THE EFFECT OF DELAY AND DIFFUSION
ON SPONTANEOUS SYMMETRY BREAKING
IN FUNCTIONAL DIFFERENTIAL EQUATIONS

JIANHONG WU

ABSTRACT. We generalize a local Hopf bifurcation theorem of Golubitsky and Stewart to retarded functional differential equations in the presence of symmetry and illustrate this generalization by discrete waves (phase-locked oscillations) in Turing rings with retarded diffusion along the sides of a polygon. The effect of delay and diffusion on the occurrence of spontaneous symmetry breaking will be demonstrated.

1. Introduction. One of the purposes of this paper is to report a result on the existence of a smooth branch of periodic solutions with prescribed symmetries bifurcating from equilibria of retarded functional differential equations in the presence of symmetry. Such an existence result was established for ordinary differential equations by Golubitsky and Stewart [**5**]. However, a proof of the analogue for functional differential equations is not an elementary exercise as the precise statement and verification of the hypotheses of this analogue require nontrivial applications of some important facts of the generalized eigenspaces of the infinitesimal generators of solution semigroups and the decomposition theory of linear retarded functional differential equations. Another purpose of this paper is to illustrate the effect of temporal delay and spatial diffusion in symmetric dynamical systems on the process of spontaneous symmetry breaking and the occurrence of various types of oscillations. In particular, we will show that in a ring of identical cells coupled by diffusion along the sides of a polygon where the state of each cell is described by one single variable, large temporal delay in the diffusion process may cause phase-locked oscillations, though such an oscillation cannot occur if the delay is ignored (see, cf., [**4, 5**]).

2. The existence of smooth branches of periodic solutions of equivariant delay differential equations. Let $r \geq 0$ be a given real number and C denote the Banach space of continuous mappings from $[-r, 0]$ into $\mathbf{R}^n$ equipped with the supremum norm $||\varphi|| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$ for $\varphi \in C$. In what follows, if $\sigma \in \mathbf{R}$, $A \geq 0$ and $x : [\sigma - r, \sigma + A] \to \mathbf{R}^n$ is a continuous mapping, then $x_t \in C$, $t \in [\sigma, \sigma + A]$, is defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

Consider the following one parameter family of retarded functional differential equations

$$(2.1) \qquad\qquad \dot{x}(t) = f(x_t, \alpha)$$

where $f : C \times \mathbf{R} \to \mathbf{R}^n$ is continuously differentiable and $f(0, \alpha) = 0$ for all $\alpha \in \mathbf{R}$. Denote by $D_\varphi f(0, \alpha)$ the derivative of $f(\varphi, \alpha)$ with respect to φ, evaluated at $(0, \alpha)$. Then there exists an $n \times n$ matrix $\eta(\alpha, \theta)$ of bounded variation in $\theta \in [-r, 0]$ such that

$$D_\varphi f(0, \alpha)\psi = \int_{-r}^{0} [d\eta(\alpha, \theta)]\psi(\theta), \qquad \psi \in C.$$

It is well known that (see [**7**]) the linear equation

$$(2.2) \qquad\qquad \dot{x}(t) = \int_{-r}^{0} [d\eta(\alpha, \theta)]x(t + \theta)$$

generates a strongly continuous semigroup of linear operators with the infinitesimal generator $A(\alpha) : \mathcal{D}(A(\alpha)) \subseteq C \to C$ given by

$$A(\alpha)\varphi = \dot{\varphi}, \qquad \varphi \in \mathcal{D}(A(\alpha)),$$
$$\mathcal{D}(A(\alpha)) := \{\varphi \in C; \dot{\varphi} \in C, \dot{\varphi}(0) = D_\varphi f(0, \alpha)\varphi\}.$$

The spectrum $\sigma(A(\alpha))$ of $A(\alpha)$ consists of eigenvalues which are solutions of the following characteristic equation

$$(2.3) \qquad\qquad \det \Delta(\alpha, \lambda) = 0$$

where

$$\Delta(\alpha, \lambda) = \lambda Id - \int_{-r}^{0} e^{\lambda\theta} [d\eta(\alpha, \theta)].$$

