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# REACTION-DIFFUSION EQUATIONS WITH INFINITE DELAY

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ABSTRACT. We have developed several results on the existence and asymptotic behavior of mild solutions to reaction-diffusion systems that have infinite delays in the nonlinear reaction terms. We find that the semiflow generated by a cooperative and irreducible reaction-diffusion system with infinite delay is not compact but set-condensing, and not strongly order-preserving but quasi strongly order-preserving. These set-condenseness and quasi strong order-preserving properties allow us to use a modification, recently given by Freedman, Miller and one of the authors of this paper, of the well-known monotone dynamical system theory due to Dancer, Hess, Hirsch, Matano, Smith, Thieme, Poláčik and Takáč to obtain some results about convergence and stability of solutions. Examples of Lotka-Volterra competition-diffusion models with distributed delay are given to illustrate the obtained results.

**1. Introduction.** A variety of mathematical models for biological processes are most appropriately framed as partial functional differential equations. For example, the reaction-diffusion logistic equation with finite delay

(1.1)
$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + ru(t,x)\left[1 - \frac{u(t-\tau,x)}{K}\right], \qquad t>0,\ x\in(0,1)$$
$$\frac{\partial u(t,x)}{\partial x} = 0, \qquad x = 0,1,$$

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where $d, r, \tau$ and $K$ are positive constants, has been used to model an one-dimensional herbivorous population and has been studied by many authors, for example, Busenberg and Huang [5], Friesecke [8, 9], Gopalsamy, He and Sun [13], Green and Stech [14], Huang [28], Lin and Khan [33], Luckhaus [35], Memory [44, 45], Morita [47], Yoshida [76] and Yoshida and Kishimoto [77], to name a few.

General partial functional differential equations with finite delay have been extensively studied. We refer to Fitzgibbon [10], Rankin [55] and Travis and Webb [68, 69] for detailed discussions on the existence and asymptotic behavior of solutions; to Kunish and Schappacher [32] for necessary conditions to generate $C_0$-semigroups; to Hale [17] for the convergence to solutions of an ordinary functional differential equation; to Fitzgibbon and Parrott [11] and Parrott [50] for the linerized stability; to He [20, 21] for periodic and almost periodic solutions; to Lin, So and Wu [34] for a center manifold theory; to Hale and Ladeira [19] for the differentiability with respect to delays, and to Rey and Mackey [57, 58] for bifurcations, traveling waves and multistability.

Recently Martin and Smith, in their three consecutive papers [36–38], have studied partial functional differential equations in a Banach space. They developed several fundamental results on the existence and asymptotic behavior of solutions to abstract semilinear functional differential equations with finite delay and then apply the results to reaction-diffusion equations which have finite time delays in the nonlinear reaction terms. By employing the monotone dynamical system theory due to Hirsch [25–27], Matano [39–42], Smith [61], and Smith and Thieme [63, 64], they established sufficient conditions for a reaction-diffusion equation with finite delay to generate a (eventually) strongly monotone semiflow on an appropriate space and concluded that almost all orbits converge to the set of equilibria. They also established the existence of an invariant rectangle and obtained certain comparing systems of ordinary functional differential equations relative to the invariant rectangle. As an application, they considered the $n$-species Lotka-Volterra model of competition with diffusion and finite delay and developed sufficient conditions for the global asymptotic stability of the coexistence state.

It is well known that distributed delay should be used to describe the stochastic element in the delayed response of a biological process. The following reaction-diffusion equation with infinite delay

$$(1.2)\qquad \begin{aligned} &\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + ru(t,x)\left[1-\frac{1}{K}\int_{-\infty}^{t} k(t-s)u(s,x)ds\right], \\ &\qquad\qquad t>0,\ x\in(0,1) \\ &\frac{\partial u(t,x)}{\partial x} = 0, \qquad x=0,1 \end{aligned}$$

with

$$d>0,\ r>0,\ K>0,\ k(s)>0 \quad \text{for } s>0$$

and

$$\int_0^\infty k(s)ds = 1$$

also has been used as a modified version of (1.1) and has been studied by Bonilla and Liñán [**3**], Britton [**4**], Redlinger [**56**], Schiaffino [**60**], Tesei [**67**] and Yamada [**73**]. Particular classes of reaction-diffusion systems with infinite delay have been investigated by Gopalsamy [**12**], Kuang and Smith [**30, 31**], Pozio [**54**], Yamada [**74**], and Yamada and Niikura [**75**]. However, there are all together very few results on general theory of reaction-diffusion systems with infinite delay.

One of the main purposes of this paper is to develop a general theory of existence, comparison, invariance, monotonicity and set-condenseness for partial functional differential equations with infinite delay and to provide some applications to reaction-diffusion systems with general distributed delays. We shall follow Martin and Smith [**36-38**] to derive general results for abstract semilinear integral equations in general phase spaces and then draw some conclusions for reaction-diffusion systems with distributed delays. We also establish an invariance principle which generalizes a well-known result due to Haddock and Terjéki [**15**] from ordinary to partial functional differential equations.

Our approaches and ideas are motivated by those in Martin and Smith [**36-38**] for reaction-diffusion systems with finite delay and in Hale and Kato [**18**] for ordinary functional differential equations with infinite delay. However, there are numerous technical difficulties in dealing with partial functional differential equations with infinite delay due to the unboundedness of the delay involved. In particular, the choice of phase space is a nontrivial job and plays an important role in establishing several inequalities which are essential to build a general theory. Secondly, the corresponding semiflow is no longer compact, but set-condensing in the sense of Nussbaum [49]. Thirdly, an example in Wu [70] indicates that the solution semiflow of a functional differential equation with infinite delay may not be (eventually) strongly monotone even if the usual quasimonotonicity and irreducibility conditions are satisfied. We shall show that the solution semiflow generated by a certain reaction-diffusion system with infinite delay is quasi strongly order-preserving in the sense described in Section 2. This enables us to apply a result due to Freedman, Miller and Wu [7], which represents some modification and generalization of those recent results due to Dancer and Hess [6], to establish some general results on the convergence and stability for a class of reaction-diffusion systems with infinite delay.

As an application of the invariance principle established in this paper, we obtain the convergence of the steady state of the $n$-species Lotka-Volterra competition-diffusion model with distributed delay. We also find that the two-species Lotka-Volterra competition-diffusion model with infinite delay generates a quasi strongly order-preserving semiflow, hence we can rule out the occurrence of pattern formation in certain situations. It is observed that the asymptotic behavior of solutions of the two-species Lotka-Volterra competition-diffusion model with infinite delay is similar to that of the two-species Lotka-Volterra competition-diffusion model without a delay or with a discrete delay.

The paper is organized as follows. Section 2 contains a short survey of some recent results about set-condensing and quasi strongly order-preserving semiflows. In Section 3, we introduce the definition and examples of a fundamental phase space which will be used throughout this paper. In Section 4, we establish a general existence theorem for the abstract semilinear integral equations. Comparison and monotonicity principles are described, in Section 5, for the abstract semilinear integral equations and, in Section 6, for the reaction-diffusion systems with infinite delay. Finally, in Section 7, we apply our general results to the Lotka-Volterra reaction-diffusion models with distributed delay.

**2. Preliminary results on order-preserving and set-condensing semiflows.** Consider the Banach space $B$ with a closed ordered cone

$B_+$. For $\phi$, $\psi \in B$, we write $\phi \geq_B \psi$ if $\phi - \psi \in B_+$, $\phi >_B \psi$ if $\phi - \psi \in B_+ \setminus \{0\}$ and $\phi \gg_B \psi$ if $\mathrm{int}(B_+) \neq \varnothing$ and $\phi - \psi \in \mathrm{int}(B_+)$. $\phi \leq_B (<_B, \ll_B)\psi$ means $\psi \geq_B (>_B, \gg_B)\phi$.

Let $R_+ = [0, \infty)$. Assume $\Phi : R_+ \times B \to B$ is a given semiflow. For each $\phi \in B$, define the positive orbit initiating from $\phi$ by $\gamma^+(\phi) = \{\Phi_t(\phi);\ t \geq 0\}$ and the omega limit set of $\gamma^+(\phi)$ by

$$\omega(\phi) = \bigcap_{t \geq 0} \overline{\gamma^+(\Phi_t(\phi))},$$

here and in what follows, $\Phi_t(\phi) = \Phi(t, \phi)$ for $(t, \phi) \in R_+ \times B$. It is well-known that if $\gamma^+(\phi)$ is relatively compact then $\omega(\phi)$ is nonempty, compact, connected and invariant.

The semiflow $\Phi$ is said to be *strictly order-preserving* if $(t, \phi)$, $(t, \psi) \in R_+ \times B$ with $\phi >_B \psi$ implies $\Phi_t(\phi) >_B \Phi_t(\psi)$ for $t > 0$. A given point $\phi \in B$ is called a *subequilibrium* of $\Phi$ if $\phi \leq_B \Phi_t(\phi)$ for $t \geq 0$, and a *strict subequilibrium* if $\phi <_B \Phi_t(\phi)$ for $t \geq 0$. Similarly, we can define a *(strict) superequilibrium*. An *entire orbit* of $\Phi$ is a mapping $u : R \to B$ such that $u(t+s) = \Phi_t(u(s))$ for $t \geq 0$ and $s \in R$.

Let $\alpha$ be a measure of noncompactness (see, cf. Nussbaum [**49**]). We say that the semiflow $\Phi_t$ is *set-condensing* on a subset $E$ of $B$ for $t > 0$ if, for every bounded subset $W$ of $E$ with $\alpha(W) \neq 0$, we have $\alpha(\Phi_t(W)) < \alpha(W)$ for every $t > 0$.

**Theorem 2.1.** *Suppose $\Phi : R_+ \times B \to B$ is strictly order-preserving. Let $\phi_1 <_B \phi_2$ be order-related equilibria of $\Phi$ and define*

$$E = [\phi_1,\ \phi_2]_B = \{\phi \in B; \phi_1 \leq_B \phi \leq_B \phi_2\}.$$

*Assume that $\Phi_t$ is set-condensing on $E$ for $t > 0$. Then one of the following statements holds:*

(i) *there exists a further fixed point $\phi$ of $\Phi$ such that $\phi_1 <_B \phi <_B \phi_2$;*

(ii) *there exists an entire orbit of strict subequilibria $u$ connecting $\phi_1$ and $\phi_2$ such that $u(t) \to \phi_1$ as $t \to -\infty$ and $u(t) \to \phi_2$ as $t \to \infty$;*

(iii) *there exists an entire orbit of strict superequilibria $u$ connecting $\phi_1$ and $\phi_2$ such that $u(t) \to \phi_2$ as $t \to -\infty$ and $u(t) \to \phi_1$ as $t \to \infty$.*

The above theorem represents a slight improvement of Theorem 8 in Matano [40] and Proposition 1.1 in Dancer and Hess [6] by relaxing the requirement on compactness and the strongly order-preserving property of $\Phi$. We refer to Smith and Thieme [64] for a related result (Proposition 3.7). Theorem 2.1 can be proved by a similar argument to that in Dancer and Hess [6] with some necessary modifications in order to deal with set-condensing semiflows in stead of compact semiflows. We refer to Freedman, Miller and Wu [7] for details.

Recall that a *strongly order-preserving* semiflow $\Phi$ is one such that $\phi <_B \psi$ implies $\Phi_t(\phi) \ll_B \Phi_t(\psi)$ for $t > 0$. The following weaker notation of strong order-preserving property of $\Phi$ was introduced in Freedman, Miller and Wu [7].

**Definition 2.2.** A semiflow $\Phi : R_+ \times B \to B$ is said to be *quasi strongly order-preserving* (QSOP) if it is order-preserving and for every sequence $\{\psi^n\}$ of equilibria and for every compact invariant subset $A \subseteq B$ such that $\lim_{n\to\infty} \psi^n = \psi <_B A$ and $\psi <_B \psi^n$ for $n = 1, 2, \ldots$, there exists an integer $n_0$ such that $\psi^{n_0} \leq_B A$.

It can be easily shown that a strongly order-preserving semiflow is QSOP, but the inverse is not true. We refer to Freedman, Miller and Wu [7] for some examples and for a sufficient condition guaranteeing an order-preserving semiflow to be QSOP.

For $r > 0$ and $\phi \in B$, define

$$\mathcal{U}_B(\phi, r) = \{\psi \in B;\ \ |\psi - \phi|_B < r\}.$$

Suppose that the semiflow $\Phi$ has a subequilibrium $\phi_1$ and a superequilibrium $\phi_2$ with $\phi_1 <_B \phi_2$. Denote $E = [\phi_1, \phi_2]_B$. An equilibrium $\phi$ in $E$ is said to be *stable with respect to* $E$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Phi_t(\psi) \in \mathcal{U}_B(\phi, \varepsilon)$ for all $\psi \in \mathcal{U}_B(\phi, \delta)$ and $t > 0$.

**Theorem 2.3.** *Suppose that $\Phi$ is a* QSOP *semiflow and that there exists a subequilibrium $\phi_1$ and a superequilibrium $\phi_2$ such that $\phi_1 <_B \phi_2$ and every equilibrium in $E = [\phi_1, \phi_2]_B$ is stable with respect to $E$. Assume further that $\Phi_t$ is set-condensing on $E$ for $t > 0$. Then every bounded positive orbit converges. That is, for every bounded positive orbit $\gamma^+(\phi)$ in $E$, $\omega(\phi)$ is a singleton.*

The above result is established by Dancer and Hess [**6**] for strongly order-preserving semiflows. It turns out (see Freedman, Miller and Wu [**7**]) that their argument applies to QSOP semiflows as well once Theorem 2.1 is justified. We refer to Alikakos, Hess and Matano [**2**], Hirsch [**27**], Paláčik [**52, 53**], Takáč [**65, 66**], Smith and Thieme [**63, 64**] for related results.

**Theorem 2.4.** *Assume $\Phi$ is a* QSOP *semiflow and $\phi_1, \phi_2$ are strict subequilibrium and strict superequilibrium of $\Phi$, respectively, with $\phi_1 \ll_B \phi_2$. Assume also that $\Phi_t$ is set-condensing for $t > 0$ on $E = [\phi_1, \phi_2]_B$. Then there exists a stable equilibrium in $E$.*

**Theorem 2.5.** *Suppose $\Phi$ is a strictly order-preserving semiflow and $\phi_1 <_B \phi_2$ are order-related equilibria. Furthermore, assume that $\Phi_t$ is set-condensing for $t > 0$ on $E = [\phi_1, \phi_2]_B$. If $\phi_1$ and $\phi_2$ are stable respect to $E$ and $\phi_1$ is an isolated equilibrium from above or $\phi_2$ is an isolated equilibrium from below, then there exists an unstable equilibrium $\phi$ such that $\phi_1 <_B \phi <_B \phi_2$.*

In the above result, $\phi_1$ ($\phi_2$) is isolated from above (below) means that there exists a small neighborhood of $\phi_1$ ($\phi_2$) in $E$ which contains no other equilibrium $\psi$ such that $\phi_1 <_B \psi$ ($\phi_2 >_B \psi$). The above two results are basically established by Dancer and Hess [**6**] (they prove these results by assuming that $\Phi_t(E)$ is relatively compact, but their argument also applies to the case where $\Phi_t$ is set-condensing in $E$). We refer to Matano [**40–42**] and Hirsch [**27**] for some earlier versions.

**3. Phase spaces and examples.** Let $R_- = (-\infty, 0]$. Assume that $X$ is a real Banach space with a norm denoted by $|\cdot|_X$ and that $\widehat{B}$ is a linear space of mappings from $R_-$ to $X$ with elements designated by $\hat{\phi}$, $\hat{\psi}, \ldots$. Suppose also there is a seminorm $|\cdot|_{\widehat{B}}$ on $\widehat{B}$ so that the quotient space $B = \widehat{B}/|\cdot|_{\widehat{B}}$ is a Banach space with a norm $|\cdot|_B$ naturally induced by $|\cdot|_{\widehat{B}}$. Throughout this paper, we shall denote by $\phi$ the corresponding equivalence class of $\hat{\phi} \in \widehat{B}$.

The Banach space $B$ will be employed as the phase space for the study of partial functional differential equations with infinite delay. The first hypothesis on $B$ is described as follows:

(A1) There exists a constant $L > 0$ such that

$$|\hat{\phi}(0)|_X \leq L|\hat{\phi}|_{\widehat{B}} \quad \text{for any } \hat{\phi} \in \widehat{B}.$$

Under the above assumption, for every equivalence class $\phi$ there is a unique $\phi(0) \in X$ defined by $\phi(0) = \hat{\phi}(0)$ for $\hat{\phi} \in \widehat{B}$ and $|\phi(0)|_X \leq L|\phi|_B$.

To describe other assumptions on the phase space, we introduce the following notation: suppose $0 \leq t \leq A < \infty$ and the mapping $u : (-\infty, A] \to X$ is given, then $u_t : R_- \to B$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in R_-$.

For any given $\hat{\phi} \in \widehat{B}$ and $A \geq 0$, we define

$$F_A(\hat{\phi}) = \{\hat{u} : (-\infty, A] \to X;\ \hat{u}_0 = \hat{\phi} \ \text{ and } \hat{u}|_{[0,A]} \text{ is continuous}\}.$$

Moreover, we set

$$F_A = \bigcup_{\hat{\phi} \in \widehat{B}} F_A(\hat{\phi}).$$

Other fundamental assumptions on the phase spaces can now be described as follows:

(A2) If $A \geq 0$ and $\hat{u} \in F_A$, then $\hat{u}_t \in \widehat{B}$ for all $t \in [0, A]$. Moreover, the mapping $t \in [0, A] \to u_t \in B$ is continuous.

(A3) There exist a continuous nondecreasing function $K : R_+ \to R_+$ and a locally bounded function $M : R_+ \to R_+$ such that

$$|u_t|_B \leq K(t) \sup_{0 \leq s \leq t} |u(s)|_X + M(t)|u_0|_B, \qquad 0 \leq t \leq A,\ u \in F_A.$$

To discuss the monotonicity of semiflows on the phase space $B$, we also assume that there exists a closed cone $X_+$ in $X$ with the property that if $u,\ -u \in X_+$, then $u = 0$. The cone $X_+$ induces a partial ordering $\geq_X$ on $X$ defined as follows

$$u \geq_X v \quad (\text{or } v \leq_X u) \quad \text{iff} \quad u - v \in X_+.$$

We assume that $X$ with the above ordering is a Banach lattice. For simplicity of notations, we shall employ the following ordering intervals:

$$\begin{aligned}
[v, w]_X &= \{u \in X;\ v \leq_X u \leq_X w\}, \\
[v, \infty)_X &= \{u \in X;\ v \leq_X u\}, \\
(-\infty, w]_X &= \{u \in X;\ u \leq_X w\}, \\
(-\infty, \infty)_X &= X,
\end{aligned}$$

where $v, w \in X$ with $v \leq_X w$.

Similarly, we assume that there exists a closed cone $B_+$ in the phase space $B$ with the property that if $\phi$, $-\phi \in B_+$, then $\phi = 0$. The partial ordering induced by $B_+$ is denoted by $\geq_B$, and ordering intervals can be defined similarly.

Finally, we require the following compatibility conditions on the orderings $\leq_X$ and $\leq_B$:

(A4) $\phi \leq_B \psi$ implies that $\phi(0) \leq_X \psi(0)$.

(A5) If $A \geq 0$ and $\hat{u} \in F_A$, $\hat{v} \in F_A$ are given such that $u_0 \leq_B v_0$ and $u(t) \leq_X v(t)$ for $t \in [0, A]$, then $u_A \leq_B v_A$.

To present an example of phase space satisfying (A1)–(A5), we suppose $g : (-\infty, 0] \to [1, \infty)$ is a function satisfying the following conditions:

(g1) $g$ is continuous, nonincreasing and $g(0) = 1$;

(g2) $g(s + \mu)/g(s) \to 1$ uniformly for $s \in (-\infty, 0]$ as $\mu \to 0^+$;

(g3) $g(s) \to \infty$ as $s \to -\infty$.

For example, if $\gamma > 0$, then the functions $g(s) = e^{-\gamma s}$ and $g(s) = (1 + |s|)^\gamma$ satisfy the above conditions.

