# AN EQUIVARIANT DEGREE WITH APPLICATIONS TO SYMMETRIC BIFURCATION PROBLEMS. PART 1: CONSTRUCTION OF THE DEGREE

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## 1. *Introduction*

In this first of a series of papers devoted to an equivariant degree theory with applications to bifurcation problems with symmetries, we present a construction of the equivariant degree.

Let $V$ be a finite-dimensional orthogonal representation of a compact Lie group $G$, and $f$: $V \times \mathbb{R}^n \to V$ be an equivariant continuous mapping such that $f(x) \neq 0$ for all $x \in \partial\Omega$, where $\Omega$ is an invariant bounded subset of $V \times \mathbb{R}^n$. The *G-degree of the mapping f* with respect to $\Omega$ which we are going to construct is a sequence of integers indexed by orbit types $(H)$ in $\Omega$ satisfying dim $W(H) = n$, where $N(H)$ denotes the normalizer of $H$ and $W(H) = N(H)/H$ is the Weyl group of $H$.

One of our major technical tools in constructing the $G$-degree is the concept of *intersection number* for a section of a smooth vector bundle $p$: $E \to M$ which is an integer when $E$ is orientable as a *manifold.* A slight (but crucial for our purpose) advantage of the intersection number, defined in the present paper, over the standard construction of the intersection numbers for mappings between smooth manifolds, and the construction of the so-called *Euler number* of an orientable vector bundle, is that in our definition $M$ is not required to be orientable and $E$ is only required to be orientable as a manifold, not as a vector bundle. Nevertheless, we should mention that our approach in defining such an intersection number is parallel to that in [**16**, **17**, **29**].

We will construct the $G$-degree, $G$-Deg$(f, \Omega)$, by induction over orbit types in $\Omega$. In the case where $\Omega$ has only one orbit type $(H)$, the construction of $G$-Deg$(f, \Omega)$ can be briefly sketched as follows. Let $V_H = \{x \in V\colon G_x = H\}$, $V^H = \{x \in V\colon H \subseteq G_x\}$ and denote by $f_H$: $V_H \times \mathbb{R}^n \to V^H$ the restriction of $f$ to $V_H \times \mathbb{R}^n$. We first observe that $W(H)$ acts freely on $V_H \times \mathbb{R}^n$ and $V_H \times \mathbb{R}^n \times V^H$, and that the $W(H)$-equivariant projection $\pi$: $V_H \times \mathbb{R}^n \times V^H \to V_H \times \mathbb{R}^n$ induces a smooth vector bundle $p$: $E \to M$ with a typical fibre $V^H$, where $E = (V_H \times \mathbb{R}^n \times V^H)/W(H)$ and $M = (V_H \times \mathbb{R}^n)/W(H)$. Next, we use the $W(H)$-equivariant mapping $x \in V_H \times \mathbb{R}^n \to (x, f_H(x)) \in V_H \times \mathbb{R}^n \times V^H$ to induce a section $s_{f,M}$: $M \to E$ and define the $(H)$-component of $G$-Deg$(f, \Omega)$ as the intersection number of $s_{f,M}$. Then we show that this definition is independent of the choice of $H$, and the $G$-Deg$(f, \Omega)$ so-defined has all the properties of a topological degree. Finally, we prove that if $W(H)$ is bi-orientable (that is, $W(H)$ has an orientation invariant under left and right translations) then $E$ is orientable as a manifold, and hence the $(H)$-component of $G$-Deg$(f, \Omega)$ is an integer. In the case where $\Omega$

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has several orbit types, we choose a minimal orbit type $\alpha = (H)$ in $\Omega$, an open invariant subset $\Omega_1 \subseteq \Omega$, and an equivariant map $f_0$: $V \oplus \mathbb{R}^n \to V$ such that

$$\overline{\Omega}_1 \subseteq \Omega \setminus (V_\alpha \oplus \mathbb{R}^n), \quad f_0^{-1}(0) \cap \Omega \subseteq (V_\alpha \oplus \mathbb{R}^n) \cup \Omega_1,$$
$$f_0 = f \quad \text{on } (V_\alpha \oplus \mathbb{R}^n) \cup \Omega_1,$$

and $f_0$ satisfies a certain '$\alpha$-normality' condition. Since $\Omega_H$ has only one orbit type and $\Omega_1$ has fewer orbit types than $\Omega$, we can define

$$G\text{-Deg}(f, \Omega) := N(H)\text{-Deg}(f_H, \Omega_H) + G\text{-Deg}(f_0, \Omega_1)$$

by induction. It then remains to show that the above definition is independent of the choice of $H$, $f_0$, and $\Omega_1$ and that the $G$-Deg$(f, \Omega)$ has all the standard properties of a topological degree.