We also assume that there exists a compact Lie group Γ acting on $\mathbf{R}^n$ such that (2.1) is Γ-equivariant, i.e.,

$$f(\gamma\varphi, \alpha) = \gamma f(\varphi, \alpha)$$

for $\gamma \in \Gamma$, $(\varphi, \alpha) \in C \times \mathbf{R}$, where $\gamma\varphi \in C$ is defined by

$$(\gamma\varphi)(\theta) = \gamma\varphi(\theta), \qquad \theta \in [-r, 0].$$

Our first hypothesis is as follows

(H1) There exists $\alpha_0 \in \mathbf{R}$ and $\beta_0 > 0$ such that

(i) $A(\alpha_0)$ has eigenvalues $\pm i\beta_0$;

(ii) the generalized eigenspace, denoted by $\mu_{i\beta_0}(A(\alpha_0))$, of these eigenvalues consists of eigenvectors of $A(\alpha_0)$ only;

(iii) all other eigenvalues of $A(\alpha_0)$ are not integer multiple of $\pm i\beta_0$.

Note that we do not require the eigenvalues $\pm i\beta_0$ to be simple. In fact, the presence of symmetry often forces those purely imaginary eigenvalues to be multiple. Hence, the standard Hopf bifurcation theorem of functional differential equations cannot be applied.

To state our next hypothesis, we recall that, under (H1), $\mu_{i\beta_0}(A(\alpha_0))$ is the real vector space consisting of $\mathrm{Re}\,(e^{i\beta_0 \cdot}b)$ and $\mathrm{Im}\,(e^{i\beta_0 \cdot}b)$ such that $b \in \mathrm{Ker}\,\Delta(\alpha_0, i\beta_0)$. Moreover, there exists a natural identification between $\mathrm{Ker}\,\Delta(\alpha_0, i\beta_0)$ and $\mathbf{R}^{2m}$, where $2m = \dim \mathrm{Ker}\,\Delta(\alpha_0, i\beta_0)$ as a real vector space. We require

(H2) There exists an m-dimensional absolutely irreducible representation V of Γ such that $\mathrm{Ker}\,\Delta(\alpha_0, i\beta_0)$ is Γ-isomorphic to $V \oplus V$.

Here a representation V of Γ is absolutely irreducible if the only linear mapping that commutes with Γ is a scalar multiple of the identity.

(H2) is the equivariant analogue of the "simple eigenvalue" assumption in the standard Hopf bifurcation theorem. A slight modification of the argument in [**6**, p. 264] shows that the nontrivial restriction (H2) on the eigenspace placed by symmetry is a generic assumption.

Under (H1) and (H2), we can obtain

Proposition 2.4. *There exists $\delta_0 > 0$ and a continuously differentiable function $\lambda : (\alpha_0 - \delta_0, \alpha_0 + \delta_0) \to \mathbf{C}$ such that $\lambda(\alpha_0) = i\beta_0$*

and for each $\alpha \in (\alpha_0 - \delta_0, \alpha_0 + \delta_0)$, $\lambda(\alpha)$ *is an eigenvalue of* $A(\alpha)$, $\mu_{\lambda(\alpha)}(A(\alpha))$ *consists of eigenvectors of* $A(\alpha)$ *and* $\dim \mu_{\lambda(\alpha)}(A(\alpha)) = \dim \mu_{i\beta_0}(A(\alpha_0))$.

The above proposition enables us to make the following transversality assumption

(H3) $$\left. \frac{d}{d\alpha} \operatorname{Re} \lambda(\alpha) \right|_{\alpha=\alpha_0} \neq 0.$$

