

Hopf Bifurcation for Parametrized Equivariant Coincidence Problems and Parabolic Equations with Delays

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§1. Introduction

In this paper, we consider the following parametrized equivariant coincidence problem

$$(1.1) \quad L_\lambda(x) = F(\lambda, x), \quad (\lambda, x) \in \mathcal{E},$$

where \mathbf{E} and \mathbf{F} are real Banach spaces which are also isometric representations of the group $S^1 = \{z \in \mathbf{C}; |z| = 1\}$, $\{L_\lambda\}_{\lambda \in \mathbf{R}^2}$ is a continuous family of equivariant Fredholm operators of index zero from \mathbf{E} to \mathbf{F} , and F is a completely continuous equivariant mapping from the locally trivial S^1 -vector bundle $\mathcal{E} := \{(\lambda, x) \in \mathbf{R}^2 \times \mathbf{E}; x \in \mathbf{E}_\lambda\}$ to \mathbf{F} , where \mathbf{E}_λ is the space $\text{Dom}(L_\lambda)$ equipped with the graph norm. We further assume that there exists a 2-dimensional submanifold $M \subseteq \mathbf{R}^2 \times \mathbf{E}^{S^1}$, where $\mathbf{E}^{S^1} = \{x \in \mathbf{E}; gx = x \text{ for all } g \in S^1\}$, such that

- (i) for every $(\lambda, x) \in M$, $L_\lambda x = F(\lambda, x)$;
- (ii) if $(\lambda_0, x_0) \in M$ then there exist open neighbourhoods U_{λ_0} of λ_0 in \mathbf{R}^2 and U_{x_0} of x_0 in \mathbf{E}^{S^1} and a C^1 -map $\eta: U_{\lambda_0} \rightarrow \mathbf{E}^{S^1}$ such that $M \cap (U_{\lambda_0} \times U_{x_0}) = \{(\lambda, \eta(\lambda)); \lambda \in U_{\lambda_0}\}$.

Under this assumption, all points $(\lambda, x) \in M$ are solutions of (1.1) (called trivial solutions). One of the main purposes of this paper is to develop a Hopf bifurcation theory which provides very sharp information about the maximum continuation of nontrivial solutions (solutions which are not in M) of (1.1).

Our approach to the bifurcation problem of (1.1) is to employ an equivariant resolvent K of $L = \{L_\lambda\}_{\lambda \in \mathbf{R}^2}$ to reduce the problem to a corresponding problem for a certain completely continuous equivariant vector field $\Theta_K(F): \mathbf{R}^2 \times \mathbf{F} \rightarrow \mathbf{F}$, and then to appeal to the method of Gęba and Marzantowicz [21] based on the notion of the S^1 -equivariant degree of [11] as well as the complementing function method of Ize (cf. [25], [26]).

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The geometric essence of our approach can be roughly described as follows: Let $(\lambda_0, x_0) \in M$ be given such that $L_{\lambda_0} - D_x F(\lambda_0, x_0): \mathbf{E}_{\lambda_0} \rightarrow \mathbf{F}$ is not an isomorphism. Applying the well-known implicit function theorem and Gleason-Tietze G -extension theorem we can obtain an open invariant neighbourhood U of (λ_0, x_0) and a continuous bounded equivariant function $\varphi: \bar{U} \rightarrow \mathbf{R}$ such that $\varphi(\lambda, x) \neq 0$ for all $(\lambda, x) \in U \cap M$ and the system of equations $x = \Theta_K(F)(\lambda, x)$, $\varphi(\lambda, x) = 0$ has no solution on ∂U . Therefore, the mapping $\Psi: (\bar{U}, \partial U) \rightarrow (\mathbf{R} \times \mathbf{F}, \mathbf{R} \times \mathbf{F} \setminus \{0\})$ defined by $\Psi(\lambda, x) = (\varphi(\lambda, x), x - \Theta_K(F)(\lambda, x))$ for $(\lambda, x) \in \bar{U}$ is an equivariant compact vector field, the S^1 -degree $S^1\text{-Deg}(\Psi, U) = \{\deg_{z_k}(\Psi, U)\}_{k=1}^\infty$ is well defined, and $S^1\text{-Deg}(\Psi, U) \neq 0$ implies that (λ_0, x_0) is a bifurcation point, i.e., in any neighbourhood of (λ_0, x_0) there exists a nontrivial solution of (1.1).

A computation formula for $S^1\text{-Deg}(\Psi, U)$ is also provided for the sake of applications. In particular, we identify \mathbf{R}^2 with \mathbf{C} , define $\alpha: S^1 \rightarrow \mathbf{R}^2$ by setting $\alpha(z) = \lambda_0 + \rho z$ for $z \in S^1$ and for a sufficiently small $\rho > 0$. Using the direct sum decomposition $\mathbf{F} = \mathbf{F}_0 \oplus \mathbf{F}_1 \oplus \dots \oplus \mathbf{F}_k \oplus \dots$, we can obtain a continuous mapping $\psi: S^1 \rightarrow GL_C^{S^1}(\mathbf{F})$ induced by $\psi(z) = T_{\alpha(z)}: \mathbf{F} \rightarrow \mathbf{F}$, and its decomposition $\psi = \psi_0 \oplus \psi_1 \oplus \dots \oplus \psi_k \oplus \dots$ such that $\psi_k: S^1 \rightarrow GL_C^{S^1}(\mathbf{F}_k)$, $k = 0, 1, \dots$, where T_λ is the linearization of $Id - \Theta_K(F)$ evaluated at $(\lambda, \eta(\lambda))$. Then we show that $S^1\text{-Deg}(\Psi, U)$ can be calculated by the well known Brouwer degree according to the following formula

$$\deg_{z_k}(\Psi, U) = \eta \deg(\det \psi_k), \quad k = 1, 2, \dots$$

where \deg denotes the classical Brouwer degree and $\eta = \pm 1$, depending on whether ψ_0 is orientation preserving or orientation reversing.

The integer $\eta \deg(\det \psi_k)$ will be called the *crossing number*. Our results indicate that if $\eta \deg(\det \psi_k)$ is not zero, then (λ_0, x_0) is a bifurcation point. Under further technical assumptions, we show that any bounded component of the closure of the set of all nontrivial solutions of (1.1) in \mathcal{E} contains only finite number of bifurcation points and the sum of the crossing numbers associated with these bifurcation points must be zero.

The established Hopf bifurcation theory for the nonlinear problem (1.1) is then applied to the problem of finding nontrivial periodic solutions for a class of parabolic partial differential equations with both delayed and advanced arguments in the nonlinear term where the elliptic operator depends on a parameter α . Following the work of [10], [19], [24], [28] and [47], we introduce the unknown period of the periodic solution as a new parameter such that the periodic problem can be reduced to the bifurcation problem of a parametrized coincidence equation of the form (1.1) in a certain Sobolev space. Using a technical result about the computation of a Brouwer degree in a parallelepiped due to [12], we show that the aforementioned crossing

number can be calculated from a characteristic equation which arises naturally from the spectrum analysis of the linearization.

One of the motivations of this research is applications in chemical reactor theory where a quotient of diffusivities in a reaction diffusion system is used as a parameter (cf. [16] and [17]). Our research is also inspired by some recent work of Memory [33], Morita [37], Yamada and Niikura [50] and Yoshida [51] about the existence and stability of non-trivial time-periodic solutions for a class of reaction-diffusion equations with delay. In particular, Memory [33] has shown that a Hopf bifurcation occurs as the diffusion coefficient decreases through a certain continuous curve. This motivates us to develop a Hopf bifurcation theory for general reaction-diffusion equations with delay where the diffusion coefficients depend on a parameter.

Our general results provide an analog of the local Hopf bifurcation theorem of Krasnosel'skii [31] and the global Hopf bifurcation theorem of Rabinowitz [44] for the parametrized S^1 -equivariant coincidence problem (1.1). The application of these general results to parabolic equations with both delayed and advanced arguments can be regarded as an extension of the Alexander-Yorke global Hopf bifurcation for ordinary differential equations (cf. [1], [2], [3], [39]), functional differential equations (cf. [7], [8], [9], [38] and [42]) and parabolic partial differential equations (cf. [17]). The novelty of our research is the broad applicability of our results (allowing dependence on a parameter in linear parts and allowing both delayed and advanced arguments in the nonlinear terms of parabolic equations) and the methodology employed (a purely topological argument based on equivariant topological degree).

It should be mentioned that multiparameter bifurcation problems of coincidence equations have been studied by a number of authors, we refer to [1], [2], [3], [20] and references therein. Moreover, the global Hopf bifurcation theory of parabolic equations without delay was discussed in [17]. Our paper provides an alternative approach to the Hopf bifurcation problem based on the S^1 -equivariant degree of [11] (we refer to [27] for another construction of the S^1 -degree) and provides some results which can be easily applied to reaction-diffusion equations with delay.

The organization of this paper is as follows: a parametrized coincidence equation is introduced in section 2, and its bifurcation problem is discussed in section 3. The application of our bifurcation theory to parabolic equations with delayed and advanced arguments is given in section 4.

§2. Parametrized equivariant coincidence problems

Let E and F be real Banach spaces which are also isometric representations

of the group $G := S^1 = \{z \in \mathbf{C}; |z| = 1\}$. For a closed equivariant Fredholm operator of index zero $N: \text{Dom}(N) \subseteq \mathbf{E} \rightarrow \mathbf{F}$, we denote by $Gr(N) = \{(x, y) \in \mathbf{E} \times \mathbf{F}; x \in \text{Dom}(N), y = Nx\}$ the *graph* of N . Clearly, $Gr(N)$ is a closed invariant subspace of $\mathbf{E} \times \mathbf{F}$, where we assume that G acts diagonally on $\mathbf{E} \times \mathbf{F}$, and we have the following commutative diagram:

$$\begin{array}{ccc} & Gr(N) & \\ pr_1 \swarrow & & \searrow pr_2 \\ \mathbf{E} \cong \text{Dom}(N) & \xrightarrow{N} & \mathbf{F} \end{array}$$

where pr_1 and pr_2 are (continuous) equivariant projections on the first and the second component, respectively. The space $\text{Dom}(N)$ equipped with the *graph norm* $\|\cdot\|_N$ is a Banach representation of G , and in what follows it will be denoted by \mathbf{E}_N . It is clear that $N: \mathbf{E}_N \rightarrow \mathbf{F}$ is a continuous equivariant Fredholm operator of index zero.

We denote by $\mathcal{F}_0^G := \mathcal{F}_0^G(\mathbf{E}, \mathbf{F})$ the set of all closed equivariant Fredholm operators of index zero from \mathbf{E} into \mathbf{F} , and by $O^G := O^G(\mathbf{E}, \mathbf{F})$ the set of all closed equivariant linear operators from \mathbf{E} to \mathbf{F} . O^G is a metric space equipped with the following metric

$$\text{dist}(N_1, N_2) = d(B(Gr(N_1)), B(Gr(N_2))), N_1, N_2 \in O^G(\mathbf{E}, \mathbf{F}),$$

where $B(V)$ denotes the unit ball in the subspace $V \subseteq \mathbf{E} \times \mathbf{F}$ and $d(\cdot, \cdot)$ is the Hausdorff metric on bounded subsets of $\mathbf{E} \times \mathbf{F}$. It has been shown that \mathcal{F}_0^G is an open subset of O^G (cf. [14]).

Throughout this section, we assume that P is a topological space, and $\{L_\lambda\}_{\lambda \in P} \subseteq O^G$ is a *continuous family of equivariant Fredholm operators of index zero, parametrized by P* i.e., for each $\lambda \in P$, $L_\lambda \in \mathcal{F}_0^G$ and the mapping $\eta: P \rightarrow O^G$ defined by $\eta(\lambda) = L_\lambda$ for $\lambda \in P$ is continuous. We now define $\pi: \xi \rightarrow P$ as follows

$$\begin{aligned} \xi &:= \{(\lambda, x, y) \in P \times \mathbf{E} \times \mathbf{F}; x \in \text{Dom}(L_\lambda), y = L_\lambda x\}, \\ \pi(\lambda, x, y) &= \lambda \quad \text{for } (\lambda, x, y) \in \xi. \end{aligned}$$

It has been shown that the Banach vector bundle $\pi: \xi \rightarrow P$ is a locally trivial G -vector bundle (cf. [14]).

Let

$$\mathcal{E} := \{(\lambda, x) \in P \times \mathbf{E}; x \in \mathbf{E}_{L_\lambda}\}$$

and let $p_1: \xi \rightarrow \mathcal{E}$ be given by $p_1(\lambda, x, y) = (\lambda, x), (\lambda, x, y) \in \xi$. Since for every $\lambda \in P$ the projection $pr_1: Gr(L_\lambda) \rightarrow \mathbf{E}_\lambda := \mathbf{E}_{L_\lambda}$ is an equivariant isometry, the mapping $p_1: \xi \rightarrow \mathcal{E}$ gives us the natural identification of the G -bundles ξ and \mathcal{E} .

Now, we can define the vector bundle morphism $L: \mathcal{E} \rightarrow \mathbf{F}$, where \mathbf{F} is viewed as a bundle over a one-point space, by

$$L(\lambda, x) = L_\lambda x, \quad (\lambda, x) \in \mathcal{E}.$$

Let X be a given subset of P . An *equivariant resolvent* of L over X is a G -vector bundle morphism $K: X \times \mathbf{E} \rightarrow \mathbf{F}$ such that

- (i) for every $\lambda \in X$, $K_\lambda: \mathbf{E} \rightarrow \mathbf{F}$ is a finite-dimensional linear operator;
- (ii) for every $\lambda \in X$, $L_\lambda + K_\lambda: \mathbf{E}_\lambda \rightarrow \mathbf{F}$ is an isomorphism.

