u)$$
$$v'(t) = F\left(q, \frac{qu + v}{1 - q^2}, \frac{u + qv}{1 - q^2}\right) + 2d(u - v). \tag{3.7}$$

Eq. (3.7) is symmetric about $u(t)$ and $v(t)$. Therefore, the diagonal $\Delta \cong \{(u, v) \in \mathbb{R}; u = v\}$ is invariant under (3.7). Since any vector field on $\Delta \cong \mathbb{R}$ cannot have nonconstant periodic solution, $(u(t), v(t) \notin \Delta$ for all $t \in \mathbb{R}$. So, without loss of generality, we may assume that

$$u(t) < v(t) \qquad \text{for all} \quad t \in \mathbb{R}. \tag{3.8}$$

Replacing t by $t - r$ in (3.8) gives

$$u(t - r) < v(t - r) \qquad \text{for all} \quad t \in \mathbb{R}. \tag{3.9}$$

On the other hand, we have

$$v(t - r) = y^k(t - r) - qx^k(t - r)$$
$$= x^k(t) - qy^k(t) = u(t)$$

and

$$u(t - r) = x^k(t - \tau) - qy^k(t - r)$$
$$= y^k(t) - qx^k(t) = v(t).$$

Therefore, it follows from (3.9) that $v(t) < u(t)$ for all $t \in \mathbb{R}$ which contradicts (3.8). This completes the proof for Case (I).

Case (II). m is even. Similarly, we need only to show the lemma for $m = 2$.

By way of contradiction, suppose that $x(t)$ is an r-periodic solution to Eq. (3.3) with $x^{k-1}(t) = x^k(t - r/2)$. Set $y^k(t) = x^k(t - r/2)$. As in Case (I), $(x^k(t), y^k(t))$ satisfies the equations

$$\frac{d}{dt} x^k(t) = \frac{F(q, x^k, x^k)}{1-q} + 2d(y^k - x^k)$$

$$\frac{d}{dt} y^k(t) = \frac{F(q, y^k, y^k)}{1-q} + 2d(x^k - y^k)$$

A similar argument to that in Case (I) leads also to a contradiction.
This completes the proof.

Remark 3.1. An analog of Lemma 3.1 for the single scalar NFDE (2.15) has been established in [39] for the case where no coupling occurs.

We will also need the following simple result.

LEMMA 3.2. *Assume that* $a > 0$, $d \geqslant 0$ *and* $xg(x) > 0$ *for all* $x \neq 0$. *Then the system of ordinary differential equations*

$$\begin{aligned} \frac{d}{dt} x^k(t) = &-ax^k(t) - g(x^k(t)) + d(x^{k+1}(t) - 2x^k(t) + x^{k-1}(t)) \\ &k = 1, 2, ..., N, \quad (mod\ N) \end{aligned} \tag{3.10}$$

has no nonconstant periodic solutions.

Proof. Suppose that $x(t) = (x^1(t), ..., x^N(t))$ is a nonconstant periodic solution of (3.10). Set

$$V(x(t)) = \tfrac{1}{2} \sum_{k=1}^{N} (x^k(t))^2.$$

We have

$$\begin{aligned} V'_{(3.10)}(x(t)) &= \sum_{k=1}^{N} x^k[-ax^k - g(x^k) + d(x^{k+1} - 2x^k + x^{k-1})] \\ &= -a \sum_{k=1}^{N} (x^k)^2 - \sum_{k=1}^{N} x^k g(x^k) + d \sum_{k=1}^{N} x^k(x^{k+1} - 2x^k + x^{k-1}) \end{aligned}$$

$$\leqslant -2aV(x) + 2d \sum_{k=1}^{N} (x^k x^{k+1} - x^k x^k)$$

$$\leqslant -2aV(x).$$

This implies that

$$V(x(t)) \leqslant V(x(0))\, e^{-2at} \to 0 \qquad \text{as} \quad t \to +\infty.$$

It follows then that $\lim_{t \to \infty} x(t) = 0$. This is impossible since $x(t)$ is a non-constant periodic solution. This proves the lemma.

In what follows, we provide a priori bounds on periodic solutions of Eq. (3.1).

LEMMA 3.3. *Assume that* $0 < a < b$ *and*

(i) $xg(x) > 0$ *for all* $x \neq 0$;

(ii) $g(x)$ *is nondecreasing*;

(iii) $\lim_{x \to \pm\infty} g(x)/x = +\infty$;

(iv) *for any* $q_0 \in [a/b, 1)$,

$$\sup_{0 < q \leqslant q_0} \overline{\lim_{x \to \pm\infty}} \frac{g(qx) - qg(x)}{qx} < -(a+b).$$

Then for any $\delta \in [a/b, 1)$, *there exists* $M = M(\delta) > 0$ *such that if* $q \in (0, \delta]$ *and* $x(t)$ *is a periodic solution of Eq.* (3.1) *with period* $p > 0$ *which satisfies* $x^{k-1}(t) = x^k(t - p/2)$, *then* $|x(t)| \leqslant M$ *for all* $t \in \mathbb{R}$.

Proof. We prove the existence of M such that $x^k(t) < M$ for any $k \in \{1, 2, ..., N\}$. The existence of M such that $x^k(t) \geqslant -M$ can be treated similarly.

Let $x(t)$ be a periodic solution of Eq. (3.1) with period $p > 0$ and $x^{k-1}(t) = x^k(t - p/2)$ for $k = 1, 2, ..., N$, (mod N). Then $t \in \mathbb{R}$ exists such that

$$x^k(t) - qx^k(t - r) = \max_{s \in \mathbb{R}} [x^k(s) - qx^k(s - r)]. \tag{3.11}$$

Therefore, for each fixed $s \in \mathbb{R}$,

$$x^k(s) \leqslant qx^k(s - r) + [x^k(t) - qx^k(t - r)]. \tag{3.12}$$

Or, equivalently,

$$D(q)\, x_s^k \leqslant D(q) x_t^k. \tag{3.13}$$

Replacing s by $s-r$ in (3.12) yields

$$x^k(s) \leqslant q^2 x^k(s-2r) + (q+1)\, D(q) x_t^k.$$

Repeating the above process m-times, we get

$$x^k(s) \leqslant q^m x^k(s-mr) + \frac{1-q^m}{1-q}\, D(q) x_t^k.$$

Therefore, letting $m \to \infty$, we have

$$x^k(s) \leqslant \frac{D(q) x_t^k}{1-q} \qquad \text{for all} \quad s \in \mathbb{R}. \tag{3.14}$$

In particular,

$$x^k(t) \leqslant \frac{x^k(t) - q x^k(t-r)}{1-q}$$

which gives

$$x^k(t) \geqslant x^k(t-r). \tag{3.15}$$

On the other hand, by (3.11), we see that $(d/dt)[x^k(t) - q x^k(t-r)] = 0$. Recall that $x(t)$ is a solution to Eq. (3.1). It follows that

$$\begin{aligned} a x^k(t) + b q x^k(t-r) = & -g(x^k(t)) + q g(x^k(t-r)) \\ & + d[D(q)\, x_t^{k+1} - 2D(q)\, x_t^k + D(q) x_t^{k-1}]. \end{aligned} \tag{3.16}$$