Suppose $\Omega \subset R^n$ is bounded, $C(\overline{\Omega}; R^m)$ is the Banach space of continuous functions from $\overline{\Omega}$ to $R^m$ with the supremum norm $|\cdot|_{C(\overline{\Omega}; R^m)}$. Define

$$C_g = \left\{ \phi;\ \phi : (-\infty, 0] \to C(\overline{\Omega}; R^m) \text{ is continuous}, \sup_{s \leq 0} \frac{|\phi(s)|_{C(\overline{\Omega}; R^m)}}{g(s)} < +\infty \right\}.$$

Then $C_g$ equipped with norm

$$|\phi|_g = |\phi|_{C_g} := \sup_{s \leq 0} \frac{|\phi(s)|_{C(\overline{\Omega}; R^m)}}{g(s)} \quad \text{for } \phi \in C_g$$

is a Banach space.

Since $g(0) = 1$, $C_g$ satisfies (A1) with $L = 1$. Moreover, as $g$ is nonincreasing, if $u : R \to C(\overline{\Omega}; R^m)$ so that $u_0 \in C_g$ and $u : [0, \infty) \to$

$C(\overline{\Omega}; R^m)$ is continuous, then $u_t \in C_g$ for all $t \geq 0$. A similar argument to that in Example 2.1 of Haddock and Terjéki [15] leads to

$$|u_t|_g \leq \sup_{0 \leq s \leq t} |u(s)|_{C(\overline{\Omega}; R^m)} + \sup_{\theta \leq -t} \frac{g(\theta + t)}{g(\theta)} |u_0|_g.$$

Hence $C_g$ satisfies (A3) with $K(t) = 1$ and $M(t) = \sup_{\theta \leq -t}\big(g(\theta + t)/g(\theta)\big)$ on $[0, \infty)$.

Obviously, $X_+ = C(\overline{\Omega}; R^m_+)$ is a closed cone in $X = C(\overline{\Omega}; R^m)$, where $R^m_+$ denotes the positive octant of $R^m$. Hence $C(\overline{\Omega}; R^m_+)$ induces a partial ordering $\geq_{C(\overline{\Omega}; R^m)}$ on $C(\overline{\Omega}; R^m)$ by

$$u \geq_{C(\overline{\Omega}; R^m)} v \quad \text{iff} \quad u - v \in C(\overline{\Omega}; R^m_+).$$

Define a closed order cone $C_g^+$ in $C_g$ by

$$C_g^+ = \{\phi \in C_g; \phi(\theta) \geq_{C(\overline{\Omega}; R^m)} 0 \text{ for } \theta \leq 0\}.$$

Similarly, $C_g^+$ induces a partial ordering $\geq_{C_g}$ on $C_g$. Obviously, if $\phi,\ \psi \in C_g^+$, then $\phi \leq_{C_g} \psi$ implies $\phi(0) \leq_{C(\overline{\Omega}; R^m)} \psi(0)$, i.e., (A4) is satisfied. Moreover, if $u, v : R \to R^n$ are given continuous functions such that $u_0,\ v_0 \in C_g$, $u_0 \leq_{C_g} v_0$ and $u(t) \leq_{C(\overline{\Omega}; R^m)} v(t)$ for $t \geq 0$, we have $u_t \leq_{C_g} v_t$ for $t \geq 0$.

Note that (A2) is not satisfied by $C_g$. We now define a subspace $UC_g$ of $C_g$ as follows

$$UC_g = \left\{\phi \in C_g;\ \frac{\phi}{g} \text{ is uniformly continuous on } R_-\right\}.$$

As a closed subspace of $C_g$, $UC_g$ is a Banach space and satisfies (A1) and (A3). Employing the same argument as that of Theorem 2.1 of Haddock and Terjéki [15], we can prove that $(UC_g, |\cdot|_g)$ satisfies (A2) as well. Similarly, we can define a closed cone $UC_g^+$ in $UC_g$. $UC_g^+ = C_g^+ \cap UC_g$ induces a partial ordering $\geq_{UC_g}$ on $UC_g$ and satisfies (A4) and (A5). Another closed subspace of $C_g$ which satisfies (A1)–(A5) is

$$C_g^0 = \{\phi \in C_g; \phi(s) \to 0 \text{ as } s \to -\infty\}.$$

It should be mentioned that the work of Wu [**71**] shows that $UC_g$ arises very naturally from reaction-diffusion equations with both discrete and distributed delays in the nonlinear reaction term which satisfies a certain *fading memory* condition. We refer to Atkinson and Haddock [**1**] for related discussions.

The following axiom on the phase space $B$ will sometime be required:

(A6) Every constant mapping from $(-\infty, 0]$ into $X$ belongs to $B$.

Denote by $\overline{w}$ the constant mapping from $(-\infty, 0]$ into $X$ with the constant value $w \in X$. We will also require the following conditions.

(A7) For any constant mapping $\overline{v}, \overline{w} \in B$, $\overline{v} \leq_B \overline{w}$ if and only if $v \leq_X w$.

(A8) There exists a constant $Q > 0$ so that for every constant mapping $\overline{w} \in B$, $|\overline{w}|_B \leq Q|w|_X$, $w \in X$.

(A1) and (A8) imply that the induced topology (from $B$) on the space of constant mappings from $(-\infty, 0]$ into $X$ is equivalent to the topology of $X$. Also

$$|\overline{\phi(0)}|_B \leq Q|\phi(0)|_X \leq QL|\phi|_B, \qquad \phi \in B.$$

It is easy to see that (A6)–(A8) are satisfied by $X = C(\overline{\Omega}; R^m)$ and $B = UC_g$ defined above. Moreover, it is trivial to show that for these two spaces, the following assumption is also satisfied:

(A9) If $\phi, \psi \in B$ and $\widehat{\phi}(\theta) \leq_X \widehat{\psi}(\theta)$ for $\theta \leq 0$, then $\phi \leq_B \psi$.

**4. Existence theorems for semilinear integral equations.** Assume $a \geq 0$ is a given constant. In this section, we shall consider an abstract integral equation in some subset of $[a, \infty) \times B$ satisfying the following properties:

(D1) $D$ is a closed subset of $[a, \infty) \times X$ and $D(t) = \{u \in X; (t, u) \in D\}$ is a nonempty subset of $X$ for each $t \geq a$.

(D2) $\mathcal{D}$ is a closed subset of $[a, \infty) \times B$ such that $\mathcal{D}(t) = \{\phi \in B;\ (t, \phi) \in \mathcal{D}\}$ is a nonempty subset of $B$ for each $t \geq a$, and for any $\widehat{u} \in F_\infty$ and $a \leq \tau \leq t$, if $u_\tau \in \mathcal{D}(\tau)$ and $u(s) \in D(s)$ for $s \in [\tau, t]$, then $u_t \in \mathcal{D}(t)$.

(D3) For each $b > a$, there exist a constant $\widehat{K}(a, b) > 0$ and a continuous nondecreasing function $\eta_{a,b} : [0, b - a) \to R_+$ such that

$\eta_{a,b}(0) = 0$ and if $a \leq t_1 < t_2 \leq b$, $u_1 \in D(t_1)$ and $u_2 \in D(t_2)$ then there is a continuous function $w : [t_1, t_2] \to X$ such that $w(t_1) = u_1$, $w(t_2) = u_2$, $w(t) \in D(t)$ for $t_1 < t < t_2$ and

$$|w(t) - w(s)|_X \leq \eta_{a,b}(|t-s|) + \widehat{K}(a,b)|t-s| \frac{|u_2 - u_1|_X}{t_2 - t_1}$$

for all $s, t \in [t_1, t_2]$.

Now we consider the following abstract semilinear integral equation

$$u(t) = S(t,a)\phi(0) + \int_a^t T(t,r)F(r,u_r)dr, \qquad t \geq a \tag{4.1}$$
$$u_a \in \phi \in \mathcal{D}(a),$$

where $F$ is a continuous mapping from an open neighborhood $\nu(\mathcal{D})$ of $\mathcal{D}$ to $X$ satisfying the following condition:

(F) for any $l \geq 0$ there exist a constant $L_{1,l} > 0$ and a continuous function $L_{2,l} : R_+ \to R_+$ with $L_{2,l}(0) = 0$ such that

$$|F(t,\phi) - F(s,\psi)|_X \leq L_{2,l}(|t-s|) + L_{1,l}|\phi - \psi|_B$$

for $(t,\phi)$, $(s,\psi) \in \mathcal{D}$ with $|\phi|_B \leq l$, $|\psi|_B \leq l$ and $a \leq t$, $s \leq a + l$.

$T = \{T(t,s);\ t \geq s \geq a\}$ is a $C_0$ linear evolution system on $X$ and $S = \{S(t,s);\ t \geq s \geq a\}$ is an affine evolution system associated with $T$. That is, the following conditions (T1)–(T3), (S1) and (S2) are satisfied.

(T1) $T(t,t)u = u$ and $T(t,s)T(s,r)u = T(t,r)u$ for all $t \geq s \geq r \geq a$, $u \in X$;

(T2) for each $u \in X$, the mapping $(t,s) \to T(t,s)u$ is continuous for $t \geq s \geq a$;

(T3) there are numbers $\widehat{M} \geq 1$ and $\omega \in R$ such that

$$|T(t,s)| = \sup\{|T(t,s)u|_X;\ |u|_X \leq 1\} \leq \widehat{M}e^{\omega(t-s)}, \qquad t, s \geq a;$$

(S1) $t \to S(t,a)0$ is continuous from $[a,\infty)$ into $X$, where $0$ is the zero of $X$;

(S2) $S(t,r)u + T(t,s)v = S(t,s)[S(s,r)u + v]$ for all $u, v \in X$ and $t \geq s \geq r \geq a$.

It can be shown that (see, cf. Martin and Smith [**36**]) conditions (S1) and (S2) are equivalent to the following

(S3) there is a continuous function $\hat{\mu} : [a, \infty) \to X$ such that

$$S(t,s)u = T(t,s)[u - \hat{\mu}(s)] + \hat{\mu}(t)$$

for all $u \in X$ and $t \geq s \geq a$.

A function $u : (-\infty, b) \to X$, $b > a$ is a *solution in* $\mathcal{D}$ of (4.1) if $u : [a,b) \to X$ is continuous, $(t, u_t) \in \mathcal{D}$ and $u$ satisfies (4.1) for all $t \in [a,b)$. To guarantee the invariance of the set $\mathcal{D}$, we also need the following *subtangential condition*:

(SC) $\lim_{h\to 0^+} (1/h) d\big(S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,r)F(t,\phi)dr;\ D(t+h)\big) = 0$ for all $(t,\phi) \in \mathcal{D}$, where $d(u; D(t)) = \inf\{|u-v|_X;\ v \in D(t)\}$ for $u \in X$ and $t \geq a$.

We refer to Martin and Smith [**36**] for detailed discussion on (D1)–(D3), (S1)–(S3), (T1)–(T3) and (SC). Under the above assumptions, we have the following basic result on the existence and uniqueness of a solution of (4.1).

**Theorem 4.1.** *Suppose that* (D1)–(D3), (T1)–(T3), (S1), (S2), (F) *and* (SC) *are satisfied. Then* (4.1) *has a unique noncontinuable solution* $u$, *denoted by* $u(t; a, \phi)$, *on an interval of the form* $[a, b)$, *where* $a < b \leq \infty$. *Moreover,* $u(t) \in D(t)$ *and* $u_t \in \mathcal{D}(t)$ *for* $a \leq t < b$ *and if* $b < \infty$, *then*

$$\limsup_{t \to b^-} |u_t|_B = \infty.$$

*Proof.* The proof is similar to that in Martin and Smith [**36**] for abstract integral equations with finite delay. However, since our argument involves certain technicalities caused by the unboundedness of the delay, we provide the details of the proof here for the sake of completeness. The basic idea of our proof is to construct a sequence of approximate solutions which converges to a solution. We divide the proof into ten steps.

*Step* 1. By assumption (D3) there exists a constant $M_1 > 0$ and a continuous nondecreasing function $M_2 : [0,1] \to R_+$ satisfying $M_2(0) = 0$ and if $a \le t_1 < t_2 \le a+1$, $u_1 \in D(t_1)$ and $u_2 \in D(t_2)$, then there exists a continuous function $w : [t_1, t_2] \to X$ such that $w(t_1) = u_1$, $w(t_2) = u_2$, $w(t) \in D(t)$ for $t_1 < t < t_2$ and

$$|w(t) - w(s)|_X \le M_2(|t-s|) + M_1|t-s|\frac{|u_2 - u_1|_X}{t_2 - t_1} \quad \text{for } s, t \in [t_1, t_2]. \tag{4.2}$$

*Step* 2. Because of the continuity of $F : \mathcal{D} \to X$, there exists $\delta_1 > 0$ such that for any $(t, \psi) \in \mathcal{D}$ with $|t - a| < \delta_1$ and $|\psi - \phi|_B < \delta_1$, we have

$$|F(t,\psi)|_X \le N := |F(a,\phi)|_X + 1. \tag{4.3}$$

On the other hand, since $D(a+1) \ne \varnothing$, by the result in Step 1, we can find a continuous function $\widetilde{w} : [a, a+1] \to X$ such that $\widetilde{w}(a) = \phi(0)$ and $\widetilde{w}(t) \in D(t)$ for $t \in [a, a+1]$. We now define $\bar{\phi} : (-\infty, a+1] \to X$ by

$$\bar{\phi}(t) = \begin{cases} \hat{\phi}(t-a), & t \le a \\ \widetilde{w}(t), & a \le t \le a+1, \end{cases}$$

where $\hat{\phi} \in \widehat{B}$ is a representing element of $\phi \in B$. By (A2) and (D2), $\bar{\phi}_t \in \mathcal{D}(t) \subseteq B$ for any $t \in [a, a+1]$ and $\bar{\phi}_t \to \phi$ as $t \to a^+$. Therefore, there exists $\delta_2 > 0$ such that $|\bar{\phi}_t - \phi|_B < \delta_1/2$ for all $t \in [a, a+\delta_2]$. Set $\delta_0 = \min\{1, \delta_1, \delta_2\}$. Then for any $u : (-\infty, a+\delta_0] \to X$ such that $u_a = \phi$, $u : [a, a+\delta_0] \to X$ is continuous, $u(s) \in D(s)$ and $|u(s) - \phi(0)|_X < \delta_1/(2[K(\delta_0)+1]+1)$ for $s \in [a, a+\delta_0]$, we have for $t \in [a, a+\delta_0]$ that

$$\begin{aligned} |u_t - \phi|_B &\le |u_t - \bar{\phi}_t|_B + |\bar{\phi}_t - \phi|_B \\ &\le K(\delta_0) \sup_{a \le s \le t} |u(s) - \phi(0)|_X + |\bar{\phi}_t - \phi|_B \\ &< K(\delta_0)\, \frac{\delta_1}{2[K(\delta_0)+1]} + \frac{1}{2}\delta_1 \\ &\le \delta_1. \end{aligned}$$

Hence, from (4.3) it follows that

$$|F(t,u_t)|_X \le N, \qquad t \in [a, a+\delta_0]. \tag{4.4}$$

*Step* 3. Because of the continuity of $S$ and $M_2$, we can find constants $\sigma > a$ and $\varepsilon_0 > 0$ such that

$$\sigma + \varepsilon_0 < a + \delta_0, \tag{4.5}$$

$$e^{|\omega|}\widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(\sigma + \varepsilon_0 - a) < \frac{\delta_1}{8[K(\delta_0)+1][2M_1+1]}, \tag{4.6}$$

$$|S(t,a)\phi(0) - \phi(0)|_X < \frac{\delta_1}{8[K(\delta_0)+1][2M_1+1]}, \quad t \in [a, \sigma+\varepsilon_0], \tag{4.7}$$

$$M_2(\sigma + \varepsilon_0 - a) < \frac{\delta_1}{4[K(\delta_0)+1][2M_1+1]}, \tag{4.8}$$

where $\omega$ and $\widehat{M}$ were defined in (T3). Now for any given $\varepsilon \in (0, \varepsilon_0]$, we construct an $\varepsilon$-approximate solution $w$ and a corresponding sequence $\{t_j\}$ as follows: set $t_0 = a$ and define $w(a+s) = \hat{\phi}(s)$ for $s \le 0$, where $\hat{\phi}$ is a fixed representative element of $\phi$. Assume that $i$ is a nonnegative integer and $w$ is constructed and continuous on $(-\infty, t_i]$ with $a \le t_i < \sigma + \varepsilon_0$ and $w(t) \in D(t)$ for $t \in [a, t_i]$. If $t_i \ge \sigma$, then set $t_{i+1} = t_i$. If $t_i < \sigma$, we define $\Gamma_i$ to be the set of all constants $\gamma \in [0, \varepsilon/2]$ such that

$$|S(t,t_i)w(t_i) - w(t_i)|_X \le \varepsilon, \qquad t_i \le t \le t_i + \gamma, \tag{4.9}$$

(4.10)

$$d\Big(S(t_i+h,t_i)w(t_i) + \int_{t_i}^{t_i+h} T(t_i+h,\theta)F(t_i,w_{t_i})d\theta,\ D(t_i+h)\Big) \le \frac{\varepsilon}{2}h,$$

where $0 \le h \le \gamma$, and

$$|w(t)-w(s)|_X \le \frac{\varepsilon}{4[M_1K(\sigma-a)+1]}, \qquad t,s \in [a,t_i],\ |t-s| \le \gamma, \tag{4.11}$$

$$K(\sigma - a)M_2(\gamma) < \varepsilon/2, \tag{4.12}$$

$$|\bar{\phi}_t - \phi|_B < \varepsilon/2, \qquad t \in [a, a+\gamma]. \tag{4.13}$$

By assumptions (S3) and (SC), $\Gamma_i$ is nonempty. We let

$$\gamma_i = \frac{3}{4}\sup\{\gamma;\ \gamma \in \Gamma_i\}, \tag{4.14}$$

and $t_{i+1} = t_i + \gamma_i$. Because of (4.10), we can find an element $w(t_{i+1}) \in D(t_{i+1})$ so that

$$\left| S(t_{i+1}, t_i)w(t_i) + \int_{t_i}^{t_{i+1}} T(t_{i+1}, \theta)F(t_i, w_{t_i})\, d\theta - w(t_{i+1}) \right|_X \le \varepsilon(t_{i+1} - t_i). \tag{4.15}$$

Finally, by Step 1, we can define $w$ on $[t_i, t_{i+1}]$ so that $w$ is continuous, $w(t) \in D(t)$ and (4.2) holds.