To illustrate the essence of our construction, we briefly recall an elementary approach to introducing the classical Brouwer degree in the non-equivariant case. Suppose that $U \subset \mathbb{R}^m$ is open and bounded, and $g$: $U \to \mathbb{R}^m$ is continuous with $g^{-1}(0) \cap \partial U = \emptyset$. Then $g$ can be approximated by a *generic* mapping $\tilde{g}$ (that is, a $C^1$-mapping for which 0 is a regular value) and deg$(g, U)$ is the 'algebraic' sum of the points in $\tilde{g}^{-1}(0)$ which is independent of the approximation $\tilde{g}$. Our construction of equivariant degrees is similar, but the generic approximation is much more complicated. Namely, to define the $(H)$-component of the $G$-degree for $f$ with respect to $\Omega$, one has to approximate the mapping $f$ by an '$\alpha$-generic' mapping $\tilde{f}$ which satisfies certain *normality conditions* and whose restriction $\tilde{f}_H$ to $\Omega_H$ is transversal to zero. This generic approximation $\tilde{f}_H$ can have only a finite number of non-degenerated orbits of zeros in $\Omega_H$. In the case where $W(H)$ is bi-orientable, these orbits are naturally oriented by $W(H)$. So to each of these orbits, we can assign a number $+1$ or $-1$, depending on whether $\tilde{f}_H$ preserves or reverses the orientations of the normal space to the orbit and $V^H$. The sum of these numbers $\pm 1$ turns out to be exactly the $(H)$-component of the $G$-Deg$(f, \Omega)$. So the $(H)$-component of $G$-Deg$(f, \Omega)$ counts the orbits of zeros of $f$ with the orbit type $(H)$, and consequently $G$-Deg$(f, \Omega)$ provides a topological invariant which can be used to determine the existence of zeros and to estimate the number of orbits of zeros as well as their orbit types.

Equivariant degrees have been the subject of much literature of which we mention [**3–5**, **7–9**, **11–15**, **18–22**, **24–28**, **30**]. In particular, a much more general $G$-equivariant degree was introduced by Ize, Massabò and Vignoli in [**19**] where they defined the equivariant degree, in the language of equivariant obstruction theory, as an element of a certain equivariant homotopy group of spheres $\pi_n^G(S^{n+k})$. Our construction expresses the fact that there is a direct factor $\bigoplus_\alpha \mathbb{Z}$ contained in $\pi_n^G(S^{n+k})$, where $\alpha = (H)$ varies over orbit types with bi-orientable Weyl group $W(H)$. This fact was also proved independently in [**26**] by a different approach based on methods of algebraic topology. While our equivariant degree is less general than that in [**19**], our elementary construction provides a simple method to compute the degree, as a sequence of integers. We should also mention the recent paper of Prieto and Ulrich (cf. [**27**]) in which the authors generalized Dold's transfer for parametrized equivariant coincidence problems and extended the notion of the equivariant fixed point index to the class of $G$-ENRs over a base space.

This paper is composed as follows. In § 2, we first modify a well-known transversality theorem for vector bundle sections, and then use the transversality technique to define an *intersection number* for a section $s$ of a smooth vector bundle $p$: $E \to M$ with respect to an open relatively compact subset $U \subset M$. The terminology of intersection numbers is borrowed from the book of M. Hirsch **[17]** and can be regarded as the intersection number between the zero section of the bundle $E$ (which is diffeomorphic to $M$) and the 'graph' of $s$ over $M$. The manifold $M$ is not required to be orientable. However, if $E$ is orientable as a manifold then the intersection number $\chi(s, U)$ is an integer with the standard properties of a topological degree. Section 3 is entirely devoted to the construction of the $G$-degree and the main result of the paper is presented in subsection 3.A. Our construction of the degree is conducted by induction over orbit types in $\Omega$. In subsection 3.B, we start the induction by considering the case of a region $\Omega$ with only one orbit type $(H)$. For such a region an equivariant mapping determines a section of a certain quotient vector bundle and the intersection number of this section is the $(H)$-component of the $G$-degree. In subsection 3.C, we complete the induction for the general case and present a verification of the properties of the $G$-degree. In § 4, we establish a reduction formula and a regular-value formula for the constructed equivariant degree.