Notice that $x^k(t-p) = x^k(t)$ and $x^{k-1}(t) = x^k(t - p/2)$ for any $k = 1$, 2, ..., N, (mod N). we have

$$\begin{aligned} D(q) x_t^{k+1} &= x^{k+1}(t) - q x^{k+1}(t-r) \\ &= x^k\left(t - \frac{p}{2}\right) - q x^k\left(t - \frac{p}{2} - r\right) = D(q) x_{t-(p/2)}^k. \end{aligned} \tag{3.17}$$

Similarly,

$$D(q) x_t^{k-1} = D(q) x_{t-(p/2)}^k. \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16), we obtain

$$ax^k(t)+bqx^k(t-r)= -g(x^k(t))+qg(x^k(t-r)) + 2d[D(q)\,x^k_{t-(p/2)}-D(q)x^k_t]. \tag{3.19}$$

We now distinguish two cases:

Case (i). $x^k(t)>0$. In this case, $x^k(t-r)<0$. For otherwise, if $x^k(t-r)\geqslant 0$, then the left hand side of (3.19) is positive, but the right hand side

$$\begin{aligned}-g(x^k(t))+qg(x^k(t-r))&+2d[D(q)\,x^k_{t-(p/2)}-D(q)x^k_t]\\ &\leqslant -g(x^k(t))+qg(x^k(t-r))\\ &=g(x^k(t))\left[-1+q\,\frac{g(x^k(t-r))}{g(x^k(t))}\right]<0\end{aligned}$$

by (3.13), (3.15), and the assumptions (i)–(ii) on g.

Now from (3.13) and (3.19), we see that

$$ax^k(t)+bqx^k(t-r)\leqslant -g(x^k(t))+qg(x^k(t-r)) \tag{3.20}$$

which implies that

$$0<ax^k(t)+g(x^k(t))\leqslant qx^k(t-r)\left[\frac{g(x^k(t-r))}{x^k(t-r)}-b\right]. \tag{3.21}$$

Since $x^k(t-r)<0$, (3.20) gives further that

$$\frac{g(x^k(t-r))}{x^k(t-r)}<b.$$

By assumption (iii), there must be a constant $M_1>0$ (independent of k) such that $x^k(t-r)\geqslant -M_1$. Substituting this into (3.21), we get

$$\begin{aligned}0<ax^k(t)+g(x^k(t))&\leqslant qg(x^k(t-r))-bqx^k(t-r)\\ &\leqslant \max_{-M_1\leqslant z\leqslant 0}\delta[g(z)-bz]\end{aligned} \tag{3.22}$$

from which another constant $M_2>0$ (independent of k) exists such that $x(t)\leqslant M_2$, due to assumption (iii). Therefore,

$$x^k(t)-qx^k(t-r)\leqslant M_2+\delta M_1.$$

This, together with (3.14), implies that $x^k(s)\leqslant (M_2+\delta M_1)/(1-\delta)$ for all $s\in\mathbb{R}$.

Case (ii). $x(t) \leqslant 0$. In this case, $x^k(t-r) \leqslant x^k(t) \leqslant 0$. If $x^k(t) - qx^k(t-r) \leqslant 0$, then, by (3.14), we are done. If $x^k(t) - qx^k(t-r) > 0$, then $x^k(t) > qx^k(t-r)$. From (3.20), we get

$$qg(x^k(t-r)) - bqx^k(t-r) \geqslant aqx^k(t-r) + g(qx^k(t-r)).$$

This implies

$$\frac{g(qx^k(t-r)) - qg(x^k(t-r))}{qx^k(t-r)} \geqslant -(a+b).$$

Therefore, by assumption (iv), there exists $M_3 > 0$ such that $x^k(t-r) \geqslant -M_3$. Repeating the argument in the last part of Case (i), we can find a constant $M > 0$ (independent of k) such that $x^k(s) \leqslant M$ for all $s \in \mathbb{R}$.

This completes the proof.

Remark 3.2. One can easily verify that all conditions (i)–(iv) are satisfied for the function $g(x) = cx^3$, $c > 0$. Physically, such a function g describes a *cubic nonlinear conductance* which can be realized with a tunnel diode or an operational amplifier (see [33, 34]). More generally, one can prove that every function $g(x) = \sum_{i=1}^{n} g_i x^{2i+1}$ with $g_1 > 0$, $g_i \geqslant 0$, $i \neq 1$ also verifies conditions (i)–(iv). For the use of higher order nonlinear conductance, we refer to [19].

4. Self-Sustained Periodic Solutions

In this section, we apply the global Hopf bifurcation theory sketched in the Appendix to study the existence of periodic solutions of Eq. (3.1). As Eq. (3.1) bears a symmetry of the cyclic group $\mathbb{Z}_N$ of order N (i.e. interchanging x^k with x^{k-1} does not change the equation), we expect that Hopf bifurcation of periodic solutions $x(t)$ satisfying $x^{k-1}(t) = x^k(t-(j/N)p)$ may occur, where p is a period of $x(t)$, k (mod N), and $0 \leqslant j \leqslant n-1$ is an integer. In the literature, these periodic solutions are called discrete waves, or synchronous oscillations (if $j=0$) and phase-locked oscillations (if $j \neq 0$).

Clearly, finding a discrete wave satisfying $x^{k-1}(t) = x^k(t-(j/N)p)$ of Eq. (3.1) is equivalent to finding a periodic solution of a period p for the following scalar neutral equation of mixed type (with both delayed and advanced arguments)

$$\frac{d}{dt} D(q)\, y_t = -ay(t) - bqy(t-r) - g(y(t)) + qg(y(t-r)) + dD(q)(y_{t+(j/N)p} - 2y_t + y_{t-(j/N)p}) \tag{4.1$_j$}$$

to which the global Hopf bifurcation theory in the Appendix can be applied.

We begin with the consideration of local Hopf bifurcations. Clearly, $y(t) \equiv 0$ is a solution of Eq. (4.1$_j$) for any $q \in [0, 1)$. The characteristic equation of the stationary point $(0, q, p)$ is $p_j(\lambda, p) = 0$, where

$$p_j(\lambda, p) = (\lambda + a_j + 2d - de^{\lambda(j/N)p} - de^{-\lambda(j/N)p})\, e^{\lambda r}$$
$$-q(\lambda - b_j + 2d - de^{\lambda(j/N)p} - de^{-\lambda(j/N)p})$$

and

$$a_j = a + dc_j, \qquad b_j = b - dc_j,$$
$$c_j = 4\sin^2(\pi j/N), \qquad j \pmod N.$$

In particular,

$$p_j\left(im\frac{2\pi}{p}, p\right) = \left(im\frac{2\pi}{p} + a_{mj}\right) e^{im(2\pi/p)r} - q\left(im\frac{2\pi}{p} - b_{mj}\right), \quad m = 1, 2, ...$$