*Step* 4. We claim that the function $w$ constructed as above satisfies the following properties:

(4.16)
$$|w(t_i) - \phi(0)|_X < \frac{\delta_1}{4[K(\delta_0)+1][2M_1+1]} \quad \text{for } i = 0, 1, 2, \dots ;$$

(4.17)
$$|w(t) - \phi(0)|_X < \frac{\delta_1}{2[K(\delta_0)+1]} \quad \text{for } t \in [a, t_i],\ i = 0, 1, 2, \dots ;$$

(4.18)
$$|F(t_i, w_{t_i})|_X \le N, \qquad i = 0, 1, 2, \dots .$$

These estimates can be proved simultaneously by induction on $i$. Clearly, (4.16), (4.17) and (4.18) hold for $i = 0$. Assume that $k \ge 0$ is given such that (4.16)-(4.18) hold for all $i = 0, 1, 2, \dots, k$. Then, from (4.15), (T3) and the fact that $t_{i+1} - t_i \le a + 1 - a = 1$, it follows that

$$\begin{aligned} |S(t_{i+1}, t_i)w(t_i) - w(t_{i+1})|_X &\le \left| \int_{t_i}^{t_{i+1}} T(t_{i+1}, \theta)F(t_i, w_{t_i})\, d\theta \right|_X + \varepsilon(t_{i+1} - t_i) \\ &\le (e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_{i+1} - t_i) \quad \text{for } i = 0, 1, 2, \dots, k. \end{aligned} \tag{4.19}$$

Therefore, by (S2), (T1), (T3) and (4.15) we have

$$\begin{aligned}
&\sup\{e^{-\omega r}|T(t_{i+1}+r,t_{i+1})[S(t_{i+1},a)\phi(0)-w(t_{i+1})]|_X;\ r\geq 0\}\\
&\leq \sup\{e^{-\omega r}|T(t_{i+1}+r,t_{i+1})[S(t_{i+1},t_i)S(t_i,a)\phi(0)\\
&\qquad\qquad -S(t_{i+1},t_i)w(t_i)]|_X;\ r\geq 0\}\\
&\quad+\sup\{e^{-\omega r}|T(t_{i+1}+r,t_{i+1})[S(t_{i+1},t_i)w(t_i)-w(t_{i+1})]|_X;\ r\geq 0\}\\
&\leq \sup\{e^{-\omega r}|T(t_{i+1}+r,t_{i+1})[S(t_i,a)\phi(0)-w(t_i)]|_X;\ r\geq 0\}\\
&\quad+\widehat{M}(e^{|\omega|}\widehat{M}N+\varepsilon_0)(t_{i+1}-t_i)\\
&\leq e^{\omega(t_{i+1}-t_i)}\sup\{e^{-\omega r}|T(t_i+r,t_i)[S(t_i,a)\phi(0)-w(t_i)]|_X;\ r\geq 0\}\\
&\quad+\widehat{M}(e^{|\omega|}\widehat{M}N+\varepsilon_0)(t_{i+1}-t_i),
\end{aligned}$$

where $i=0,1,2,\dots,k$. Using the above inequality for $i=0,1,2,\dots,k$, we can easily get

$$\begin{aligned}
&\sup\{e^{-\omega r}|T(t_{k+1}+r,t_{k+1})[S(t_{k+1},a)\phi(0)-w(t_{k+1})]|_X;\ r\geq 0\}\\
&\qquad\qquad\leq e^{|\omega|}\widehat{M}(e^{|\omega|}\widehat{M}N+\varepsilon_0)(t_{k+1}-a).
\end{aligned}$$

Consequently, by (4.6), we obtain

$$\begin{aligned}
(4.20)\quad &|S(t_{k+1},a)\phi(0)-w(t_{k+1})|_X\\
&\leq \sup\{e^{-\omega r}|T(t_{k+1}+r,t_{k+1})[S(t_{k+1},a)\phi(0)-w(t_{k+1})]|_X;\ r\geq 0\}\\
&\leq e^{|\omega|}\widehat{M}(e^{|\omega|}\widehat{M}N+\varepsilon_0)(t_{k+1}-a)\\
&<\frac{\delta_1}{8[K(\delta_0)+1][2M_1+1]}.
\end{aligned}$$

Hence, by (4.7) we get

$$\begin{aligned}
|w(t_{k+1})-\phi(0)|_X &\leq |w(t_{k+1})-S(t_{k+1},a)\phi(0)|_X\\
&\quad+|S(t_{k+1},a)\phi(0)-\phi(0)|_X\\
&<\frac{\delta_1}{4[K(\delta_0)+1][2M_1+1]}.
\end{aligned}$$

Moreover, by (4.2) and (4.8), for $t \in [t_k,\ t_{k+1}]$ we have

$$\begin{aligned}
|w(t) - \phi(0)|_X &\le |w(t) - w(t_k)|_X + |w(t_k) - \phi(0)|_X \\
&\le M_2(|t - t_k|) + M_1 \frac{|w(t_{k+1}) - w(t_k)|_X}{t_{k+1} - t_k} (t - t_k) \\
&\quad + |w(t_k) - \phi(0)|_X \\
&\le M_2(\sigma + \varepsilon_0 - a) + M_1[|w(t_{k+1}) - \phi(0)|_X \\
&\qquad\qquad + |w(t_k) - \phi(0)|_X] + |w(t_k) - \phi(0)|_X \\
&< \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]} + M_1 \frac{\delta_1}{2[K(\delta_0) + 1][2M_1 + 1]} \\
&\quad + \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]} \\
&< \frac{\delta_1}{2[K(\delta_0) + 1]}.
\end{aligned}$$

Therefore, by the result of Step 2, $|F(t_{k+1}, w_{t_{k+1}})|_X \le N$.

*Step* 5. Let $\rho = \lim_{i\to\infty} t_i$. Obviously, $\rho$ exists and $w$ is defined on $[a, \rho)$ with $w(t) \in D(t)$ for all $t \in [a, \rho)$. We claim that $z = \lim_{t\to\rho^-} w(t)$ exists and $z \in D(\rho)$. Indeed, using a similar argument as that for (4.20), we obtain for $j \ge k$,

(4.21)
$$\begin{aligned}
|S(t_j, t_k)w(t_k) - w(t_j)|_X &\le e^{|\omega|(t_j - t_k)} \widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_j - t_k) \\
&\le \widehat{N}(t_j - t_k),
\end{aligned}$$

where $\widehat{N} = e^{|\omega|(\rho - a)}\widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)$. Let $\bar{\varepsilon} > 0$ be given and choose $k > 0$ so that $2\widehat{N}(\rho - t_k) < \bar{\varepsilon}/2$. By the continuity of $S$, we can find an integer $n(\bar{\varepsilon}) > k$ such that

$$|S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)|_X \le \bar{\varepsilon}/2 \quad \text{for } i, j \ge n(\bar{\varepsilon}). \tag{4.22}$$

Therefore, for $i, j \ge n(\bar{\varepsilon})$ we have

$$\begin{aligned}
|w(t_i) - w(t_j)|_X &\le |w(t_i) - S(t_i, t_k)w(t_k)|_X \\
&\quad + |S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)|_X \\
&\quad + |S(t_j, t_k)w(t_k) - w(t_j)|_X \\
&\le \widehat{N}(\rho - t_k) + \frac{\bar{\varepsilon}}{2} + \widehat{N}(\rho - t_k) < \bar{\varepsilon}.
\end{aligned}$$

Hence, $z = \lim_{i\to\infty} w(t_i)$ exists. Since $(t_i, w(t_i)) \in D$ and $D$ is closed, it follows that $(\rho, z) \in D$. Moreover, by (4.2) we have

$$|w(t) - w(t_i)|_X \le M_2(|t - t_i|) + M_1|t - t_i|\frac{|w(t_{i+1}) - w(t_i)|_X}{t_{i+1} - t_i}$$
$$\le M_2(|t_{i+1} - t_i|) + M_1|w(t_{i+1}) - w(t_i)|_X,$$
$$t_i \le t \le t_{i+1}.$$

This implies that $w(t) \to z$ as $t \to \rho^-$.

*Step* 6. We show that the $\varepsilon$-approximate solution constructed above is defined in $[a, \sigma]$, i.e., there is an integer $n = n(\varepsilon)$ so that $t_n \ge \sigma$. By contradiction, if no such $n$ exists, then $t_i < \sigma$ for all $i \ge 1$, $\lim_{i\to\infty} t_i = \rho \le \sigma$ and $\lim_{i\to\infty} w(t_i) = z$. Let $w(\rho) = z$. Clearly, $w_\rho \in \mathcal{D}(\rho)$. By the subtangential condition (SC), there exists a constant $\delta > 0$ independent of $i$, such that (4.9), (4.11), (4.12) and (4.13) hold with $\gamma$ being replaced by $\delta$. Moreover,

$$d\Big(S(\rho+\eta,\rho)z + \int_\rho^{\rho+\eta} T(\rho+\eta,\theta)F(\rho, w_\rho)d\theta;\ D(\rho+\eta)\Big) \le \frac{\varepsilon}{4}\eta$$

for all $\eta \in (0, \delta]$. (A2) implies $\lim_{i\to\infty} w_{t_i} = w_\rho$. Therefore, we have

$$d\Big(S(t_i+\eta, t_i)w(t_i) + \int_{t_i}^{t_i+\eta} T(t_i+\eta,\theta)F(t_i, w_{t_i})d\theta;\ D(t_i+\eta)\Big) \le \frac{\varepsilon}{2}\eta$$

for sufficiently large $i$. This implies that $\delta \in \Gamma_i$, hence $\delta \le 4\gamma_i/3 = 4(t_{i+1} - t_i)/3 \to 0$ as $i \to \infty$, a contradiction to the independence of $\delta$ on $i$.

*Step* 7. We now construct a sequence of approximate solutions. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a decreasing sequence in $[0, \varepsilon_0)$ such that $\varepsilon_n \to 0$ as $n \to \infty$, and for each $n \ge 1$ let $w^n$ and $\{t_i^n\}_{i=1}^\infty$ be as constructed above with $\varepsilon = \varepsilon_n$, $t_i = t_i^n$ and $w = w^n$. Therefore, by the result in Step 6, for each $n \ge 1$ there exists an $n_0 = n_0(n)$ such that $t_{n_0}^n \ge \sigma$. For convenience, we define a comparison function $v^n$ for $w^n$ as follows

(4.23)
$$v^n(t) = S(t,a)\phi(0) + \int_a^t T(t,\theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\,d\theta, \qquad t \in [a,\sigma],$$
$$v^n(a+\theta) = \widehat{\phi}(\theta) \quad \text{for } \theta \le 0,$$

where $\gamma^n : [a,\sigma] \to [a,\sigma]$ is given by $\gamma^n(t) = t_i^n$ for $t_i^n \le t < t_{i+1}^n$. We claim that there exists a constant $P > 0$ independent of $n$, such that

$$|v^n(t) - w^n(t)|_X \le P \max\{\varepsilon_n, M_2(\varepsilon_n)\}, \qquad a \le t \le \sigma,\ n = 1, 2, \dots . \tag{4.24}$$

Indeed, by (S2) we obtain, for $a \le s \le t \le \sigma$, that

$$\begin{aligned} v^n(t) &= S(t,s)S(s,a)\phi(0) \\ &\quad + T(t,s)\int_a^s T(s,\theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta \\ &\quad + \int_s^t T(t,\theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta \\ &= S(t,s)v^n(s) + \int_s^t T(t,\theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta. \end{aligned} \tag{4.25}$$

Note that for each $n \ge 1$ and $i = 1, 2, \dots$, we have

$$\begin{aligned} &\sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_{i+1}^n)]|_X;\ r \ge 0\} \\ &= \sup\Big\{e^{-\omega r}\Big|T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)v^n(t_i^n) \\ &\qquad + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta - w^n(t_{i+1}^n)]\Big|_X;\ r \ge 0\Big\} \\ &\le \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)v^n(t_i^n) \\ &\qquad\qquad - S(t_{i+1}^n, t_i^n)w^n(t_i^n)]|_X;\ r \ge 0\} \\ &\quad + \sup\left\{e^{-\omega r}\Big|T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)w^n(t_i^n) \right. \\ &\qquad \left. + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta - w^n(t_{i+1}^n)]\Big|_X;\ r \ge 0\right\} \\ &\le \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)T(t_{i+1}^n, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X;\ r \ge 0\} \\ &\quad + \widehat{M}\Big|S(t_{i+1}^n, t_i^n)w^n(t_i^n) + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\, d\theta \\ &\qquad\qquad - w^n(t_{i+1}^n)\Big|_X \\ &= \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X;\ r \ge 0\} \end{aligned}$$

$$+ \widehat{M}\left| S(t_{i+1}^n, t_i^n)w^n(t_i^n) + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n)\, d\theta \right.$$
$$\left. - w^n(t_{i+1}^n)\right|_X .$$

Therefore, by (4.15), we have

$$\begin{aligned}
&\sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_i^n)]|_X;\ r \geq 0\} \\
&\quad \leq e^{|\omega|(t_{i+1}^n - t_i^n)} \sup\{e^{-\omega r}|T(t_i^n + r, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X;\ r \geq 0\} \\
&\qquad + \widehat{M}\varepsilon_n(t_{i+1}^n - t_i^n), \qquad i = 0, 1, 2, \ldots .
\end{aligned}$$

Applying the above inequality at $0, 1, 2, \ldots, i$ and using $e^{|\omega|(t_{i+1}^n - t_i^n)} \leq e^{|\omega|(t_{i+1}^n - a)}$, we get

$$\begin{aligned}
&|v^n(t_{i+1}^n) - w^n(t_{i+1}^n)|_X \\
&\qquad \leq \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_{i+1}^n)|_X;\ r \geq 0\} \\
&\qquad \leq e^{|\omega|(t_{i+1}^n - a)}\widehat{M}\varepsilon_n(t_{i+1}^n - a) \\
&\qquad \leq \widehat{Q}\varepsilon_n,
\end{aligned}$$

where $\widehat{Q} = e^{|\omega|(\sigma - a)}\widehat{M}(\sigma - a)$. If $t_i^n \leq t \leq t_{i+1}^n \leq \sigma$, then by using (4.25) we get

$$\begin{aligned}
|v^n(t) - w^n(t)|_X \leq &\left| S(t, t_i^n)v^n(t_i^n) \right. \\
&\left. + \int_{t_i^n}^{t} T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n)d\theta - S(t, t_i^n)w(t_i^n)\right|_X \\
&+ |S(t, t_i^n)w^n(t_i^n) - w^n(t_i^n)|_X + |w^n(t_i^n) - w^n(t)|_X .
\end{aligned}$$

By (S3), (T3), (4.2) and (4.9), we have

$$\begin{aligned}
|v^n(t) - w^n(t)|_X \leq\ &\widehat{M}e^{|\omega|(\sigma - a)}|v^n(t_i^n) - w^n(t_i^n)|_X \\
&+ \left| \int_{t_i^n}^{t} T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n)\, d\theta \right|_X \\
&+ \varepsilon_n + M_2(t_{i+1}^n - t_i^n) + M_1|w^n(t_{i+1}^n) - w^n(t_i^n)|_X \\
\leq\ &\widehat{M}e^{|\omega|(\sigma - a)}\widehat{Q}\varepsilon_n + \widehat{M}e^{|\omega|(\sigma - a)}N\varepsilon_n \\
&+ \varepsilon_n + M_2(\varepsilon_n) + M_1|w^n(t_{i+1}^n) - w^n(t_i^n)|_X ,
\end{aligned}$$

where we used the fact that $t - t_i^n \le t_{i+1}^n - t_i^n \le \varepsilon_n$. By (4.21) and (4.9), we obtain

$$
\begin{aligned}
|w^n(t_{i+1}^n) - w^n(t_i^n)|_X &\le |w^n(t_{i+1}^n) - S(t_{i+1}^n, t_i^n)w^n(t_i^n)|_X \\
&\quad + |S(t_{i+1}^n, t_i^n)w^n(t_i^n) - w^n(t_i^n)|_X \\
&\le \widetilde{N}(t_{i+1}^n - t_i^n) + \varepsilon_n \\
&\le (\widetilde{N} + 1)\varepsilon_n,
\end{aligned} \tag{4.26}
$$

where $\widetilde{N} = e^{|\omega|(\sigma - a)}(e^{|\omega|}\widehat{M}N + \varepsilon_0)$. Therefore

$$
|v^n(t) - w^n(t)|_X \le [\widehat{M}e^{|\omega|(\sigma-a)}(\widehat{Q} + N) + 1 + M_1(\widetilde{N} + 1)]\varepsilon_n + M_2(\varepsilon_n).
$$

This proves (4.24).

*Step* 8. We claim that there exists a constant $Q > 0$ independent of $t \in [a, \sigma]$ and $n \ge 1$, such that

$$
|w_t^n - w_{\gamma^n(t)}^n)|_B \le Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \quad \text{for } t \in [a, \sigma],\ n = 1, 2, \dots .
$$

Indeed, for all $n \ge 1$ and $t_i^n \le t \le t_{i+1}^n \le \sigma$, by (A3) we have

$$
\begin{aligned}
|w_t^n - w_{\gamma^n(t)}^n|_B &= |w_t^n - w_{t_i^n}^n|_B \\
&\le K(t_i^n - a) \sup_{a \le s \le t_i^n} |w^n(t - t_i^n + s) - w^n(s)|_X \\
&\quad + M(t_i^n - a)|w_{t - t_i^n + a}^n - \phi|_B.
\end{aligned} \tag{4.27}
$$

On the other hand, by (A3) we get

$$
\begin{aligned}
|w_{t-t_i^n+a}^n - \phi|_B &\le |w_{t-t_i^n+a}^n - \overline{\phi}_{t-t_i^n+a}|_B + |\overline{\phi}_{t-t_i^n+a} - \phi|_B \\
&\le K(t - t_i^n + a) \sup_{a \le s \le t - t_i^n + a} |w(s) - \phi(0)|_X \\
&\quad + |\overline{\phi}_{t-t_i^n+a} - \phi|_B.
\end{aligned}
$$

Therefore, (4.11) and (4.13) imply that

$$
|w_{t-t_i^n+a}^n - \phi|_B \le K(\sigma - a)\frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]} + \frac{\varepsilon_n}{2} \le \frac{3}{4}\varepsilon_n. \tag{4.28}
$$

Moreover, if $t - t_i^n + s \le t_i^n$, then from (4.11) it follows that

$$|w^n(t - t_i^n + s) - w^n(s)|_X \le \frac{\varepsilon_n}{4[M_1 K(\sigma - a) + 1]}, \tag{4.29}$$

and if $s \le t_i^n$, $t - t_i^n + s > t_i^n$, then

$$|w^n(t_i^n) - w^n(s)|_X \le \frac{\varepsilon_n}{4[M_1 K(\sigma - a) + 1]}$$

and by (4.2) we have

$$\begin{aligned} &|w^n(t - t_i^n + s) - w^n(s)|_X \\ &\quad \le |w^n(t - t^n + s) - w^n(t_i^n)|_X + |w^n(t_i^n) - w^n(s)|_X \\ &\quad \le M_2(\varepsilon_n) + M_1|w^n(t_{i+1}^n) - w^n(t_i^n)|_X + \frac{\varepsilon_n}{4[M_1 K(\sigma - a) + 1]}. \end{aligned}$$

This, together with (4.26), implies that

$$|w^n(t - t_i^n + s) - w^n(s)|_X \le M_2(\varepsilon_n) + M_1(\tilde{N} + 1)\varepsilon_n + \frac{\varepsilon_n}{4[M_1 K(\sigma - a) + 1]}. \tag{4.30}$$

Substituting (4.28), (4.29) and (4.30) into (4.27), we obtain

$$\begin{aligned} |w_t^n - w_{\gamma^n(t)}^n|_B \le K(\sigma - a)\Big\{ & M_2(\varepsilon_n) + M_1(\tilde{N} + 1)\varepsilon_n \\ & + \frac{\varepsilon_n}{4[M_1 K(\sigma - a) + 1]}\Big\} \\ & + \sup_{a \le s \le \sigma} M(s - a) \cdot \frac{3}{4}\varepsilon_n. \end{aligned}$$

*Step* 9. We show that $\{w^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a, \sigma]$. In fact, by (4.17), we have

$$|w^n(t) - \phi(0)|_X \le \frac{\delta_1}{2[K(\delta_0) + 1]}$$

and hence from the argument in Step 2, we have

$$|w_t^n - \phi|_B \le \delta_1.$$

This implies that

$$|w_t^n|_B \leq |\phi|_B + \delta_1.$$

Therefore, by the Lipschitz condition of $F$, there exist a constant $L_1 > 0$ and a continuous function $L_2 : R_+ \to R_+$ such that $L_2(0) = 0$, and for all $\theta \in [a, \sigma]$, $n, m \geq 1$, we have

$$\begin{aligned}|F(\gamma^n(\theta), w^n_{\gamma^n(\theta)}) - F(\gamma^m(\theta), w^m_{\gamma^m(\theta)})|_X \\ \leq L_1|w^n_{\gamma^n(\theta)} - w^m_{\gamma^m(\theta)}|_B + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|).\end{aligned}$$

By the results in Step 7 and Step 8 as well as assumption (A3), we get

$$\begin{aligned}|F(\gamma^n(\theta), w^n_{\gamma^n(\theta)}) &- F(\gamma^m(\theta), w^m_{\gamma^m(\theta)})|_X \\ &\leq L_1|w^n_\theta - w^m_\theta|_B + L_1 Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ &\quad + L_1 Q \max\{\varepsilon_m, M_2(\varepsilon_m)\} + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|) \\ &\leq L_1 K(\sigma - a) \sup_{a \leq s \leq \theta} |w^n(s) - w^m(s)|_X \\ &\quad + L_1 Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ &\quad + L_1 Q \max\{\varepsilon_m, M_2(\varepsilon_m)\} + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|) \\ &\leq L_1 K(\sigma - a) \sup_{a \leq s \leq \theta} |v^n(s) - v^m(s)|_X \\ &\quad + L_1[K(\sigma - a)P + Q] \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ &\quad + L_1[K(\sigma - a)P + Q] \max\{\varepsilon_m, M_2(\varepsilon_m)\} \\ &\quad + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|).\end{aligned}$$

By the definition of $v^n$, it follows that

$$\begin{aligned}|v^n(t) &- v^m(t)|_X \\ &\leq \int_a^t |T(t,\theta)| F(\gamma^n(\theta), w^n_{\gamma^n(\theta)}) - F(\gamma^m(\theta), w^m_{\gamma^m(\theta)})|_X \, d\theta \\ &\leq L_1 K(\sigma - a)\widehat{M} e^{|\omega|(\sigma - a)} \int_a^t \sup_{a \leq s \leq \theta} |v^n(s) - v^m(s)|_X \, d\theta + \eta_{n,m},\end{aligned}$$

where

$$\begin{aligned}\eta_{n,m} = L_1 \widehat{M} e^{|\omega|(\sigma - a)}(\sigma - a)[K(\sigma - a)P + Q]\big(\max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ + \max\{\varepsilon_m, M_2(\varepsilon_m)\}\big) \\ + \widehat{M} e^{|\omega|(\sigma - a)}(\sigma - a) \max_{a \leq r \leq \sigma} L_2(|\gamma^n(r) - \gamma^m(r)|).\end{aligned}$$

Hence, by the well-known Gronwall integral inequality, we have

$$\sup_{a\leq s\leq t} |v^n(s) - v^m(s)|_X \leq \eta_{n,m} \exp\big(L_1 K(\sigma - a)\widehat{M} e^{|\omega|(\sigma-a)(t-a)}\big) \to 0$$

uniformly for $t \in [a,\sigma]$ as $n,m \to \infty$, since $\eta_{n,m} \to 0$ as $n,m \to \infty$. Therefore, $\{v^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a,\sigma]$, and so is $\{w^n(t)\}_{n=1}^{\infty}$ by (4.24).