Under further assumption, it is possible to apply the result of Chow, Mallet-Paret and Yorke [**2**] on Hopf bifurcations from multiple eigenvalues to establish the existence of small-amplitude periodic solutions. However, our primary interest here is the existence of periodic solutions with prescribed symmetry. To specify the prescribed symmetry, we let $\omega = 2\pi/\beta_0$ and denote by P_ω the Banach space of all continuous ω-periodic function $x : \mathbf{R} \to \mathbf{R}^n$. Identify S^1 with $\mathbf{R}/\omega\mathbf{Z}$ and define an action $\Gamma \times S^1$ on P_ω by

$$(\gamma, \theta)x(t) = \gamma x(t + \theta)$$

for $(\gamma, \theta) \in \Gamma \times S^1$, $x \in P_\omega$, $t \in \mathbf{R}$. The isotropy subgroup of an element $x \in P_\omega$ is defined by

$$\Sigma_x = \{(\gamma, \theta); \ (\gamma, \theta)x = x\}$$

which is also called the group of the symmetry of x. Clearly, Σ_x is a combination of the spatial symmetry Γ and the temporal dynamic-shift symmetry S^1. It can be shown that the subspace SP_ω of P_ω consisting of all ω-periodic solutions of the linearization (2.2) with $\alpha = \alpha_0$ is $\Gamma \times S^1$-invariant, i.e., $(\gamma, \theta)SP_\omega \subseteq SP_\omega$ for all $(\gamma, \theta) \in \Gamma \times S^1$. Therefore, for every isotropy group $\Sigma \leq \Gamma \times S^1$, the fixed point set

$$\operatorname{Fix}(\Sigma, SP_\omega) := \{x \in SP_\omega; \ (\gamma, \theta)x = x \text{ for } (\gamma, \theta) \in \Sigma\}$$

is a subspace.

Now we are in a position to state a local Hopf bifurcation theorem for (2.1).

Theorem 2.5. *Assume that* (H1)–(H3) *are satisfied. If there exists a subgroup* $\Sigma \leq \Gamma \times S^1$ *such that*

$$(2.6) \qquad\qquad \dim \operatorname{Fix}(\Sigma, SP_\omega) = 2,$$

then there exists a unique branch of small-amplitude periodic solutions of (2.1) *with period near* $2\pi/\beta_0$, *having* Σ *as the group of symmetry. More precisely, for a chosen basis* $\{\sigma_1, \sigma_2\}$ *of* $\operatorname{Fix}(\Sigma, SP_\omega)$ *there are constants* $a_0 > 0$, $a_0^* > 0$, $\delta_0 > 0$, *functions* $\alpha : \mathbf{R}^2 \to \mathbf{R}$, $\omega^* :$ $\mathbf{R}^2 \to (0, \infty)$ *and* $x^* : \mathbf{R}^2 \to \mathbf{R}$, *with all functions being continuously differentiable in* $a \in \mathbf{R}^2$ *with* $|a| < a_0$, *such that* $x^*(a)$ *is an* $\omega(a)$-*periodic solution of* (2.1) *with* $\alpha = \alpha(a)$, *and*

$$\gamma x^*(a)(t) = x^*(a)\left(t - \frac{\omega(a)}{\omega}\theta\right), \qquad (\gamma, \theta) \in \Sigma,$$

$$x^*(0) = 0, \qquad \omega^*(0) = \omega, \qquad \alpha(0) = \alpha_0,$$

$$x^*(a) = (\sigma_1, \sigma_2)a + z^*(a)$$

$$z^*(a) = o(|a|) \quad \text{as } |a| \to 0.$$

Furthermore, for $|\alpha - \alpha_0| < \alpha_0^*$, $|\omega^* - 2\pi/\beta_0| < \delta_0$, *every* ω^*-*periodic solution of* (2.1) *with* $||x_t|| < \delta_0$, $\gamma x(t) = x(t - (\omega^*/\omega)\theta)$ *for* $(\gamma, \theta) \in \Sigma$, $t \in \mathbf{R}$, *must be of the above type.*

The above local Hopf bifurcation theorem says that, under certain nonresonance, genericity and transversality assumptions maximal isotropy groups with minimal dimensional fixed-point subspace lead to periodic solutions with a certain spatial-temporal symmetry. This represents an analogue for functional differential equations of the Golubitsky-Stewart theory. The detailed proofs and related discussion as well as some results on the stability of the obtained periodic solutions can be found in [**12**]. A similar result was also obtained in [**3**] via a topological approach based on an equivariant degree theory. However, the smoothness of the branch of periodic solutions with respect to the parameter cannot be described due to the topological nature of the argument.