So, we first consider the equation

$$(\lambda + a_j)\, e^{\lambda r} - q(\lambda - b_j) = 0. \tag{4.2}$$

Lemma 4.1. *If $0 < a_j < b_j$, for some $j \in \{0, 1, 2, ..., [(N-1)/2]\}$, then*

(i) *the equation*

$$\tan \beta r = \frac{(a+b)\, \beta}{\beta^2 - a_j b_j} \tag{4.3}$$

has infinitely many positive solutions $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots \to \infty$ as $n \to \infty$, such that

(a) *if $\sqrt{a_j b_j} = \pi/2r$, then $2r/(n+1) < 2\pi/\beta_n < 2r/n \leqslant 2r$ for $n \geqslant 1$;*

(b) *if $\sqrt{a_j b_j} = \pi/2r + m\pi/r$ for some positive integer m, then $2r < 2\pi/\beta_1 < 4r$, $2r/n < 2\pi/\beta_n < 2r/(n-1) \leqslant 2r$ for $2 \leqslant n \leqslant m$ (when $m \geqslant 2$), $2r/(n+1) < 2\pi/\beta_n < 2r/n \leqslant 2\tau$ for $n \geqslant m+1$;*

(c) *if $r\sqrt{a_j b_j}/\pi - 1/2$ is not an integer, then $2\pi/\beta_1 > 2r$ and $2r/n < 2\pi/\beta_n < 2r/(n-1) \leqslant 2r$ for $n \geqslant 2$;*

(ii) *when $q = q_{\pm n}$, (4.2) has one and only one pair of purely imaginary zeros which are given by $\pm i\beta_n$, where*

$$q_{\pm n} = \pm\sqrt{\frac{\beta_n^2 + a_j^2}{\beta_n^2 + b_j^2}},$$

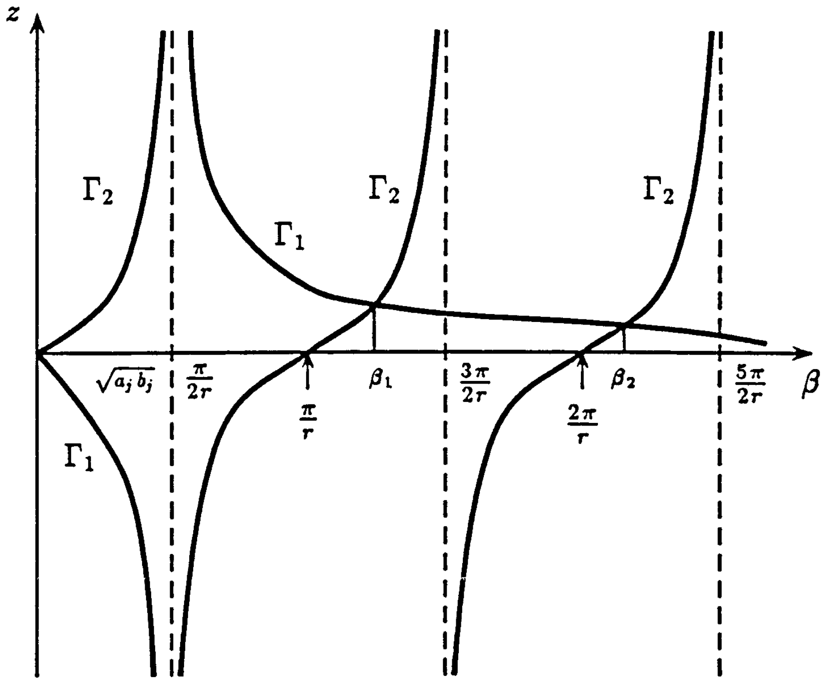

FIGURE 4.1

and when $q > 0$ *and* $q \neq q_n$, $n = 1, 2, \ldots$, (4.2) *has no purely imaginary zeros*;

(iii) *Let* $\lambda_n(q) = u_n(q) + iv_n(q)$ *be the root of* (4.2), *where* q *is close to* q_n *such that* $u_n(q_n) = 0$ *and* $v_n(q_n) = \beta_n$. *Then* $(d/dq)\, u_n(q)|_{q=q_n} > 0$.

Proof. We consider the graphs Γ_1 and Γ_2 in the region $\{(\beta, z) : \beta > 0\}$ of $\beta - z$ plane for the function $z = (a+b)\beta/(\beta^2 - a_j b_j)$ and $z = \tan \beta r$, respectively. If $\sqrt{a_j b_j} = \pi/2r$, then, as Fig. 4.1 shows, Γ_1 and Γ_2 have infinitely many intersections (β_n, z_n) such that

$$\frac{n\pi}{r} < \beta_n < \frac{(2n+1)\pi}{2r}, \quad n = 1, 2, 3, \ldots.$$

This gives

$$\frac{2r}{n+1} < \frac{2\pi}{\beta_n} < \frac{2r}{n} \leqslant 2r, \quad n = 1, 2, 3, \ldots.$$

If $\sqrt{a_j b_j} = \pi/2r + m\pi/r$ for a positive integer m, then Γ_1 and Γ_2 have infinitely many intersection points (β_n, z_n) such that

$$\frac{(2n-1)\pi}{2r} < \beta_n < \frac{n\pi}{r}, \quad n = 1, 2, \ldots, m$$

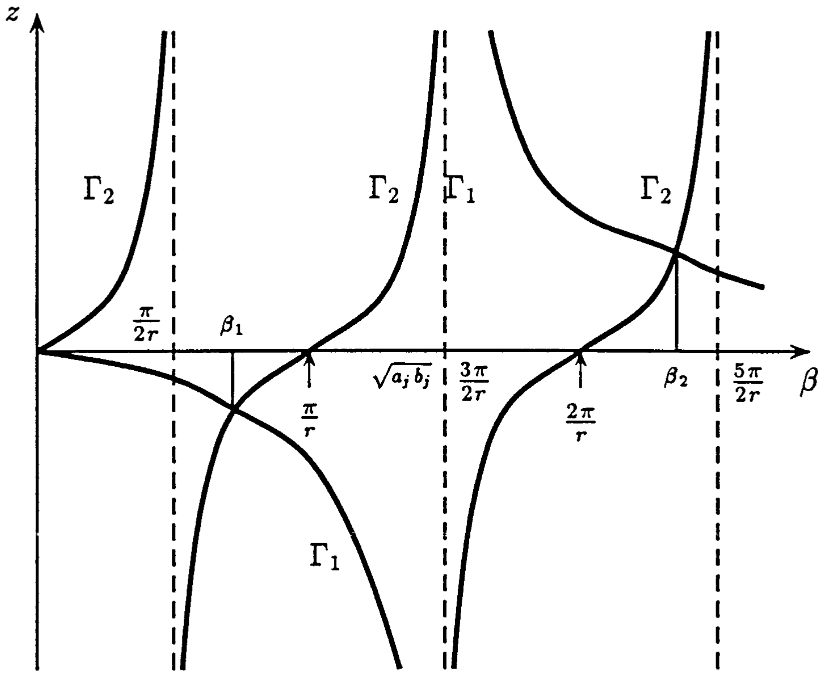

FIGURE 4.2

and

$$\frac{(m+k)\pi}{r} < \beta_{m+k} < \frac{2(m+k)+1}{2r}\,\pi, \quad k = 1, 2, ...$$