*Step* 10. Let $u(t) = \lim_{n\to\infty} w^n(t)$ uniformly for $t \in [a,\sigma]$ and $u(t) = \tilde{\phi}(t-a)$ for $t \leq a$. Clearly $(t,u(t)) \in D$ since $(t,w^n(t)) \in D$ and $D$ is closed. We claim that $u(t)$ is a solution of (4.1). In fact, by the result in Step 8 and assumption (A3), we have

$$\begin{aligned} |w^n_{\gamma^n(t)} - u_t|_B &\leq |w^n_{\gamma^n(t)} - w^n_t|_B + |w^n_t - u_t|_B \\ &\leq Q\max\{\varepsilon_n, M_2(\varepsilon_n)\} + K(\sigma - a)\sup_{a\leq s\leq t}|w^n(s) - u(s)|_X. \end{aligned}$$

This implies $w^n_{\gamma^n(t)} \to u_t$ in $B$ uniformly for $t \in [a,\sigma]$ as $n \to \infty$, and hence

$$F(\gamma^n(t), w^n_{\gamma^n(t)}) \to F(t,u_t)$$

uniformly for $t \in [a,\sigma]$ as $n \to \infty$. Therefore

$$\begin{aligned} u(t) &= \lim_{n\to\infty}\left\{S(t,a)\phi(0) + \int_a^t T(t,\theta)F(\gamma^n(\theta), w^n_{\gamma^n(\theta)})\,d\theta\right\} \\ &= S(t,a)\phi(0) + \int_a^t T(t,\theta)F(\theta,u_\theta)d\theta. \end{aligned}$$

This establishes the existence of a solution of (4.1) on $[a,\sigma]$.

The uniqueness and continuation can be proved by using the standard technique. The proof is complete. □

As a consequence of Theorem 4.1, we obtain the following invariant property:

**Corollary 4.2.** *Suppose that $K_1$ and $K_2$ are nonempty closed, convex subsets of $X$ and $B$, respectively, such that* (D2), (T1)–(T3), *(S1), (S2) and (F) are satisfied with $D = [a,\infty) \times K_1$ and $\mathcal{D} = [a,\infty) \times K_2$. Moreover, assume that*

(i) *$S(t,s)K_1 \subseteq K_1$ for $t \geq s \geq a$;*

(ii) *$\lim_{h\to 0^+}(1/h)d\big(\phi(0) + hF(t,\phi); K_1\big) = 0$ for $(t,\phi) \in R \times K_2$.*

*Then the unique noncontinuable solution $u : (-\infty, b) \to B$, $b > a$ of (4.1) satisfies that $u(t) \in K_1$ and $u_t \in K_2$ for all $t \in [a,b)$.*

*Proof.* Obviously, (D3) holds since $D(t) = K_1$ is convex. To prove Corollary 4.2. it suffices to verify that

$$\lim_{h\to 0^+} \frac{1}{h} d\bigg( S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,\theta)F(t,\phi)d\theta;\ K_1\bigg) = 0$$

for $(t,\phi) \in [a,\infty)\times K_2$. Indeed, for given $h > 0$ and $(t,\phi) \in [a,\infty)\times K_2$, define $\psi : (-\infty, t+h] \to X$ by $\psi_t = \phi$ and $\psi(\theta) = S(\theta,t)\phi(0)$ for $\theta \in [t,t+h]$. Since $S(\theta,t)K_1 \subseteq K_1$ for $t \leq \theta \leq t+h$, by assumption (D2) we have $\psi_{t+h} \in K_2$ and $\psi_{t+h}$ is continuous in $h \in R_+$. Therefore, the set $\{(t,\psi_{t+h});\ 0 \leq h \leq 1\}$ is a compact set of $[a,\infty)\times K_2$. This, together with the convexity of the function $h \in R_+ \to d\big(\phi(0) + hF(t,\phi);\ K_1\big) \in R_+$ as well as the continuity of $F$, shows that assumption (ii) implies that $\lim_{h\to 0^+}(1/h)d\big(\psi(t+h) + hF(t,\psi_{t+h});\ K_1\big) = 0$.

On the other hand, by (A3), we have

$$\begin{aligned}|\psi_{t+h} - \psi|_B &\leq |\psi_{t+h} - \bar{\phi}_{t+h}|_B + |\bar{\phi}_{t+h} - \phi|_B \\ &\leq K(h) \sup_{t\leq\theta\leq t+h} |S(\theta,t)\phi(0) - \phi(0)|_X \\ &\quad + |\bar{\phi}_{t+h} - \phi|_B \to 0 \quad \text{as } h \to 0^+\end{aligned}$$

and for any $\varepsilon_1 > 0$ there exists $h_1 > 0$ such that if $0 < h < h_1$, then

$$|T(t+h,\theta)F(t,\phi) - F(t,\phi)|_X < \varepsilon_1, \qquad t \leq \theta \leq t+h.$$

Hence, if $0 < h < h_1$, we obtain

$$\begin{aligned}&\frac{1}{h} d\bigg( S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,\theta)F(t,\phi)d\theta;\ K_1\bigg) \\ &\qquad\qquad \leq \frac{1}{h} d\big(S(t+h,t)\phi(0) + hF(t,\phi);\ K_1\big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{h}\left|\int_t^{t+h}[T(t+h,\theta)F(t,\phi)-F(t,\phi)]\,d\theta\right|_X \\
&\le \frac{1}{h}d\big(S(t+h,t)\phi(0)+hF(t,\phi);\ K_1\big)+\varepsilon_1 \\
&\le \frac{1}{h}d\big(\psi(t+h)+hF(t,\psi_{t+h});\ K_1\big) \\
&\quad +|F(t,\psi_{t+h})-F(t,\phi)|_X+\varepsilon_1 \\
&\to 0 \quad \text{as } h\to 0.
\end{aligned}
$$

This completes the proof. □

**5. Comparison and monotonicity principles for semilinear integral equations.** In this section, we show how to apply the general results in Section 4 to obtain some comparison principles.

Let $S^+ = \{S^+(t,s);\ t\ge s\ge a\}$ and $S^- = \{S^-(t,s);\ t\ge s\ge a\}$ be given families of mappings from $X$ to $X$ which satisfy the following conditions:

(C1) $S^+$ and $S^-$ satisfy (S1) and (S2) with $S$ replaced by $S^+$ and $S^-$, respectively;

(C2) $S^-(t,s)x\le S(t,s)x\le S^+(t,s)x$ for all $t\ge s\ge a$ and $x\in X$.

In addition to (T1), (T2) and (T3), we require that

(T4) $T(t,s)X_+\subseteq X_+$ for $t\ge s\ge a$.

Suppose that $v^\pm : (-\infty,b]\to X$, $b>a$ are given such that $v_a^\pm\in B$, $v^\pm : [a,b]\to X$ are continuous and the following condition is satisfied:

(C3) $v_a^-\le_B v_a^+$ and $v^-(t)\le_X v^+(t)$ for $t\in[a,b]$.

Let

$$D(t)=[v^-(t),\ v^+(t)]_X,\qquad \mathcal{D}(t)=[v_t^-,\ v_t^+]_B\quad \text{for } t\ge a.$$

Define

$$D=\{(t,x);\ t\ge a, x\in D(t)\}\quad\text{and}\quad \mathcal{D}=\{(t,\phi);\ t\ge a,\phi\in\mathcal{D}(t)\}.$$

Clearly, (A5) implies that (D2) is satisfied. We assume that there exist continuous mappings $F^\pm:\mathcal{D}\to X$ such that

(C4) $v^+(t+h) \geq S^+(t+h,t)v^+(t) + \int_t^{t+h} T(t+h,\theta)F^+(\theta, v_\theta^+)d\theta$, $a \leq t < t+h < b$;

(C5) $v^-(t+h) \leq S^-(t+h,t)v^-(t) + \int_t^{t+h} T(t+h,\theta)F^-(\theta, v_\theta^-)d\theta$, $a \leq t < t+h < b$;

(C6) $\lim_{h\to 0^+}(1/h)d\big(v^+(0) - \phi(0) + h[F^+(t,v_t^+) - F(t,\phi)];\ X_+\big) = 0$, $a \leq t \leq b$, $(t,\phi) \in \mathcal{D}$;

(C7) $\lim_{h\to 0^+}(1/h)d\big(\phi(0) - v^-(0) + h[F(t,\phi) - F^-(t,v_t^-)];\ X_+\big) = 0$, $a \leq t \leq b$, $(t,\phi) \in \mathcal{D}$.

Now we can state and prove the following general comparison principle:

**Theorem 5.1.** *Suppose that* (T1)–(T4), (S1), (S2), (F), *and* (C1)–(C7) *are satisfied. If* $v_a^- \leq_B \phi \leq_B v_a^+$, *then the abstract semilinear integral equation* (4.1) *has a solution* $u$ *on* $[a, \bar{b})$ *for some* $\bar{b} \in (a, b]$ *such that*

$$v^-(t) \leq_X u(t) \leq_X v^+(t) \quad \textit{and} \quad v_t^- \leq_B u_t \leq_B v_t^+$$

*for* $t \in [a, \bar{b})$.

*Proof.* It has been shown in Martin and Smith [**36**] that (D3) is satisfied with $D(t) = [v^-(t), v^+(t)]_X$ on $[a, b]$. By using assumptions (C2), (C4), (S2) and (T3), we can show that if $t \in [a, b]$, $\phi \in \mathcal{D}(t)$ and $h > 0$ is sufficiently small, then

$$\begin{aligned}
d\Big(v^+(t+h) - S(t+h,t)\phi(0) - \int_t^{t+h} T(t+h,\theta)F(t,\phi)\,d\theta;\ X_+\Big) \\
\leq d\Big(S^+(t+h,t)v^+(t) - S(t+h,t)\phi(0) \\
+ \int_t^{t+h} T(t+h,\theta)[F^+(\theta,v_\theta^+) - F(t,\phi)]\,d\theta;\ X_+\Big) \\
\leq d\Big(S(t+h,t)v^+(t) - S(t+h,t)\phi(0) \\
+ \int_t^{t+h} T(t+h,\theta)[F^+(\theta,v_\theta^+) - F(t,\phi)]\,d\theta;\ X_+\Big)
\end{aligned}$$

$$
\begin{aligned}
&= d\Big(T(t+h,t)[v^+(t)-\phi(0)] \\
&\qquad\qquad + \int_t^{t+h} T(t+h,\theta)[F^+(\theta,v_\theta^+)-F(t,\phi)]d\theta;\ X_+\Big) \\
&\le d\big(T(t+h,t)\big([v^+(t)-\phi(0)]+h[F^+(t,v_t^+)-F(t,\phi)]\big);\ X_+\big) \\
&\quad + \Big| \int_t^{t+h} T(t+h,\theta)[F^+(\theta,v_\theta^+)-F(t,\phi)]\,d\theta \\
&\qquad - hT(t+h,t)[F^+(t,v_t^+)-F(t,\phi)]\Big|_X \\
&\le \widehat{M}e^{\omega h} d\big(v^+(t)-\phi(0)+h[F^+(t,v_t^+)-F(t,\phi)];\ X_+\big) \\
&\quad + \Big| \int_t^{t+h} T(t+h,\theta)[F^+(\theta,v_\theta^+)-F(t,\phi)]\,d\theta \\
&\qquad\qquad - hT(t+h,t)[F^+(t,v_t^+)-F(t,\phi)]\Big|_X .
\end{aligned}
$$

Therefore, by assumption (C6), we get

$$
\lim_{h\to 0^+} \frac{1}{h} d\Big(v^+(t+h) - S(t+h,t)\phi(0) - \int_t^{t+h} T(t+h,r)F(t,\phi)dr;\ X_+\Big) = 0.
$$

That is,

$$
\lim_{h\to 0^+} \frac{1}{h} d\Big(S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,\theta)F(t,\phi)d\theta;\ (-\infty, v^+(t+h)]_X\Big) = 0.
$$

Similarly, we can prove that

$$
\lim_{h\to 0^+} \frac{1}{h} d\Big(S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,r)F(t,\phi)dr;\ [v^-(t+h),\infty)_X\Big) = 0.
$$

These imply that

$$\lim_{h\to 0^+} \frac{1}{h} d\Big( S(t+h,t)\phi(0) + \int_t^{t+h} T(t+h,r)F(t,\phi)\,dr; \\ [v^-(t+h), v^+(t+h)]_X \Big) = 0,$$

which completes the proof, by Theorem 4.1. □

As an immediate consequence of the general comparison principle, we obtain the following monotonicity principle:

**Theorem 5.2.** *Suppose that* (T1)–(T4), (S1), (S2), (F) *and* (C3)–(C5) *are satisfied with* $S^+ = S^- = S$ *and* $F^+ = F^- = F$. *Moreover, suppose that* $F$ *satisfies the following quasimonotonicity condition*

(QM) $\lim_{h\to 0^+}(1/h)d\big(\psi(0) - \phi(0) + h[F(t,\psi) - F(t,\phi)];\ X_+\big) = 0$ *for all* $(t,\phi)$, $(t,\psi) \in \mathcal{D}$ *with* $\phi \le \psi$.

*Then for each* $\phi \in \mathcal{D}(a)$ *with* $v_a^- \le_B \phi \le_B v_a^+$, *equation* (4.1) *has a unique solution* $u(t;a,\phi)$ *on* $[a,\bar{b})$, *where* $a < \bar{b} = \bar{b}(\phi)$. *Furthermore, if* $v_a^- \le_B \phi \le_B \psi \le_B v_a^+$, *then*

$$v^-(t) \le_X u(t;a,\phi) \le_X u(t;a,\psi) \le_X v^+(t)$$

*and*

$$v_t^- \le_B u_t(a,\phi) \le_B u_t(a,\psi) \le_B v_t^+$$

*for all* $t \in [a,\hat{b})$, *where* $\hat{b} = \min\{\bar{b}(\phi), \bar{b}(\psi)\}$.

Throughout the remainder of this section, in order to establish some strict inequalities leading to the so-called quasi strongly order-preserving property, we assume that the Banach lattice $X$ is a product space $X = \prod_{i=1}^m X_i$, where $m$ is a positive integer, $X_i$ is a Banach space with a cone $X_i^+ \subseteq X_i$ such that $X_+ = \prod_{i=1}^m X_i^+$. We shall use $\ge_{X_i}$ and $\ge_X$ to denote the partial order on $X_i$ and $X$ induced by $X_i^+$ and $X_+$, respectively. Therefore, if $x = (x_i)^m$, $y = (y_i)^m \in X$, then $x \ge_X y$ if and only if $x_i \ge_{X_i} y_i$ for $i \in \{1, 2, \dots, m\}$.

We first assume that (T1)–(T3), (S1) and (S2) are satisfied by $T$ and $S$ which are defined as follows:

(CT1) $T(t,s)x = \left(T_i(t,s)x_i\right)_1^m$, where $T_i(t,s) : X_i \to X_i$ for all $t \ge s \ge a$ and $x = (x_i)_1^m \in X$;

(CT2) $S(t,s)x = \left(S_i(t,s)x_i\right)_1^m$, where $S_i(t,s) : X_i \to X_i$ for all $t \ge s \ge a$ and $x = (x_i)_1^m \in X$.

Moreover, we assume that (C1)–(C7) and (F) are satisfied with $v^\pm = \left(v_i^\pm\right)_1^m$, $F = \left(F_i\right)_1^m$, $F^\pm = \left(F_i^\pm\right)_1^m$, where $v_i^\pm$ maps $(-\infty, b)$ into $X_i$, $F_i^\pm$ maps $\mathcal{D}$ into $X_i$, and $F_i$ maps a neighborhood of $\mathcal{D}$ into $X_i$.

Under these assumptions, system (4.1) can be reformulated as

$$(5.1)\qquad \begin{aligned} &u_i(t) = S_i(t,a)\phi_i(0) + \int_a^t T_i(t,r)F_i(r,u_r)dr, \qquad t \ge a \\ &u_i(a+\theta) = \widehat{\phi}_i(\theta), \qquad \theta \le 0, \end{aligned}$$

where $\hat{\phi}_i = \left(\hat{\phi}_i\right)_1^m$ and $u = \left(u_i\right)_1^m$ is the solution of (4.1). From Theorem 5.1, if $v_a^- \le_B \phi \le_B v_a^+$, then $v_i^-(t) \le_{X_i} u_i(t) \le_{X_i} v_i^+(t)$ and $v_t^- \le_B u_t \le_B v_t^+$ for $t \in [a, \bar{b})$ and $i = 1, 2, \dots, m$.

The following one side Lipschitz condition will be useful in establishing strict inequalities:

(L1) for each $l > 0$ there exists $L_l > 0$ such that

$$F_i(t,\phi) - F_i^-(t, v_t^-) \ge -L_l[\phi_i(0) - v_i^-(t)]$$

for all $i \in \{1, 2, \dots, m\}$ and $(t,\phi) \in [a, a+l] \times B$ with $v_t^- \le_B \phi \le_B v_t^+$ and $|\phi|_B \le l$.

Under this assumption, it can be shown (by using the same argument as that for Lemma 3.1 in Martin and Smith [36]) that, if $u = \left(u_i\right)_1^m$ is the solution of (4.1), $b \in (a, \bar{b})$ with $b - a \le l$ and $|u_t|_B \le l$ for all $t \in [a, b]$, then

$$(5.2)\qquad u_i(t) - v_i^-(t) \ge_{X_i} e^{-L_l(t-t_0)} T_i(t,t_0)[u_i(t_0) - v_i^-(t_0)]$$

for all $a \le t_0 \le t \le b$ and $i = 1, 2, \dots, m$.

For each $i \in \{1, 2, \dots, m\}$, let $X_i^*$ be the dual space of $X_i$ and

$$P_i^* = \{\Phi_i \in X_i^*;\ \Phi_i(x_i) \ge 0 \text{ for all } x_i \in X_i^+\}.$$

Following Martin and Smith [36], to achieve generality we introduce a nonempty index set $\Lambda_i$ and assume that for each $\rho \in \Lambda_i$ there is a given $\Phi_i^\rho \in P_i^*$. Let

$$\Lambda_i^* = \{\Phi_i^\rho;\ \rho \in \Lambda_i\}.$$

The inequality (5.2) motivates the following abstract "maximum principle":

(MP) if $i \in \{1, 2, \dots, m\}$ and $x_i \in X_i^+$, then $\Phi_i^\sigma(x_i) > 0$ for some $\sigma \in \Lambda_i$ implies that $\Phi_i^\rho(T_i(t, t_0)x_i) > 0$ for all $t > t_0 \geq a$ and all $\rho \in \Lambda_i$.