In subsequent papers, we will discuss the application of the equivariant degree to the (local, global) Hopf bifurcation problems of dynamical systems with symmetries and bifurcation problems of time-reversible dynamical systems.

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## 2. *Intersection numbers of bundle sections*

We start this section by recalling a well-known transversality theorem for mappings between smooth manifolds.

DEFINITION 2.1. Suppose that $N_1$ and $N_2$ are smooth manifolds, $f$: $N_1 \to N_2$ a $C^1$-map, $A \subseteq N_2$ a submanifold, and $K \subseteq N_1$ a subset. We say that $f$ is *transversal to A along K* if

$$T_{f(x)}N_2 = T_{f(x)}A + T_xf(T_xN_1), \quad \text{for } x \in K \cap f^{-1}(A).$$

In what follows, we will denote by $f \pitchfork_K A$ the fact that $f$ is transversal to $A$ along $K$. Moreover, for every $r \geqslant 1$, we define

$$\pitchfork^r_K(N_1, N_2; A) = \{f \in C^r(N_1, N_2): f \pitchfork_K A\}.$$

The following transversality theorem for mappings between manifolds is a well-known fact. For details, we refer to **[17]**.

THEOREM 2.2. *Let $N_1, N_2$ be two smooth manifolds, $A \subseteq N_2$ a closed smooth* *submanifold and $K \subseteq N_1$ a compact subset. Then for each $r \geqslant 1$, $\pitchfork_K^r(N_1, N_2; A)$ is dense in $C(N_1, N_2)$, and open and dense in $C^r(N_1, N_2)$ equipped with the compact-open topology.*

Suppose that $p\colon E \to M$ is a smooth vector bundle equipped with a Riemannian metric $\langle\cdot,\cdot\rangle_x$ for $x \in M$. The norm induced by $\langle\cdot,\cdot\rangle_x$ will be denoted by $\|\cdot\|_x$, or simply $\|\cdot\|$. A continuous map $s\colon M \to E$ is called a *section* of $E$ if $p(s(x)) = x$ for every $x \in M$. The set of all sections of $E$ will be denoted by $\Gamma(E)$, and for every $r \geqslant 1$ we define

$$\Gamma^r(E) = \{s \in \Gamma(E)\colon\ s \in C^r(M, E)\}.$$

A section $s \in \Gamma^r(E)$ is called a *$C^r$-section.* The zero section of $p\colon E \to M$ is the section $z\colon M \to E$ such that, for every $x \in M$, $z(x)$ is the zero element of the fibre $E_x := p^{-1}(x)$. In what follows, we will often identify $M$ with $z(M)$ via $z$, and for a given section $s \in \Gamma(E)$, we will use $s(x) = 0$, for $x \in M$, to denote $s(x) = z(x)$.

Let $\Gamma^r_\pitchfork(K, E) := \Gamma(E) \cap \pitchfork^r_K(M, E; z(M))$ for every given subset $K \subseteq M$ and $r \geqslant 1$. The following theorem can be regarded as a version of the transversality theorem for bundle sections.