See Fig. 4.2. Therefore,

$$2r < \frac{2\pi}{\beta_1} < 4r,$$

$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1/2} < \frac{2r}{n-1}, \quad n = 2, 3, ..., m, \quad \text{when } m \geqslant 2,$$

and

$$\frac{2r}{m+k+1} < \frac{2\pi}{\beta_{m+k}} < \frac{2r}{m+k}, \quad k = 1, 2, 3, ...$$

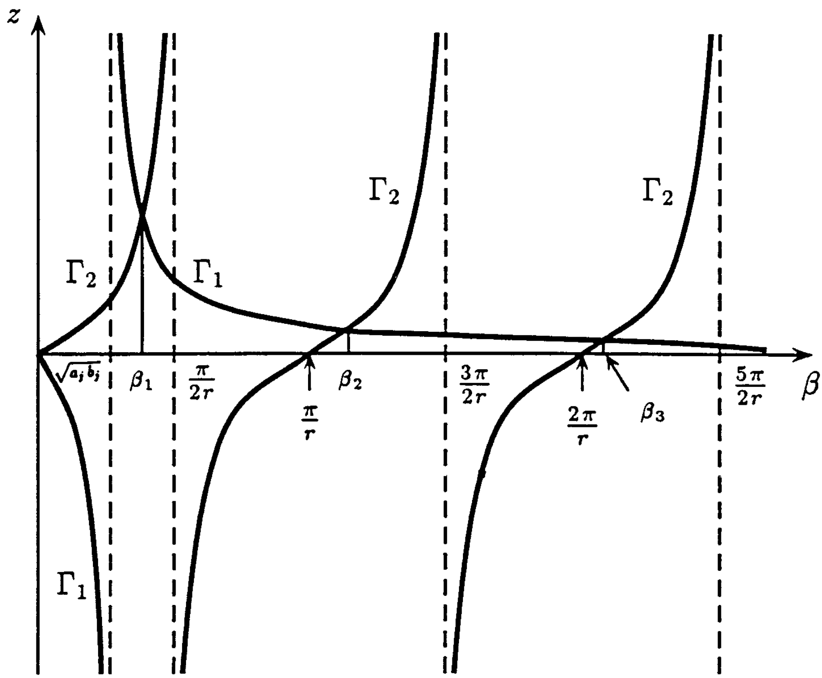

FIGURE 4.3

If $\pi/2r + (m-1)\pi/r < \sqrt{a_j b_j} < \pi/2r + m\pi/r$ for some nonnegative integer m, then Γ_1 and Γ_2 have infinitely many intersection points (β_n, z_n) such that

$$\frac{(n-1)\pi}{r} < \beta_n < \frac{(2n-1)\pi}{2r}, \quad n = 1, 2, \ldots$$

in case $m = 0$ (see Fig. 4.3), and

$$\frac{(2n-1)\pi}{2r} < \beta_n < \frac{n\pi}{r}, \quad n = 1, 2, \ldots, m$$

$$\frac{m+k+1}{r}\pi < \beta_{m+k} < \frac{2(m+k)-1}{2r}\pi, \quad k = 1, 2, 3, \ldots$$

in case $m \geqslant 1$ (see Fig. 4.4). Therefore, we have

$$\frac{2\pi}{\beta_1} > 4r,$$

$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1} \leqslant 2r, \qquad \text{for} \quad n \geqslant 2$$

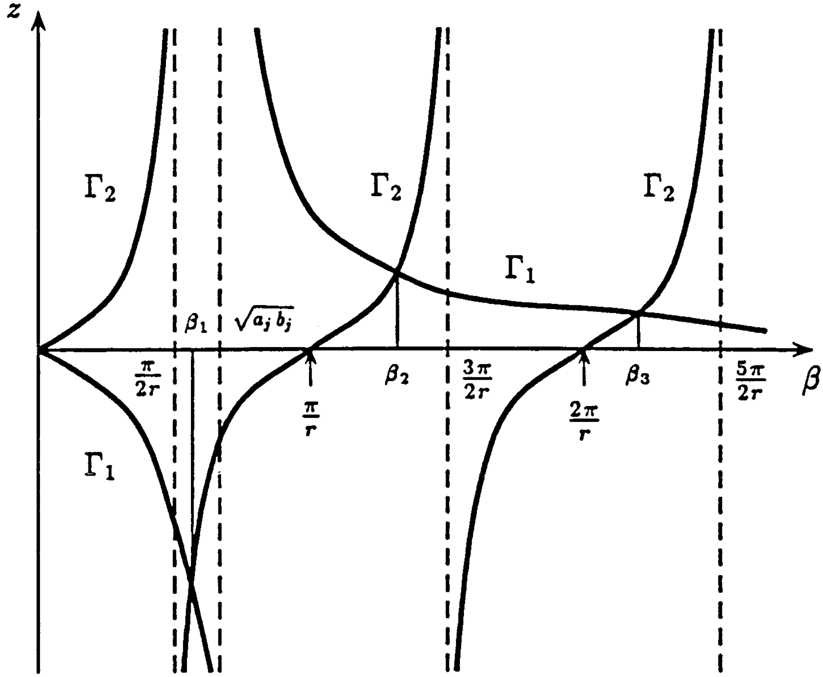

FIGURE 4.4

if $m=0$ and

$$2r < \frac{2\pi}{\beta_1} < 4r,$$

$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1/2} < \frac{2r}{n-1} \leqslant 2r, \quad n=2, 3, ..., m, \quad \text{when } m \geqslant 2$$

$$\frac{2r}{m+k} < \frac{2r}{m+k-1/2} < \frac{2\pi}{\beta_{m+k}} < \frac{2r}{m+k-1} \leqslant 2r, \quad k=1, 2, ...$$

if $m \geqslant 1$. This completes the proof of (i).

To prove (ii), we substitute $\lambda = i\beta$ in (4.2) and get

$$(i\beta + a_j)\, e^{i\beta r} = q(i\beta - b_j)$$

which is equivalent to

$$-a_j \cos \beta r + \beta \sin \beta r = qb_j$$

$$\beta \cos \beta r + a_j \sin \beta r = q\beta.$$

Thus

$$\tan \beta r = \beta \frac{a+b}{\beta^2 - a_j b_j}$$

$$\beta^2 = \frac{q^2 b_j^2 - a_j^2}{1-q^2}$$

from which (ii) follows immediately.