3. Application to Turing rings with delayed coupling. As an illustrative example, we consider the following system of delay-

differential equations

$$(3.1) \quad \dot{y}_i(t) = -\alpha y_i(t) + \alpha h(y_i(t))[g(y_{i-1}(t-1)) \\ + g(y_{i+1}(t-1)) - 2g(y_i(t-1))]$$

where $i \pmod n$, $\alpha > 0$ is a positive constant, $h, g : \mathbf{R} \to \mathbf{R}$ are continuously differentiable functions with

$$g(0) = 0, \qquad g'(0) \neq 0$$
$$h(x) \neq 0 \quad \text{for all } x \in \mathbf{R},$$
$$\mu := h(0)g'(0) > 0.$$

Equation (3.1) can be obtained, by rescaling the time and making a change of variables, from the following system

$$(3.2) \quad \dot{v}_i(t) = -f(v_i(t)) + d[v_{i+1}(t-\alpha) + v_{i-1}(t-\alpha) - 2v_i(t-\alpha)]$$

which arises from the study of a ring of n identical cells with delayed coupling between adjacent cells, where the kinetics of each cell is described by a simple scalar ordinary differential equation $\dot{v}_i = -f(v_i)$ and the coupling of cells is nearest-neighbor, symmetric and delayed.

The linearization of (3.1) at the zero solution is

$$\dot{x}_i(t) = -\alpha x_i(t) + \alpha\mu[x_{i-1}(t-1) + x_{i+1}(t-1) - 2x_i(t-1)]$$

and the characteristic equation becomes

$$\det \Delta(\alpha, \lambda) = 0, \qquad \Delta(\alpha, \lambda) = (\lambda + \alpha)\mathrm{Id} - \alpha\mu e^{-\lambda}\delta,$$

where δ is defined by

$$(\delta x)_i = x_{i+1} + x_{i-1} - 2x_i, \qquad i \pmod n, x \in \mathbf{R}^n.$$

Let $\xi = e^{i2\pi/n}$ and

$$\mathbf{C}_r = \{(1, \xi^r, \dots, \xi^{(n-1)r},)^T a; \ a \in \mathbf{C}\}, \qquad 0 \le r \le n-1.$$

Then

$$\mathbf{C}^n = \mathbf{C}_0 \oplus \mathbf{C}_1 \oplus \cdots \oplus \mathbf{C}_{n-1}, \qquad \Delta(\alpha, \lambda)\mathbf{C}_r \subseteq \mathbf{C}_r,$$

$$\Delta(\alpha,\lambda)|_{\mathbf{C}_r} = \Delta_r(\alpha,\lambda) := \lambda + \alpha + 4\alpha\mu e^{-\lambda}\sin^2 \pi r/n,$$
$$0 \le r \le n-1.$$

So

$$\det \Delta(\alpha,\lambda) = \prod_{r=0}^{n-1}[\lambda + \alpha + 4\alpha\mu e^{-\lambda}\sin^2(\pi r/n)].$$

For simplicity, we only consider the case where $n = 2k+1$ is an odd number. Assume

$$\mu > \frac{1}{4\sin^2(k\pi/(2k+1))}$$

and define $\beta_0 \in (\pi/2, \pi)$ as the unique solution of

$$\cos \beta_0 = -\frac{1}{4\mu\sin^2(k\pi/(2k+1))}.$$

Let

$$\alpha_0 = -\frac{\beta_0}{\tan \beta_0}.$$

Then $\Delta_r(\alpha_0, i\beta_0) = 0$ for $r = k$ and $k+1$. Moreover, if $\lambda(\alpha)$ is a smooth curve of zeros of $\Delta_r(\alpha,\lambda)$ with $\lambda(\alpha_0) = i\beta_0$, then $(d/d\alpha)\mathrm{Re}\,\lambda(\alpha)|_{\alpha=\alpha_0} > 0$. Consequently,

$$\dim \mu_{i\beta_0}(A(\alpha_0)) = 4.$$

Equation (3.1) is Γ-equivariant, where $\Gamma = \mathbf{D}_n$ acts on $\mathbf{R}^n$ by

$$(e^{i2\pi/n} \cdot x)_j = x_{j-1}, \qquad (\kappa \cdot x)_j = x_{n-j}, \qquad x \in \mathbf{R}^n, j \pmod{n}.$$