By $FR^G(L, X)$ we denote the set of all equivariant resolvents of L over X . Contrary to the non-equivariant case, it may happen that an equivariant Fredholm operator of index zero has no equivariant resolvent. Therefore, in general, $FR^G(L, X)$ may be an empty set. Nevertheless, as the following result shows, in certain situations the existence of an equivariant resolvent of L at one point $\lambda_0 \in X$ implies that $FR^G(L, X) \neq \emptyset$.

Lemma 2.1. *Let X be a compact contractible subset of P such that $FR^G(L, \{\lambda_0\}) \neq \emptyset$ for some $\lambda_0 \in X$. Then $FR^G(L, X) \neq \emptyset$.*

Proof: Suppose that $K_{\lambda_0} \in FR^G(L, \{\lambda_0\})$, i.e., $K_{\lambda_0}: \mathbf{E} \rightarrow \mathbf{F}$ is an equivariant finite dimensional operator such that $L_{\lambda_0} + K_{\lambda_0}: \mathbf{E}_{\lambda_0} \rightarrow \mathbf{F}$ is an isomorphism. We notice that since G is a compact Lie group, if $\tilde{p}: \mathbf{F} \rightarrow \mathbf{F}$ is a projection onto an invariant finite-dimensional subspace \mathbf{F} , then

$$(2.1) \quad p(x) = \int_G g^{-1} \tilde{p}(gx) d\mu(g), \quad x \in \mathbf{F},$$

where $d\mu(g)$ denotes the Haar measure on G , is an equivariant projection onto $\tilde{p}(\mathbf{F})$. Therefore there exists a finite-dimensional invariant subspace $\mathbf{F}_0 \subseteq \mathbf{F}$ such that $\text{Im } K_{\lambda_0} \subseteq \mathbf{F}_0$ and $\text{Im } L_\lambda + \mathbf{F}_0 = \mathbf{F}$ for all $\lambda \in X$.

In fact, for any $\lambda^* \in X$, since $\text{Im } L_{\lambda^*}$ is an invariant subspace of finite codimension, there exists a projection $\tilde{p}_{\lambda^*}: \mathbf{F} \rightarrow \text{Im } L_{\lambda^*}$. By using the average (2.1), we can get an equivariant projection $p_{\lambda^*}: \mathbf{F} \rightarrow \text{Im } L_{\lambda^*}$ and an invariant subspace $\text{Ker } p_{\lambda^*}$ of \mathbf{F} such that $\text{Ker } p_{\lambda^*} \oplus \text{Im } L_{\lambda^*} = \mathbf{F}$. By using the local trivialization of \mathcal{E} and the fact that the set of isomorphisms is open with respect to the operator topology, we can find a neighbourhood $U(\lambda^*)$ of λ^* such that $\text{Ker } p_{\lambda^*} \oplus \text{Im } L_\lambda = \mathbf{F}$ for every $\lambda \in U(\lambda^*)$. Now because of the compactness of X , we can construct a finite covering, say, $U(\lambda_1), U(\lambda_2), \dots, U(\lambda_n)$, and finite dimensional invariant subspaces $\text{Ker } p_{\lambda_1}, \dots, \text{Ker } p_{\lambda_n}$ such that $\text{Ker } p_{\lambda_i} \oplus \text{Im } L_\lambda = \mathbf{F}$ for every $\lambda \in U(\lambda_i)$ and $i = 1, \dots, n$. Then it is easy to verify that the space \mathbf{F}_0 defined by

$$\mathbf{F}_0 := \text{Ker } p_{\lambda_1} + \dots + \text{Ker } p_{\lambda_n} + \text{Im } K_{\lambda_0}$$

satisfies the required properties.

Let $I - Q: \mathbf{F} \rightarrow \mathbf{F}$ be an equivariant projection onto \mathbf{F}_0 , hence $Q: \mathbf{F} \rightarrow \mathbf{F}$ is an equivariant Fredholm operator of index zero. It follows that the so called *index bundle* over X ,

$$\begin{aligned} \mathbf{B} := \mathbf{B}(L, X) &= \{(\lambda, x) \in \mathcal{E}|_X; x \in \text{Ker } Q \circ L_\lambda\} \\ &= \{(\lambda, x) \in \mathcal{E}|_X; L_\lambda x \in \text{Ker } Q = \mathbf{F}_0\} \end{aligned}$$

is a locally trivial and finite-dimensional ($\dim \mathbf{B} = \dim \mathbf{F}_0$) G -vector bundle over X . Since X is contractible to the point λ_0 , there exists a G -equivariant trivialization of the bundle \mathbf{B} , namely

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow[\cong]{\tilde{\psi}} & X \times \mathbf{B}_{\lambda_0} \\ \pi \searrow & & \swarrow pr_1 \\ & X & \end{array}$$

We notice that for every $x \in \mathbf{B}_{\lambda_0}$,

$$\begin{aligned} (L_{\lambda_0} + K_{\lambda_0})(x) &= (I - Q)(L_{\lambda_0} + K_{\lambda_0})(x) + QL_{\lambda_0}x + QK_{\lambda_0}x \\ &= (I - Q)(L_{\lambda_0} + K_{\lambda_0})(x) \in \text{Ker } Q = \mathbf{F}_0. \end{aligned}$$

Therefore, since $L_{\lambda_0} + K_{\lambda_0}: \mathbf{E}_{\lambda_0} \rightarrow \mathbf{F}$ is an equivariant isomorphism, we know that $(L_{\lambda_0} + K_{\lambda_0})|_{\mathbf{E}_{\lambda_0}}: \mathbf{B}_{\lambda_0} \rightarrow \mathbf{F}_0$ is an equivariant isomorphism. Consequently, we obtain the following equivariant trivialization of the vector bundle \mathbf{B}

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow[\cong]{\psi} & X \times \mathbf{F}_0 \\ \pi \searrow & & \swarrow pr_1 \\ & X & \end{array}$$

Since \mathbf{B} is a G -subbundle of $X \times \mathbf{E}$, by using the averaging (2.1) over G , one can easily construct a G -vector bundle morphism $p: X \times \mathbf{E} \rightarrow X \times \mathbf{E}$ which is a fiberwise equivariant projection onto \mathbf{B} .

Now, we can give an explicit formula for the resolvent $K \in FR^G(L, X)$. We put for $(\lambda, x) \in X \times \mathbf{E}$,

$$K_\lambda(x) = \psi_\lambda(p_\lambda(x)) - L_\lambda(p_\lambda(x)).$$

It is clear that $K_\lambda: \mathbf{E} \rightarrow \mathbf{F}$ is a finite-dimensional equivariant operator. Moreover, for any $\lambda \in X$ we have

$$\begin{aligned}
(L_\lambda + K_\lambda)x &= Q(L_\lambda + K_\lambda)x + (I - Q)(L_\lambda + K_\lambda)x \\
&= QL_\lambda(I - p_\lambda)x + (I - Q)L_\lambda x + \psi_\lambda \circ p_\lambda(x) - (I - Q)L_{\lambda_0} \circ p_\lambda(x) \\
&= QL_\lambda(I - p_\lambda)x + \psi_\lambda \circ p_\lambda(x) + (I - Q)L_\lambda(I - p_\lambda)(x).
\end{aligned}$$

Thus $L_\lambda + K_\lambda$ is an equivariant isomorphism. This completes the proof. \square

In the remaining part of this section we assume that $X \subseteq P$ is a compact contractible set such that $FR^G(L, X) \neq \emptyset$. We also fix an equivariant resolvent $K \in FR^G(L, X)$.

Let $F: \mathcal{E} \rightarrow \mathbf{F}$ be an equivariant mapping. We are interested in the following *parametrized equivariant coincidence problem*

$$(2.2) \quad L_\lambda x = F(\lambda, x), \quad (\lambda, x) \in \mathcal{E}|_X.$$

It follows from the assumption that the problem (2.2) can be reduced to the following equivariant fixed point problem:

$$(2.3) \quad y = \Theta_K(F)(\lambda, y), \quad (\lambda, y) \in X \times \mathbf{F},$$

where $R_\lambda := (L_\lambda + K_\lambda)^{-1}: \mathbf{F} \rightarrow \mathbf{E}_\lambda$ is an equivariant isomorphism and $\Theta_K(F): X \times \mathbf{F} \rightarrow \mathbf{F}$ is given by

$$(2.4) \quad \Theta_K(F)(\lambda, y) = F(\lambda, R_\lambda(y)) + K_\lambda(R_\lambda(y)), \quad (\lambda, y) \in X \times \mathbf{F}.$$

Evidently, $\Theta_K(F)$ is a completely continuous mapping (i.e. $\Theta_K(F)$ is continuous and $\Theta_K(F)(X \times B)$ is relatively compact in \mathbf{F} for every bounded subset $B \subseteq \mathbf{F}$) if and only if $F: \mathcal{E} \rightarrow \mathbf{F}$ is completely continuous.

Summarizing the above discussion, we conclude that if F is completely continuous and $FR^G(L, X) \neq \emptyset$, then the parametrized equivariant coincidence problem (2.2) can be reduced to the fixed point problem (2.3) for the parametrized equivariant completely continuous mapping $\Theta_K(F)$.

§3. S^1 -degree and the bifurcation theory

Throughout this section, \mathbf{E} and \mathbf{F} are given real Banach isometric representations of the group $G := S^1$, $P = \mathbf{R}^2$ and $\{L_\lambda\}_{\lambda \in P}$ is a continuous family of equivariant Fredholm operators of index zero which satisfies the following condition

$$(A.1) \quad \text{there exists } \lambda^* \in P \text{ such that } FR^G(L, \{\lambda^*\}) \neq \emptyset.$$

Assume that there is an isometric Banach representation $\widehat{\mathbf{E}}$ such that we have the following injective morphism of G -vector bundles $J: \mathcal{E} \rightarrow P \times \widehat{\mathbf{E}}$ such that for every $\lambda \in P$, $J_\lambda: \mathbf{E}_\lambda \rightarrow \widehat{\mathbf{E}}$ is an equivariant compact operator.

Assume also that $\hat{F}: P \times \hat{\mathbf{E}} \rightarrow \mathbf{F}$ is an equivariant C^1 -map. We define $F: \mathcal{E} \rightarrow \mathbf{F}$ as the composition $F = \hat{F} \circ J$. Obviously, F is *completely continuous* and equivariant.

Furthermore, we assume that there exists a 2-dimensional submanifold $M \subset \mathbf{R}^2 \times \mathbf{E}^G$, where $\mathbf{E}^G := \{x \in \mathbf{E}; gx = x \text{ for all } g \in S^1\}$, satisfying the following conditions

- (A) For every $(\lambda, x) \in M$, $x \in \text{Dom}(L_\lambda)$, and $L_\lambda x = F(\lambda, x)$,
 (B) If $(\lambda_0, x_0) \in M$, then there exist open neighbourhoods U_{λ_0} of λ_0 in \mathbf{R}^2 and U_{x_0} of x_0 in \mathbf{E}^G and a C^1 -map $\eta: U_{\lambda_0} \rightarrow \mathbf{E}^G$ such that

$$M \cap (U_{\lambda_0} \times U_{x_0}) = \{(\lambda, \eta(\lambda)); \lambda \in U_{\lambda_0}\}.$$

By (A), all points $(\lambda, x) \in M$ are solutions to the following coincidence problem

$$(3.1) \quad L_\lambda x = F(\lambda, x), \quad (\lambda, x) \in \mathcal{E}.$$

We call all these points *trivial solutions*. All other solutions of (3.1) will be called *nontrivial*. A point $(\lambda_0, x_0) \in M$ is called a *bifurcation point* if in any neighbourhood of (λ_0, x_0) there exists a nontrivial solution for (3.1).

As discussed in the previous section, the problem (3.1) restricted to any given compact subset X of P can be reduced to the following parametrized fixed point problem

$$(3.2) \quad y = \Theta_K(F)(\lambda, y), \quad (\lambda, y) \in X \times \mathbf{F},$$

where $K \in FR^G(L, X)$. We define $\tilde{M}_X := \{(\lambda, (L_\lambda + K_\lambda)x); (\lambda, x) \in M \cap (X \times \mathbf{E})\}$. Evidently, $(\lambda, y) \in \tilde{M}_X$, iff $(\lambda, R_\lambda y) \in M \cap (X \times \mathbf{E})$. Thus the points from \tilde{M}_X represent the trivial solutions to the problem (3.1) restricted to the subbundle $\mathcal{E}|_X$. We define a mapping $f: X \times \mathbf{F} \rightarrow \mathbf{F}$ by $f(\lambda, y) = y - \Theta_K(F)(\lambda, y)$, $(\lambda, y) \in X \times \mathbf{F}$. Clearly, f is an equivariant compact field of class C^1 and

$$D_y f(\lambda, y) = Id - \Theta_K(D_x F(\lambda, x)), \quad x = R_\lambda(y),$$

where D_y and D_x denote the derivatives with respect to y and x , respectively. Evidently, $D_y f(\lambda, y)$ is a Fredholm operator of index zero, and if $(\lambda, y) \in X \times \mathbf{E}^G$, (in particular, if $(\lambda, y) \in \tilde{M}_X$), then $D_y f(\lambda, y)$ is also an equivariant operator. It follows from the implicit function theorem that if $(\lambda_0, x_0) \in M \cap (X \times \mathbf{E})$ is a bifurcation point, then the derivative $D_y f(\lambda_0, y_0)$, $x_0 = R_{\lambda_0}(y_0)$, is not an isomorphism of \mathbf{F} and this is equivalent to the fact that $L_{\lambda_0} - D_x F(\lambda_0, x_0): \mathbf{E}_{\lambda_0} \rightarrow \mathbf{F}$, $x_0 = R_{\lambda_0}(y_0)$, is not an isomorphism. In this case we say that $(\lambda_0, x_0) \in M$ is *L-singular*. We set

$$A = \{(\lambda, x) \in M; (\lambda, x) \text{ is } L\text{-singular}\}.$$

In order to obtain bifurcation results for the problem (3.1), we are going to apply the method of Gęba and Marzantowicz (cf. [21]) based on the notion of the S^1 -degree (cf. [11]) and the complementary function method of Ize (cf. [25], [26]).