Finally, we prove (iii). By viewing λ as a function of q, we differentiate both sides of (4.2). It follows that

$$\frac{d\lambda}{dq} = \frac{\lambda - b_j}{[1 + r(\lambda + a_j)]\, e^{\lambda r} - q}. \tag{4.4}$$

Note that (4.2) implies

$$\lambda - b_j = \frac{(\lambda + a_j) e^{\lambda r}}{q}. \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain

$$\begin{aligned} \frac{1}{q}\left(\frac{d\lambda}{dq}\right)^{-1} &= \frac{[1 + r(\lambda + a_j)]\, e^{\lambda r} - q}{(\lambda + a_j) e^{\lambda r}} \\ &= \frac{1}{\lambda + a_j} + r - \frac{1}{\lambda - b_j}. \end{aligned} \tag{4.6}$$

Therefore, with (4.6) in mind, we have

$$\begin{aligned} \text{sign}\left\{\frac{d}{dq} u_n(q)\right\}\bigg|_{q=q_n} &= \text{Sign}\left\{\frac{d}{dq}\,\text{Re}\,\lambda\right\}\bigg|_{q=q_n} \\ &= \text{sign}\left\{\text{Re}\left(\frac{d\lambda}{dq}\right)\right\}\bigg|_{q=q_n} = \text{sign}\left\{\text{Re}\left(\frac{d\lambda}{dq}\right)^{-1}\right\}\bigg|_{q=q_n} \\ &= \text{sign}\left\{\text{Re}\left(\frac{1}{\lambda + a_j} + r - \frac{1}{\lambda - b_j}\right)\right\}\bigg|_{\substack{\lambda = i\beta_n \\ q = q_n}} \\ &= \text{sign}\left\{r + \frac{a_j}{a_j^2 + \beta_n^2} + \frac{b_j}{b_j^2 + \beta_n^2}\right\} = 1 > 0. \end{aligned}$$

This proves (iii) and completes the proof of Lemma 4.1.

Translating the results of Lemma 4.1 in terms of the notions in the Appendix, we know that for each $j \pmod N$ with $0 < a_j < b_j$ and for every integer $n \geqslant 1$, $(0, q_n, 2\pi/\beta_n)$ is an isolated center of (4.1_j), $1 \in J(0, q_n, 2\pi/\beta_n)$ and $\gamma_1(0, q_n, 2\pi/\beta_n) = -1$. Consequently, (4.1_j) has a local bifurcation of periodic solutions of periods near $2\pi/\beta_n$. This implies that the original system (3.1) of neutral equations has a (local) bifurcation of non-constant discrete waves $x(t)$ with period p close to $2\pi/\beta_n$ and $x^{k-1}(t)=x^k(t-(j/N)p)$, k (mod N) and $t\in\mathbb{R}$.

To investigate the maximal continua of the above (local) branch of discrete waves, we now apply the global Hopf bifurcation theorem in the Appendix in conjunction with the two lemmas in Section 3. The following results are typical and should easily be generalized to the case where N is not necessarily even and $j\neq N/2$.

THEOREM 4.2. *Suppose* $0<a_j<b_j$, *where* N *is even and* $j=N/2$. *Assume that* g *satisfies the conditions* (i)–(iv) *in Lemma* 3.3.

(i) *If* $\sqrt{a_j b_j}=\pi/2r$, *then for any* $n\geqslant 1$ *and* $q\in(q_n,1)$, *Eq.* (3.1) *has* n *phase-locked periodic solutions* $\{x^k_{l,q}(t)\}_{k=1}^N$ *whose periods* $p_{l,q}$ *satisfy* $2r/(l+1)<p_{l,q}<2r/l$, $l=1,2,...,n$ *and* $x^{k-1}_{l,q}(t)=x^k_{l,q}(t-(1/2)p_{l,q})$, $k=1,2,...,N$, (*mod* N);

(ii) *If* $\sqrt{a_j b_j}=\pi/2r+m\pi/r$ *for some positive integer* m, *then for any* $n\geqslant 2$ *and* $q\in(q_n,1)$, *Eq.* (3.1) *has* $n-1$ *phase-locked periodic solutions* $\{x^k_{l,q}(t)\}_{k=1}^N$ *whose periods* $p_{l,q}$ *satisfy* $2r/l<p_{l,q}<2r/(l-1)$ *for* $2\leqslant l\leqslant m$ (*when* $m\geqslant 2$), $2r/(l+1)<p_{l,q}<2r/l$ *for* $m+1\leqslant l\leqslant n$, *and* $x^{k-1}_{l,q}(t)=x^k_{l,q}(t-(1/2)p_{l,q})$, $k=1,2,...,N$, (*mod* N);

(iii) *If* $r\sqrt{a_j b_j}/\pi-1/2$ *is not an integer, then for any* $n\geqslant 2$ *and* $q\in(q_n,1)$, *Eq.* (3.1) *has* $n-1$ *phase-locked periodic solutions* $\{x^k_{l,q}(t)\}_{k=1}^N$ *where periods* $p_{l,q}$ *satisfy* $2r/l<p_{l,q}<2r/(l-1)$ *for* $l=2,3,...,n$ *and* $x^{k-1}_{l,q}(t)=x^k_{l,q}(t-(1/2)p_{l,q})$, $k=1,2,...,N$, (*mod* N).

Proof. We only give the proof for (iii). Other cases can be proved analogously.

For any fixed positive integer n, we consider the following neutral equations

$$\begin{aligned}\frac{d}{dt}D(Q_n(\alpha))\,y_t=&-ay(t)-bQ_n(\alpha)\,y(t-r)-g(y(t))+Q_n(\alpha)\,g(y(t-r))\\&+dD(Q_n(\alpha))[y_{t+(p/2)}-2y_t+y_{t-(p/2)}]\end{aligned}\tag{4.7}$$

where

$$Q_n(\alpha)=\frac{q_{n+1}+a/b}{\pi}\left(\arctan\alpha+\frac{\pi}{2}\right)-\frac{a}{b}.$$

Note that $Q_n(\alpha)$ is an increasing function with $\lim_{\alpha\to\infty}Q_n(\alpha)=q_{n+1}$ and $\lim_{\alpha\to-\infty}Q_n(\alpha)=-a/b$. The map b: $X\times\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ defined by

$$b(\varphi,\alpha,p)=Q_n(\alpha)\,\varphi(-r)$$

for $(\varphi, \alpha, p) \in X \times \mathbb{R} \times \mathbb{R}_+$ satisfies a Lipschitz condition with Lipschitz constant $L = q_{n+1} < 1$, where X is the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Under the assumption (i) of Lemma 3.3, we can easily show that for any $(\alpha, p) \in \mathbb{R} \times \mathbb{R}_+$, $(0, \alpha, p)$ is the only stationary solution of (4.7). Moreover, using (4.2), we can show that 0 is never a characteristic value of (4.7) since $Q_n(\alpha) > -a/b$. That is, (H2) of Theorem B in the Appendix is satisfied. Let $\alpha_l = Q_n^{-1}(q_l)$ for $l = 2, 3, ..., n$, where Q_n^{-1} denotes the inverse function of Q_n. Then $(0, \alpha_l, 2\pi/\beta_l)$ are isolated centers of (4.7) for each $2 \leqslant l \leqslant n$ by Lemma 4.1. Fix now $n \geqslant 2$ and consider the set

$$\mathcal{S} = \mathcal{C}\ell\{(y, \alpha, p);\ y(t) \quad \text{is a } p\text{-periodic solution of (4.7)}\}$$

Let $\mathcal{C}(0, \alpha_l, 2\pi/\beta_l)$ denote the connected component of $\mathcal{S}$ containing $(0, \alpha_l, 2\pi/\beta_l)$. By the statements after Lemma 4.1, $\mathcal{C}(0, \alpha_l, 2\pi/\beta_l)$ is nonempty. Moreover, the global Hopf bifurcation theorem, Theorem B in the Appendix, implies that $\mathcal{C}(0, \alpha_l, 2\pi/\beta_l)$ must be unbounded, as $\gamma(0, \alpha_l, 2\pi/\beta_l) = -1$.