THEOREM 2.3. *Let $p\colon E \to M$ be a smooth vector bundle, $K \subseteq M$ a compact subset, and $r \geqslant 1$. Then $\Gamma^r_\pitchfork(K, E)$ is dense in $\Gamma(E)$, and open and dense in $\Gamma^r(E)$ equipped with the compact-open topology.*

*Proof.* Notice that if $K = \bigcup_{i=1}^n K_i$, then

$$\Gamma^r_\pitchfork(K, E) = \bigcap_{i=1}^n \Gamma^r_\pitchfork(K_i, E).$$

Therefore, it suffices to prove the theorem in the case where $K \subseteq U$ and $U$ is a local chart for the bundle $p\colon E \to M$. The inclusion $i\colon U \hookrightarrow M$ induces the following commutative diagram:

$$\begin{array}{ccccc}
\Gamma^r(E) & \longrightarrow & \Gamma^r(E|_U) & \cong & C^r(U, F) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma^r_\pitchfork(K, E) & \longrightarrow & \Gamma^r_\pitchfork(K, E|_U) & \cong & \pitchfork^r_K(U, F; \{0\})
\end{array}$$

where $F$ denotes the standard fibre of $E$. Thus, by Theorem 2.2, $\Gamma^r_\pitchfork(K, E)$ is open in $\Gamma^r(E)$. The density of $\Gamma^r_\pitchfork(K, E)$ can be verified by using a cut-off function for the pair $K \subseteq U$ and, again, by using Theorem 2.2.

Throughout the remaining part of this section, we assume that $p\colon E \to M$ is a smooth $n$-dimensional vector bundle over a smooth $n$-dimensional manifold $M$, and $\Omega \subseteq M$ is an open subset with compact closure.

DEFINITION 2.4. We say that $s \in \Gamma(E)$ is an *$\Omega$-admissible section* if $s(x) \neq 0$ for every $x \in \partial\Omega$.

DEFINITION 2.5. A continuous map $h\colon M \times [0, 1] \to E$ is called a *homotopy in*

$\Gamma(E)$ if $h_t \in \Gamma(E)$ for every $t \in [0, 1]$, where $h_t(x) = h(x, t)$ for $(x, t) \in M \times [0, 1]$. A homotopy $h$: $M \times [0, 1] \to E$ is said to be *$\Omega$-admissible*, if $h_t$ is $\Omega$-admissible for every $t \in [0, 1]$.

The goal in the next subsection is to assign to every $\Omega$-admissible section $s$: $M \to E$ a topological invariant which is an integer when $E$ is orientable as a manifold.

2.A. *The case where E is orientable*

In this subsection, we assume that $E$ is orientable. Identifying $M$ with $z(M)$ via the zero section $z$: $M \to E$, we obtain the following exact sequence of vector bundles over $M$:

$$TM \to TE|_M \to E,$$

where the first map is induced by the inclusion and the second is the quotient map.

DEFINITION 2.6. Let $s$: $M \to E$ be a $C^1$-section. We say that $x \in M$ is a *regular zero* of $s$ if $s(x) = 0$ and $D_v s(x)$: $T_x M \to E_x$, defined by the composition

$$T_x M \xrightarrow{T_x s} T_x E \longrightarrow E_x,$$

is an isomorphism of vector spaces.

We now consider a section $s \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$. Clearly, the set $s^{-1}(M) \cap \bar{\Omega}$ is finite and composed of regular zeros of $s$ only. Since $E$ is oriented, for every given $x \in s^{-1}(M) \cap \bar{\Omega}$ a chosen orientation of $T_x M$ determines an orientation of $E_x$ so that the identification $T_{z(x)}E \cong T_x M \oplus E_x$ preserves the orientations. Define

$$n(s, x) = \begin{cases} +1 & \text{if } D_v s(x) \text{ preserves the orientations,} \\ -1 & \text{if } D_v s(x) \text{ reverses the orientations.} \end{cases}$$

It is easy to verify that the above definition of $n(s, x)$ does not depend on the choice of the orientation of $T_x M$. This justifies the following definition.