For the sake of completion, we briefly describe here the infinite-dimensional version of the S^1 -degree. For details, we refer to [11] and [12]. Consider an infinite-dimensional Banach isometric representation V of $G = S^1$. Let $U \subseteq \mathbf{R} \times V$ be an open bounded invariant subset and $f: (\bar{U}, \partial U) \rightarrow (V, V \setminus \{0\})$ be an equivariant compact vector field on \bar{U} . Then there is defined the S^1 -degree of f with respect to U given by

$$S^1\text{-Deg}(f, U) := \{\deg_H(f, U)\},$$

where H runs through the family of all closed proper subgroups of S^1 and $\deg_H(f, U) \in \mathbf{Z}$. The basic property of S^1 -degree is that the inequality $S^1\text{-Deg}(f, U) \neq 0$ implies the existence of a solution in U to the equation $f(\mu, x) = 0$, $(\mu, x) \in U$. More precisely, we have the following standard properties of the S^1 -degree:

- (i) **(Existence Property).** If $\deg_H(f, U) \neq 0$, then $f^{-1}(0) \cap U^H \neq \emptyset$, where $U^H = \{y \in U; G_y \supseteq H\}$ and G_y denotes the isotropy group of y ;
- (ii) **(Additivity Property).** If U_1, U_2 are two open invariant subsets of U such that $U_1 \cap U_2 = \emptyset$ and $f^{-1}(0) \subseteq U_1 \cup U_2$, then

$$S^1\text{-Deg}(f, U) = S^1\text{-Deg}(f, U_1) + S^1\text{-Deg}(f, U_2);$$

- (iii) **(Homotopy Invariance Property).** If $h: (\bar{U} \times [0, 1], \partial U \times [0, 1]) \rightarrow (V, V \setminus \{0\})$ is an S^1 -equivariant homotopy of compact vector fields, then $S^1\text{-Deg}(h_0, U) = S^1\text{-Deg}(h_1, U)$;

- (iv) **(Product Property).** Suppose that W is another Banach isometric representation of S^1 and Ω is an open bounded invariant subset of W such that $0 \in \Omega$. Define $g: \bar{U} \times \bar{\Omega} \rightarrow V \oplus W$ by $g(x, y) = (f(x), y)$. Then $S^1\text{-Deg}(g, U \times \Omega) = S^1\text{-Deg}(f, U)$.

In what follows, we want to show that the S^1 -degree for equivariant compact vector fields can be extended to the S^1 -degree for a continuous family of Fredholm operators $\{L_\lambda\}_{\lambda \in P}$ associated with the problem of finding nontrivial solutions to the equation (3.1) for $\lambda \in X$, where $X \subseteq P$ is the closure of a bounded open subset of \mathbf{R}^2 .

For this purpose, we use Lemma 2.1 to find $K \in FR^G(L, X)$ such that the problem (3.1) can be reduced to the problem of the existence of nontrivial solutions to (3.2). Let U be an open invariant subset of $P \times \mathbf{F}$ such that

$\bar{U} \subseteq X \times \mathbf{F}$, and $\varphi: \bar{U} \rightarrow \mathbf{R}$ be a continuous bounded equivariant function such that $\varphi(\lambda, y) \neq 0$ for all $(\lambda, y) \in \bar{U} \cap \tilde{M}_X$. Such a function is called a *complementing function*. Clearly, if $\varphi: \bar{U} \rightarrow \mathbf{R}$ is a complementing function, then all solutions to the following parametrized system of equations

$$(3.3) \quad \begin{cases} y = \Theta_K(F)(\lambda, y), & (\lambda, y) \in U \\ \varphi(\lambda, y) = 0 \end{cases}$$

are nontrivial solutions of the equation (3.2). Consequently, if (λ, y) is a solution to (3.3), then $(\lambda, R_\lambda y)$ is a nontrivial solution to (3.1) in the set $\tilde{U} := \{(\lambda, R_\lambda y); (\lambda, y) \in U\}$. If we further assume that the system (3.3) has no solution $(\lambda, y) \in \partial U$, then we can define the S^1 -coincidence degree as follows

$$S^1\text{-Deg}([L, F, \varphi], \tilde{U}) := S^1\text{-Deg}(\Psi, U),$$

where $\Psi: \bar{U} \rightarrow \mathbf{R} \times \mathbf{F}$, $U \subseteq P \times \mathbf{F} \subseteq \mathbf{R} \times (\mathbf{R} \times \mathbf{F})$, is given by

$$\Psi(\lambda, y) = (\varphi(\lambda, y), y - \Theta_K(F)(\lambda, y)), \quad (\lambda, y) \in \bar{U}.$$

Therefore, by the Existence Property, if $S^1\text{-Deg}([L, F, \varphi], \tilde{U}) \neq 0$, then we can conclude that there exists in \tilde{U} a nontrivial solution of the problem (3.1).

Obviously, the important part of the above approach is the construction of a complementing function and the computation of $S^1\text{-Deg}([L, F, \varphi], \tilde{U})$ which we are going to achieve in the remaining part of this section. First, let U_0 be an open bounded subset of M whose closure is contained in a local neighbourhood of the type described in the condition (B), i.e. $\bar{U}_0 \subseteq \{(\lambda, \eta(\lambda)); \lambda \in U_{\lambda_0}\} \subset M$. We are interested in finding a bifurcation point in U_0 . For this purpose, we assume that $\Gamma := U_0 \cap A \neq \emptyset$, and $\partial U_0 \cap A = \emptyset$. Next, we define a *special neighbourhood* of the set $\tilde{\Gamma} := \{(\lambda, \tilde{\eta}(\lambda)); (\lambda, \eta(\lambda)) \in \Gamma\}$, where $\tilde{\eta}(\lambda) = (L_\lambda + K_\lambda)\eta(\lambda)$, by

$$U(r) := \{(\lambda, y) \in \mathbf{R}^2 \times \mathbf{F}; \lambda \in V_{\lambda_0}, \|y - \tilde{\eta}(\lambda)\| < r\}$$

where $r > 0$ and V_{λ_0} is the projection of U_0 to \mathbf{R}^2 . By the implicit function theorem, there exists a sufficiently small $r > 0$ such that $y \neq \Theta_K(F)(\lambda, y)$ for all $(\lambda, y) \in \overline{U(r)}$ satisfying $\lambda \in \partial V_{\lambda_0}$ and $\|y - \tilde{\eta}(\lambda)\| \neq 0$.

Let $\rho > 0$ be given and suppose that $\tau_0: \bar{U}_0 \rightarrow [0, \rho]$ is a continuous function such that $\tau_0^{-1}(0) = \Gamma$ and $\tau_0^{-1}(\rho) = \partial U_0$.

Definition 3.1: We say that a G -equivariant function $\theta: \overline{U(r)} \rightarrow \mathbf{R}$ is an *auxiliary function* if

- (i) $\theta(\lambda, y) = -\tau_0(\lambda, R_\lambda(y))$ for all $(\lambda, R_\lambda(y)) \in \bar{U}_0$
- (ii) $\theta(\lambda, y) = r$ if $\|y - \tilde{\eta}(\lambda)\| = r$.

The existence of an auxiliary function is guaranteed by the well-known Gleason-Tietze G -extension theorem (cf. [5]).

Note that if θ is an auxiliary function, then the system

$$\begin{cases} y = \Theta_K(F)(\lambda, y), & (\lambda, y) \in \overline{U(r)} \\ \theta(\lambda, y) = 0 \end{cases}$$

has no solution in $\partial U(r)$, and consequently we can define the S^1 -coincidence degree

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) := S^1\text{-Deg}(\Psi_\theta, U(r))$$

where

$$\Psi_\theta(\lambda, y) = (\theta(\lambda, y), y - \Theta_K(F)(\lambda, y)), \quad (\lambda, y) \in U(r).$$

It should be mentioned that the $S^1\text{-Deg}([L, F, \theta], \tilde{U}(r))$ does not depend on the choice of τ_0 or the auxiliary function θ . This is due to the homotopy invariance of S^1 -degree and the fact that if θ^1 and θ^2 are two auxiliary functions, then for every $t \in [0, 1]$, $t\theta^1 + (1-t)\theta^2$ is also an auxiliary function.

Proposition 3.1. *Assume that the hypotheses (A.1), (A) and (B) are satisfied. Let $U_0 \subset M$ be defined as above and $r > 0$ be a sufficiently small number. If $S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) \neq 0$ then the set $\Gamma = U_0 \cap \Lambda$ contains a bifurcation point for the equation (3.1). More precisely, if $\deg_H([L, F, \theta], \tilde{U}(r)) \neq 0$, then the equation (3.1) has a sequence of non-trivial solutions in $\tilde{U}(r)^H$ bifurcating from Γ .*

Proof: We first show that there exists a complementing function $\varphi: \overline{U(r)} \rightarrow \mathbf{R}$ such that

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) = S^1\text{-Deg}([L, F, \varphi], \tilde{U}(r)).$$

Indeed, by applying the Homotopy Invariance Property of S^1 -degree to the equivariant homotopy

$$(h(\lambda, y, t), y - \Theta_K(F)(\lambda, y))$$

where $\varepsilon > 0$ is a sufficiently small number and

$$h(\lambda, y, t) = \theta(\lambda, y) - t\varepsilon, \quad (\lambda, y) \in \tilde{U}(r), \quad t \in [0, 1],$$

we obtain $S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) = S^1\text{-Deg}([L, F, h_t], \tilde{U}(r))$ for all $t \in [0, 1]$. In particular, we have

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) = S^1\text{-Deg}([L, F, \varphi], \tilde{U}(r))$$

where $\varphi = h_1$ is a complementing function.

We then remark that for a sufficiently small number $r > 0$

$$S^1\text{-Deg} ([L, F, \theta], \tilde{U}(r)) = S^1\text{-Deg} (\tilde{\Psi}_\theta, U(r))$$

where

$$\tilde{\Psi}_\theta(\lambda, y) = (\theta(\lambda, y), y - \Theta_K(D_x F(\lambda, \tilde{\eta}(\lambda))y), \quad (\lambda, y) \in U(r).$$

Therefore $S^1\text{-Deg} ([L, F, \theta], \tilde{U}(r))$ does not depend on r , and thus the bifurcation result follows. \square

In order to be able to use Proposition 3.1, we need to develop a computation formula for $S^1\text{-Deg} ([L, F, \theta], \tilde{U}(r))$. For this purpose, we make the following assumption:

(A.2) There exists a diffeomorphism $\kappa: D = \{z \in \mathbf{C}; |z| \leq 1\} \rightarrow \bar{V}_{\lambda_0}$ such that $\bar{U}_0 = \{(\kappa(z), \eta(\kappa(z))); z \in D\}$.

It is well known that the representation \mathbf{F} has the following direct sum decomposition

$$\mathbf{F} = \mathbf{F}_0 \oplus \mathbf{F}_1 \oplus \dots \oplus \mathbf{F}_k \oplus \dots,$$

where $\mathbf{F}_k, k > 0$, are isotypical representations of $G = S^1$ such that $\mathbf{F}_0 = \mathbf{F}^G$ and for all $x \in \mathbf{F}_k \setminus \{0\}, G_x = \mathbf{Z}_k$. The spaces \mathbf{F}_k are closed invariant subspaces of \mathbf{F} , and if $k > 0$, then \mathbf{F}_k can be endowed with a natural complex structure. Indeed, for $x \in \mathbf{F}_k$, we can define the multiplication of x by the number i by

$$i * x = e^{i\pi/2k} x.$$

We put $\mathbf{F}^\perp := \bigoplus_{k=1}^\infty \mathbf{F}_k$. The space \mathbf{F}^\perp has the natural complex structure as we described above and moreover, an \mathbf{R} -linear operator $B: \mathbf{F}^\perp \rightarrow \mathbf{F}^\perp$ is S^1 -equivariant if and only if it is a \mathbf{C} -linear operator (with respect to the above complex structure on \mathbf{F}^\perp) and $B(\mathbf{F}_k) \subset \mathbf{F}_k$. We denote by $GL_c(\mathbf{F}^\perp)$ the group of all linear automorphisms of \mathbf{F}^\perp of the type "Identity + a compact linear operator", and by $GL_c^G(\mathbf{F}^\perp)$ we denote the subgroup of $GL_c(\mathbf{F}^\perp)$ of all equivariant automorphisms. It has been shown (cf. [43]) that if $\dim \mathbf{F}_k = \infty, k = 1, 2, \dots$, then there is the following homotopy equivalence

$$i: \lim_{m \rightarrow \infty} GL(m, \mathbf{C}) \hookrightarrow GL_c^G(\mathbf{F}_k) = GL_c^G(\mathbf{F}_k).$$

Therefore, the fundamental group $\pi_1(GL_c^G(\mathbf{F}_k)) = \mathbf{Z}$. Since we can identify the set of all homotopy classes of continuous maps from S^1 into $GL_c^G(\mathbf{F}_k)$, denoted by $[S^1, GL_c^G(\mathbf{F}_k)]$, with $\pi_1(GL_c^G(\mathbf{F}_k)) = \mathbf{Z}$, we have the following isomorphism

$$\mathcal{V}_k: [S^1, GL_c^G(\mathbf{F}_k)] \longrightarrow \mathbf{Z}.$$

Moreover, it is well known that restriction of \mathcal{V}_k to the subset $[S^1, GL(m, \mathbf{C})]$ can be described by

$$\mathcal{V}_k[A] = \deg(\det(A)),$$

where $A: S^1 \rightarrow GL(m, \mathbf{C})$, $\det: GL(m, \mathbf{C}) \rightarrow \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the usual determinant homomorphism and \deg denotes the classical Brouwer degree.