Note that $Q_n(\alpha)$ increases from $-a/b$ to q_{n+1}, there is ξ_n such that $Q_n(\xi_n) = 0$. At $\alpha = \xi_n$, Eq. (4.7) reduces to the following equations

$$\frac{d}{dt} y(t) = -ay(t) - g(y(t)) + d\left[y\left(t + \frac{p}{2}\right) - 2y(t) + y\left(t - \frac{p}{2}\right)\right]. \quad (4.8)$$

Recall that a p-periodic solution of (4.8) gives a p-periodic solution $x(t)$ of (3.10) satisfying $x^{k-1}(t) = x^k(t - (p/2))$. So by Lemma 3.2, Eq. (4.8) has no nonconstant periodic solutions. Recall that $2r/l < 2\pi/\beta_l < 2r/(l-1)$. By Lemmas 3.1 and 3.3, we conclude that there exists a constant $M_n = M_n(q_{n+1}) > 0$ such that

$$\mathcal{C}\left(0, \alpha_l, \frac{2\pi}{\beta_l}\right) \subset BC(M_n) \times (\xi_n, \infty) \times \left[\frac{2r}{l}, \frac{2r}{l-1}\right],$$

where

$$BC(M_n) = \{ y \in X; \sup_{t \in \mathbb{R}} |y(t)| < M_n\}.$$

Since $\mathcal{C}(0, \alpha_l, 2\pi/\beta_l)$ is unbounded, the projection of $\mathcal{C}(0, \alpha, 2\pi/\beta_l)$ onto the parameter (α)-space must be unbounded above. This implies that for every $\alpha > \alpha_l$, Eq. (4.7) has a nonconstant periodic solution $y_{l,\alpha}(t)$ with period

$p_{l,\alpha} \in (2r/l, 2r/(l-1))$. This, in turn, implies that for all $q \in (q_l, q_{n+1})$, Eq. (3.1) has a nonconstant discrete waves $\{x^K_{l,q}(t)\}^N_{k=1}$ with period $p_{l,q} \in (2r/l, 2r/(l-1))$ and such that $x^{k-1}_{l,q}(t) = x^k_{l,q}(t-(p_{l,q}/2))$, $k = 1, 2, ..., N$, (mod N). This completes the proof.

We end this section with several remarks.

Remark 4.1. We note that the existence of phase-locked periodic solutions of periods less than $2r$ for Eq. (3.1) with $q \in (q_2, 1)$ has been guaranteed by Theorem 4.2 in all cases. We call these solutions *rapidly oscillating solutions*. It has been observed, both numerically and theoretically, that rapidly oscillating periodic solutions appear to be unstable for many retarded equations. It is still a question that whether the same phenomenon happens to the neutral equations.

Remark 4.2. We are unable to obtain global results on the existence of phase-locked periodic solutions with period greater than $2r$. This is because we are unable to exclude the existence of phase-locked periodic solution with period equal to nr, where $n \geqslant 2$ is an integer. However, the local Hopf bifucation theorem guarantees that phase-locked periodic solutions with period greater than $2r$ do exist for *q near* q_1 in the case (ii) and (iii). We call these periodic solutions *slowly oscillating solutions*. It is also an interesting question that whether these periodic solutions are stable.

Remark 4.3. If $d = 0$ in Eq. (3.1), i.e. there is no coupling between lines, then Theorem 4.2 gives also a global branch of synchronous oscillating solutions. Physically, this can be interpreted as each terminal voltage oscillating in the same way (each voltage is of identically the same amplitude at any time) so that there is no current flowing through the coupling resistor R (in this case, we can view $R = \infty$ and $d = 0$).

Remark 4.4. It follows from Theorem 4.2 that if $0 < a_j < b_j$ for $j = N/2$ (when N is even), then the system (3.1) has large amplitude periodic solutions. It is not difficult to see that $0 < a_j < b_j$ for $j = N/2$ is equivalent to

$$0 < \gamma RC - 4 < R/Z. \tag{4.9}$$

It follows that, if $ZC > 1/\gamma$, choosing large coupling resistance R will guarantee (4.9). This also implies that if the lumped parallel capacitance C or the characteristic impedance Z is large, there likely exist phase-locked oscillations. And the synchronous oscillations always exist in the system (by taking $R = \infty$, see Remark 4.3). This analysis seems in agreement with that obtained by Shimura [55].

5. Conclusions and Discussions

In this paper, we have studied the ring structured, resistively-coupled lossless transmission lines. The telegrapher equation is reduced to a symmetric neutral system. We have proved, under fairly general conditions, the existence of *large amplitude* phase-locked and synchronous periodic solutions. To the best of our knowledge, it is the first global result on the existence of periodic solutions for n-dimensional autonomous neutral functional differential systems. This is due to the symmetry of the equations in question and our global Hopf bifurcation theorem.

As electric circuits, the transmission lines can be coupled *resistively*, *inductively* (magnetically) or *capacitively* (electrostatically). In this paper, only resistive coupling (by a common resitor R) is discussed. The same problem for inductive coupling or capacitive coupling should also be addressed. But differential equations governing the transmission lines will be more complicated. This is beyond the scope of this paper and we shall consider them elsewhere.

Note also that the inductive coupling may not be electrically connected at all. In this case, the coupling is affected through *mutual inductance* of the lines in nearest neighbours. Moreover, a combination of the above couplings is also possible. We refer to [7, 25, 56, 58] for more details on circuit couplings.

We are only concerned with the *existence* of symmetric periodic solutions which describe phase-locked or synchronous oscillations. The *stability* of these periodic solutions is an important issue and remains unsolved. We will address this problem in our future investigations. (We have recently received a preprint of Hale [30] where an idea of how to determine the stability of periodic solutions of *neutral equations* is presented.)

A natural question also arises here. Since electric circuits are widely used to simulate biological rhythms, it is plausible to question the applicability of the lossless transmission line equations presented in this paper to problems of oscillations in biology. The simulations of the classical van del Pol relaxation oscillator in various disciplines are well-known [3–7, 17–19, 25, 34, 40–42, 47, 62, 63, 67]. In neuron electro-physiology, numerous electric circuits have been built to model the nerve activities [15, 22, 31–34]. In particular, Hodgkin–Huxley theory [31] on nerve conduction has represented a non-myelinated axon membrance as a one-dimensional transmission line. Although the electrical characterization of the membrance is very different from the (lossless) transmission line we described (the membrance has a distributed constant resistance but has no inductance), it is still interesting to construct a specific realization of the dynamical system (3.1) in mathematical biology.