DEFINITION 2.7. Let $s \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$ be an $\Omega$-admissible section. The *intersection number* of $s$ with respect to $\Omega$ is defined by

$$\chi(s, \Omega) = \sum_{x \in s^{-1}(M) \cap \Omega} n(s, x).$$

LEMMA 2.8. *Suppose that $h$: $M \times [0, 1] \to E$ is an $\Omega$-admissible homotopy in $\Gamma(E)$ such that $h_0, h_1 \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$. Then $\chi(h_0, \Omega) = \chi(h_1, \Omega)$.*

*Proof.* Without loss of generality, we may assume that $M$ is connected. If $M$ is orientable, then the conclusion of the lemma is a well-known fact (see [**17**]). We will now prove the lemma in the case where $M$ is non-orientable, by using arguments similar to those in [**16**]. Let $q\colon M_0 \to M$ be the orientable double cover of $M$ and $p_0\colon E_0 \to M_0$ the induced vector bundle. Then we have the following commutative diagram:

$$\begin{array}{ccc} E_0 & \longrightarrow & E \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M_0 & \underset{q}{\longrightarrow} & M \end{array}$$

Set $\Omega_0 = q^{-1}(\Omega)$. An admissible section $s\colon \bar{\Omega} \to E$ determines uniquely an admissible section $s_0\colon \bar{\Omega}_0 \to E_0$ such that

$$s(q(x)) = q^*(s_0(x)).$$

Moreover, if $s \in \Gamma^1_{\pitchfork}(\bar{\Omega}, E)$, then $s_0 \in \Gamma^1_{\pitchfork}(\bar{\Omega}_0, E)$ and

$$\chi(s_0, \Omega_0) = 2\chi(s, \Omega).$$

Similarly, an $\Omega$-admissible homotopy determines an $\Omega_0$-admissible homotopy. Consequently, the non-orientable case follows from the orientable case.

With Lemma 2.8, we are now able to extend the intersection number to every $\Omega$-admissible section.

Definition 2.9. Suppose that $s \in \Gamma(E)$ is an $\Omega$-admissible section. The *intersection number* of $s$ is defined by

$$\chi(s, \Omega) = \chi(\bar{s}, \Omega),$$

where $\bar{s} \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$ is an $\Omega$-admissible section such that

$$\sup_{x \in \bar{\Omega}} \|s(x) - \bar{s}(x)\|_x < \inf_{x \in \partial\Omega} \|s(x)\|_x.$$

Remark 2.10. The existence of $\bar{s}$ is guaranteed by Theorem 2.3. The fact that the definition $\chi(s, \Omega)$ is independent of the choice of $\bar{s}$ can easily be verified by using Lemma 2.8. This justifies the above definition.

From the above construction, we can employ standard arguments in the topological degree theory (see [**23**]) to obtain the following fundamental properties of intersection numbers.

Theorem 2.11. *The intersection number of bundle sections has the following properties.*

(i) (*Existence*) *If $s \in \Gamma(E)$ is an $\Omega$-admissible section and $\chi(s, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $s(x) = 0$.*

(ii) (*Excision*) *If $\Omega_1 \subseteq \Omega$ is an open subset and $s \in \Gamma(E)$ is an $\Omega$-admissible section such that $s(x) \neq 0$ for $x \in \overline{\Omega \setminus \Omega_1}$, then $\chi(s, \Omega) = \chi(s, \Omega_1)$.*

(iii) (*Additivity*) *If $\Omega_1, \Omega_2$ are two disjoint open subsets of $\Omega$ and $s \in \Gamma(E)$ is an $\Omega$-admissible section such that $s(x) \neq 0$ for $x \in \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)}$, then*

$$\chi(s, \Omega) = \chi(s, \Omega_1) + \chi(s, \Omega_2).$$

(iv) (*Homotopy Invariance*) *If* $h: M \times [0,1] \to E$ *is an* $\Omega$-*admissible homotopy, then* $\chi(h_t, \Omega)$ *is a constant independent of* $t \in [0,1]$.

(v) (*Product Property*) *Assume that* $V$ *is a finite-dimensional linear space and* $\mathcal{U} \subseteq V$ *is an open subset containing* 0. *For* $s \in \Gamma(E)$, *let* $\sigma: M \times \mathcal{U} \to E \times V \times V$ *denote the section of the product bundle* $E \times V \times V \to M \times V$ *defined by* $\sigma(v, x) = (s(v), x, x)$. *If* $s$ *is an* $\Omega$-*admissible section, then* $\sigma$ *is* $\Omega \times \mathcal{U}$-*admissible and* $\chi(\sigma, \Omega \times \mathcal{U}) = \chi(s, \Omega)$.

Note that the sign of $\chi(s, \Omega)$ depends on the choice of orientation of $E$. Note also that in (v) we choose an orientation of $V$ and then orient $V \times V$ by using the product orientation.