In the remaining part of this section, we assume that the hypotheses (A.1), (A), (B) and (A.2) are satisfied.

For every $\lambda \in \overline{U_{\lambda_0}}$, we define a linear equivariant operator $T_\lambda: \mathbf{F} \rightarrow \mathbf{F}$ by

$$T_\lambda y = y - \Theta_K(D_x F(\lambda, \eta(\lambda)))(\lambda, y), \quad y \in \mathbf{F}.$$

Therefore, by the definition, $A \cap U_0 = \{(\lambda, \eta(\lambda)) \in U_0; T_\lambda \notin GL_c(\mathbf{F})\}$. We define $T_\lambda^k: \mathbf{F}_k \rightarrow \mathbf{F}_k$ by $T_\lambda^k := T_\lambda|_{\mathbf{F}_k}$, $k = 0, 1, 2, \dots$

We identify \mathbf{R}^2 with \mathbf{C} , and define $\alpha: S^1 \rightarrow M$ by setting $\alpha := \kappa|_{S^1}$. Then the formula

$$\psi(z) = T_{\alpha(z)}: \mathbf{F} \longrightarrow \mathbf{F}$$

defines a continuous mapping $\psi: S^1 \rightarrow GL_c^G(\mathbf{F})$. It follows from $T_\lambda^k(\mathbf{F}_k) \subseteq \mathbf{F}_k$ that

$$\psi = \psi_0 \oplus \psi_1 \oplus \dots \oplus \psi_k \oplus \dots,$$

where

$$\psi_k: S^1 \longrightarrow GL_c^G(\mathbf{F}_k), \quad k = 0, 1, 2, \dots$$

Since $GL_c(\mathbf{F}_0) = GL_c^G(\mathbf{F}_0)$ has exactly two connected components: $GL_c^+(\mathbf{F}_0)$ (containing the identity) and $GL_c^-(\mathbf{F}_0)$, we can define

$$\mathcal{E}_0 := \begin{cases} 1 & \text{if } \psi_0(z) \in GL_c^+(\mathbf{F}_0) \\ -1 & \text{if } \psi_0(z) \in GL_c^-(\mathbf{F}_0). \end{cases}$$

Finally, for $k = 1, 2, \dots$, put

$$\gamma_k = \gamma_k(\Gamma) := \mathcal{E}_0 \mathcal{V}_k([\psi_k]).$$

We can now state a computational formula for S^1 -Deg $([L, F, \theta], \tilde{U}(r))$.

Proposition 3.2. *Assume that the hypotheses (A.1), (A), (B) and (A.2) are satisfied. Let U_0 be defined as above and $r > 0$ be a sufficiently small number. Then*

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(r)) = \{\deg_{\mathbf{Z}^k}([L, F, \theta], \tilde{U}(r))\}_{k \in \mathbf{N}},$$

satisfies

$$\text{deg}_{\mathbf{Z}_k} ([L, F, \theta], \tilde{U}(r)) = \gamma_k(\Gamma)$$

for $k = 1, 2, 3, \dots$

Proof: By the Homotopy Invariance Property of the S^1 -degree and by the assumptions (A), (B), (A.1) and (A.2), we obtain that

$$S^1\text{-Deg} ([L, F, \theta], \tilde{U}(r)) = S^1\text{-Deg} (\Psi, U(r)),$$

where $\Psi(\lambda, y) = (\theta(\lambda, y), T_\lambda y)$. Since the space $GL_c^G(\mathbf{F}^\perp)$ is homotopically equivalent to $\lim_{m \rightarrow \infty} GL(m, \mathbf{C})$, by the Product Property of S^1 -degree, we can

find a family $\tilde{T}_\lambda: \mathbf{F} \rightarrow \mathbf{F}$, $\lambda \in \bar{V}_{\lambda_0}$, of the form $\tilde{T}_\lambda = I - C_\lambda$, such that $\text{Im } C_\lambda$ is contained in a fixed invariant finite-dimensional subspace $\tilde{\mathbf{F}}$ of \mathbf{F} , and

$$S^1\text{-Deg} (\Psi, U(r)) = S^1\text{-Deg} (\tilde{\Psi}, \tilde{U}),$$

where $\tilde{U} = U(r) \cap (\mathbf{R}^2 \times \tilde{\mathbf{F}})$ and

$$\tilde{\Psi}: \overline{U(r)} \cap (\mathbf{R}^2 \times \tilde{\mathbf{F}}) \longrightarrow \mathbf{R} \times \tilde{\mathbf{F}}, \quad \tilde{\Psi}(\lambda, y) = (\theta(\lambda, y), \tilde{T}_\lambda y)$$

for $(\lambda, y) \in \overline{U(r)} \cap (\mathbf{R}^2 \times \tilde{\mathbf{F}})$.

Similarly to the infinite-dimensional case we have the following isogenic direct sum decomposition of the space $\tilde{\mathbf{F}}$:

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_0 \oplus \tilde{\mathbf{F}}_1 \oplus \tilde{\mathbf{F}}_2 \oplus \dots \oplus \tilde{\mathbf{F}}_m,$$

where $G_y = S^1$ for $y \in \tilde{\mathbf{F}}_0$ and $G_y = \mathbf{Z}_k$ for all $y \in \tilde{\mathbf{F}}_k \setminus \{0\}$, $k = 1, \dots, m$. By the construction of the S^1 -degree (cf. [11]),

$$S^1\text{-Deg} (\tilde{\Psi}, U(r) \cap (\mathbf{R}^2 \times \tilde{\mathbf{F}})) = S^1\text{-Deg} (\tilde{\Psi} \circ \tilde{\kappa}, U_1(r) \cap (\mathbf{R}^2 \times \tilde{\mathbf{F}})),$$

where $U_1(r) := \{(\lambda, y) \in \mathbf{R}^2 \times \tilde{\mathbf{F}} : |\lambda| < 1, \|y\| < r\}$ and $\tilde{\kappa}: \overline{U_1(r)} \rightarrow \overline{U(r)}$ is given by $\tilde{\kappa}(\lambda, y) = (\kappa(\lambda), \tilde{\eta}(\lambda) + y)$, $\tilde{\eta}(\lambda) := (L_\lambda + K_\lambda)\eta(\lambda)$. We assume here that $r > 0$ is a sufficiently small number so that $\tilde{\kappa}$ is a C^1 -diffeomorphism. In order to simplify the notation we will denote by $\tilde{\Psi}$ the mapping $\tilde{\Psi} \circ \tilde{\kappa}$. We have that for every $z \in S^1 \subset \mathbf{C}$, $\tilde{T}_{\kappa(z)}|_{\tilde{\mathbf{F}}_k}: \tilde{\mathbf{F}}_k \rightarrow \tilde{\mathbf{F}}_k$ is an isomorphism. Put $\tilde{\Phi}_\kappa(z) := \tilde{T}_{\kappa(z)}|_{\tilde{\mathbf{F}}_k}$. Then

$$\tilde{\Phi}_\kappa: S^1 \longrightarrow GL^G(\tilde{\mathbf{F}}_k) = GL(n_k, \mathbf{C}), \quad k = 1, \dots, m,$$

where $n_k = \dim(\tilde{\mathbf{F}}_k)$. Since $\gamma_k = \varepsilon_0 \mathcal{V}_k([\tilde{\Phi}_\kappa])$, by applying an appropriate deformation to $\tilde{\Phi}$ we can obtain a modified mapping, denoted by Φ , such that for all $k = 1, 2, \dots, m$,

$$\Phi_k: S^1 \longrightarrow GL^G(\tilde{\mathbf{F}}_k) = GL(n_k, \mathbf{C})$$

is exactly given by the formula

$$\Phi_k(z)(z_1, z_2, \dots, z_{n_k}) = (z^{\gamma_k} z_1, z_2, \dots, z_{n_k}), \quad z \in S^1,$$

Now, we can apply the results from [11] to conclude that

$$\deg_{z_k}(\tilde{\Psi}, \tilde{U}) = \gamma_k. \quad \square$$

As a corollary of Propositions 3.1 and 3.2, we obtain the following:

Theorem 3.1 (Local Bifurcation Theorem). *Assume that the hypotheses (A), (B) and (A.1) are satisfied and let U_0 be an open bounded subset of M such that $\partial U_0 \cap A = \emptyset$, $\Gamma = U_0 \cap A \neq \emptyset$ and the assumption (A.2) is satisfied. If $\gamma_k(\Gamma) \neq 0$, then Γ contains a bifurcation point for the equation (3.1). Moreover, there exists a sequence $\{(\lambda_n, x_n)\}$ of non-trivial solutions of (3.1) such that the isotropy group of x_n contains Z_k and (λ_n, x_n) converges (in the topology of \mathcal{E}) to a trivial solution $(\lambda_0, x_0) \in \Gamma$. \square*

For a global Hopf bifurcation, we need to assume

(A.3) *The submanifold M is complete and every L -singular point (λ, x) of M is isolated in M .*

In this case for a given $(\lambda_0, x_0) \in A$ we introduce the following notation:

Definition 3.2: Let (λ_0, x_0) be an L -singular point. We say that

$$U(\rho, r) = \{(\lambda, y) \in \mathbf{R}^2 \times \mathbf{F}; |\lambda - \lambda_0| < \rho, \|\tilde{\eta}(\lambda) - y\| < r\}$$

where $\rho, r > 0$, is a *special neighbourhood* of (λ_0, x_0) , if (λ_0, x_0) is the only L -singular point in $\tilde{U}(\rho, r)$, and $y \neq \Theta_K(F)(\lambda, y)$ for $(\lambda, y) \in \tilde{U}(\rho, r)$ satisfying $|\lambda - \lambda_0| = \rho$ and $y \neq \tilde{\eta}(\lambda)$.

In this case we can also introduce the definition of an *auxiliary function*, namely, a G -equivariant function $\theta: \overline{U(\rho, r)} \rightarrow \mathbf{R}$ satisfying the following conditions:

- (i) $\theta(\lambda, \tilde{\eta}(\lambda)) = -|\lambda - \lambda_0|$ for all λ such that $|\lambda - \lambda_0| \leq \rho$
- (ii) $\theta(\lambda, y) = r$ if $\|\tilde{\eta}(\lambda) - y\| = r$.

The existence of θ follows immediately from the Gleason-Tietze G -extension theorem (cf. [5]). Note that if θ is an auxiliary function, then the system

$$(3.4) \quad \begin{cases} y = \Theta_K(F)(\lambda, y), & (\lambda, y) \in \overline{U(\rho, r)} \\ \theta(\lambda, y) = 0 \end{cases}$$

has no solution in $\partial U(\rho, r)$, and consequently we can define again the S^1 -coincidence degree

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(\rho, r)) := S^1\text{-Deg}(\Psi_\theta, U(\rho, r))$$

where

$$\Psi_\theta(\lambda, y) = (\theta(\lambda, y), y - \Theta_K(F)(\lambda, y)), \quad (\lambda, y) \in \overline{U(\rho, r)}.$$

Definition 3.3: Suppose that $U(\rho, r)$ is a special neighbourhood of (λ_0, x_0) . The function $\theta_0: \overline{U(\rho, r)} \rightarrow \mathbf{R}$ defined by

$$\theta_0(\lambda, y) = \frac{\rho^2}{4} - |\lambda - \lambda_0|^2$$

is called an *Ize function*.