From inequality (5.2) it easily follows that assumptions (L1) and (MP) imply that

(5.3) if $\Phi_k^\sigma(u_k(t_0)) > \Phi_k^\sigma(v_k^-(t_0))$ for some $t_0 \in [a, b)$ and $\sigma \in \Lambda_k$, then $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $t \in (t_0, b]$ and $\rho \in \Lambda_k$.

In order to obtain strict inequalities for other components, we need the following "irreducibility" condition:

(I1) there exists a constant $\tau_1 \geq 0$ such that if $t_1$ and $t_2$ are given constants such that $a \leq t_1 < t_1 + \tau_1 < t_2$, $\Sigma$ is a proper nonempty subset of $\{1, 2, \dots, m\}$ and $w = \left(w_j\right)_1^m : (-\infty, t_2] \to B$ is given such that $w : [t_1, t_2] \to B$ is continuous, $w_{t_1} \in B$ and

(a) $v_{t_1}^- \leq_B w_{t_1} \leq_B v_{t_1}^+$ and $v^-(t) \leq_X w(t) \leq_X v^+(t)$ for all $t \in [t_1, t_2]$;

(b) $\Phi_j^\rho(w_j(t)) = \Phi_j^\rho(v_j^-(t))$ for all $j \in \Sigma^c$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

(c) $\Phi_j^\rho(w_j(t)) > \Phi_j^\rho(v_j^-(t))$ for all $j \in \Sigma$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

then there exist a $k \in \Sigma^c$ and a $\sigma \in \Lambda_k$ such that

$$\sup\{\Phi_k^\sigma(F_k(t, w_t)) - \Phi_k^\sigma(F_k^-(t, v_t^-));\ t_1 \leq t \leq s\} > 0$$

for all $t_1 + \tau < s \leq t_2$.

We are now in the position to state our first result on strict inequalities relative to $v^-(t)$. Similar results hold for $v^+(t)$.

**Theorem 5.3.** *Suppose that, in addition to all conditions of Theorem* 5.1, (MP), (L1) *and* (I1) *are satisfied. Consider the solution $u$ of* (4.1) *defined on $[a, \bar{b})$. If there exists a $t_1 \geq a$ such that $t_1 + (m-1)\tau_1 < \bar{b}$ and $\Phi_j^\sigma(u_j(t_1)) > \Phi_j^\sigma(v_j^-(t_1))$ for some $j \in \{1, 2, \dots, m\}$ and some $\sigma \in \Lambda_j$, then there exists a $t_m \in [t_1, t_1 + (m-1)\tau_1]$ such that*

$$\Phi_i^\rho(u_i(t)) > \Phi_i^\rho(v_i^-(t))$$

*for all* $t \in (t_m, \bar{b})$, $i \in \{1, 2, \ldots, m\}$ *and* $\rho \in \Lambda_i$.

*Proof.* Let $\Sigma = \{j\}$. If $m = 1$, then by (5.3) we are done. If $m > 1$, then $\Sigma$ is a proper and nonempty subset of $\{1, 2, \ldots, m\}$. We now fix $\varepsilon \in (0, \bar{b} - t_1 - \tau_1)$ and claim that there exists $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and some $t_2 \in [t_1, t_1 + \tau_1 + \varepsilon]$ such that $\Phi_k^\sigma(u_k(t_2)) > \Phi_k^\sigma(v_k^-(t_2))$. By the way of contradiction, if the claim is false, then $\Phi_k^\rho(u_k(t)) = \Phi_k^\rho(v_k^-(t))$ for all $k \in \Sigma^c$, $\rho \in \Lambda_k$ and $t \in [t_1, t_1 + \tau_1 + \varepsilon]$. By (I1), we can find $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and a sequence $\{\varepsilon_j\}_1^\infty$ in $(0, \varepsilon)$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$\Phi_k^\sigma(F_k(t_1 + \tau_1 + \varepsilon_j, u_{t_1+\tau_1+\varepsilon_j})) - \Phi_k^\sigma(F_k^-(t_1 + \tau_1 + \varepsilon_j, v^-_{t_1+\tau_1+\varepsilon_j})) > 0.$$

By continuity, one can find $\bar{t}_j \in [t_1 + \tau_1, t_1 + \tau_1 + \varepsilon_j]$ such that

$$\Phi_k^\sigma(T_k(t_1 + \tau_1 + \varepsilon_j, \theta)[F_k(\theta, u_\theta) - F_k^-(\theta, u_\theta)]) > 0$$

for $\bar{t}_j \le \theta \le t_1 + \tau_1 + \varepsilon_j$. Therefore

$$\begin{aligned}
&\Phi_k^\sigma(u_k(t_1 + \tau_1 + \varepsilon_j) - v_k^-(t_1 + \tau_1 + \varepsilon_j)) \\
&\quad \ge \Phi_k^\sigma(S(t_1 + \tau_1 + \varepsilon_j, \bar{t}_j)[u_k(\bar{t}_j) - v_k^-(\bar{t}_j)]) \\
&\quad + \int_{\bar{t}_j}^{\bar{t}_j+\tau_1+\varepsilon_j} \Phi_k^\sigma(T(t_1 + \tau_1 + \varepsilon_j, t\theta)[F_k(\theta, u_\theta) - F_k^-(\theta, u_\theta)])\, d\theta \\
&\quad > 0,
\end{aligned}$$

a contradiction to the assumption that $\Phi_k^\sigma(u_k(t)) = \Phi_k^\sigma(v_k^-(t))$ for all $t \in [t_1, t_1 + \tau_1 + \varepsilon]$..

Therefore, there exist $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and some $t_2 \in [t_1, t_1 + \tau_1 + \varepsilon]$ such that $\Phi_k^\sigma(u_k(t_2)) > \Phi_k^\sigma(v_k^-(t_2))$. By (5.3), we have $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $t > t_2$ and thus for $t > t_1 + \tau_1 + \varepsilon$. Because of the arbitrary choice of $\varepsilon$, we have $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $\rho \in \Lambda_k$ and $t > t_2$, where $t_2$ is some number in $[t_1, t_1 + \tau_1]$.

If $m = 2$, then we are done. If $m > 2$, by repeating the above argument $m$ times, we will arrive at the conclusion. □

In order to obtain strict inequalities between compatible solutions of (4.1), we assume

(L2) for each $l > 0$, there is an $L_l > 0$ such that

$$F_i(t,\psi) - F_i(t,\phi) \geq -L_l[\psi_i(0) - \phi_i(0)]$$

for all $i = 1, 2, \ldots, m$ and $(t,\phi)$, $(t,\psi) \in [a, a+l] \times B$ with $v_t^- \leq_B \phi \leq_B \psi \leq_B v_t^+$ and $|\phi|_B$, $|\psi|_B \leq l$;

(I2) there exists a constant $\tau_1 > 0$ such that if $t_1$ and $t_2$ are given constants with $a \leq t_1 < t_1 + \tau_1 < t_2$, $\Sigma$ is a nonempty and proper subset of $\{1, 2, \ldots, m\}$ and $w^\pm = \left(w_j^\pm\right)_1^m : (-\infty, t_2] \to X$ are given such that $w^\pm : [t_1, t_2] \to X$ are continuous, $w_{t_1}^\pm \in B$ and

(a) $v_{t_1}^- \leq_B w_{t_1}^- \leq_B w_{t_1}^+ \leq_B v_{t_1}^+$ and $v^-(t) \leq_X w^-(t) \leq_X w^+(t) \leq_X v^+(t)$ for all $t \in [t_1, t_2]$;

(b) $\Phi_j^\rho(w_j^+(t)) = \Phi_j^\rho(w_j^-(t))$ for all $j \in \Sigma^c$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

(c) $\Phi_j^\rho(w_j^+(t)) > \Phi_j^\rho(w_j^-(t))$ for all $j \in \Sigma$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

then there exist a $k \in \Sigma^c$ and a $\sigma \in \Lambda_k$ such that

$$\sup\{\Phi_k^\sigma(F_k(t, w_t^+)) - \Phi_k^\sigma(F_k(t, w_t^-));\ t_1 \leq t \leq s\} > 0$$

for all $t_1 + \tau_1 < s \leq t_2$.

**Theorem 5.4.** *Suppose that, in addition to the assumptions for Theorem 5.2, (L2) and (I2) are satisfied. Then for $\psi^\pm \in B$ with $v_a^- \leq_B \psi^- <_B \psi^+ \leq_B v_a^+$, if $u^\pm$ denotes the solutions of (4.1) on $[a, \bar{b}(\psi^\pm))$ with $\phi = \psi^\pm$, respectively, then*

$$v^-(t) \leq_X u^-(t) \leq_X u^+(t) \leq_X v^+(t),$$
$$v_t^- \leq_B u_t^- \leq_B u_t^+ \leq_B v_t^+, \qquad a \leq t \leq \hat{b},$$

*where $\hat{b} = \min\{\bar{b}(\psi^-), \bar{b}(\psi^+)\}$. Also, if there is a $t_1 \geq a$ such that $t_1 + (m-1)\tau_1 < \hat{b}$ and $\Phi_j^\sigma(u_j^+(t_1)) > \Phi_j^\sigma(u_j^-(t_1))$ for some $j \in \{1, 2, \ldots, m\}$ and some $\sigma \in \Lambda_j$, then there is a $t_m \in [t_1, t_1 + (m-1)\tau_1]$ such that*

$$\Phi_i^\rho(u_i^+(t)) > \Phi_i^\rho(u_i^-(t))$$

*for all $t \in (t_m, \hat{b})$, $i \in \{1, 2, \ldots, m\}$ and $\rho \in \Lambda_i$.*

*Proof.* This is an immediate consequence of Theorems 5.1 and 5.3. □

**6. Applications to reaction-diffusion equations with distributed delay.** Suppose $\Omega$ is a bounded region in $R^n$ with $\partial\Omega$ smooth, $C(\overline{\Omega}; R^m)$ is the Banach space of continuous functions from $\overline{\Omega}$ to $R^m$ with the supremum norm, $\Delta$ is the Laplacian operator on $\Omega$ and $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$. Let $X = C(\overline{\Omega};\ R^m)$ and $B$ be a phase space satisfying (A1)–(A5) with $X_+ = C(\overline{\Omega};\ R^m_+)$ as specified in Section 3. Consider the following nonlinear reaction-diffusion system with infinite delay

$$
(6.1)\quad
\begin{aligned}
&\frac{\partial}{\partial t}u^i(t,x) = F_i(t,u_t)(x), && t > a,\ x \in \Omega,\ i \in \Sigma_0\\
&u^i_a(.,x) = \hat{\phi}^i(.,x), && x \in \overline{\Omega},\ i \in \Sigma_0\\
&\frac{\partial}{\partial t}u^i(t,x) = d_i\Delta u^i(t,x) + F_i(t,u_t)(x), && t > a,\ x \in \Omega,\ i \in \Sigma_0^c\\
&\alpha_i(x)u^i(t,x) + \frac{\partial}{\partial n}u^i(t,x) = \beta_i(t,x), && t > a,\ x \in \partial\Omega,\ i \in \Sigma_0^c\\
&u^i_a(.,x) = \hat{\phi}^i(.,x), && x \in \overline{\Omega},\ i \in \Sigma_0^c,
\end{aligned}
$$

where

(H1) $\Sigma_0$ is a given subset of $\{1,2,\dots,m\}$ and $d_i > 0$, $\alpha_i \in C^1(\overline{\Omega};\ R_+)$ and $\beta_i \in C^2(R_+ \times \overline{\Omega};\ R)$ for $i \in \Sigma_0^c$;

(H2) $a \geq 0$ is a constant, $\phi = (\phi_i)_1^m \in B$ is a given initial function and $\hat{\phi}$ is a representing element of $\phi$;

(H3) $F = (F_i)_1^m : R_+ \times B \to C(\overline{\Omega}; R^m)$ is continuous and for each $l > 0$, there exists $L_{1,l} > 0$ and $L_{2,l} : R_+ \to R_+$ with $L_{2,l}(0) = 0$ such that

$$|F(t,\phi) - F(s,\psi)|_{C(\overline{\Omega};R^m)} \leq L_{1,l}|\phi - \psi|_B + L_{2,l}(|t-s|)$$

for all $0 \leq t,\ s \leq l$ and $\phi,\ \psi \in B$ with $|\phi|_B,\ |\psi|_B \leq l$.

For $i \in \Sigma_0^c$, let $\mathcal{A}^i : \operatorname{dom}(\mathcal{A}^i) \subseteq C(\overline{\Omega}; R) \to C(\overline{\Omega}; R)$ be the linear operator defined by

$$\operatorname{dom}(\mathcal{A}^i) = \left(u^i \in C^2(\Omega; R) \cap C^1(\overline{\Omega}; R);\ \alpha_i u^i + \frac{\partial}{\partial n}u^i = 0 \text{ on } \partial\Omega\right),$$

$$\mathcal{A}^i u^i = d_i \Delta u^i, \qquad u^i \in \operatorname{dom}(\mathcal{A}^i).$$

Then the closure of $\mathcal{A}^i$ generates an analytic, nonexpansive and positive semigroup $\{T_i(t)\}_{t\geq 0}$ on $C(\overline{\Omega}; R)$ (see, cf. Mora [46] and Rothe [59]). Assume that $\gamma^i : [0,\infty) \times \overline{\Omega} \to R$ is a smooth mapping satisfying

$$\alpha_i(x)\gamma^i(t,x) + \frac{\partial}{\partial n}\gamma^i(t,x) = \beta_i(t,x) \quad \text{on } (0,\infty) \times \partial\Omega.$$

Define

$$S_i(t,s)w_0^i = T_i(t-s)[w_0^i - \hat{\mu}_i(s)] + \hat{\mu}_i(t), \qquad t \geq s \geq 0,\ w_0^i \in C(\overline{\Omega}; R),$$

where

$$\hat{\mu}_i(t) = \gamma^i(t) + \int_0^t T_i(t-s)\left[d_i\Delta\gamma^i(s) - \frac{\partial}{\partial t}\gamma^i(s)\right]ds.$$

For $i \in \Sigma_0$ and $v_0^i, w_0^i \in C(\overline{\Omega}; R)$, define

$$T_i(t)v_0^i = v_0^i, \qquad t \geq 0,$$
$$S_i(t,s)w_0^i = w_0^i, \qquad t \geq s \geq 0.$$

Let

$$v^i(t;\cdot,v_0^i) = T_i(t)v_0^i, \qquad v_0^i \in C(\overline{\Omega}; R),\ t \geq 0, \tag{6.2}$$

and

$$w^i(t,s,\cdot,w_0^i) = S_i(t,s)w_0^i, \qquad w_0^i \in C(\overline{\Omega}; R),\ t > s \geq 0. \tag{6.3}$$

Then $v^i(t;x,v_0^i)$, $i \in \Sigma_0^c$, is a solution of the following linear system (6.4)

$$\begin{aligned} \frac{\partial}{\partial t}v^i(t,x) &= d_i\Delta v^i(t,x), && t > 0,\ x \in \Omega,\ i \in \Sigma_0^c \\ \alpha_i(x)v^i(t,x) + \frac{\partial}{\partial n}v^i(t,x) &= 0, && t > 0,\ x \in \partial\Omega,\ i \in \Sigma_0^c \\ v^i(0,x) &= v_0^i(x), && x \in \overline{\Omega},\ i \in \Sigma_0^c, \end{aligned}$$

and $w^i(t,s,x,w_0^i)$, $i \in \Sigma_0^c$, is a solution of the following nonhomogeneous system (6.5)

$$\begin{aligned} \frac{\partial}{\partial t}w^i(t,x) &= d_i\Delta w^i(t,x), && t > s \geq 0,\ x \in \Omega,\ i \in \Sigma_0^c \\ \alpha_i(x)w^i(t,x) + \frac{\partial}{\partial n}w^i(t,x) &= \beta_i(t,x), && t > s \geq 0,\ x \in \partial\Omega,\ i \in \Sigma_0^c \\ w^i(s,x) &= w^i(x), && s \geq 0,\ x \in \overline{\Omega},\ i \in \Sigma_0^c. \end{aligned}$$

It is known that (T1)–(T4) and (S1)–(S2) are satisfied (see, cf. Martin and Smith [**36, 37**]). We will call a solution of the abstract integral equation

$$(6.6)\qquad \begin{aligned} u(t) &= S(t,a)\phi(0) + \int_a^t T(t-\theta)F(\theta, u_\theta)d\theta, \quad t \geq a \\ u_a &\in \phi, \quad (a,\phi) \in R_+ \times B \end{aligned}$$

a *mild solution* of (6.1). Hence by Theorem 4.1, for given closed subsets $D \subset [a,\infty) \times X$ and $\mathcal{D} \subset [a,\infty) \times B$ satisfying (D1)–(D3), we have the following result about the existence and uniqueness of a solution of the general nonlinear reaction-diffusion system (6.1).

**Theorem 6.1** (Existence). *Suppose that the operators* $T = \{T(t)\}_{t\geq 0}$ *and* $S = \{S(t,s)\}_{t\geq s\geq 0}$ *defined above satisfy* (SC). *Then system* (6.1) *has a unique noncontinuable mild solution* $u$, *denoted by* $u(t;a,\phi)$, *on an interval of the form* $[a,b)$, *where* $a < b \leq \infty$. *Moreover,* $u(t) \in D(t)$ *and* $u_t \in \mathcal{D}(t)$ *for* $t \in [a,b)$ *and if* $b < \infty$, *then*

$$\limsup_{t\to b^-} |u_t|_B = \infty.$$

We now show the existence of solutions of the nonlinear equation (6.1) relative to given upper and lower solutions. Suppose that $F^\pm = (F_i^\pm)_1^m : [a,\infty) \times B \to C(\overline{\Omega};\ R^m)$ are continuous functions, $v^\pm = (v_i^\pm)_1^m : (-\infty, c) \to C(\overline{\Omega};\ R^m)$ are given mappings, $a < c \leq \infty$, such that

(UL1) $v_a^\pm \in B$ with $v_a^- \leq_B v_a^+$ and $v^\pm(t) \in X$ with $v^-(t) \leq_{C(\overline{\Omega};R^m)} v^+(t)$ for $a \leq t < c$;

(UL2) if $i \in \Sigma_0^c$, $(\partial/\partial t)v_i^\pm$, $(\partial^2/\partial x^2)v_i^\pm : (a,c) \to C(\Omega; R)$ are continuous;

(UL3) if $i \in \Sigma_0$, $(\partial/\partial t)v_i^\pm : [a,c) \to C(\overline{\Omega}; R)$ is continuous;

(UL4) there exist $\beta_i^\pm \in C^2(R_+ \times \overline{\Omega};\ R)$ for $i \in \Sigma_0^c$ such that

$$\begin{gathered} \frac{\partial}{\partial t} v_i^+(t,x) \geq d_i \Delta v_i^+(t,x) + F_i^+(t, v_t^+)(x), \\ a < t < c,\ x \in \Omega,\ i \in \Sigma_0^c \end{gathered}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t} v_i^+(t,x) \geq F_i^+(t,v_t^+)(x), \\
&\qquad a \leq t < c,\ x \in \overline{\Omega},\ i \in \Sigma_0 \\
&\alpha_i(x) v_i^+(t,x) + \frac{\partial}{\partial n} v_i^+(t,x) = \beta_i^+(t,x) \geq \beta_i(t,x), \\
&\qquad a < t < c,\ x \in \partial\Omega,\ i \in \Sigma_0^c
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t} v_i^-(t,x) \leq d_i \Delta v_i^-(t,x) + F_i^-(t,v_t^-)(x), \\
&\qquad a < t < c,\ x \in \Omega,\ i \in \Sigma_0^c \\
&\frac{\partial}{\partial t} v_i^-(t,x) \leq F_i^-(t,v_t^-)(x), \\
&\qquad a \leq t < c,\ x \in \overline{\Omega},\ i \in \Sigma_0 \\
&\alpha_i(x) v_i^-(t,x) + \frac{\partial}{\partial n} v_i^-(t,x) = \beta_i^-(t,x) \leq \beta_i(t,x), \\
&\qquad a < t < c,\ x \in \partial\Omega,\ i \in \Sigma_0^c.
\end{aligned}
$$

We have the following result about the existence of a solution for equation (6.1) relative to an upper solution $(v^+)$ and a lower solution $(v^-)$.