As

$$\mathrm{Ker}\,\Delta(\alpha_0, i\beta_0) = \{(z_1 + iz_2)\varepsilon + (z_3 + iz_4)\bar{\varepsilon};\ z_i \in \mathbf{R}, i = 1,\dots,4\},$$

where

$$\varepsilon = (1, \xi^k, \xi^{2k}, \dots, \xi^{(n-1)k})^T,$$
$$\bar{\varepsilon} = (1, \xi^{-ik}, \xi^{-2k}, \dots, \xi^{-(n-1)k})^T,$$

we can easily show that the restricted representation of Γ on $\mathrm{Ker}\,\Delta(\alpha_0, i\beta_0)$ is Γ-isomorphic to $\mathbf{R}^2 \oplus \mathbf{R}^2$, where the action of Γ on $\mathbf{R}^2$ is absolutely irreducible.

Let $\sin^{\beta_0}, \cos^{\beta_0}$ denote the $2\pi/\beta_0$-periodic function from $\mathbf{R}$ to $\mathbf{R}$ defined by

$$(\sin^{\beta_0})t = \sin\beta_0 t, \qquad (\cos)^{\beta_0} t = \cos\beta_0 t, \qquad t \in \mathbf{R}.$$

Denote by $\omega = 2\pi/\beta_0$. Then SP_ω is spanned by

$$\begin{cases} \varepsilon_1 = \sin^{\beta_0} u + \cos^{\beta_0} v, \\ \varepsilon_2 = \sin^{\beta_0} u - \cos^{\beta_0} v, \\ \varepsilon_3 = \cos^{\beta_0} u - \sin^{\beta_0} v, \\ \varepsilon_4 = \cos^{\beta_0} u + \sin^{\beta_0} v, \end{cases}$$

where

$$u = \left(1, \cos\frac{2k\pi}{2k+1}, \dots, \cos\frac{2k(n-1)\pi}{2k+1}\right)^T,$$
$$v = \left(0, \sin\frac{2k\pi}{2k+1}, \dots, \sin\frac{2k(n-1)\pi}{2k+1}\right)^T.$$

Clearly, for every $\theta \in [0, \omega)$,

$$\Sigma^\theta := \{(e^{i\cdot(2\pi/(2k+1))j}, e^{i\theta j}); \; j = 0, 1, \dots, 2k\}$$

is a subgroup of $\Gamma \times S^1$. After a straightforward but tedious computation, we have

$$\mathrm{Fix}\,(\Sigma^\theta, SP_\omega) = \begin{cases} \{z_1\varepsilon_1 + z_3\varepsilon_3; \; z_1, z_3 \in \mathbf{R}\} & \text{if } \theta = (k/(2k+1))\omega, \\ \{z_2\varepsilon_2 + z_4\varepsilon_4; \; z_2, z_4 \in \mathbf{R}\} & \text{if } \theta = ((k+1)/(2k+1))\theta, \\ \{0\} & \text{otherwise.} \end{cases}$$

So

$$\dim\mathrm{Fix}\,(\Sigma^\theta, SP_\omega) = \begin{cases} 2 & \text{if } \theta = k/(2k\pi) \text{ or } \theta = ((k+1)/(2k+1))\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 2.4, we obtain

Theorem 3.3. *There exists a unique branch of small-amplitude periodic solutions of* (3.1) *with period near $2\pi/\beta_0$, having $\Sigma^{(k/(2k+1))\theta}$ as the group of symmetry. More precisely, there exist constants $a_0 > 0$,*