By the implicit function theorem, it can be easily shown that if r is sufficiently small, then $y \neq \Theta_K(F)(\lambda, y)$ for $(\lambda, y) \in \overline{U(\rho, r)}$ with $\|y\| = r$ and $|\lambda - \lambda_0| \geq \frac{1}{2}\rho$. Therefore the system

$$(3.5) \quad \begin{cases} y = \Theta_K(F)(\lambda, y), & (\lambda, y) \in \overline{U(\rho, r)} \\ \theta_0(\lambda, y) = 0 \end{cases}$$

has no solution in $\partial U(\rho, r)$, and consequently we can define the following S^1 -coincidence degree

$$S^1\text{-Deg}([L, F, \theta_0], \tilde{U}(\rho, r)) := S^1\text{-Deg}(\Psi_{\theta_0}, U(\rho, r))$$

where

$$\Psi_{\theta_0}(\lambda, y) = (\theta_0(\lambda, y), y - \Theta_K(F)(\lambda, y)), \quad (\lambda, y) \in \overline{U(\rho, r)}.$$

By using a homotopy argument, we can verify that the degree $S^1\text{-Deg}([L, F, \theta], \tilde{U}(\rho, r))$ does not depend on the choice of the special neighbourhood of (λ_0, y_0) , $y_0 = (L_{\lambda_0} + K_{\lambda_0})x_0$, and the auxiliary function θ . Moreover, we can prove the following equality

$$S^1\text{-Deg}([L, F, \theta], \tilde{U}(\rho, r)) = S^1\text{-Deg}([L, F, \theta_0], \tilde{U}(\rho, r)).$$

For details, we refer to [21].

Denoting by $\gamma_k(\lambda_0, x_0)$ the element $\gamma_k(\{\lambda_0, x_0\})$ constructed, as before, for the special neighbourhood $U(\rho, r)$, we are now in the position to state the following:

Theorem 3.2 (Global Bifurcation Theorem). *Assume that the hypotheses (A), (A.1) and (A.3) are satisfied. Let \mathcal{S} denote the closure of the set of all*

nontrivial solutions of (3.1) in \mathcal{E} . Then for each bounded component \mathcal{C} of \mathcal{S} , the set $\mathcal{C} \cap M$ is finite. Moreover, if $\mathcal{C} \cap M = \{(\lambda_1, x_1), \dots, (\lambda_q, x_q)\}$, then for every $k \in N$,

$$(3.6) \quad \gamma_k(\lambda_1, x_1) + \gamma_k(\lambda_2, x_2) + \dots + \gamma_k(\lambda_q, x_q) = 0.$$

Proof: The first conclusion is an immediate consequence of the assumption (A.3). We put $\tilde{\mathcal{C}} := (L + K)(\mathcal{C}) \subseteq \mathbf{R}^2 \times \mathbf{F}$, and choose $r, \rho > 0$ such that for $i = 1, \dots, q$, $\tilde{U}_i = U_i(\rho, r)$ is a special neighbourhood of (λ_i, x_i) and $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ for $i \neq j$. Let $\tilde{U} = \tilde{U}_1 \cup \dots \cup \tilde{U}_q$ and $\Omega_1 \subseteq \mathbf{R}^2 \times \mathbf{F}$ be an open invariant subset such that $\tilde{\mathcal{C}} \setminus \tilde{U} \subseteq \Omega_1$ and $\Omega_1 \cap M = \emptyset$.

Next, we find an open invariant subset Ω of $\mathbf{R}^2 \times \mathbf{F}$ such that $\tilde{\mathcal{C}} \subseteq \Omega \subseteq \Omega_1 \cup \tilde{U}$ and $\partial\Omega \cap \tilde{\mathcal{S}} = \emptyset$, where $\tilde{\mathcal{S}} := (L + K)(\mathcal{S})$. The subset Ω is bounded. By the implicit function theorem, there exists $r_0, \rho_0 > 0$ such that $0 < r_0 < r$, $0 < \rho_0 < \rho$, $U_i := U_i(\rho_0, r_0) \subseteq \Omega$, $\tilde{\mathcal{S}} \cap U_i(\rho, r_0) \subset U_i(\rho_0, r_0)$, $i = 1, 2, \dots, q$. We set $U = U_1 \cup \dots \cup U_q$, and construct a continuous equivariant function $\theta: \bar{\Omega} \cup \bar{U} \rightarrow \mathbf{R}$ such that

- (i) $\theta(\lambda, y) = -|\lambda - \lambda_i|$ for $(\lambda, y) \in \tilde{U}_i \cap \tilde{M}$, $i = 1, 2, \dots, q$;
- (ii) $\theta(\lambda, y) = r_0$ for $(\lambda, y) \in \bar{\Omega} \setminus U$.

Define $\Phi: \bar{\Omega} \rightarrow \mathbf{R} \times \mathbf{F}$ by

$$\Phi(\lambda, y) = (\theta(\lambda, y), y - \Theta_K(F)(\lambda, y)).$$

By the definition, $\Phi^{-1}(0) \subseteq \tilde{\mathcal{S}}$, thus since $\partial\Omega \cap \tilde{\mathcal{S}} = \emptyset$, $S^1\text{-Deg}(\Phi, \Omega)$ is well defined. We now define the following homotopy

$$H: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \longrightarrow (\mathbf{R} \times \mathbf{F}, \mathbf{R} \times \mathbf{F} \setminus \{0\})$$

by

$$H(\lambda, y, t) = (\chi(\lambda, y, t), y - \Theta_K(F)(\lambda, y)),$$

where

$$\chi(\lambda, y, t) = (1 - t)\theta(\lambda, y) + t\rho.$$

Clearly, $H_0(\lambda, y) = \Phi(\lambda, y)$ and $H_1(\lambda, y) \neq 0$ for all $(\lambda, y) \in \bar{\Omega}$. Therefore $S^1\text{-Deg}(\Phi, \Omega) = 0$. On the other hand, $\Phi^{-1}(0) \subseteq \mathcal{S} \cap U \subseteq \mathcal{S} \cap \Omega$. Therefore

$$S^1\text{-Deg}(\Phi, \Omega) = S^1\text{-Deg}(\Phi, U) = 0,$$

and consequently, by the Additivity Property of S^1 -degree,

$$\gamma_k(\lambda_1, x_1) + \dots + \gamma_k(\lambda_q, x_q) = 0, \quad k \in N.$$

This completes the proof. \square

Remark 3.1. Theorem 3.2 still holds if we replace $P = \mathbf{R}^2$ by $\mathbf{R} \times \mathbf{R}_+$, where $\mathbf{R}_+ = (0, \infty)$. In this case, a component C of S is said to be bounded if

$$\sup \{ \text{dist}(\lambda, \partial(\mathbf{R} \times \mathbf{R}_+)) + |\lambda| + |x|_{L^\lambda}; (\lambda, x) \in C \} < \infty.$$

§4. Hopf bifurcation for functional parabolic partial differential equations

4.1. Description of the problem. Suppose that Ω is a bounded regular region in \mathbf{R}^n . For a scalar-valued function $v: \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ and $t \in \mathbf{R}$, v_t denotes the function $v_t(\tau, x) = v(t + \tau, x)$ for $(\tau, x) \in \mathbf{R} \times \Omega$.

Let m be a given positive integer. We consider the following system of functional parabolic partial differential equations

$$(4.1) \quad \begin{cases} \frac{\partial}{\partial t} u^i(t, x) + P_i(\alpha, x, D)u^i = f_i(u_t^1, \dots, u_t^m, \alpha)(x) \text{ in } \mathbf{R} \times \Omega, \\ B_i(\alpha, x)u^i = 0 \text{ on } \mathbf{R} \times \partial\Omega, i = 1, \dots, m, \alpha \in \mathbf{R}, \end{cases}$$

where

$$(EP) \quad P_i(\alpha, x, D)u^i = - \sum_{k, \ell=1}^n \frac{\partial}{\partial x_k} \left(a_{k\ell}^i(\alpha, x) \frac{\partial}{\partial x_\ell} u^i(t, x) \right) + a_0^i(\alpha, x)u^i(t, x)$$

is an elliptic operator of the second order, i.e. there exists a constant $c > 0$ such that

$$\sum_{k, \ell=1}^m a_{k, \ell}^i(\alpha, x) \xi_k \xi_\ell \geq c \|\xi\|^2$$

for all $x \in \bar{\Omega}$, $\alpha \in \mathbf{R}$, $i = 1, \dots, m$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$. We assume that all coefficient functions $a_{k, \ell}^i, a_0^i \in C^2(\mathbf{R} \times \bar{\Omega}; \mathbf{R})$;

(BD) The boundary operators $B_i(\alpha, x)$ are given by either

$$B_i(\alpha, x)u^i(t, x) = u^i(t, x)$$

or

$$B_i(\alpha, x)u^i(t, x) = \gamma(\alpha, x)u^i(t, x) + \frac{\partial}{\partial n^i} u^i(t, x),$$

where $\gamma_i \in C^1(\mathbf{R} \times \partial\Omega; \mathbf{R})$, $\frac{\partial}{\partial n^i} = \sum_{k, \ell=1}^n a_{k\ell}^i(\alpha, x)v_\ell(x) \frac{\partial}{\partial x_k}$ and $v_\ell(x)$, $\ell = 1, \dots, n$, are

the components of the outward normal $v(x)$ to $\partial\Omega$;

(FZ) The functions $f_i: [C(\mathbf{R}; L^2(\Omega, \mathbf{R}))]^m \times \mathbf{R} \rightarrow L^2(\Omega; \mathbf{R})$ are of class C^1 and bounded on bounded sets, $i = 1, \dots, m$.

It will be convenient to write (4.1) in the more succinct vector notation:

$$(4.2) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) + P(\alpha, x, D)u = f(u_t, \alpha)(x) & \text{in } \mathbf{R} \times \Omega, \\ B(\alpha, x)u = 0 & \text{on } \mathbf{R} \times \partial\Omega, \alpha \in \mathbf{R}, \end{cases}$$

where $u = \text{col}(u^1, u^2, \dots, u^m)$, $u_t = \text{col}(u_t^1, \dots, u_t^m)$, $P(\alpha, x, D)u = \text{col}(P_1(\alpha, x, D)u^1, \dots, P_m(\alpha, x, D)u^m)$, $B(\alpha, x)u = \text{col}(B_1(\alpha, x)u^1, \dots, B_m(\alpha, x)u^m)$ and $f: C(\mathbf{R}; L^2(\Omega; \mathbf{R}^m)) \times \mathbf{R} \rightarrow L^2(\Omega; \mathbf{R}^m)$ is defined by $f(u_t, \alpha)(x) = \text{col}(f_1(u_t^1, \dots, u_t^m, \alpha)(x), \dots, f_m(u_t^1, \dots, u_t^m, \alpha)(x))$.

Observe that we allow the case where $f(u_t, \alpha)(x)$ may depend on the values of $u(s, \cdot)$ for $s \geq t$, though in typical applications, the functional is nonanticipatory.

Our purpose is to find nontrivial periodic solutions of (4.2) when α varies over \mathbf{R} . Following [10], [19], [24], [28] and [47], we introduce the unknown period β explicitly as a new parameter in the equation and normalize the period by making the following change of variable

$$v(t, x) = u\left(\frac{1}{\beta}t, x\right).$$

Then the original periodic problem is reduced to finding a nontrivial family (α, β, v) which satisfies

$$(4.3) \quad \begin{cases} \frac{\partial}{\partial t} v(t, x) + \frac{1}{\beta} P(\alpha, x, D)v = \frac{1}{\beta} f(v_{t, \beta}, \alpha)(x) & \text{in } \mathbf{R} \times \Omega, \\ B(\alpha, x)v = 0 & \text{on } \mathbf{R} \times \partial\Omega, \alpha \in \mathbf{R}, \\ v(t, x) = v(t + 2\pi, x) & \text{in } \mathbf{R} \times \Omega, \end{cases}$$

where

$$v_{t, \beta}(\tau, x) = v(t + \beta\tau, x) \quad \text{for } (\tau, x) \in \mathbf{R} \times \Omega.$$

4.2. Abstract formulation of the problem. We put $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ and introduce the following spaces

$$\mathcal{H}_{B_i(\alpha)}^{1,2} = \{\varphi \in H^{1,2}(S^1 \times \Omega); B_i(\alpha)\varphi = 0\},$$

where $H^{k,\ell}(S^1 \times \Omega)$ is a Sobolev space of functions with weak derivatives of order k in S^1 and weak derivatives of order ℓ in Ω (cf. [48]),

$$\begin{aligned} \text{Dom}(L_{(\alpha, \beta)}) &= \{u = \text{col}(u^1, \dots, u^m) \in L^2(S^1 \times \Omega; \mathbf{R}^m); \\ &u^i \in \mathcal{H}_{B_i(\alpha)}^{1,2} \text{ for } i = 1, \dots, m\}. \end{aligned}$$

For every $(\alpha, \beta) \in \mathbf{R} \times \mathbf{R}_+$ we define the operator

$$L_{(\alpha, \beta)}: \text{Dom}(L_{(\alpha, \beta)}) \subset L^2(S^1 \times \Omega; \mathbf{R}^m) \longrightarrow L^2(S^1 \times \Omega; \mathbf{R}^m)$$

by

$$L_{(\alpha, \beta)}v(t, x) = \frac{\partial}{\partial t}v(t, x) + \frac{1}{\beta}P(\alpha, x, D)v.$$

It is well known (cf. [32], p. 80, Theorem 1.31) that for every $(\alpha, \beta) \in P := \mathbf{R} \times \mathbf{R}_+$ the linear operator $L_{(\alpha, \beta)}$ is a closed Fredholm operator of index zero.

In order to make our notation compatible with those in Section 3, we put

$$\mathbf{E} = \mathbf{F} = L^2(S^1 \times \Omega; \mathbf{R}^m)$$

and

$$\hat{\mathbf{E}} = C(S^1; L^2(\Omega; \mathbf{R}^m)).$$

As was verified in Section 2, for every $\lambda = (\alpha, \beta) \in \mathbf{R} \times \mathbf{R}_+$, the operator $L_\lambda = L_{(\alpha, \beta)}: \text{Dom}(L_\lambda) \subseteq \mathbf{E} \rightarrow \mathbf{F}$ gives rise to a vector bundle \mathcal{E} over $\mathbf{R} \times \mathbf{R}_+$, and a vector bundle morphism $L: \mathcal{E} \rightarrow \mathbf{F}$ is defined by $L(\lambda, v) = L_\lambda v$ for $(\lambda, v) \in \mathcal{E}$.