2.B. *The case where E is non-orientable*

We now consider the case where $E$ is not orientable.

DEFINITION 2.12. Suppose that $s \in \Gamma(E)$ is an $\Omega$-admissible section. The *modulo 2 intersection number* of $s$ with respect to $\Omega$ is an element of the group $\mathbb{Z}_2 = \{0, 1\}$ defined by

$$\chi_2(s, \Omega) = n(\bar{s}^{-1}(M) \cap \Omega) \pmod 2,$$

where $\bar{s} \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$ is chosen so that

$$\sup_{x \in \bar{\Omega}} \|s(x) - \bar{s}(x)\|_x < \inf_{x \in \partial\Omega} \|s(x)\|_x,$$

and $n(\bar{s}^{-1}(M) \cap \Omega)$ denotes the number of elements in the set $\bar{s}^{-1}(M) \cap \Omega$.

REMARK 2.13. Clearly, the set $\bar{s}^{-1}(M) \cap \Omega$ is finite. Using an argument similar to that for Lemma 2.8, one can show that $\chi_2(s, \Omega)$ is independent of the choice of $\bar{s} \in \Gamma^2_{\pitchfork}(\bar{\Omega}, E)$. Therefore, $\chi_2(s, \Omega)$ is well defined. Moreover, we can easily verify that the fundamental properties in Theorem 2.11 are still true for $\chi_2(s, \Omega)$ (mod 2).

## 3. *Equivariant degree theory*

3.A. *Main results*

Suppose that $G$ is a fixed compact Lie group. We say that two closed subgroups $H$ and $K$ are *conjugate* in $G$, denoted by $H \sim K$, if there exists $g \in G$ such that $H = gKg^{-1}$. The relation $H \sim K$ is an equivalence relation. The equivalence class of $H$ is called a *conjugacy class* of $H$ in $G$ and will be denoted by $(H)$.

Let $O(G)$ stand for the set of all conjugacy classes of closed subgroups of $G$. The set $O(G)$ is partially ordered under the following order relation:

$\alpha \leq \beta$ for $\alpha, \beta \in O(G)$ if and only if there exist closed subgroups $H$ and $K$ of $G$ such that $\alpha = (H)$, $\beta = (K)$, and $K$ is conjugate to a subgroup of $H$.

For a closed subgroup $H$ of $G$, we use $N(H)$ to denote the *normalizer* of $H$ in $G$, and $W(H)$ to denote the *Weyl group* $N(H)/H$ of $H$ in $G$. For every $n \in \mathbb{N}$, we put

$$O_n(G) := \{(H) \in O(G): \dim W(H) = n\}.$$

The following terminology is borrowed from **[26]**.

DEFINITION 3.1. A compact Lie group is said to be *bi-orientable* if it has an orientation which is invariant under all left and right translations.

It is clear from the definition that every connected or abelian compact Lie group is bi-orientable. A finite group is also (trivially) bi-orientable. It is not too difficult to check that the group $O(2)$ is not bi-orientable.

We will use the following notation:

$$\begin{aligned} OA_n(G) &:= \{(H) \in O_n(G)\colon\ W(H) \text{ is bi-orientable}\};\\ OB_n(G) &:= \{(H) \in O_n(G)\colon\ W(H) \text{ is not bi-orientable}\};\\ A_n[G] &= \bigoplus \{\mathbb{Z}\colon\ (H) \in OA_n(G)\};\\ B_n[G] &= \bigoplus \{\mathbb{Z}_2\colon\ (H) \in OB_n(G)\};\\ AB_n[G] &= A_n[G] \oplus B_n[G]. \end{aligned}$$

An element of $AB_n[G]$ will be written as $\gamma = \sum_\alpha \gamma_\alpha \cdot \alpha$ where

$$\gamma_\alpha \in \begin{cases} \mathbb{Z} & \text{if } \alpha \in OA_n(G),\\ \mathbb{Z}_2 & \text{if } \alpha \in OB_n(G). \end{cases}$$

Assume that $V$ is a real finite-dimensional orthogonal representation of the Lie group $G$. We consider the product space $V \oplus \mathbb{R}^n$, where we assume that $G$ acts trivially on the second component.