$\alpha_0^* > 0$, $\delta_0 > 0$ *functions* $\alpha(z_1, z_2)$, $\omega(z_1, z_2)$ *and an* $\omega(z_1, z_2)$-*periodic function* $x^*(z_1, z_2)$, *with all functions being continuously differentiable in* $(z_1, z_2)^T \in \mathbf{R}^2$ *with* $|z_1| + |z_2| < a_0$, *such that* $x^*(z_1, z_2)$ *is a solution of* (3.1), *when* $\alpha = \alpha(z_1, z_2)$ *and*

$$x_{i-1}^*(z_1, z_2)(t) = x_i^*(z_1, z_2)\left(t - \frac{k}{2k+1}\omega(z_1, z_2)\right),$$

$$x^*(0,0) = 0, \qquad \omega^*(0,0) = 2\pi/\beta_0, \qquad \alpha^*(0,0) = \alpha_0,$$

$$x^*(z_1, z_2) = (z_1 \sin^{\beta_0} + z_2 \cos^{\beta_0})u + (z_1 \cos^{\beta_0} - z_2 \sin^{\beta_0})v$$

$$+ o(|z_1| + |z_2|) \quad as \ |z_1| + |z_2| \to 0.$$

Furthermore, for $|\alpha - \alpha_0| < \alpha_0^*$, $|\omega^* - 2\pi/\beta_0| < \delta_0$, *every* ω^*-*periodic solution of* (3.1) *with* $||x_t|| < \delta_0$ *and* $x_{i-1}(t) = x_i(t - (k/(2k+1))\omega^*)$ *for* $t \in \mathbf{R}$ *must be of the above type.*

Remark 3.4. The periodic solution obtained in Theorem 3.3 is a special form of discrete waves and is called phase-locked oscillation [**1**] where each cell oscillates like others except not in phase with each other.

Note that the symmetry of the bifurcated branch of periodic solutions decreases to proper subgroup of $\Gamma \times S^1$ of the equilibrium. This feature of spontaneous symmetry breaking has been extensively investigated for ordinary differential equations and partial differential equations. But, to the best of my knowledge, little has been done about the effect of temporal delay on the symmetry breaking.

Remark 3.5. The phase-locked oscillation observed in Theorem 3.3 is totally attributed to the temporal delay in the diffusion process. In fact, if $\alpha = 0$ then (3.2) generates a strongly monotone dynamical system. According to a result of Hirsh [**8**], each solution of the linearization of (3.2) either converges to zero or infinity as $t \to \infty$ and, consequently, (3.2) has no Hopf bifurcation. In general, it has been shown that a ring of identical cells coupled by instant diffustion between adjacent cells does not exhibit Hopf bifurcation if the state of each cell is described by a single variable (see [**4, 5, 6**]). Consequently, it is the temporal delay that causes the oscillation of (3.2).
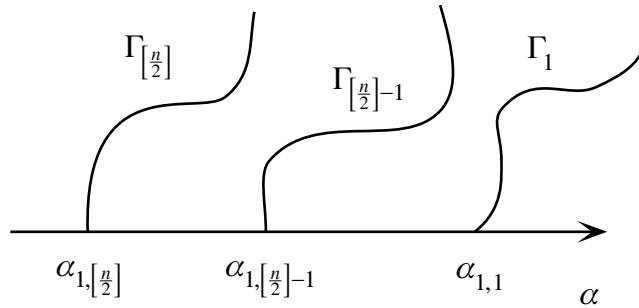
FIGURE 1. Each branch Γ_r represents a branch of nontrivial periodic solutions with the symmetry $x_{j-1}(t) = x_j(t - (r/n)\omega)$, $t \in R$, where the period ω near $2\pi/\beta_{1,r} \in (2,4)$.