It is easy to prove that for any $(\alpha, \beta, v) \in P \times \hat{\mathbf{E}}$ and $t \in S^1$, we have $u_{t, \beta} \in [C(\mathbf{R}; L^2(\Omega; \mathbf{R}))]^m$ and the mapping $(t, \beta) \in S^1 \times (0, \infty) \rightarrow v_{t, \beta} \in [C(\mathbf{R}; L^2(\Omega; \mathbf{R}))]^m$ is continuous. Therefore by the assumption (FZ) on $f, \frac{1}{\beta}f(v_{t, \beta}, \alpha)(\cdot) \in L^2(\Omega; \mathbf{R}^m)$ is well defined and the mapping $t \in S^1 \rightarrow \frac{1}{\beta}f(v_{t, \beta}, \alpha)(\cdot) \in L^2(\Omega; \mathbf{R}^m)$ defines an element $\bar{F}_{\alpha, \beta, v}$ in $C(S^1; L^2(\Omega; \mathbf{R}^m))$. Let i denote the natural imbedding $C(S^1; L^2(\Omega; \mathbf{R}^m)) \hookrightarrow L^2(S^1 \times \Omega; \mathbf{R}^m)$, we can now define a mapping $\hat{F}: \mathbf{R} \times \mathbf{R}_+ \times \hat{\mathbf{E}} \rightarrow L^2(S^1 \times \Omega; \mathbf{R}^m) = \mathbf{F}$ as follows

$$\hat{F}(\alpha, \beta, v)(t, x) = i \circ \bar{F}_{\alpha, \beta, v}(t)(x) = \frac{1}{\beta}f(v_{t, \beta}, \alpha)(x).$$

The continuity of f as a mapping from $[C(\mathbf{R}; L^2(\Omega, \mathbf{R}))]^m \times \mathbf{R}$ to $L^2(\Omega; \mathbf{R}^m)$ and the continuity of the imbedding i imply that \hat{F} is continuous.

Let us notice that we have the following natural imbedding

$$j: \mathcal{E} \hookrightarrow \mathbf{R} \times \mathbf{R}_+ \times H^{1,2}(S^1 \times \Omega; \mathbf{R}^m).$$

On the other hand, it is well known that the composition of the following imbeddings

$$H^{1,2}(S^1 \times \Omega; \mathbf{R}^m) \hookrightarrow H^{\frac{2}{3},0}(S^1 \times \Omega; \mathbf{R}^m) \hookrightarrow C(S^1; L^2(\Omega; \mathbf{R}^m))$$

is compact. Therefore the natural injective morphism of vector bundles

$$J: \mathcal{E} \longrightarrow \mathbf{R} \times \mathbf{R}_+ \times \hat{\mathbf{E}}$$

satisfies that $J_\lambda: \mathbf{E}_\lambda \rightarrow \hat{\mathbf{E}}$ is a compact operator for each $\lambda \in \mathbf{R} \times \mathbf{R}_+$. This shows that the mapping $F: \mathcal{E} \rightarrow \mathbf{F}$ defined by $F = \hat{F} \circ J$ is completely continuous.

Proposition 4.1. *Finding a periodic solution $v \in H^{1,2}(S^1 \times \Omega; \mathbf{R}^m)$ of the system (4.3) is equivalent to solving the following parametrized coincidence problem*

$$(4.4) \quad L_{(\alpha, \beta)} v = F(\alpha, \beta, v), \quad (\alpha, \beta) \in \mathbf{R} \times \mathbf{R}_+.$$

Let us point out that all of the spaces introduced above, i.e. \mathbf{E} , \mathbf{F} , $\hat{\mathbf{E}}$ and $H^{1,2}(S^1 \times \Omega; \mathbf{R}^m)$ are Banach isometric representations of $G = S^1$ which act on functions by shifting the t -argument. Since linear operator $L_{(\alpha, \beta)}$ and the nonlinear perturbation F are equivariant with respect to this group action, the coincidence problem (4.4) is exactly a parametrized coincidence problem of the type studied in the previous sections.

To continue our discussion, we further assume the nonlinearity $f: [C(\mathbf{R}; L^2(\Omega; \mathbf{R}))]^m \times \mathbf{R} \rightarrow L^2(\Omega; \mathbf{R}^m)$ satisfies the following assumption:

(H.1) *there exists a number $s > 0$ such that the map f has a C^1 -factorization $\bar{f}: [C((-\infty, s]; L^2(\Omega; \mathbf{R}))]^m \times \mathbf{R} \rightarrow L^2(\Omega; \mathbf{R}^m)$, i.e. we have the following commutative diagram*

$$\begin{array}{ccc} [C(\mathbf{R}; L^2(\Omega; \mathbf{R}))]^m \times \mathbf{R} & \xrightarrow{f} & L^2(\Omega; \mathbf{R}^m) \\ & \searrow r & \nearrow \bar{f} \\ & [C((-\infty, s]; L^2(\Omega; \mathbf{R}))]^m \times \mathbf{R} & \end{array}$$

where r denotes the "restriction of domain" operator.

The above assumption means simply that we admit functionals with both unbounded delayed argument and bounded advanced argument.

Let $u_0 \in L^2(\Omega; \mathbf{R}^m) \subset L^2(\Omega; \mathbf{C}^m)$ be a t -constant function. Since $(u_0)_t = u_0$ for all $t \in \mathbf{R}$, u_0 is a solution of (4.1) with $\alpha = \alpha_0$ if $u_0 \in X_{\alpha_0} := \{\omega \in H^2(\Omega; \mathbf{C}^m); B(\alpha_0, x)\omega = 0\}$ and the following equation is satisfied

$$\begin{cases} P(\alpha_0, x, D)u_0 = f(u_0, \alpha_0)(x) & \text{in } \Omega, \\ B(\alpha_0, x)u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

We will call (u_0, α_0) a *stationary solution* of (4.1) with $\alpha = \alpha_0$.

Assume that (u_0, α_0) is a stationary solution of (4.1) with $\alpha = \alpha_0$. We put $L(\alpha_0) := P(\alpha_0, x, D) - D_u \bar{f}(u_0, \alpha_0): X_{\alpha_0} \subset L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$, where $D_u \bar{f}(u_0, \alpha_0)$ denotes the complexification of the restriction of the derivative of f with respect to u to the space X_{α_0} . We say that the stationary point (u_0, α_0) is *nonsingular* if $0 \notin \sigma(L(\alpha_0))$, where $\sigma(L(\alpha_0))$ denotes the spectrum of $L(\alpha_0)$.

Let (u_0, α_0) be a stationary solution of (4.1). We define the mapping $\mathcal{F}_{\alpha_0}: C := C((-\infty, s]; L^2(\Omega; \mathbf{C}^m)) \rightarrow L^2(\Omega; \mathbf{C}^m)$ by the formula

$$\varphi \in C \longmapsto \mathcal{F}_{\alpha_0}(\varphi) := D_u \bar{f}(u_0, \alpha_0) \varphi \in L^2(\Omega; \mathbf{C}^m),$$

where $D_u \bar{f}(u_0, \alpha_0)$ denotes simply the complexification of the derivative of \bar{f} with respect to u . If (u_0, α_0) is a nonsingular stationary point, i.e., $L(\alpha_0): X_{\alpha_0} \rightarrow L^2(\Omega; \mathbf{C}^m)$ is an isomorphism from X_{α_0} (equipped with the graph norm) to $L^2(\Omega; \mathbf{C}^m)$, then by the implicit function theorem, there is a continuously differentiable function $u(\alpha)$ for α near α_0 such that $(u(\alpha), \alpha)$ is a stationary point for each α . To simplify the notation, we put

$$\mathcal{F}_\alpha(\varphi) := D_u \bar{f}(u(\alpha), \alpha) \varphi, \quad \varphi \in C.$$

4.3. Linearization and characteristic equations. The spectrum analysis of the linearization of the system (4.2) at a stationary solution (u, α) leads to the following *characteristic equation*

$$(4.5) \quad \Delta_\alpha(\lambda)w := \lambda w + P(\alpha, x, D)w - \mathcal{F}_\alpha(e^{\lambda \cdot} w) = 0,$$

where $w \in X_\alpha := \{w \in H^2(\Omega; \mathbf{C}^m); B(\alpha, x)w = 0\}$, $\alpha \in \mathbf{R}$. As the linear operator $w \rightarrow \mathcal{F}_\alpha(e^{\lambda \cdot} w)$ is well defined and bounded only for $\lambda \in \mathbf{C}$ with $\text{Re } \lambda \geq 0$, the above equation has meaning only for those complex numbers.

Let us notice that $\Delta_\alpha(\lambda)$ is an unbounded operator in the space $L^2(\Omega; \mathbf{C}^m)$ such that $\text{Dom } (\Delta_\alpha(\lambda)) = X_\alpha$. Since $P(\alpha, x, D): X_\alpha \subseteq L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$ is an elliptic self-adjoint operator, $\Delta_\alpha(\lambda)$ is a closed Fredholm operator of index zero.

Let $S: L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$ be defined by

$$Sw = irw, \quad w \in L^2(\Omega; \mathbf{C}^m),$$

where $r > 0$ is any given constant. Since $P(\alpha, x, D)$ is self-adjoint, the inverse $\tilde{R}_{\alpha,r} := [P(\alpha, x, D) + S]^{-1}: L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$ exists for all $\alpha \in \mathbf{R}$. Moreover, since the inclusion $H^2(\Omega; \mathbf{C}^m) \hookrightarrow L^2(\Omega; \mathbf{C}^m)$ is a compact operator, the operator $\tilde{R}_{\alpha,r}$ is compact.

Now, the equation (4.5) can be rewritten as follows

$$(4.6) \quad \tilde{\Delta}_{\alpha,r}(\lambda)w := w - \mathcal{F}_\alpha(e^{\lambda \cdot} \tilde{R}_{\alpha,r}(w)) + (\lambda - ir)\tilde{R}_{\alpha,r}(w) = 0.$$

It is clear that $\tilde{\Delta}_{\alpha,r}(\lambda)$ is an analytic function of λ for $\text{Re } \lambda > 0$. Moreover, $\tilde{\Delta}_{\alpha,r}(\lambda)$ is a bounded operator of the type “identity + compact operator”, thus it is a bounded Fredholm operator of index zero.

Definition 4.1: λ is a *characteristic value* of (4.5) if, $\text{Ker } \Delta_{\alpha,r}(\lambda) \neq \{0\}$ (or equivalently, $\text{Ker } \tilde{\Delta}_{\alpha,r}(\lambda) \neq \{0\}$). The *multiplicity* of a characteristic value λ , denoted by $m(\lambda)$, is the dimension of the generalized kernel of $\tilde{\Delta}_{\alpha,r}(\lambda)$, i.e.,

$$m(\lambda) = \dim \bigcup_{k=1}^{\infty} \text{Ker} [\tilde{A}_{\alpha,r}(\lambda)]^k.$$

As an immediate consequence of the above definition, $m(\lambda) < \infty$ for all characteristic values λ such that $\text{Re } \lambda \geq 0$. Moreover, if $\text{Re } \lambda > 0$, then λ is an isolated characteristic value.

Definition 4.2: A nonsingular stationary solution (u_0, α_0) of (4.1) is a *center* if the linearization of (4.1) at (u_0, α_0)

$$L(\alpha_0) := P(\alpha_0, x, D) - D_u f(u_0, \alpha_0)$$

has purely imaginary eigenvalues. We will call (u_0, α_0) an *isolated center* if it is the only center in some neighbourhood of (u_0, α_0) in $L^2(\Omega; \mathbf{R}^m) \times \mathbf{R}$.

We denote by $\sigma_\alpha \subset \mathbf{R}$ the spectrum of the self-adjoint operator $P(\alpha, x, D): X_\alpha \subseteq L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$. Since $P(\alpha, x, D)$ is an elliptic differential operator, the spectrum σ_α is discrete and all eigenvalues μ_i^α are of finite multiplicity such that

$$\mu_0^\alpha < \mu_1^\alpha < \dots < \mu_j^\alpha < \dots$$

We denote by E_j^α the generalized eigensubspace of $P(\alpha, x, D)$ corresponding to $\mu_j^\alpha \in \sigma_\alpha$, and let $\rho_j^\alpha: L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$ be the orthogonal projection onto E_j^α . Consequently, for every $w \in L^2(\Omega; \mathbf{C}^m)$ we have $w = \sum_{j=0}^{\infty} \rho_j^\alpha(w)$. Substituting $w = \sum_{j=0}^{\infty} \rho_j^\alpha(w)$ into (4.6), we obtain

$$(4.7) \quad \sum_{j=0}^{\infty} \left[\rho_j^\alpha(w) - \frac{1}{\mu_j^\alpha + ir} \mathcal{F}_\alpha(e^{\lambda \cdot} \rho_j^\alpha(w)) + \frac{\lambda - ir}{\mu_j^\alpha + ir} \rho_j^\alpha(w) \right] = 0.$$

We denote by F_j^α the subspace of C_c spanned by functions of the type $t \rightarrow \varphi(t)w$, where $\varphi \in C((-\infty, s]; \mathbf{C})$ and $w \in E_j^\alpha$. The following hypothesis (see, cf. [35] for explanation) will be needed for the presentation of our Hopf bifurcation theory.