**Theorem 6.2.** *Suppose that conditions* (UL1)–(UL4) *are satisfied and*

(FC) *for any* $(t,\phi) \in [a,\infty) \times B$ *with* $v_t^- \leq_B \phi \leq_B v_t^+$, *we have*

(a) *if* $\phi_k(0)(x) = v_k^+(t)(x)$ *for some* $k \in \{1,2,\dots,m\}$ *and* $x \in \overline{\Omega}$, *then* $F_k(t,\phi)(x) \leq F_k^+(t,v_t^+)(x)$;

(b) *if* $\phi_k(0)(x) = v_k^-(t)(x)$ *for some* $k \in \{1,2,\dots,m\}$ *and* $x \in \overline{\Omega}$, *then* $F_k(t,\phi)(x) \geq F_k^-(t,v_t^-)(x)$.

*Then the nonlinear system* (6.1) *has a unique noncontinuable mild solution* $u$ *on* $[a,b)$, *where* $b \geq c$, *and this solution satisfies*

$$
v^-(t) \leq_{C(\overline{\Omega};R^m)} u(t) \leq_{C(\overline{\Omega};R^m)} v^+(t) \quad \text{and} \quad v_t^- \leq_B u_t \leq_B v_t^+
$$

*for all* $t \in [a,c)$.

*Proof.* As $\{T(t)\}_{t\geq 0}$ is a positive semigroup on $C(\overline{\Omega}; R^m)$, (T4) is satisfied. Moreover, (UL4) implies (C4) and (C5), and (FC) implies (C6) and (C7). Therefore, the theorem follows from Theorem 5.1 immediately. □

One side inequality in Theorem 6.2 holds when the corresponding condition (a) or (b) of (FC) is satisfied. In particular, we have the following nonnegative property of solutions of equation (6.1).

**Theorem 6.3** (Nonnegativeness). *Suppose that*

(i) $\beta_i \geq 0$ *on* $\partial\Omega \times R_+$ *for all* $i \in \Sigma_0^c$;

(ii) $F$ *is quasipositive in the sense that if* $k \in \{1, 2, \ldots, m\}$ *and* $(t, \phi) \in R_+ \times B_+$, *then* $\phi_k(0)(x) = 0$ *at some* $x \in \overline{\Omega}$ *implies that* $F(t, \phi)(x) \geq 0$.

*Then for each* $(t, \phi) \in [a, \infty) \times B_+$, *system* (6.1) *has a unique noncontinuable mild solution* $u : (-\infty, b) \to C(\overline{\Omega}; R_+^m)$, $b > a$, *such that* $u(t) \in C(\overline{\Omega}; R_+^m)$ *and* $u_t \in B_+$ *for all* $t \in [a, b)$.

To state a criterion for invariant rectangles, we will require that the phase space $B$ satisfies (A6) in Section 3. Recall that we denote by $\bar{w}$ the constant mapping from $(-\infty, 0]$ into $X$ with the constant value $w \in X$.

**Theorem 6.4** (Invariant Rectangle Criterion). *Suppose that* $M = (M_i)_i^m$, $N = (N_i)_i^m \in R^m$ *with* $-\infty \leq M_i < N_i \leq \infty$ *for* $i \in \{1, 2, \ldots, m\}$ *and* $\overline{M} \leq_B \overline{N}$ *are given so that* $\alpha_i(x)M_i \leq \beta_i(t, x) \leq \alpha_i(x)N_i$ *for* $i \in \Sigma_0^c$, $x \in \overline{\Omega}$ *and that for any* $(t, \phi) \in [a, \infty) \times B$ *with* $\overline{M} \leq_B \phi \leq_B \overline{N}$, *we have*

(a) *if* $\phi_k(0)(x) = M_k$ *for some* $k \in \{1, 2, \ldots, m\}$, $x \in \overline{\Omega}$, *then* $F_k(t, \phi)(x) \geq 0$;

(b) *if* $\phi_k(0)(x) = N_k$ *for some* $k \in \{1, 2, \ldots, m\}$, $x \in \overline{\Omega}$, *then* $F_k(t, \phi)(x) \leq 0$.

*Then for each* $(t, \phi) \in [a, \infty) \times B$ *such that* $\overline{M} \leq_B \phi \leq_B \overline{N}$, *system* (6.1) *has a unique noncontinuable mild solution* $u : (-\infty, b) \to C(\overline{\Omega}; R_+^m)$, $b > a$, *such that* $\overline{M} \leq_B u_t \leq_B \overline{N}$ *for* $t \in [a, b)$.

Let us assume that all conditions of Theorem 6.4 are satisfied. We want to construct comparing systems for (6.1) relative to the rectangle $[\overline{M},\overline{N}]_B$. For $\phi \in [\overline{M},\overline{N}]_B$, $x \in \overline{\Omega}$ and $i \in \{1,2,...,m\}$, define

$$h_i(t,\phi)(x) = \inf\{F_i(t,\psi)(x);\ \phi \le_B \psi \le_B \overline{N},\ \phi_i(0)(x) = \psi_i(0)(x)\},$$
$$H_i(t,\phi)(x) = \sup\{F_i(t,\psi)(x);\ \overline{M} \le_B \psi \le_B \phi,\ \phi_i(0)(x) = \psi_i(0)(x)\}.$$

We assume that $h, H : R_+ \times [\overline{M},\overline{N}]_B \to C(\overline{\Omega}; R^m)$ are continuous, can be extended to a neighborhood of $R_+ \times [\overline{M},\overline{N}]_B$, and satisfy (H3) with $F$ replaced by $h$ and $H$, respectively and $R_+ \times B$ replaced by $R_+ \times [\overline{M},\overline{N}]_B$. Consider the following comparing systems

(6.7)
$$\begin{aligned}
&\frac{\partial}{\partial t}v^i(t,x) = h_i(t,v_t)(x), && t>a,\ x\in\overline{\Omega},\ i\in\Sigma_0\\
&\frac{\partial}{\partial t}v^i(t,x) = d_i\Delta v^i(t,x) + h_i(t,v_t)(x), && t>a,\ x\in\Omega,\ i\in\Sigma_0^c\\
&\alpha_i(x)v^i(t,x) + \frac{\partial}{\partial n}v^i(t,=x) = \beta_i(t,x), && t>a,\ x\in\partial\Omega,\ i\in\Sigma_0^c\\
&v_a^i(s,x) = \phi_i^-(s,x), && i\in\{1,2,\dots,m\},\ -\infty<s\le 0,;x\in\overline{\Omega}
\end{aligned}$$

and

(6.8)
$$\begin{aligned}
&\frac{\partial}{\partial t}w^i(t,x) = H_i(t,w_t)(x), && t>a,\ x\in\overline{\Omega},\ i\in\Sigma_0\\
&\frac{\partial}{\partial t}w^i(t,x) = d_i\Delta w^i(t,x) + H_i(t,w_t)(x), && t>a,\ x\in\Omega,\ i\in\Sigma_0^c\\
&\alpha_i(x)w^i(t,x) + \frac{\partial}{\partial n}w^i(t,x) = \beta_i(t,x), && t>a,\ x\in\partial\Omega,\ i\in\Sigma_0^c\\
&w_a^i(s,x) = \phi_i^+(s,x), && i\in\{1,2,\dots,m\},\ -\infty<s\le 0,\ x\in\overline{\Omega}.
\end{aligned}$$

By Theorem 6.2, we have the following comparison theorem

**Theorem 6.5** (Comparison Principle). *Assume that all conditions of Theorem* 6.4 *are satisfied. If* $\overline{M} \le_B \phi^- \le_B \phi \le_B \phi^+ \le_B \overline{N}$, *then* $\overline{M} \le_B v_t \le_B u_t(a,\phi) \le_B w_t \le_B \overline{N}$ *for all* $t \in [a,b)$ *provided* $u, v$ *and* $w$ *are defined on* $[a,b)$.

We now discuss the monotonicity of solutions of system (6.1). By Theorem 6.2, we have

**Theorem 6.6** (Monotonicity Principle). *Suppose that all conditions of Theorem 6.2 hold and $F : R_+ \times B \to C(\overline{\Omega}; R^m)$ satisfies the following quasi-monotonicity property*

(QMP) *if $k \in \{1, 2, \ldots, m\}$ and $(t, \phi)$, $(t, \psi) \in R_+ \times B$ are given such that $\phi \leq_B \psi$ and $\phi_k(0)(x) = \psi_k(0)(x)$ at some $x \in \overline{\Omega}$, then $F_k(t, \phi)(x) \leq F_k(t, \psi)(x)$.*

*Then for any $(a, \phi)$, $(a, \psi) \in R_+ \times B$ with $\phi \leq_B \psi$, we have*

$$u(t; a, \phi) \leq_{C(\overline{\Omega}; R^m)} u(t; a, \psi) \quad \text{and} \quad u_t(a, \phi) \leq_B u_t(a, \psi)$$

*for all $t \in [a, \min\{b(\phi), b(\psi)\})$.*

In order to obtain strict inequalities of solutions of (6.1), we suppose $F$ satisfies the following one-side Lipschitz condition, ignition condition and irreducibility condition:

(LR) for each $l > 0$, there is $L_l > 0$ such that

$$F_i(t, \psi) - F_i(t, \phi) \geq_{C(\overline{\Omega}; R^m)} -L_l[\psi_i(0) - \phi_i(0)]$$

for all $i = 1, 2, \ldots, m$ and $(t, \phi), (t, \psi) \in [a, a+l] \times B$ with $v_t^- \leq_B \phi \leq_B \psi \leq_B v_t^+$ and $|\phi|_B, |\psi|_B \leq l$;

(IG) there exists $\tau_0 > 0$ such that for any continuous functions $u, v : [a, a + \tau_0] \to C(\overline{\Omega}; R^m)$ with $u_a <_B v_a$ and $u(t) = v(t)$ for $t \in [a, a + \tau_0]$, there exists $k \in \{1, 2, \ldots, m\}$ and $x \in \overline{\Omega}$ such that

$$\sup\{F_k(t, v_t)(x) - F_k(t, u_t)(x);\ a \leq t \leq a + \tau_0\} > 0;$$

(IR) there exists a constant $\tau_1 > 0$ such that if $\Sigma$ is a proper, nonempty subset of $\{1, 2, \ldots, m\}$, $\tau > a + \tau_1$ and $u, v : (-\infty, \tau] \to C(\overline{\Omega}; R^m)$ are given so that

(a) $u_j(t)(x) < v_j(t)(x)$ for all $j \in \Sigma, x \in \overline{\Omega}$ and $t \in [\tau - \tau_1, \tau]$;

(b) $u_j(t)(x) = v_j(t)(x)$ for all $j \in \Sigma^c, x \in \overline{\Omega}$ and $t \in [\tau - \tau_1, \tau]$;

(c) $v_t^- \leq_B u_t \leq_B v_t \leq_B v_t^+$ for $t \in [a, \tau - \tau_1]$;

then there exists a $k \in \Sigma^c$ and $x \in \overline{\Omega}$ such that

$$\sup\{F_k(t, v_t)(x) - F_k(t, u_t)(x);\ \tau - \tau_1 \leq t \leq \tau\} > 0.$$

Applying Theorem 5.4 and the classical maximal principle to system (6.1) with $\Lambda_i = \overline{\Omega}$ and $\Phi_i^x(u_i) = u_i(x)$ for $x \in \overline{\Omega}$, $i \in \{1, 2, \ldots, m\}$ and $u_i \in C(\overline{\Omega}; R)$, we get the following result.

**Theorem 6.7** (Strict Inequality Principle). *Suppose that, in addition to the conditions of Theorem 6.6, (LR), (IG) and (IR) are satisfied. Assume that $\phi, \psi \in B$ are given so that $v_a^- \leq_B \phi <_B \psi \leq_B v_a^+$, and $u(t; a, \phi)$ and $u(t; a, \psi)$ are defined on $[a, b)$ with $b > a + \tau_0 + (m-1)\tau_1$. Then*

$$u^i(t; a, \phi)(x) < u^i(t; a, \psi)(x)$$

*for all $i \in \{1, 2, \ldots, m\}$, $x \in \overline{\Omega}$ and $t \in (a + \tau_0 + (m-1)\tau_1, b)$.*

In the following, we suppose that $F(t, u_t)(x) \equiv F(u_t)(x)$, i.e., we consider the following autonomous reaction-diffusion system with infinite delay

(6.9)

$$\frac{\partial}{\partial t} u^i(t, x) = F_i(u_t)(x), \qquad t > a,\ x \in \Omega,\ i \in \Sigma_0$$

$$u_a^i(., x) = \hat{\phi}^i(., x), \qquad x \in \overline{\Omega},\ i \in \Sigma_0$$

$$\frac{\partial}{\partial t} u^i(t, x) = d_i \Delta u^i(t, x) + F_i(u_t)(x),$$

$$t > a,\ x \in \Omega,\ i \in \Sigma_0^c$$

$$\alpha_i(x) u^i(t, x) + \frac{\partial}{\partial n} u^i(t, x) = \beta_i(t, x),$$

$$t > a,\ x \in \partial\Omega,\ i \in \Sigma_0^c$$

$$u_a^i(., x) = \widehat{\phi}^i(., x), \qquad x \in \overline{\Omega},\ i \in \Sigma_0^c,$$

Clearly, all the above theorems hold for system (6.9) with necessary modifications. Let $u(t; a, \phi)$ be a solution of the autonomous system (6.9). In the following, we always assume that all solutions of system (6.9) can be extended to infinity. Let $\Phi : [0, \infty) \times B \to B$ be defined by $\Phi(t, \phi) = u_t(\phi) = u_t(0, \phi)$ for all $(t, \phi) \in [0, \infty) \times B$. Then $\Phi$ is a semiflow generated by (6.9). The above strict inequality principle (Theorem 6.7) implies the following quasi strongly order-preserving property for the semiflow $\Phi$.

**Theorem 6.8** (Quasi Strongly Order-Preserving Principle). *Assume that all the conditions of Theorem 6.7 hold for autonomous system (6.9) and assume axioms (A7) and (A9) are satisfied by the phase space $B$. Then the solution semiflow $\Phi$ is quasi strongly order-preserving.*

*Proof.* For any $\phi, \psi \in B$ with $\phi <_B \psi$, by Theorems 6.6 and 6.7, we have

$$u(t,\phi) \leq_{C(\overline{\Omega};R^m)} u(t,\psi), \ u_t(\phi) \leq_B u_t(\psi) \quad \text{for } t \geq 0$$

and

$$u(t,\phi) \ll_{C(\overline{\Omega};R^m)} u(t,\psi) \quad \text{for all } t \geq \tau_0 + (m-1)\tau_1,$$

provided both solutions are defined.

Let $E_0$ denote the space of all constant mappings from $(-\infty, 0]$ into $C(\overline{\Omega}; R^m)$. Clearly, $E_0$ can be identified with $C(\overline{\Omega}; R^m)$, and all equilibria of $\Phi$ belong to $E_0$, the norm of $E_0$ is weaker than the induced topology from $B$, and by axiom (A7), for any $u, v \in E_0$, we have $u(t) \leq_{C(\overline{\Omega};R^m)} v(t)$ if and only if $u_t \leq_B v_t$.

Suppose $v \in E_0$ and $A$ is a given compact set invariant with respect to the semiflow and such that $v <_B A$.

**Claim I.** $v = v(t) \ll_{C(\overline{\Omega};R^m)} \psi(t)$ *for every* $\psi \in A$ *and* $t \leq 0$.

If the claim is false, then there exist $t^* \leq 0$, $k \in \{1, 2, \ldots, m\}, x \in \overline{\Omega}$ and $\psi \in A$ such that

$$v_k(x) = v_k(t^*)(x) = \psi_k(t^*)(x).$$

Let $\tau = 1 + \tau_0 + (m-1)\tau_1 - t^*$. Since $A$ is invariant, there exists $\psi^* \in A$ such that $\psi = u_\tau(\psi^*)$, so

$$u(\tau + t^*, \psi^*) = u_\tau(\psi^*)(t^*) = \psi(t^*).$$

On the other hand, since $v <_B \psi^*$, we have

$$\begin{aligned} v_k(y) &= u_k(\tau + t^*; v)(y) < u_k(\tau + t^*; \psi^*)(y) \\ &= \psi_k(t^*)(y) \quad \text{for all } y \in \overline{\Omega}, \end{aligned}$$

which contradicts the choice of $k, t^*$ and $x \in \overline{\Omega}$ such that $v_k(x) = \psi_k(t^*)(x)$.

**Claim II.** *There exists $\delta > 0$ such that*

$$v(t) + \delta \hat{e} \leq_{C(\overline{\Omega};R^m)} \psi(t)$$

*for every $t \leq 0$ and $\psi \in A$, where $\hat{e} \in C(\overline{\Omega};R^m)$ has the constant value $\hat{e} = (1, 1, \ldots, 1)^T$.*

In fact, if Claim II is not true, there must exist sequences $\{t_k\} \subset R_-$, $\{x_k\} \subseteq \overline{\Omega}$, $\{\psi^k\} \subseteq A$ and an integer $l \in \{1, 2, \ldots, m\}$ such that

$$v_l(x_k) + \frac{1}{k} \geq \psi_l^k(t_k)(x_k)$$

for infinitely many $k$. Since $\psi_{t_k}^k \in A$ and $A$ is compact, there exists a subsequence, also denoted by $\{\psi_{t_k}^k\}$, such that $\psi_{t_k}^k \to \psi \in A$. So $\psi^k(t_k) \to \psi(0)$ in $C(\overline{\Omega};R^m)$. We can also assume, without loss of generality, that $x_k \to x \in \overline{\Omega}$. Hence, by taking $k \to \infty$ in the above inequality, we get $v_l(x) \geq \psi_l(0)(x)$, a contradiction to Claim I.

Let $v_0 = v + \delta\hat{e}/2$. Then $v_0 \in E_0$ and $v \ll_{C(\overline{\Omega};R^m)} v_0$, and $v_0 \leq_B A$ by using (A9).

Now we can prove the quasi strongly order-preserving property of $\Phi$. Suppose that $\{\psi^n\}$ is a sequence of equilibria of $\Phi$ such that

$$\lim_{n\to\infty} \psi^n = \psi <_B A \quad \text{and} \quad \psi <_B \psi^n \quad \text{for } n = 1, 2, \ldots .$$

By Claim II, there exists $\psi^0 \in E_0$ such that

$$\psi \ll_{C(\overline{\Omega};R^m)} \psi^0, \quad \text{and} \quad \psi^0 \leq_B A.$$

On the other hand, since $\psi^n \to \psi$ in $E_0$, there exists an integer $n_0$ such that $\psi^{n_0} \ll_{C(\overline{\Omega};R^m)} \psi^0$. Hence, by (A7), we get

$$\psi <_B \psi^{n_0} \leq_B \psi^0 \leq_B A.$$

This completes the proof. □

We now consider the set-condensing property of the semiflow $\Phi_t : B \to B$, $t > 0$.

**Theorem 6.9** (Set-Condensing Principle). *Assume that the following fading memory condition is satisfied*

(FM) $M(t) < 1$ *for all* $t > 0$.

*Assume further that* $E$ *is a given subset of* $B$ *such that*

(i) $F$ *maps bounded subsets of* $E$ *into bounded subsets of* $X$;

(ii) *For all* $t > 0$, $\Phi_t$ *is defined and maps bounded subsets of* $E$ *into bounded subsets of* $X$.

*Then* $\Phi_t$ *is set-condensing on* $E$ *for* $t > 0$ *with respect to the Kuratowski measure of noncompactness.*

*Proof.* Clearly, $\Phi_t(\phi) = P_1(t)\phi + P_2(t)\phi$ for $(t,\phi) \in R_+ \times E$, where

$$(P_1(t)\phi)(\theta) = \begin{cases} S(t+\theta,0)\phi(0), & -t \le \theta \le 0 \\ \phi(t+\theta), & \theta < -t, \end{cases}$$

$$(P_2(t)\phi)(\theta) = \begin{cases} \int_0^{t+\theta} T(t+\theta-s)F(\Phi_s(\phi))\,ds, & -t \le \theta \le 0 \\ 0, & \theta < -t. \end{cases}$$

By using the same argument as that in Travis and Webb [**69**], we can show that $P_2(t) : E \to B$ is compact for $t > 0$. Moreover, using (A3) we can prove that if (FM) holds, then $P_1(t) : E \to B$ is a set-condensing mapping with respect to the Kuratowski measure of noncompactness for each $t > 0$. Therefore, the conclusion follows. □

Note that if $B = UC_g$ with

$$\sup_{\theta \le -t} \frac{g(t+\theta)}{g(\theta)} < 1, \qquad t > 0, \tag{6.10}$$

then (FM) is satisfied.