Let us pursue this line of thought somewhat further before we leave this discussion. Actually, we have been led to a well-developed theory of *dynamical analogies* [7, 51, 53]. It is well known that *any electrical system* can be replaced by an analogous *mechanical system*, and conversely. Under this analogy, we conclude that the electric circuit considered in this paper can be a simulator of almost all stringed instruments, where the tunnel diode in the transmission line corresponds to the Coulomb friction in the string, the voltage-current corresponds to frictional force-relative velocity and the inductance corresponds to the mass (see, for example, [55]). It is the main purpose of the theory of dynamical analogies to study those systems which are utterly diverse in character, yet there is a precise sense in which certain pairs of diverse systems may be considered dynamically equivalent.

Appendix

Let X denote the Banach space of bounded continuous mappings $x: \mathbb{R} \to \mathbb{R}^n$ equipped with the supremum norm. For reasons stated in Section 4, we need to consider neutral functional differential equations of mixed type (with both delayed and advanced arguments). Therefore, for $x \in X$ and $t \in \mathbb{R}$, we will use x_t to denote the element in X defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in \mathbb{R}$.

Consider the following neutral functional differential equation

$$\frac{d}{dt}[x(t) - b(x_t, \alpha, p)] = f(x_t, \alpha, p) \tag{NFDE}$$

parametrized by two real numbers $(\alpha, p) \in \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, $f: X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is completely continuous, $b: X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous and there exists a constant $L \in [0, 1)$ such that

$$|b(\varphi, \alpha, p) - b(\psi, \alpha, p)| \leqslant L \sup_{\theta \in \mathbb{R}} |\varphi(\theta) - \psi(\theta)|$$

for all $(\alpha, p) \in \mathbb{R} \times \mathbb{R}_+$ and $\varphi, \psi \in X$. Identifying the subspace of X consisting of all constant mappings with $\mathbb{R}^n$, we obtain the restricted mappings $\hat{f} := f|_{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+}$ and $\hat{b} := b|_{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$. We need the smoothness of these two mappings.

(H1) $\hat{b}$ and $\hat{f}$ are twicely continuously differentiable.

Denote by $\hat{x}_0$ the constant mapping with the value $x_0 \in \mathbb{R}^n$. We call $(\hat{x}_0, \alpha_0, p_0)$ a stationary solution of (NFDE) if $\hat{f}(x_0, \alpha_0, p_0) = 0$. We also propose the following assumption:

(H2) At each stationary solution $(\hat{x}_0, \alpha_0, p_0)$, the derivative of $\hat{f}(\hat{x}, \alpha, p)$ with respect to the first variable x, evaluated at (x_0, α_0, p_0), is an isomorphism of $\mathbb{R}^n$.

Therefore, for each stationary solution $(\hat{x}_0, \alpha_0, p_0)$ there exist $\varepsilon_0 > 0$ and a continuously differentiable mapping $y: B_{\varepsilon 0}(\alpha_0, p_0) \to \mathbb{R}^n$ such that $\hat{f}(y(\alpha, p), \alpha, p) = 0$ for

$$(\alpha, p) \in B_\varepsilon(\alpha_0, p_0) := (\alpha_0 - \varepsilon_0, \alpha_0 + \varepsilon_0) \times (p_0 - \varepsilon_0, p_0 + \varepsilon_0).$$

Concerning the characteristic function, the following assumption is natural.

(H3) $b(\varphi, \alpha, p)$ and $f(\varphi, \alpha, p)$ are differentiable with respect to φ, and the $n \times n$ complex matrix function $\Delta_{(y(\alpha, p), \alpha, p)}(\lambda)$ is continuous in $(\alpha, p, \lambda) \in B_{\varepsilon 0}(\alpha_0, p_0) \times \mathbb{C}$.

Here, for each stationary solution $(\hat{x}_0, \alpha_0, p_0)$,

$$\Delta_{(\hat{x}_0, \alpha_0, p_0)}(\lambda) \triangleq \lambda[Id - D_\varphi b(\hat{x}_0, \alpha_0, p_0)(e^{\lambda\cdot} Id)] \\ - D_\varphi f(\hat{x}_0, \alpha_0, p_0)(e^{\lambda\cdot} Id)$$

is called the characteristic matrix of $(\hat{x}_0, \alpha_0\, p_0)$ and the zeros of $\det \Delta_{(\hat{x}_0, \alpha_0, p_0)}(\lambda) = 0$ are called *characteristic values* of $(\hat{x}_0, \alpha_0, p_0)$. A stationary solution $(\hat{x}_0, \alpha_0, p_0)$ is said to be a *center* if it has purely imaginary characteristic values of the form $im(2\pi/p_0)$ for some positive integer m. A center $(\hat{x}_0, \alpha_0, p_0)$ is *isolated* if (i): it is the only center in some neighbourhood of $(\hat{x}_0, \alpha_0, p_0)$ and (ii): it has only finitely purely imaginary characteristic values of the form $im(2\pi/p_0)$, m is an integer.

Let $(\hat{x}_0, \alpha_0, p_0)$ be an isolated center. We set $J(\hat{x}_0, \alpha_0, p_0) = \{m;\, m$ is a positive integer and $im(2\pi/p_0)$ is characteristic value of $(\hat{x}_0, \alpha_0, p_0)\}$. We also assume the following:

(H4) For some $m \in J(\hat{x}_0, \alpha_0, p_0)$ there exist $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \varepsilon_0)$ so that on $[\alpha_0 - \delta, \alpha_0 + \delta] \times \partial\Omega_{\varepsilon, p_0}$, $\det \Delta_{(y(\alpha, p), \alpha, p)}(u + m(2\pi/p)i) = 0$ if and only if $\alpha = \alpha_0$, $u = 0$ and $p = p_0$, where $\Omega_{\varepsilon, p_0} = \{(u, p);\ 0 < u < \varepsilon, p_0 - \varepsilon < p < p_0 + \varepsilon\}$.

Let

$$H_m^{\pm}(\hat{x}_0, \alpha_0, p_0)(u, p) \triangleq \det \Delta_{(y(\alpha_0 \pm \delta, p), \alpha_0 \pm \delta, p)}\left(u + i\frac{2m\pi}{p}\right),$$

$$\gamma_m(\hat{x}_0, \alpha_0, p_0) \triangleq \deg_B(H_m^-(\hat{x}_0, \alpha_0, p_0), \Omega_{\varepsilon, p_0}) \\ - \deg_B(H_m^+(\hat{x}_0, \alpha_0, p_0), \Omega_{\varepsilon, p_0}),$$

where $\deg_B$ denotes the Brouwer degree. We can now state the local and global Hopf bifurcation theorems.

THEOREM A. *Assume that* (H1)–(H4) *hold for some isolated center* $(\hat{x}_0, \alpha_0, p_0)$ *and some* $m \in J(\hat{x}_0, \alpha_0, p_0)$. *If* $\gamma_m(\hat{x}_0, \alpha_0, p_0) \neq 0$ *then there exists a sequence* $(\alpha_k, p_k) \in \mathbb{R} \times \mathbb{R}_+$ *with* $(\alpha_k, p_k) \to (\alpha_0, p_0)$ *as* $k \to \infty$ *and such that at each* (α_k, p_k), *(NFDE) has a nonconstant periodic solution* $x_k(t)$ *with a period* p_k/m *and* $x_k(t) \to \hat{x}_0$ *uniformly for* $t \in \mathbb{R}$ *as* $k \to \infty$.