For a given $x \in V \oplus \mathbb{R}^n$, we denote by $G_x := \{g \in G\colon\ gx = x\}$ the *isotropy group of* $x$. The conjugacy class $(G_x)$ will be called the *orbit type of* $x$. For an invariant subset $X \subseteq V \oplus \mathbb{R}^n$, a closed subgroup $H$ of $G$, and a conjugacy class $\alpha \in O(G)$, we put

$$\begin{aligned} X^H &:= \{x \in X\colon\ hx = x \text{ for all } h \in H\};\\ X_H &:= \{x \in X\colon\ G_x = H\};\\ X^\alpha &:= \{x \in X\colon\ (G_x) \leq \alpha\};\\ X_\alpha &:= \{x \in X\colon\ (G_x) = \alpha\}. \end{aligned}$$

DEFINITION 3.2. For an open bounded invariant subset $\Omega$ of $V \oplus \mathbb{R}^n$, an equivariant continuous map $f\colon V \oplus \mathbb{R}^n \to V$ is said to be *$\Omega$-admissible* if $f(x) \neq 0$ for $x \in \partial\Omega$. An equivariant continuous map $h\colon V \oplus \mathbb{R}^n \times [0, 1] \to V$, where $G$ acts on $[0, 1]$ trivially, is called an *equivariant homotopy*. An equivariant homotopy $h\colon V \oplus \mathbb{R}^n \times [0, 1] \to V$ is said to be *$\Omega$-admissible* if $h_t$, defined by $h_t(x) = h(x, t)$ for $(x, t) \in V \oplus \mathbb{R}^n \times [0, 1]$, is $\Omega$-admissible for every $t \in [0, 1]$. For an $\Omega$-admissible homotopy $h\colon V \oplus \mathbb{R}^n \times [0, 1] \to V$, we say that $h_0$ and $h_1$ are *$\Omega$-homotopic*.

Our major result in this section can be formulated as follows.

THEOREM 3.3. *For every $\Omega$-admissible map $f\colon V \oplus \mathbb{R}^n \to V$ we can assign an element $G$-Deg$(f, \Omega) \in AB_n[G]$ such that the following properties are satisfied.*

(P1) (*Existence*) *If $G$-Deg$(f, \Omega) = \sum_\alpha \gamma_\alpha \cdot \alpha \neq 0$, that is, there is an $\alpha \in O_n(G)$ such that $\gamma_\alpha \neq 0$, then there exists $x \in \Omega \cap f^{-1}(0)$ such that $(G_x) \leq \alpha$.*

(P2) (*Homotopy Invariance*) *If* $h$ *is an* $\Omega$*-admissible homotopy, then* $G$-$\mathrm{Deg}(h_t, \Omega)$ *does not depend on* $t \in [0, 1]$.

(P3) (*Excision*) *If* $\Omega_0 \subseteq \Omega$ *is an open and invariant subset and* $f^{-1}(0) \cap \Omega \subseteq \Omega_0$, *then* $G$-$\mathrm{Deg}(f, \Omega) = G$-$\mathrm{Deg}(f, \Omega_0)$.

(P4) (*Additivity*) *If* $\Omega_1$ *and* $\Omega_2$ *are two open invariant subsets of* $\Omega$ *such that* $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$ *and* $f^{-1}(0) \cap \Omega \subseteq \Omega_1 \cup \Omega_2$, *then*

$$G\text{-}\mathrm{Deg}(f, \Omega) = G\text{-}\mathrm{Deg}(f, \Omega_1) + G\text{-}\mathrm{Deg}(f, \Omega_2).$$

(P5) (*Product Formula*) *If* $U$ *is another orthogonal representation of* $G$, $\mathcal{U}$ *is an open invariant subset of* $U$ *such that* $0 \in \mathcal{U}$, *and* $g\colon V \oplus \mathbb{R}^n \times U \to V \times U$ *is defined by* $g(x, y) = (f(x), y)$ *for* $(x, y) \in V \oplus \mathbb{R}^n \times U$, *then* $G$-$\mathrm{Deg}(g, \Omega \times \mathcal{U}) = G$-$\mathrm{Deg}(f, \Omega)$.