We are now in the position to discuss some applications of Theorem 2.1, 2.3–2.5 to the reaction-diffusion system (6.9).

**Theorem 6.10.** *Assume that* $F$ *satisfies* (QMP) *and* $\phi <_B \psi$ *are order-related equilibria of* $\Phi$ *such that there is no equilibrium in* $[\phi,\psi]_B$ *except* $\phi$ *and* $\psi$, *and all conditions of Theorem* 6.9 *are satisfied with* $E = [\phi,\psi]_B$. *Then there exists a monotone orbit connecting* $\phi$ *and* $\psi$.

**Theorem 6.11.** *Suppose that all conditions of Theorem 6.8 are satisfied, and there exist a subequilibrium $\phi$ and a superequilibrium $\psi$ of $\Phi$ with $\phi <_B \psi$, such that all conditions of Theorem 6.9 are satisfied with $E = [\phi, \psi]_B$. Then*

(a) *If all equilibria of $\Phi$ in $E$ are stable with respect to $E$, then every bounded solution in $E$ converges.*

(b) *If $\phi$ and $\psi$ are strict subequilibrium and superequilibrium of $\Phi$, respectively, then there exists a stable equilibrium in $E$.*

(c) *If $\phi$ and $\psi$ are equilibria stable with respect to $E$ and $\phi$ is isolated from above or $\psi$ is isolated from below, then there exists an unstable equilibrium in $E$.*

We conclude this section with an invariance principle for system (6.9) of Liapunov-Razumikhin type. We will state our results for $B = UC_g$ with $g$ satisfying (g1)–(g3) and (6.10). We leave extensions of this invariance principle in general phase spaces to a further paper.

Let $V : C(\overline{\Omega}; R^m) \to R$ be a given continuous function. The derivative of $V$ with respect to a solution $u$ of (6.9) is defined by

$$V'_{(6.9)}(u(t)) = \limsup_{h\to 0^+} \frac{1}{-h}\{V(u(t-h)) - V(u(t))\}, \qquad t > 0.$$

Define

$$\begin{aligned} D(W) &= \{\phi \in UC_g; V(\phi(\cdot)) : (-\infty, 0] \to R \quad \text{is bounded}\}, \\ W(\phi) &= \sup_{\theta\le 0} \frac{V(\phi(\theta))}{g(\theta)}, \qquad \phi \in D(W). \end{aligned}$$

For a constant $C$, define

$$\begin{aligned} G_C &= \{\phi \in D(W); W(u_t(\phi)) \equiv W(\phi) = C,\ t \ge 0\} \\ &= \left\{\phi \in D(W); \sup_{\theta\le 0} \frac{V(u(\phi)(t+\theta))}{g(\theta)} \equiv \sup_{\theta\le 0} \frac{V(\phi(\theta))}{g(\theta)} = C,\ t \ge 0\right\}, \end{aligned}$$

$M_C =$ the largest subset of $G_C$ that is invariant with respect to(6.9).

**Theorem 6.12** (Invariance Principle). *Suppose that there exists a continuous function $V : C(\overline{\Omega}; R^m) \to R$ such that for all $\phi \in D(W)$*

*and $t > 0$ with $V(u(t,\phi)) = W(u_t(\phi))$, $V'_{(6.9)}(u(t,\phi)) \le 0$. Assume further that $F$ maps bounded subsets of $D(W)$ into bounded subsets of $X$, and that $\Phi_t$ is defined for $t > 0$ and maps bounded subsets of $D(W)$ into bounded subsets of $X$. Then for any given $\phi \in D(W)$ such that $u_t(\phi)$ and $V(u_t(\phi))$ are bounded and $Cl\{u_t(\phi); t \ge 0\} \subseteq D(W)$, there exists $C \in R$ such that*

$$u_t(\phi) \to M_C \quad \text{as } t \to \infty.$$

*Proof.* By Theorem 6.10, the semiflow $\Phi_t$ is set-condensing on $D(W)$ for $t > 0$. So, for any $\phi \in D(W)$ such that $u_t(\phi)$ is bounded and $Cl\{u_t(\phi); t \ge 0\} \subseteq D(W), \omega(\phi)$ is nonempty, compact, connected and invariant. We claim that the map $t \to W(u_t)$ is nonincreasing on $[0, \infty)$, where $u_t = u_t(\phi)$.

In fact, if it is not nonincreasing on $[0, \infty)$, then for some $t_0 > 0$,

$$\liminf_{h \to 0^+} \frac{1}{-h}\{W(u_{t_0-h}) - W(u_{t_0})\} > 0,$$

hence there exist $h_n \to 0^+$ as $n \to \infty$ and $\alpha > 0$ such that

$$\frac{W(u_{t_0-h_n}) - W(u_{t_0})}{-h_n} \ge \alpha, \qquad n \ge 1,$$

which implies that $W(u_{t_0}) > W(u_{t_0-h_n})$. We shall show that $W(u_{t_0}) = V(u(t_0))$.

Suppose that $W(u_{t_0}) > V(u(t_0))$. Then the boundedness of $V(u(\cdot)) : (-\infty, 0] \to R$ and $\lim_{s \to -\infty} g(s) = \infty$ guarantee that there exists $\theta_0 < 0$ such that

$$W(u_{t_0}) = \frac{V(u(t_0 + \theta_0))}{g(\theta_0)}.$$

Choose sufficiently large $n$ so that $\theta_0 + h_n < 0$. Hence

$$\begin{aligned} W(u_{t_0-h_n}) &\ge \frac{V(u(t_0+\theta_0))}{g(\theta_0+h_n)} \\ &= \frac{V(u(t_0+\theta_0))}{g(\theta_0)} \cdot \frac{g(\theta_0)}{g(\theta_0+h_n)} \end{aligned}$$

$$\geq \frac{V(u(t_0+\theta_0))}{g(\theta_0)}$$
$$\geq W(u_{t_0}),$$

a contradiction to $W(u_{t_0}) > W(u_{t_0-h_n})$.

By assumption, we have

$$\limsup_{h\to 0^+} \frac{V(u(t_0-h)) - V(u(t_0))}{-h} \leq 0.$$

On the other hand, we have

$$\begin{aligned} V(u(t_0-h_n)) - V(u(t_0)) &= V(u(t_0-h_n)) - W(u_{t_0}) \\ &\leq W(u_{t_0-h_n}) - W(u_{t_0}), \end{aligned}$$

which implies that

$$\frac{V(u(t_0-h_n)) - V(u(t_0))}{-h_n} \geq \frac{W(u_{t_0-h_n}) - W(u_{t_0})}{-h_n} \geq \alpha,$$

a contradiction. This proves that $W(u_t)$ is nonincreasing on $[0,\infty)$.

As $V(u_t(\phi))$ is bounded for all $t \geq 0$, $\inf_{t\geq 0} W(u_t) \geq \inf_{t\geq 0} V(u_t) > -\infty$. Therefore,

$$\lim_{t\to\infty} W(u_t) = C \quad \text{exists.}$$

Let $\psi \in \omega(\phi)$. Then there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $u_{t_n}(\phi) \to \psi$ as $n \to \infty$. This leads to

$$W(\psi) = \lim_{n\to\infty} W(u_{t_n}) = C.$$

Thus

$$W(u_t(\psi)) = W(\psi) = C \quad \text{for all } t \geq 0,$$

which implies that $\psi \in G_C$ and we have $\omega(\phi) \subseteq G_C$. It follows that $u_t(\phi) \to M_C$ as $t \to \infty$ and the proof is complete. □

The above result generalizes the invariance principle of Haddock and Terjéki [**15**] for functional differential equations with infinite delay to reaction-diffusion systems with distributed delay.

**7. Lotka-Volterra models with diffusion and distributed delay.** Consider the Lotka-Volterra competition-diffusion model with distributed delay
(7.1)
$$\frac{\partial}{\partial t}u^i = d_i\Delta u^i+u^i\left[r_i-a_{ii}u^i-\sum_{j=1}^{m}b_{ij}\int^{t}-\infty k_{ij}(t-s)u^j(s,x)ds\right],$$
$$t>0,\ x\in\Omega$$
$$\frac{\partial}{\partial n}u^i = 0, \qquad t>0,\ x\in\partial\Omega$$
$$u^i(s,x)=\phi_i(s,x), \qquad s\in(-\infty,0],\ x\in\Omega,\ i\in\{1,2,\dots,m\},$$

where $\Omega$ is a bounded region in $R^n$ with smooth boundary $\partial\Omega$, the parameters satisfy

$$d_i>0, \qquad r_i>0, \qquad a_{ii}>0, \qquad b_{ij}\geq 0,$$

and each $k_{ij}:[0,\infty)\to R_+$ is a nonincreasing continuous function and can be normalized so that

$$\int_0^\infty k_{ij}(s)ds = 1.$$

Let $\mathcal{A}:\text{dom}(\mathcal{A})\subseteq C(\overline{\Omega};R^m)\to C(\overline{\Omega};R^m)$ be the linear operator defined by

$$\text{dom}(\mathcal{A})=\left\{u\in C^2(\Omega;R^m)\cap C^1(\overline{\Omega};R^m),\ \frac{\partial}{\partial n}u^i=0 \text{ on } \partial\Omega,\ 1\leq i\leq m\right\},$$

$$\mathcal{A}u=(d_i\Delta u^i)_1^m, \quad u\in\text{dom}\,(\mathcal{A}).$$

It is well known that (see, cf. Rothe [59]) the closure of the operator $\mathcal{A}$ generates an analytic semigroup $T=\{T(t)\}_{t\geq 0}$ in the space $C(\overline{\Omega};R^m)$. In [1], it has shown that there exists $g:(-\infty,0]\to[1,\infty)$ satisfying conditions (g1)–(g3) in Section 3 and such that

$$\int_\infty^0 k_{ij}(-s)g(s)ds<\infty, \qquad i,j\in\{1,2,\dots,m\}.$$

Clearly, the mapping $F = (F_i)_1^m : UC_g \to C(\overline{\Omega}; R^m)$ defined by

$$F_i(\phi) = \phi_i(0)\left[r_i - a_{ii}\phi_i(0) - \sum_{j=1}^{m} b_{ij} \int_{-\infty}^{0} k_{ij}(-s)\phi_j(s)ds\right] \tag{7.2}$$

satisfies (H3) in Section 6.

We now consider the abstract integral equation

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\theta)F(u_\theta)d\theta, \qquad t \geq s \geq 0. \tag{7.3}$$

Theorem 6.1 implies that for any $\phi \in UC_g$, there exists a unique solution $u(t,\phi)$ of (7.3) subject to the initial condition

$$u_0 = \phi. \tag{7.4}$$

For any $R = (R_i)_1^m \in \text{int}R_+^m$, we set

$$K_R = \prod_{i=1}^{m}[0, R_i] \subseteq R_+^m,$$
$$UC_{g,R} = \{\phi \in UC_g; \phi(\theta) \in C(\overline{\Omega}; K_R) \text{ for } \theta \leq 0\}.$$

**Lemma 7.1.** *If $R_i \geq r_i/a_{ii}$ for $i \in \{1, 2, \ldots, m\}$, then $UC_{g,R}$ is positively invariant for system* (7.1). *That is, if $\phi \in UC_{g,R}$, then $u_t(\phi) \in UC_{g,R}$ for all $t \geq 0$.*

*Proof.* Assume that $\phi \in UC_{g,R}$. If $\phi_i(0)(x) = 0$ for some $x \in \overline{\Omega}$, then clearly $F_i(\phi)(x) = 0$. If $\phi_i(0)(x) = R_i$, then $F_i(\phi)(x) \leq \phi_i(0)(x)[r_i - a_{ii}\phi_i(0)(x)] \leq 0$. Consequently, the conclusion follows from the invariant rectangular criterion (Theorem 6.4). □

For given $\phi \in UC_{g,R}$ with $R_i \geq r_i/a_{ii}$ for $i \in \{1,2,\dots,m\}$, define

$$\begin{aligned} F_i^-(\phi)(x) &= \inf\Big\{\psi_i(0)(x)\Big[r_i - a_{ii}\psi_i(0)(x) - \sum_{j=1}^m b_{ij}\int_{-\infty}^0 k_{ij}(-s)\psi_j(s)(x)ds\Big]; \\ &\qquad \phi \leq_{UC_g} \psi \leq_{UC_g} \overline{R},\ \psi_i(0)(x) = \phi_i(0)(x)\Big\} \\ &= \phi_i(0)(x)\Big[r_i - a_{ii}\phi_i(0)(x) - \sum_{j=1}^m b_{ij}R_j\Big] \end{aligned}$$

and

$$\begin{aligned} F_i^+(\phi)(x) &= \sup\Big\{\psi_i(0)(x)\Big[r_i - a_{ii}\psi_i(0)(x) - \sum_{j=1}^m b_{ij}\int_{-\infty}^0 k_{ij}(-s)\psi_j(s)(x)\,ds\Big]; \\ &\qquad 0 \leq_{UC_g} \psi \leq_{UC_g} \phi,\ \psi_i(0)(x) = \phi_i(0)(x)\Big\} \\ &= \phi_i(0)(x)[r_i - a_{ii}\phi_i(0)(x)]. \end{aligned}$$

Let $v^i(t,\phi)$ and $w^i(t,\phi)$ be the solutions of the following systems

$$\begin{aligned} &\frac{\partial}{\partial t}v^i = d_i\Delta v^i - v^i\Big[\Big(-r_i + \sum_{j=1}^m b_{ij}R_j\Big) + a_{ii}v^i\Big], \qquad x \in \Omega \\ &\frac{\partial}{\partial n}v^i = 0, \qquad x \in \partial\Omega,\ 1 \leq i \leq m \\ &v^i(0,\phi)(x) = \phi_i(0,x), \qquad x \in \Omega \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &\frac{dw^i}{dt} = w^i[r_i - a_{ii}w^i], \qquad 1 \leq i \leq n \\ &w^i(0,\phi) = \sup_{x\in\Omega}\phi_i(0,x), \end{aligned} \tag{7.6}$$

respectively. We have the following result.

**Proposition 7.2.** *If $u(t,\phi)$ is a solution of system* (7.3) *such that $\phi \in UC_{g,R}$ with $R_i \geq r_i/a_{ii}$ for $i \in \{1,2,\dots,m\}$, then*

$$v^i(t,\phi)(x) \leq u^i(t,\phi)(x) \leq w^i(t,\phi) \tag{7.7}$$

*for all $i \in \{1,2,\dots,m\}$, $x \in \overline{\Omega}$ and $t > 0$. In particular, if $\phi_i(0)(\bar{x}) > 0$ for some $\bar{x} \in \Omega$, then*

$$u^i(t,\phi)(x) > 0 \tag{7.8}$$

*for all $i \in \{1,2,\dots,m\}$, $x \in \Omega$ and $t > 0$. Moreover, $u(\cdot,\phi) : R_+ \to C(\overline{\Omega}; R^m)$ is bounded.*

*Proof.* The inequality (7.7) is an immediate consequence of the comparison principle (Theorem 6.5). If $\phi_i(0)(x) > 0$ for some $x \in \Omega$, then the classical maximum principle implies that $v^i(t,\phi)(x) > 0$ for all $(t,x) \in (0,\infty) \times \Omega$, thus $u^i(t,\phi)(x) > 0$ for all $(t,x) \in (0,\infty) \times \Omega$.

Clearly, $\lim_{t\to\infty} w^i(t,\phi) \leq r_i/a_{ii} < \infty$ implies that

$$\limsup_{t\to\infty} u^i(t,\phi)(x) \leq r_i/a_{ii}, \qquad 1 \leq i \leq m.$$

This implies the boundedness of the solutions $u(t,\phi)$ of (7.3). □

**Theorem 7.3.** *Suppose that system* (7.1) *has a unique, spatially homogeneous steady state solution $u^* = (u^{*i})_1^m \in \operatorname{int} R_+^m$. Furthermore, assume that*

$$a_{ii} > \sum_{j=1}^{m} b_{ij} \int_{-\infty}^{0} k_{ij}(-s) g(s) ds, \qquad 1 \leq i \leq m. \tag{7.9}$$

*Then for any $\phi \in UC_{g,R}$ so that $R_i \geq r_i/a_{ii}$ and $\phi_i(0) >_{C(\overline{\Omega};R)} 0$, $i = 1,2,\dots,m$, we have $u_t(\phi) \to u^*$ as $t \to \infty$.*

*Proof.* Throughout the proof, we assume, without loss of generality, that $u^{*i} \neq u^{*j}$ for $i \neq j$. For otherwise we can choose $\delta_i > 0$ such that $\delta_i u^{*i} \neq \delta_j u^{*j}$, $i \neq j$, and make the change of variables $\overline{u}^i = \delta_i u^i$ in system (7.1).

For $\xi \in R^m$, define

$$V_1(\xi) = \max\{|\xi^i - u^{*i}| : 1 \le i \le m\},\ I(\xi) = \{i : V_1(\xi) = |\xi^i - u^{*i}|\}.$$

Also, for $u \in C(\overline{\Omega}, R^m)$, define

$$V(u) = \max\{V_1(u(x)) : x \in \overline{\Omega}\},\ J(u) = \{x : V(u) = V_1(u(x))\}.$$

It is easy to check that if

$$D_-V(u)(v) = \limsup_{h\to 0+} \frac{V(u-hv) - V(u)}{-h}$$

for all $u, v \in C(\overline{\Omega}, R^m)$, then for $u(x) \not\equiv u^*$ on $\overline{\Omega}$, we have
(7.10)

$$D_-V(u)(v) = \max\{\text{sign}\,[u^i(x) - u^{*i}]v^i(x) : x \in J(u),\ i \in I(u(x))\},$$

where

$$\text{sign}\,(r) = \begin{cases} 1 & \text{if } r > 0 \\ -1 & \text{if } r < 0. \end{cases}$$

Note that $T(t)u^* \equiv u^*$ for $t \ge 0$. Applying the maximum principle, we can show that

$$V(T(h)u) \le V(u) \quad \text{for all } h \ge 0 \quad \text{and} \quad u \in C(\overline{\Omega}; R^m).$$

Assume $\phi \in UC_{g,R}$ and denote $u(t) = u(t, \phi)$. Then

$$u(t) - hF(u_t) = T(h)u(t-h) + o(h) \quad \text{for } t \ge 0,$$

where $h^{-1}|o(h)| \to 0$ as $h \to 0^+$. Hence,

$$V(u(t) - hF(u_t)) \le V(T(h)u(t-h)) + o(h) \le V(u(t-h)) + o(h).$$

Consequently,

$$V'_{(7.3)}(u(t))) \le D_-V(u(t))(F(u_t)). \tag{7.11}$$

Now for $\phi \in UC_{g,R}$, define

$$W(\phi) = \sup_{\theta \le 0} \frac{V(\phi(\theta))}{g(\theta)}. \tag{7.12}$$

We want to show that if $W(u_t) = V(u(t))$, then

$$V'_{(7.3)}(u(t)) \leq 0. \tag{7.13}$$

Pick up $i \in \{1, 2, \ldots, m\}$ and $x_0 \in \overline{\Omega}$ so that $V(u(t)) = |u^i(t)(x_0) - u^{*i}|$. Assume at this moment that $V(u(t)) \neq 0$. We consider

$$\begin{aligned}
&\operatorname{sign}[u^i(t)(x_0) - u^{*i}]F_i(u_t)(x_0) \\
&= \operatorname{sign}[u^i(t)(x_0) - u^{*i}]u^i(t)(x_0) \\
&\quad \times \left\{ r_i - a_{ii}u^i(t)(x_0) - \sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s)u^j(s)(x_0)\,ds \right\} \\
&= \operatorname{sign}[u^i(t)(x_0) - u^{*i}]u^i(t)(x_0) \\
&\quad \times \left\{ - a_{ii}[u^i(t)(x_0) - u^{*i}] - \sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s)[u^j(s)(x_0) - u^{*i}]\,ds \right\} \\
&= u^i(t)(x_0)|u^i(t)(x_0) - u^{*i}| \\
&\quad \times \left\{ - a_{ii} - \frac{\sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s)[u^j(s)(x_0) - u^{j*}]\,ds}{u^i(t)(x_0) - u^{*i}} \right\}.
\end{aligned}$$

As $W(u_t) = V(u(t))$, we have

$$|u^j(s)(x_0) - u^{j*}| \leq |u^i(t)(x_0) - u^{*i}|g(s-t)$$

for all $j \in \{1, 2, \ldots, m\}$ and $s \leq t$. Therefore,

$$\begin{aligned}
-\frac{b_{ij} \int_{-\infty}^t k_{ij}(t-s)[u^j(s)(x_0) - u^{*i}]ds}{u^i(t)(x_0) - u^{*i}} &\leq b_{ij} \int_{-\infty}^t k_{ij}(t-s)g(s-t)ds \\
&= b_{ij} \int_{\infty}^0 k_{ij}(-s)g(s)ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\operatorname{sign}[u^i(t)(x_0) - u^{*i}]F_i(u_t)(x_0) \\
&\leq u^i(t)(x_0)|u^i(t)(x_0) - u^{*i}| \left\{ - a_{ii} + \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s)g(s)ds \right\}.
\end{aligned}$$

This, together with (7.9)–(7.11), implies (7.13).