Remark 3.6. The argument for Theorem 3.3 can also be applied to obtain the existence of a unique branch of small-amplitude periodic solution of (3.1) with period near $2\pi/\beta_0$ and the symmetry

$$x_{i-1}(t) = x_i(t - \omega/2), \qquad t \in \mathbf{R},$$

in the case where n is even. One can also obtain several other branches of periodic solutions for arbitrary n. More precisely, for any n, m, r with $0 \le r \le n - 1$, let

$$\cos \beta_{1,r} = -\frac{1}{4\mu \sin^2(\pi r/n)}, \qquad \beta_{1,r} \in (\pi/2, \pi),$$

$$\beta_{m,r} = \beta_{1,r} + (m-1)2\pi,$$

$$\alpha_{m,r} = -\frac{\beta_{m,r}}{\tan \beta_{m,r}}.$$

If

$$\left(4\mu \sin^2 \frac{\pi pr}{n} \right)^2 \neq p^2 \left[\left(4\mu \sin^2 \frac{\pi r}{n} \right) - 1 \right] \quad \text{for all } p \ge 1$$

then there exists a unique branch of small-amplitude periodic solutions of (3.1) with period ω near $2\pi/\beta_{m,r}$ and having the following symmetry

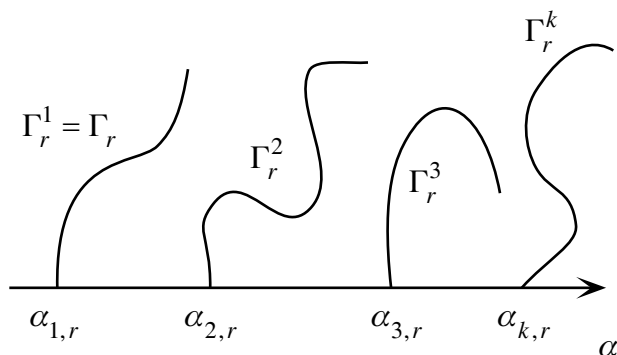$$x_{i-1}(t) = x_i\left(t - \frac{r}{n}\omega \right).$$

FIGURE 2. Each Γ_r^m, $1 \leq m < \infty$, represents a branch of nontrivial periodic solutions with the symmetry $x_{j-1}(t) = x_j(t-(r/n)\omega)$, $t \in R$, where the period ω near $2\pi/\beta_{m,r} \in (2/(2m-1), 2/(2m-3/2))$.

These results can be depicted in Figure 1 and Figure 2. The description of the maximal continuation of these branches requires a global bifurcation theory developed in [**3**] and the stability of the obtained periodic solutions is still an open problem.

REFERENCES

1. J.C. Alexander and G. Auchmuty, *Global bifurcations of phase-locked oscillations*, Arch. Rational Mech. Anal. **93** (1986), 253–270.

2. S.-N. Chow, J. Mallet-Paret and J. Yorke, *Global Hopf bifurcation from a multiple eigenvalue*, Nonlinear Anal. **2** (1978), 753–763.

3. K. Geba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems*, III: *Hopf bifurcation theorems of functional differential equations with symmetries*, preprint.

4. M. Golubitsky and I.N. Stewart, *Hopf bifuration with dihedral group symmetry*: *Coupled nonlinear oscillators*, in *Multiparameter bifurcation theory* (C.M. Golubitsky and J. Guckenheimer, eds.), 1985, 131–173.

5. ———, *Hopf bifurcation in the presence of symmetry*, Arch. Rational Mech. Anal. **817** (1985), 107–165.

6. M. Golubitsky, I. Stewart and D.G. Schaeffer, *Singularities and groups in bifurcation theory*, Vol. 2, Springer-Verlag, New York, Berlin, 1988.

7. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, Berlin 1977.

8. M.W. Hirsch, *The dynamical systems approach to differential equations*, Bull. Amer. Math. Soc. **11** (1984), 1–64.

9. W. Krawcewicz and J. Wu, *Discrete waves and phase-locked oscillations in the growth of a single-species populations over a patchy environment*, Open Systems and Information Dynamics **1** (1992), 127–147.

10. D.H. Sattinger, *Spontaneous symmetry breaking in bifurcation problems*, in *Systems in science* (C.B. Gruker and R.S. Millman, eds.), Plenum Press, New York, 1980, 365–383.

11. A. Vanderbauwhede, *Local bifuration and symmetries*, Research Notes in Math. **75**, Pitman, London, 1982.

12. J. Wu, *The existence and stability of symmetric periodic solutions in functional differential equations*, preprint.

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, NORTH YORK, ON, CANADA M3J 1P3