THEOREM B. *Assume that* (H1)–(H3) *hold and*

(H5) *All centers of* (*NFDE*) *are isolated and* (H4) *holds for every isolated center* $(\hat{x}_0, \alpha_0, p_0)$ *and* $m \in J(\hat{x}_0, \alpha_0, p_0)$;

(H6) *For every bounded set* $W \subset X \times \mathbb{R} \times \mathbb{R}_+$, *there exists* $M > 0$ *so that* $|f(\varphi, \alpha, p) - f(\psi, \alpha, p)| \leqslant M \sup_{\theta \in \mathbb{R}} |\varphi(\theta) - \psi(\theta)|$ *for all* (φ, α, p), $(\psi, \alpha, p) \in W$.

Let

$$\sum(b, f) = \mathscr{C}\ell\{(x(t), \alpha, p);\ x(t) \text{ is a } p\text{-periodic solution of } (NFDE)\};$$

$$\mathcal{N}(b, f) = \{(\hat{x}, \alpha, p); f(\hat{x}, \alpha, p) = 0\};$$

$$\mathscr{C}(\hat{x}_0, \alpha_0, p_0) = \text{the connected component of } (\hat{x}_0, \alpha_0, p_0) \text{ in } \sum(b, f).$$

Then either

(i) $\mathscr{C}(\hat{x}_0, \alpha_0, p_0)$ *is unbounded, or*

(ii) $\mathscr{C}(\hat{x}_0, \alpha_0, p_0)$ *is bounded,* $\mathscr{C}(\hat{x}_0, \alpha_0, p_0) \cap \mathcal{N}(b, f)$ *is finite and*

$$\sum_{(\hat{x}, \alpha, p) \in \mathscr{C}(\hat{x}_0, \alpha_0, p_0) \cap \mathcal{N}(b, f)} \gamma_m(\hat{x}, \alpha, p) = 0,$$

for all $m = 1, 2, \ldots$, *where* $\gamma_m(\hat{x}, \alpha, p)$ *is defined as above if* $m \in J(\hat{x}, \alpha, p)$, *and* $\gamma_m(\hat{x}, \alpha, p) = 0$ *otherwise.*

Proof of Theorems A *and* B. We give only a sketch of the proof. For the detailed argument and related references, we refer to [39].

Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{E} = C(S^1; \mathbb{R}^n)$, $\mathbb{F} = L^2(S^1; \mathbb{R}^n)$ and define $L\colon \mathrm{Dom}(L) = H^1(S^1; \mathbb{R}^n) \subset \mathbb{E} \to \mathbb{F}$, $B, N\colon \mathbb{E} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{F}$ by

$$(Lz)(t) = \dot{z}(t), \qquad N(z, \alpha, p)(t) = \frac{p}{2\pi} f(z_{t,p}, \alpha, p),$$

$$B(z, \alpha, p)(t) = b(z_{t,p}, \alpha, p),$$

where $z_{t,p}(\theta) = z(t + (2\pi/p)\theta)$, $\theta \in \mathbb{R}$. It then follows that $x(t)$ is a p-periodic solution of (NFDE) if and only if $z(t) = x((p/2\pi)t)$ is a solution in $\mathbb{E}$ of the operator equation $L[z - B(z, \alpha, p)] = N(z, \alpha, p)$, called composite coincidence equation.

Let the group S^1 act on $\mathbb{E}$ and $\mathbb{F}$ by shifting the argument. L is then an equivariant bounded linear Fredholm operator of index zero with an equivariant compact resolvent K. Similarly, N is an equivariant compact mapping and B is an equivariant condensing mapping. Moreover, at $(y(\alpha, p), \alpha, p)$,

$$D_z N(y(\alpha, p), \alpha, p)\, z(t) = \frac{p}{2\pi}\, Df(y(\alpha, p), \alpha, p) z_{t,p},$$

$$D_z B(y(\alpha, p), \alpha, p)\, z(t) = Db(y(\alpha, p), \alpha, p) z_{t,p},$$

where $(\alpha, p) \in \mathscr{D} = (\alpha_0 - \delta, \alpha_0 + \delta) \times (p_0 - \varepsilon, p_0 + \varepsilon)$. Identify $\partial\mathscr{D}$ with the unit circle S^1. It follows that the mapping $U := Id - D_z B(y(\alpha, p), \alpha, p) - (L + K)^{-1}\ [D_z N(y(\alpha, p), \alpha, p) + K(Id - D_z B(y(\alpha, p), \alpha, p))]$ is an isomorphism of $\mathbb{E}$ for $(\alpha, p) \in \partial\mathscr{D}$ and the mapping $\Psi: S^1 \to GL(\mathbb{E})$ given by $(\alpha, p) \in S^1 = \partial\mathscr{D} \to U(\alpha, p) \in GL(\mathbb{E})$ is continuous.

$\mathbb{E}$ has the (isotypical) decomposition $\mathbb{E} = \bigoplus_{k=0}^{\infty} \mathbb{E}_k$, where $\mathbb{E}_0 \cong \mathbb{R}^n$ and for each $k \geqslant 1$, $\mathbb{E}_k$ is spanned by $\cos(kt)\varepsilon_j$ and $\sin(kt)\varepsilon_j$, $1 \leqslant j \leqslant n$, where $\{\varepsilon_1, ..., \varepsilon_n\}$ is the standard basis of $\mathbb{R}^n$. So, we have $\Psi(\alpha, p)\, \mathbb{E}_k \subseteq \mathbb{E}_k$, which leads to a well-defined mapping $\Psi_k(\alpha, p) = \Psi(\alpha, p)|_{\mathbb{E}_k}$, for each $k \geqslant 0$. Now a direct computation shows that

$$\Psi_k(\alpha, p) = \frac{p}{2k\pi i}\, \Delta_{(y(\alpha, p), \alpha, p)}\left(ik\,\frac{2\pi}{p}\right).$$

Let

$$\varepsilon = \text{sign det } \Psi_0(\alpha, p), \qquad (\alpha, p) \in \partial\mathscr{D},$$

$$n_k(\hat{x}_0, \alpha_0, p_0) = \varepsilon \deg_B(\det \Psi_k, \partial\mathscr{D}), \quad k = 1, 2, ...$$

The corresponding S^1-equivariant degree relative to $\mathscr{D}$ for the nonlinear composite coincidence equation $L[z - B(z, \alpha, p)] = N(z, \alpha, p)$, as shown in [39], is a sequence of integers whose kth component is exactly $n_k(\hat{x}_0, \alpha_0, p_0)$. Moreover, it is also known from [39] that $n_k(\hat{x}_0, \alpha_0, p_0) = \gamma_k(\hat{x}_0, \alpha_0, p_0)$, $k = 1, 2, ...$. Consequently, Theorem A and B follow from the existence and additivity properties of the S^1-degree (see [39] for details). This completes the proof.

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