(H.2) $\mathcal{F}_\alpha(F_j^\alpha) \subset E_j^\alpha$ for all stationary solutions (u, α) and $j = 0, 1, 2, \dots$. Under the above hypothesis, (4.7) becomes the sequence of equations

$$(4.8) \quad \rho_j^\alpha(w) - \frac{1}{\mu_j^\alpha + ir} \mathcal{F}_\alpha(e^{\lambda \cdot} \rho_j^\alpha(w)) + \frac{\lambda - ir}{\mu_j^\alpha + ir} \rho_j^\alpha(w) = 0, \quad j = 0, 1, \dots$$

4.4. Crossing number and its computation. Suppose that λ_0 is a characteristic value of (4.5) for $\alpha = \alpha_0$, i.e. $\text{Ker } \tilde{A}_{\alpha_0,r}(\lambda_0) \neq \{0\}$. Since the multiplicity $m(\lambda_0)$ is finite, there exists a number k such that for $j > k$ the equations (4.8) for $\lambda = \lambda_0$ and $\alpha = \alpha_0$ have no nontrivial solution. Moreover, one can find a

neighbourhood V of λ_0 in \mathbf{C} such that for all $\lambda \in V$, with $\operatorname{Re} \lambda \geq 0$, the equations (4.8) for $j > k$ and $\alpha = \alpha_0$, have no non-trivial solution. Consequently, we can put $E^{\alpha_0, k} := \bigoplus_{i=0}^k E_i^{\alpha_0}$ and $\tilde{A}_{\alpha_0, r}^*(\lambda) := \tilde{A}_{\alpha_0, r}(\lambda)|_{E^{\alpha_0, k}}: E^{\alpha_0, k} \rightarrow E^{\alpha_0, k}$. Then $\lambda \in V$ with $\operatorname{Re} \lambda \geq 0$ is a characteristic value of (4.6) for $\alpha = \alpha_0$ if and only if $\det \tilde{A}_{\alpha_0, r}^*(\lambda) = 0$.

Denote by $p_{\alpha_0}: L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$ the orthogonal projection onto the space $E^{\alpha_0, k}$. By using the spectral integral we can find an open neighbourhood W of α_0 and a continuous mapping $p: W \rightarrow \mathcal{L}(L^2(\Omega; \mathbf{C}^m))$ such that $p_\alpha := p(\alpha)$ is exactly the orthogonal projection on the eigensubspace $E^{\alpha, k(\alpha)}$ of the operator $P(\alpha, x, D)$ corresponding to the part of the spectrum σ_α that is related to the eigenvalues $\mu_0^{\alpha_0}, \dots, \mu_k^{\alpha_0}$. To achieve this we take a contour containing the eigenvalues $\mu_0^{\alpha_0}, \dots, \mu_k^{\alpha_0}$ and the spectral integral over this area, for α close to α_0 (cf. [29] and [45]). Moreover, we can take the subset $W \subseteq \mathbf{R}$ sufficiently small such that

$$\tilde{A}_{\alpha, r}(\lambda)|_{(E^{\alpha, k(\alpha)})^\perp}: (E^{\alpha, k(\alpha)})^\perp \longrightarrow (E^{\alpha, k(\alpha)})^\perp$$

is an isomorphism for $\lambda \in V$ with $\operatorname{Re} \lambda \geq 0$, and $\alpha \in W$.

In what follows, we put, for $\alpha \in W$,

$$\tilde{A}_{\alpha, r}^*(\lambda) := \tilde{A}_{\alpha, r}(\lambda)|_{E^{\alpha, k(\alpha)}}: E^{\alpha, k(\alpha)} \longrightarrow E^{\alpha, k(\alpha)}.$$

For the local Hopf bifurcation, we assume the following:

- (H.3) there exist a stationary solution $(u_0, \alpha_0) \in L^2(\Omega; \mathbf{R}^m) \times \mathbf{R}$ and $\beta_0 > 0$ such that $i\beta_0$ is an isolated characteristic value of (4.5) for $\alpha = \alpha_0$. Moreover, there is $\varepsilon > 0$ such that for $0 < |\alpha - \alpha_0| < \varepsilon$, $\operatorname{Ker} \tilde{A}_{\alpha, r}(i\beta) = \{0\}$ for every $\beta \in \mathbf{R}$, and $\operatorname{Ker} \tilde{A}_{\alpha, r}(0) = \{0\}$ for all $\alpha \in \mathbf{R}$.

Under this assumption, by using the above construction with $\lambda_0 = i\beta_0$, we can find the intervals $[\alpha_0 - \delta_0, \alpha_0 + \delta_0] \subseteq W$ and a neighbourhood $V \subseteq \mathbf{C}$ of λ_0 such that the numbers

$$\gamma_\pm(u_0, \alpha_0, \beta_0) := \operatorname{deg}(\det \tilde{A}_{\alpha_0 \pm \delta, r}^*, V_+)$$

are well defined for every $\delta > 0$ such that $0 < \delta < \delta_0$, where $V_+ := V \cap \{\lambda; \operatorname{Re} \lambda > 0\}$ does not contain other zeros of $\det \tilde{A}_{\alpha_0, r}^*$ and deg denotes the usual Brouwer degree. We now introduce the number

$$\gamma(u_0, \alpha_0, \beta_0) := \gamma_-(u_0, \alpha_0, \beta_0) - \gamma_+(u_0, \alpha_0, \beta_0)$$

which will be called the *crossing number* of $(\alpha_0, i\beta_0, u_0)$. The crossing number $\gamma(u_0, \alpha_0, \beta_0)$ counts the number of characteristic values (with multiplicity) that enter the subset V_+ when α crosses the value α_0 .

To compute such a crossing number, we recall a result due to [12].

Suppose $a_1, a_2, b, c_1, c_2 \in \mathbf{R}$, $a_1 < a_2$, $b > 0$, $c_1 < c_2$. Let $\mathcal{D} = (-b, b) \times (c_1, c_2) \subseteq \mathbf{R}^2$, and let $\Phi: [a_1, a_2] \times \bar{\mathcal{D}} \rightarrow \mathbf{R}^2$ be a continuous mapping. For every $\alpha \in [a_1, a_2]$, we put

$$\Phi_\alpha(u, v) = \Phi(\alpha, u, v), \quad (u, v) \in \bar{\mathcal{D}}.$$

We assume

(C.1) $\Phi(\alpha, u, v) \neq 0$ for all $\alpha \in [a_1, a_2]$ and $(u, v) \in \partial\mathcal{D}$;

(C.2) if $(u, v) \in \mathcal{D}$, $\alpha = a_j$ ($j = 1, 2$) and $\Phi(a_j, u, v) = 0$, then $u \neq 0$.

Set $\mathcal{D}_+ = (0, b) \times (c_1, c_2)$. Define $\Phi^+, \Phi^-: \bar{\mathcal{D}}_+ \rightarrow \mathbf{R}^2$ by $\Phi^+ := \Phi_{a_2}|_{\mathcal{D}_+}$, $\Phi^- := \Phi_{a_1}|_{\mathcal{D}_+}$. The conditions (C.1) and (C.2) imply that Φ^+, Φ_- have no zero on $\partial\mathcal{D}_+$. Consequently, we can define the integers $\gamma_\pm := \deg(\Phi^\pm, \mathcal{D}_+)$ and the *crossing number* γ for the family $\{\Phi_\alpha\}_{\alpha \in [a_1, a_2]}$ by

$$\gamma := \gamma_- - \gamma_+.$$

The following result provides a computation formula for γ .

Lemma 4.1. *Suppose that $\Phi: [a_1, a_2] \times \bar{\mathcal{D}} \rightarrow \mathbf{R}^2$ is continuous and satisfies the conditions (C.1) and (C.2). We put $\mathcal{D}_1 := (a_1, a_2) \times (c_1, c_2)$ and define $\psi: \bar{\mathcal{D}}_1 \rightarrow \mathbf{R}^2$ by $\psi(\alpha, v) = \Phi(\alpha, 0, v)$ for $\alpha \in [a_1, a_2]$ and $v \in [c_1, c_2]$. Then $\psi(\alpha, v) \neq 0$ for $(\alpha, v) \in \partial\mathcal{D}_1$ and*

$$\gamma = \deg(\psi, \mathcal{D}_1).$$

The proof can be found in [12].

4.5. Statement of main results. We are now in the position to state our local and global Hopf bifurcation theorems.

Theorem 4.1. *Under the assumptions (H.1), (H.2) and (H.3), if $\gamma(u_0, \alpha_0, \beta_0) \neq 0$, then the point (α_0, β_0, u_0) is a bifurcation point of the parametrized coincidence problem (4.4), i.e. there exists a sequence $\{(\alpha_n, \beta_n, u_n(t, x))\}_{n=1}^\infty$ such that $\alpha_n \rightarrow \alpha_0$, $\beta_n \rightarrow \beta_0$, $u_n \rightarrow u_0$ as $n \rightarrow \infty$ and $u_n(\cdot, x)$ is a nonconstant periodic solution of the system (4.2) for $\alpha = \alpha_n$ with a period $\frac{2\pi}{\beta_n}$.*

Remark 4.1: One can easily verify that the reaction-diffusion equations with delay considered in the literature [33], [35], [37], [50] and [51] satisfy conditions (FZ) and (H.1). (H.3) is a standard assumption in the Hopf bifurcation theory and is satisfied under the “transversality condition” (see, cf. [17, 33, 50, 51]). (H.2) is required mainly for the sake of simplicity. This assumption is motivated by the work of Memory [33], [34], [35] and is satisfied by the reaction-diffusion logistic equation with delay (see, cf. [35]).

For the global bifurcation problem, we make the following assumption:

(H.4) All stationary solutions of (4.1) are nonsingular and all centers of (4.1) are isolated.

Under this assumption, zero is a regular value of the restriction $G_0 := (L - F)|_{\mathcal{E}^G}: \mathcal{E}^G \rightarrow \mathbf{F}^G$. Consequently, $M := G_0^{-1}(0)$ is a 2-dimensional submanifold of $\mathcal{E}^G \subseteq \mathbf{R} \times \mathbf{R}_+ \times \mathbf{E}^G$. It is easy to see that M satisfies conditions (A) and (B).

Let S denote the closure of the set of all nontrivial t -periodic solutions of (4.1) in the space $\mathbf{R} \times \mathbf{R}_+ \times H^{1,2}(S^1 \times \Omega; \mathbf{R}^m)$, and let $C(\alpha_0, \beta_0, u_0)$ denote the connected component of a bifurcation point $(\alpha_0, \beta_0, u_0) \in S$. We are now in the position to state the following global Hopf bifurcation theorem:

Theorem 4.2. *Under the assumptions (H.1), (H.2) and (H.4), if $(\alpha_0, \beta_0, u_0) \in M$ is a bifurcation point of (4.4), then the connected component $C(\alpha_0, \beta_0, u_0)$ of (α_0, β_0, u_0) in S is either unbounded or the number of bifurcation points in $C(\alpha_0, \beta_0, u_0)$ is finite, i.e.,*

$$C(\alpha_0, \beta_0, u_0) \cap M = \{(\alpha_0, \beta_0, u_0), (\alpha_1, \beta_1, u_1), \dots, (\alpha_q, \beta_q, u_q)\}.$$

In the later case, we have the following equality:

$$\gamma(u_0, \alpha_0, \beta_0) + \gamma(u_1, \alpha_1, \beta_1) + \dots + \gamma(u_q, \alpha_q, \beta_q) = 0.$$

□

We emphasize that Theorem 4.1 can be extended to a more general version of a local bifurcation theorem. In fact, if all stationary solutions of (4.1) are nonsingular, then we have again a well defined 2-dimensional submanifold $M = G_0^{-1}(0)$ which satisfies the assumptions (A) and (B). We denote by $A \subset M$ the set of all centers.

Let U_0 be an open bounded subset of M whose closure is contained in a local neighbourhood of the type described in the condition (B), i.e. $\overline{U_0} \subseteq \{(\lambda, \eta(\lambda)); \lambda \in U_{\lambda_0}\} \subset M$, and such that $\partial U_0 \cap A = \emptyset$, $\Gamma := U_0 \cap A \neq \emptyset$, where $U_{\lambda_0} = (\alpha_-, \alpha_+) \times (b_1, b_2)$, $\alpha_- < \alpha_+$, $0 < b_1 < b_2$. Let $d > 0$, we denote by \mathcal{P}_0 the paralleloiped

$$\mathcal{P}_0 := \{(\alpha, \tau, \beta): \alpha_- \leq \alpha \leq \alpha_+, 0 \leq \tau \leq d, b_1 \leq \beta \leq b_2\}.$$

Now we can make the following hypothesis.

(H.3) $\text{Ker } \tilde{A}_{\alpha,r}(0) = \{0\}$ for all $\alpha = (\alpha_-, \alpha_+)$, and for every $(\alpha, \tau, \beta) \in \partial \mathcal{P}_0$, if $\text{Ker } \tilde{A}_{\alpha,r}(\tau + i\beta) \neq \{0\}$ then either $\tau = 0$ and $(\alpha, \beta) \in U_{\lambda_0}$ or $\alpha \in \{\alpha_-, \alpha_+\}$ and $(\tau, \beta) \in (0, d) \times (b_1, b_2)$.