By the invariance principle (Theorem 6.12), we know that there exists $C \in R$ such that

$$u_t \to M_C \subseteq G_C \quad \text{as } t \to \infty.$$

We will show that $C \equiv 0$.

For the sake of contradiction, we suppose that $C > 0$. Since $\omega(\phi)$ of $u_t(\phi)$ is nonempty, compact and invariant, if $\psi \in \omega(\phi)$ and $v(t) = v(t, \psi)$ is the solution of (7.3), then

$$W(v_t) = C > 0 \quad \text{for all } t \geq 0. \tag{7.14}$$

For each $t > 0$, there exists $x^t \in \overline{\Omega}$ and $i \in \{1, 2, \dots, m\}$ so that

$$V(v(t)) = |v^i(t, \psi)(x^t) - u^{*i}|.$$

**Claim I.** *For any $t > 0$, if $V(v(t)) = C = W(v_t)$ then $v^i(t)(x^t) = 0$.*

In fact, if $V(v(t)) = W(v_t)$ and $v^i(t)(x^t) > 0$ hold simultaneously at some $t > 0$, then the above argument for (7.13) implies that

$$\limsup_{h \to 0^+} \frac{V(v(t-h)) - V(v(t))}{-h} < 0.$$

Consequently, for some sufficiently small $h > 0$ with $t - h > 0$, we have

$$W(v_{t-h}) \geq V(v(t-h)) > V(v(t)) = C,$$

a contradiction to $W(v_s) \equiv C$ for all $s \geq 0$. This justifies the claim.

**Claim II.** *It is impossible for $V(v(t)) < C$ for all $t > 0$.*

By the way of contradiction, we assume that $V(v(t)) < C$ for all $t > 0$. Fix $s_1 < 0$ so that $g(s_1) > 1$. Hence, for a fixed $t > -s_1$, we have

$$\overline{C} := \max\{V(v(s)); t + s_1 \leq s \leq t\} < C$$

and

$$W(v_t) \leq \max\{\overline{C}, C/g(s_1)\} < C,$$

a contradiction to $W(v_s) \equiv C$ for all $s \geq 0$.

Putting Claim I and Claim II together, we obtain $t_0 > 0$, $i_0 \in \{1, 2, \ldots, m\}$ and $x^{t_0} \in \overline{\Omega}$ so that $V(v(t_0)) = W(v_{t_0}) = C = |v^{i_0}(t_0)(x^{t_0}) - u^{i_0 *}|$ and $v^{i_0}(t_0)(x^{t_0}) = 0$. Hence, $C = u^{i_0 *}$.

**Claim III.** $|v^j(t)(x) - u^{*j}| < C = u^{*i_0}$ *for all* $t > 0$, $x \in \overline{\Omega}$ *and* $j \neq i_0$.

For otherwise, there exists $t^* > 0$, $x^* \in \overline{\Omega}$ and $j \neq i_0$ so that $V(v(t^*)) = |v^j(t^*)(x^*) - u^{*j}| = C = W(v_{t^*})$. By the result in Claim I, $v^j(t^*)(x^*) = 0$ and thus, $u^{*j} = C = u^{*i_0}$, a contradiction to our assumption $u^{*i} \neq u^{*j}$ for $i \neq j$.

Since $g(s) \to +\infty$ as $s \to -\infty$, we can choose $\sigma > 0$ so that $R_i/g(-\sigma) < C/2$. For this chosen $\sigma > t_0$, $v_\sigma \in \omega(\phi)$ and hence there exists a sequence $\{t_k\}$ so that $t_k \to \infty$ and $u_{t_k}(\phi) \to v_\sigma$ in $UC_g$ as $k \to \infty$. Consequently, we can use Claim III to find $K_1 > 0$ so that

$$\max\{|u^j(t_k + \theta)(x) - u^{*j}|;\ \theta \in [-\sigma, 0],\ x \in \overline{\Omega},\ j \neq i_0\} < u^{*i_0} = C$$

for $k \geq K_1$.

On the other hand, by Proposition 7.2, we derive from $v^{i_0}(t_0)(x^{t_0}) = 0$ that $v^{i_0}(\sigma + s)(x) = 0$ for all $s < 0$ (and hence for $s \leq 0$ by continuity) and $x \in \overline{\Omega}$. Since $u_{t_k}(\phi) \to v_\sigma$ as $k \to \infty$, we can find $K_2 > 0$ so that $0 < u^{i_0}(t_k + \theta)(x) \leq u^{*i_0}$ for all $x \in \overline{\Omega}, \theta \in [-\sigma, 0]$ and $k \geq K_2$. Hence, for $k \geq K_2$, we have

$$\max\{|u^{i_0}(t_k + \theta)(x) - u^{*i_0}|;\ \theta \in [-\sigma, 0],\ x \in \overline{\Omega}\} < u^{*i_0} = C.$$

Therefore,

$$\max\{|u^j(t_k + \theta)(x) - u^{*j}|;\ \theta \in [-\sigma, 0],\ x \in \overline{\Omega},\ j = 1, 2, \ldots, m\} < C$$

and

$$W(u_{t_k}) \leq \max\Big\{\max\{|u^j(t_k + \theta)(x) - u^{*j}|;\ \theta \in [-\sigma, 0],\ x \in \overline{\Omega},\ j = 1, 2, \ldots, m\},\ \frac{2R_i}{g(-\sigma)}\Big\} < C.$$

Therefore, $W(u_t) \leq W(u_{t_{K_2}}) < C$ for all $t \geq t_{K_2}$ and $\lim_{t\to\infty} W(u_t) < C$, a contradiction to $\lim_{t\to\infty} W(u_t) = C$.

Hence $C \equiv 0$ and then $G_C = \{u^*\}$. This completes the proof. □

In general, system (7.1) does not satisfy the quasimonotone hypothesis even allowing the nonstandard partial ordering. However, if $m = 2$, then system (7.1) generates a monotone semiflow in the sense described below. For $u, v \in R^2$, define $u \leq_Q v$ if $u_1 \leq v_1$ and $u_2 \geq v_2$. The subscript $Q$ indicates that the order is generated by the second quadrant of $R^2$. This generates a new partial ordering in $UC_g$ in the obvious pointwise sense, that is, $\phi = (\phi_1, \phi_2) \leq_Q (\psi_1, \psi_2) = \psi$ in $UC_g$ if and only if $\phi_1(s) \leq \psi_1(s)$ and $\phi_2(s) \geq \psi_2(s)$ for all $s \in (-\infty, 0]$.

Now we consider the following two-species Lotka-Volterra diffusion-competition model with infinite delay

(7.15)
$$\begin{aligned}
&\frac{\partial}{\partial t}u_1 = d_1\Delta u_1 + u_1\left[r_1 - a_1u_1 - b_1\int_{-\infty}^{t} k_1(t-s)u_2(s)ds\right], \qquad x \in \Omega\\
&\frac{\partial}{\partial t}u_2 = d_2\Delta u_2 + u_2\left[r_2 - b_2\int_{-\infty}^{t} k_2(t-s)u_1(s)ds - a_2u_2\right], \qquad x \in \Omega\\
&\frac{\partial}{\partial n}u_1 = \frac{\partial}{\partial n}u_2 = 0, \qquad x \in \partial\Omega\\
&u_1(s,x) = \phi_1(s,x) \geq 0, \qquad u_2(s,x) = \phi_2(s,x) \geq 0,\\
&\qquad s \in (-\infty, 0],\ x \in \Omega.
\end{aligned}$$

**Proposition 7.4.** *If $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ are elements of $UC_g^+$ and $\phi \leq_Q \psi$, then $u(t, x; \phi) \leq_Q u(t, x; \psi)$ for all $t \geq 0$, $x \in \Omega$.*

*Proof.* Setting $v_1 = u_1$ and $v_2 = -u_2$ in (7.15), we obtain

$$\text{(7.16)}\quad \begin{aligned}
&\frac{\partial}{\partial t}v_1 = d_1\Delta v_1 + v_1\left[r_1 - a_1v_1 + b_1\int_{-\infty}^{t} k_1(t-s)v_2(s,x)ds\right]\\
&\frac{\partial}{\partial t}v_2 = d_2\Delta v_2 + v_2\left[r_2 - b_2\int_{-\infty}^{t} k_2(t-s)v_1(s,x)ds + a_2v_2\right].
\end{aligned}$$

It can be easily verified that (QMP) in Section 6 holds for the standard partial ordering, hence the conclusion follows by the monotonicity principle (Theorem 6.6). □

System (7.15) has two boundary equilibria $(r_1/a_1, 0)$, $(0, r_2/a_2)$ and an interior equilibrium

$$(u_1^*, u_2^*) = \left(\frac{r_1a_2 - r_2b_1}{a_1a_2 - b_2b_1}, \frac{r_2a_1 - r_1b_2}{a_1a_2 - b_2b_1}\right) \quad \text{if } a_1a_2 - b_2b_1 \neq 0.$$

**Proposition 7.5.** *Suppose*

$$\max\{a_1/b_2,\ b_1/a_2\} < r_1/r_2. \tag{7.17}$$

*If* $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ *with* $R_1 = R_2 \geq \max\{\frac{r_1}{a_1}, \frac{r_2}{a_2}\}$ *and* $\phi_1(0, x) > 0$ *for some* $x \in \Omega$, *then*

$$(u_1(t,x),\ u_2(t,x)) \to (r_1/a_1, 0)$$

*uniformly in* $x \in \Omega$ *as* $t \to \infty$.

*Proof.* By (7.17) and the fact that $\int_0^\infty k_i(s)ds = 1$, $i = 1, 2$, for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$b_1 \int_{T_\varepsilon}^\infty k_1(s)ds < R_1^{-1}\varepsilon, \qquad \frac{a_2}{r_2} > \frac{b_1 \int_0^{T_\varepsilon} k_1(s)ds}{r_1 - \varepsilon},$$
$$\frac{a_1}{r_1 - \varepsilon} < \frac{b_2 \int_0^{T_\varepsilon} k_1(s)ds}{r_2}.$$

By Lemma 7.1 and Proposition 7.2, $0 < u_1(t,x) \leq R_1$ and $u_2(t,x) \leq R_2$ for all $t > 0$ and $x \in \overline{\Omega}$. Note that

$$\begin{aligned}
u_1&\left[r_1 - a_1u_1 - b_1 \int_{-\infty}^t k_1(t-s)u_2(s)\,ds\right] \\
&= u_1\left[r_1 - a_1u_1 - b_1 \int_{-\infty}^{t-T_\varepsilon} k_1(t-s)u_2(s)\,ds \right.\\
&\qquad \left. - b_1 \int_{t-T_\varepsilon}^t k_1(t-s)u_2(s)\,ds\right] \\
&\geq u_1(r_1-\varepsilon)\left[1 - \frac{a_1}{r_1-\varepsilon}u_1 - \frac{b_1}{r_1-\varepsilon}\int_{t-T_\varepsilon}^t k_1(t-s)u_2(s)\,ds\right]
\end{aligned}$$

and similarly

$$
\begin{aligned}
u_2\left[r_2 - a_2u_2 - b_2\int_{-\infty}^{t} k_2(t-s)u_1(s)\,ds\right]&\\
&= u_2\left[r_2 - a_2u_2 - b_2\int_{-\infty}^{t-T_\varepsilon} k_2(t-s)u_1(s)\,ds\right.\\
&\qquad\left. - b_2\int_{t-T_\varepsilon}^{t} k_2(t-s)u_1(s)\,ds\right]\\
&\le u_2\left[r_2 - a_2u_2 - b_2\int_{t-T_\varepsilon}^{t} k_2(t-s)u_1(s)\,ds\right]\\
&= u_2r_2\left[1 - \frac{a_2}{r_2}u_2 - \frac{b_2}{r_2}\int_{t-T_\varepsilon}^{t} k_2(t-s)u_1(s)ds\right].
\end{aligned}
$$

Therefore, by Theorem 6.2 we get

$$u_1(t,x) \ge u_1^*(t,x), \qquad u_2(t,x) \le u_2^*(t,x)$$

for all $t \ge 0$ and $x \in \overline{\Omega}$, where $(u_1^*, u_2^*)$ solves the following reaction-diffusion equation with delay

(7.18)

$$
\begin{aligned}
&\partial_t u_1^* = d_1\Delta u_1^* + (r_1-\varepsilon)u_1^*\left[1 - \frac{a_1}{r_1-\varepsilon}u_1^*\right.\\
&\qquad\qquad\left. - \frac{b_1}{r_1-\varepsilon}\int_{t-T_\varepsilon}^{t} k_1(t-s)u_2^*(s)\,ds\right], \qquad x\in\overline{\Omega}\\
&\partial_t u_2^* = d_2\Delta u_2^* + r_2u_2^*\left[1 - \frac{a_2}{r_2}u_2^* - \frac{b_2}{r_2}\int_{t-T_\varepsilon}^{t} k_2(t-s)u_1^*(s)\,ds\right], \qquad x\in\overline{\Omega}\\
&\partial_n u_1^* = \partial_n u_2^* = 0, \qquad x\in\partial\Omega\\
&u_1^*(s) = \phi_1(s), \qquad u_2^*(s) = \phi_2(s), \qquad s\in[-T_\varepsilon, 0].
\end{aligned}
$$

Now, Proposition 5.6 in [**37**] can be applied to (7.18) (as delay $T_\varepsilon$ is finite). It follows that

$$\lim_{t\to\infty} u_1^*(t,x) = \frac{r_1-\varepsilon}{a_1}, \qquad \lim_{t\to\infty} u_2^*(t,x) = 0$$

uniformly for $x \in \overline{\Omega}$. Since $u_2(t,x) \le u_2^*(t,x)$, we have that $\lim_{t\to\infty} u_2(t,x) = 0$ uniformly for $x\in\overline{\Omega}$. On the other hand, by Proposition 7.2 we get $\limsup_{t\to\infty} u_1(t,x) = r_1/a_1$, and by $u_1(t,x) \ge u_1^*(t,x)$

we have $\liminf_{t\to\infty} u_1(t,x) = (r_1 - \varepsilon)/a_1$. Consequently, due to the arbitrary choice of $\varepsilon$, we have $\lim_{t\to\infty} u_1(t,x) = r_1/a_1$ uniformly for $x \in \overline{\Omega}$. This completes the proof. □

Similarly we can prove the following result.

**Proposition 7.6.** *Suppose*

$$\min\{a_1/b_2, b_1/a_2\} > r_1/r_2. \tag{7.19}$$

*If* $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ *with* $R_1 = R_2 \geq \max\{r_1/a_1, r_2/a_2\}$ *and* $\phi_2(0,x) > 0$ *for some* $x \in \Omega$, *then*

$$(u_1(t,x),\ u_2(t,x)) \to (0, r_2/a_2)$$

*uniformly in* $x \in \Omega$ *as* $t \to \infty$.

Propositions 7.5 and 7.6 indicate that if the intraspecies competition is weaker than the interspecies competition, then one of the two competitors will go to extinction, that is, competition exclusion principle applies to system (7.15). For the positive interior equilibrium, we can employ a similar argument to establish the following result.

**Proposition 7.7.** *Suppose*

$$b_1/a_2 < r_1/r_2 < a_1/b_2. \tag{7.20}$$

*If* $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ *with* $R_1 = R_2 \geq \max\{r_1/a_1, r_2/a_2\}$ *and* $\phi_1(0,x) > 0$, $\phi_2(0,x) > 0$ *for some* $x \in \Omega$, *then*

$$(u_1(t,x),\ u_2(t,x)) \to (u_1^*, u_2^*)$$

*uniformly in* $x \in \Omega$ *as* $t \to \infty$.

The above propositions demonstrate that if one of the conditions (7.17), (7.19) and (7.20) holds, then the asymptotic behavior of (7.16) is very similar to that of the ordinary differential equation model

$$\begin{aligned} \frac{du_1}{dt} &= u_1[r_1 - a_1 u_1 - b_1 u_2] \\ \frac{du_2}{dt} &= u_2[r_2 - b_2 u_1 - a_2 u_2] \end{aligned} \tag{7.21}$$

and the reaction-diffusion model

$$
\begin{aligned}
&\frac{\partial}{\partial t}u_1 = d_1\Delta u_1 + u_1[r_1 - a_1u_1 - b_1u_2], \qquad x \in \Omega \\
&\frac{\partial}{\partial t}u_2 = d_2\Delta u_2 + u_2[r_2 - b_2u_1 - a_2u_2], \qquad x \in \Omega \\
&\frac{\partial}{\partial n}u_1 = \frac{\partial}{\partial n}u_2 = 0, \qquad x \in \partial\Omega.
\end{aligned}
\tag{7.22}
$$

In particular, there are no stable spatially inhomogeneous solutions.

In the remaining case that

$$a_1/b_2 < r_1/r_2 < b_1/a_2, \tag{7.23}$$

Matano and Mimura [**43**] showed that there may exist a stable spatially inhomogeneous solution for the reaction-diffusion model (7.22) provided the domain $\Omega$ is suitably nonconvex, for example, dumbbell shaped with a narrow handle. Matano and Mimura only considered the reaction-diffusion model where there are no delays, but we believe that their conclusion is still true for (7.16), the reaction-diffusion model with infinite delay. This should be proved by following Matano and Mimura's procedure and by applying the techniques and results in this paper.

However, if the domain $\Omega$ is convex and condition (7.23) holds, then there cannot exist stable spatially inhomogeneous solutions (for general reaction-diffusion systems, this was proved by Kishimoto and Weinberger [**29**]). In fact, we have the following results when (7.23) is satisfied.

**Proposition 7.8.** *Suppose that* (7.23) *holds.*

(i) *If $\phi$ satisfies that $u_1^* < \phi_1(t,x) < R_1$, $0 < \phi_2(t,x) < u_2^*$ for $(t,x) \in (-\infty, 0] \times \overline{\Omega}$ with $R_1 \geq r_1/a_1$, then*

$$(u_1(t,x),\ u_2(t,x)) \to (r_1/a_1, 0)$$

*uniformly in $x \in \Omega$ as $t \to \infty$.*

(ii) *If $\phi$ satisfies that $0 < \phi_1(t,x) < u_1^*$, $u_2^* < \phi_2(t,x) < R_2$ for $(t,x) \in (-\infty, 0] \times \overline{\Omega}$ with $R_2 \geq r_2/a_2$, then*

$$(u_1(t,x),\ u_2(t,x)) \to (0, r_2/a_2)$$

*uniformly in* $x \in \Omega$ *as* $t \to \infty$.

(iii) $(u_1^*, u_2^*)$ *is unstable.*

*Proof.* (i) and (ii) can be proved following the previous procedure. To see (iii), we know that

$$(0, r_2/a_2) <_Q (u_1^*, u_2^*) <_Q (r_1/a_1, 0),$$

where $(0, r_2/a_2)$ and $(r_1/a_1, 0)$ are (locally) stable with respect to $E = [(0, r_2/a_2), \ (r_1/a_1, 0)]_Q$, and $(0, r_2/a_2)$ is isolated from above, Theorem 6.11 implies that there is an unstable equilibrium in $E$, while $(u_1^*, u_2^*)$ is the only steady state in $E$, hence it is unstable. □

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