The above assumption means that a characteristic value λ , for $\alpha_- \leq \alpha \leq \alpha_+$

can “escape” from the set $(0, d) \times (b_1, b_2)$ only through the side $\{0\} \times (b_1, b_2)$ lying on the imaginary axis. Clearly, the assumption (H.3) is a special case of (H.3)′.

Now, we denote by Σ the set of all pairs (α, λ_α) , where λ_α is a characteristic value in $[0, d] \times [b_1, b_2]$ for $\alpha_- \leq \alpha \leq \alpha_+$, i.e.

$$\Sigma := \{(\alpha, \tau, \beta) \in \bar{\mathcal{P}}_0 : \text{Ker } \tilde{A}_{\alpha,r}(\tau + i\beta) \neq \{0\}\}.$$

The set Σ is compact. Therefore from the subspaces $E^{\alpha, \kappa(\alpha)}$ defined in previous subsections we can construct a finite dimensional subspace E^α , depending on $\alpha \in [\alpha_-, \alpha_+]$, such that the operator

$$\tilde{A}_{\alpha,r}^*(\lambda) := \tilde{A}_{\alpha,r}(\lambda)|_{E^\alpha} : E^\alpha \longrightarrow E^\alpha$$

is well defined and

$$\text{Ker } \tilde{A}_{\alpha,r}^*(\lambda) = \text{Ker } \tilde{A}_{\alpha,r}(\lambda).$$

Now, we can put $V_+ = (0, d) \times (b_1, b_2)$ and we define again

$$\gamma_\pm(\Gamma, \alpha_\pm) = \text{deg}(\det \tilde{A}_{\alpha_\pm,r}^*, V_+)$$

and

$$\gamma(\Gamma, \alpha_-, \alpha_+) := \gamma_-(\Gamma, \alpha_-) - \gamma_+(\Gamma, \alpha_+).$$

The number $\gamma(\Gamma, \alpha_-, \alpha_+)$ will be called the *crossing number* of the set Γ with respect to α_- and α_+ . This crossing number expresses the difference of the number of characteristic values (with multiplicity) in the set V_+ for $\alpha = \alpha_-$ and $\alpha = \alpha_+$.

Theorem 4.3. *Under the assumptions (H.1), (H.2) and (H.3)′, if $\gamma(\Gamma, \alpha_-, \alpha_+) \neq 0$, then the set Γ contains a bifurcation point of the parametrized coincidence problem (4.4), i.e. there exists a sequence $\{(\alpha_n, \beta_n, u_n(t, x))\}_{n=1}^\infty$ such that $(\alpha_n, \beta_n, u_n) \rightarrow (\alpha_0, \beta_0, u_0)$ as $n \rightarrow \infty$ for some $(\alpha_0, \beta_0, u_0) \in \Gamma$, and $u_n(\cdot, x)$ is a nonconstant periodic solution of the system (4.2) for $\alpha = \alpha_n$ with a period $\frac{2\pi}{\beta_n}$.*

Finally, we should point out that the relationship between the crossing number and the center index introduced by Mallet-Paret and Yorke [39] as well as the Hopf index defined by Fiedler [17] was discussed in [12] for (ordinary) functional differential equations. It can be shown that such a relationship still holds for the parabolic partial differential equations discussed in this section. For details, we refer to [12].

4.6. Proof of main results.

Proof of Theorem 4.1: We begin the proof with some remarks concerning the operators $L_{(\alpha,\beta)}: \text{Dom}(L_{(\alpha,\beta)}) \subseteq L^2(S^1 \times \Omega; \mathbf{R}^m) \rightarrow L^2(S^1 \times \Omega; \mathbf{R}^m)$. Let us recall that we denote by $\sigma_\alpha = \{\mu_0^\alpha, \mu_1^\alpha, \dots\}$ the spectrum of the self-adjoint operator $P(\alpha, x, D): X_\alpha \subseteq L^2(\Omega; \mathbf{C}^m) \rightarrow L^2(\Omega; \mathbf{C}^m)$, where $\mu_0^\alpha < \mu_1^\alpha < \dots < \mu_j^\alpha < \dots$. By applying the separation of variable to the operator $L_{(\alpha,\beta)}$ we obtain that the spectrum $\sigma(L_{(\alpha,\beta)})$ of the operator $L_{(\alpha,\beta)}$ is the set $\{\beta^{-1}\mu_j^\alpha \pm ik; j, k \in \mathbf{N}\}$, where $\mathbf{N} = \{0, 1, \dots\}$.

We put $\mathbf{F} := L^2(S^1 \times \Omega; \mathbf{R}^m)$. The space \mathbf{F} is an orthogonal representation of $G := S^1$ acting on functions $u(t, x)$ by shifting the t -argument, i.e. for $\gamma \in S^1$,

$$(\gamma \cdot u)(z, x) = u(\gamma \cdot z, x), \quad \text{where } z \in S^1, x \in \Omega.$$

The space \mathbf{F} has the following direct sum decomposition

$$\mathbf{F} = \mathbf{F}_0 \oplus \mathbf{F}_1 \oplus \dots \oplus \mathbf{F}_k \oplus \dots$$

where $\mathbf{F}_k = \mathcal{F}_k \otimes L^2(\Omega; \mathbf{R}^m)$, $\mathcal{F}_k = \text{span}\{\cos kt, \sin kt\} \subset L^2(S^1; \mathbf{R})$. It is easy to prove that \mathbf{F}_k is an isotypical summand for the action of G such that for every $x \in \mathbf{F}_k \setminus \{0\}$ and $k > 0$, $G_x = \mathbf{Z}_k$. Moreover

$$\sigma(L_{(\alpha,\beta)}|_{\mathbf{F}_k}) = \{\beta^{-1}\mu_j^\alpha \pm ik; j \in \mathbf{N}\}.$$

This means that the restriction of $L_{(\alpha,\beta)}$ to the subspace \mathbf{F}_k is a bijective operator from $\text{Dom}(L_{(\alpha,\beta)}) \cap \mathbf{F}_k$ onto \mathbf{F}_k . We also recall that the subspace \mathbf{F}_k , $k > 0$, has the natural complex structure induced by the S^1 action on \mathbf{F}_k . By using such a structure, we can represent any function $u \in \mathbf{F}_k$ by $u(t, x) = \exp(ikt)\varphi(x)$, where $\varphi \in L^2(\Omega; \mathbf{R}^m)$.

We consider a sufficiently small neighbourhood \mathcal{V} of (α_0, β_0) in P (we will specify it more precisely later). Since $\text{Ker } L_{(\alpha,\beta)} \subseteq \mathbf{E}^G = \mathbf{F}_0$ and $\text{Coker } L_{(\alpha,\beta)} \subseteq \mathbf{E}^G = \mathbf{F}_0$, there exists an equivariant resolvent K of L over \mathcal{V} . Moreover, we can assume that for any $(\alpha, \beta) \in \mathcal{V}$, $\text{Ker } K_{(\alpha,\beta)} \supseteq \bigoplus_{i=1}^{\infty} \mathbf{F}_k$ and $\text{Im } K_{(\alpha,\beta)} \subseteq \mathbf{F}_0$.

Consequently, for all $(\alpha, \beta) \in \mathcal{V}$, $R_{(\alpha,\beta)}|_{\mathbf{F}_k}: \mathbf{F}_k \rightarrow \mathbf{F}_k$ is simply $\{L_{(\alpha,\beta)}|_{\mathbf{F}_k}\}^{-1}$.

We are now in the position to find the exact formula for the linear equivariant operator $T_{(\alpha,\beta)}^k: \mathbf{F}_k \rightarrow \mathbf{F}_k$ given by

$$\begin{aligned} T_{(\alpha,\beta)}^k u &= u - D_u F(\alpha, \beta, u_0) \circ R_{(\alpha,\beta)}(u) - K_{(\alpha,\beta)} \circ R_{(\alpha,\beta)}(u) \\ &= u - D_u F(\alpha, \beta, u_0) \circ [L_{(\alpha,\beta)}]^{-1} u, \quad u \in \mathbf{F}_k. \end{aligned}$$

By the definition,

$$D_u F(\alpha, \beta, u_0)w = \frac{1}{\beta} \mathcal{F}_\alpha(w_{\cdot,\beta}).$$

Therefore

$$T_{(\alpha,\beta)}^k u = u - \frac{1}{\beta} \mathcal{F}_\alpha([L_{(\alpha,\beta)}]^{-1}u)_{\cdot,\beta}.$$

We define $\Omega_1 := (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$, where $\delta > 0$ and $c > 0$ are such that $\bar{\Omega}_1 \subseteq \mathcal{V}$. Evidently, there is a homeomorphism $\tilde{\alpha}: S^1 \rightarrow \partial\Omega_1$ (preserving the orientation) which we are going to consider as an identification of S^1 with $\partial\Omega_1$. We now define

$$\psi_k: \partial\Omega_1 \longrightarrow GL_c^C(\mathbf{F}_k), \quad k = 1, 2, \dots$$

by

$$\psi_k(\alpha, \beta) = T_{(\alpha,\beta)}^k: \mathbf{F}_k \rightarrow \mathbf{F}_k, \quad (\alpha, \beta) \in \partial\Omega_1.$$

Let $e_k: L^2(\Omega; \mathbf{C}^m) \rightarrow \mathbf{F}_k$ be defined by $e_k(w_1 + iw_2) = \cos(k \cdot)w_1 + i \sin(k \cdot)w_2$, where $w_1, w_2 \in L^2(\Omega; \mathbf{R}^m)$. Then, e_k is evidently an isomorphism which preserves the complex structure. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{F}_k \cap \text{Dom}(L_{(\alpha,\beta)}) & \xrightarrow{L_{(\alpha,\beta)}} & \mathbf{F}_k \\ \uparrow e_k & & \downarrow e_k^{-1} \\ \text{Dom}(P(\alpha, x, D)) & \xrightarrow{P_{(\alpha,\beta)}^k} & L^2(\Omega; \mathbf{C}^m) \end{array}$$

where $P_{(\alpha,\beta)}^k u = ik u + \frac{1}{\beta} P(\alpha, x, D)u = \frac{1}{\beta} [ik\beta u + P(\alpha, x, D)u]$. Therefore

$$e_k^{-1}[L_{(\alpha,\beta)}]^{-1}e_k = \beta \tilde{R}_{\alpha,k\beta}.$$

Consequently, we have

$$\begin{aligned} \tilde{T}_{(\alpha,\beta)}^k w &:= e_k^{-1} T_{(\alpha,\beta)}^k (e_k \cdot w) \\ &= w - \frac{1}{\beta} e_k^{-1} \mathcal{F}_\alpha([L_{(\alpha,\beta)}]^{-1}e_k w)_{\cdot,\beta} \\ &= w - \frac{1}{\beta} e_k^{-1} \mathcal{F}_\alpha((e_k \beta \tilde{R}_{\alpha,k\beta} w)_{\cdot,\beta}) \\ &= w - e_k^{-1} e^{ik \cdot} \mathcal{F}_\alpha(e^{ik\beta \cdot} \tilde{R}_{\alpha,k\beta}(w)) \\ &= w - \mathcal{F}_\alpha(e^{ik\beta \cdot} \tilde{R}_{\alpha,k\beta}(w)). \end{aligned}$$

Therefore, we prove that

$$\tilde{\psi}_k(\alpha, \beta) := \tilde{T}_{(\alpha,\beta)}^k: L^2(\Omega; \mathbf{C}^m) \longrightarrow L^2(\Omega; \mathbf{C}^m)$$

is given by

$$\tilde{T}_{(\alpha,\beta)} w = w - \mathcal{F}_\alpha(e^{ik\beta} \cdot \tilde{R}_{\alpha,k\beta}(w)) = \tilde{A}_{\alpha,k\beta}(ik\beta).$$

Now we can specify what we meant by the sufficiently small neighbourhood \mathcal{V} of (α_0, β_0) . Namely, we assume that for $(\alpha, \beta) \in \mathcal{V}$, there exists no $\lambda \in \{z; |\operatorname{Re} z| \leq b, |\operatorname{Im} z - \beta_0| < c\}$ such that the equation $\tilde{A}_{\alpha,\beta}(\lambda)w = 0$ has non-zero solution $w \in (E^{\alpha,k(\alpha)})^\perp$, where $b = b(\alpha_0, \beta_0) > 0$ and $c = c(\alpha_0, \beta_0) > 0$ are sufficiently small numbers.

Note that $\mathcal{V}_1([\psi_1]) = \mathcal{V}_1([\tilde{\psi}_1]) = \operatorname{deg}(\hat{\psi}_1, \Omega_1)$ and $\hat{\psi}(\alpha, \beta) := \det \tilde{A}_{\alpha,\beta}^*(i\beta)$ is evidently homotopic to $\tilde{\psi}_1(\alpha, \beta) := \det \tilde{A}_{\alpha,\beta_0}^*(i\beta)$. Therefore, it follows from Lemma 4.1 that

$$\mathcal{V}_1([\psi_1]) = \gamma(u_0, \alpha_0, \beta_0) \neq 0.$$

Thus our conclusion is a consequence of Theorem 3.1. This completes the proof. \square

Theorem 4.2 is an immediate consequence of Theorem 3.2.

The proof of Theorem 4.3 is similar to the proof of Theorem 4.1.

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