The element $G$-$\mathrm{Deg}(f, \Omega) \in AB_n[G]$ will be called the *G-(equivariant) degree* of the map $f$ with respect to the set $\Omega$. The construction of such a degree will be conducted by induction on orbit types of $\Omega$ in the following two subsections.

3.B. *Construction of G-degree: one orbit type*

We need the following technical lemma in order to construct the $G$-degree in the case of one orbit type.

LEMMA 3.4. *Suppose that* $\Gamma$ *is a bi-orientable compact Lie group,* $W$ *is a real finite-dimensional orthogonal representation of* $\Gamma$, *and* $\mathcal{U}$ *is an open invariant subset of* $W$ *such that*

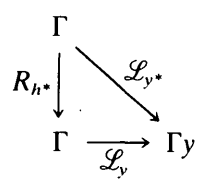
(i) $\Gamma$ *acts freely on* $\mathcal{U}$;

(ii) *the action of* $\Gamma$ *preserves the orientation of* $W$; *more precisely, for every* $h \in \Gamma$, *the map* $\varphi_h\colon W \to W$, *defined by* $\varphi_h(x) = hx$ *for* $x \in W$, *preserves the orientation of* $W$.

*Then* $\mathcal{U}/\Gamma$ *is orientable, and the orientation of* $\mathcal{U}/\Gamma$ *is determined by the choice of orientations of* $\Gamma$ *and* $W$.

*Proof.* We fix an orientation of $T_e\Gamma$ and use the left translation $L_g\colon \Gamma \to \Gamma$, $L_g(h) = gh$, for $g, h \in \Gamma$, to choose an orientation of the space $T_g\Gamma$ for any given $g \in \Gamma$. Since $\Gamma$ is bi-orientable, the right translation $R_g\colon \Gamma \to \Gamma$, $R_g(h) = hg^{-1}$, where $g, h \in \Gamma$, preserves these orientations.

Let $y \in \mathcal{U}$. Since $\Gamma$ acts on $\mathcal{U}$ freely, the map $\mathcal{L}_y\colon \Gamma \to \Gamma y$ defined by $\mathcal{L}_y(h) = hy$, with $h \in \Gamma$, is a diffeomorphism. This allows us to choose an orientation on $\Gamma y$. Note that if $y^* \in \Gamma y$ then there exists $h^* \in \Gamma$ such that $y^* = h^{*-1}y$. Since $R_{h^*}$ preserves the orientation of $\Gamma$, it follows from the commutative diagram




that the orientation of $\Gamma y$ determined above does not depend on the choice of $y^*$ in the orbit $\Gamma y$.

For $y \in \mathcal{U}$, let $[y]$ denote the orbit $\Gamma y$ in $\mathcal{U}/\Gamma$. Choose an orientation of $W$. Then $T_y\mathcal{U}$ and $T_y(\Gamma y)$ are oriented and determine an orientation of

$$T_{[y]}(\mathcal{U}/\Gamma) \cong T_y\mathcal{U}/T_y(\Gamma y) \cong S_y,$$

where $S_y = \{x \in W: x - y \perp T_y(\Gamma y)\}$ is the linear slice at $y$. Therefore $\mathcal{U}/\Gamma$ is orientable and a choice of orientations of $W$ and $\Gamma$ determines the orientation of $\mathcal{U}/\Gamma$.

Throughout the rest of this subsection, we assume that $G$ is a compact Lie group, $V$ is a real finite-dimensional orthogonal representation of $G$, $\Omega$ is an open bounded invariant subset of $V \oplus \mathbb{R}^n$, and there exists a closed subgroup $H$ of $G$ such that $(G_x) = (H)$ for all $x \in \bar{\Omega}$. That is, all points of $\bar{\Omega}$ are of the same orbit type $(H)$.

It can easily be shown that $N(H)$ acts on $V_H \times \mathbb{R}^n = (V \oplus \mathbb{R}^n)_H$, and consequently induces a free action of $W(H)$ on $V_H \times \mathbb{R}^n$. Similarly, we obtain a free diagonal action of $W(H)$ on the product space $V_H \times \mathbb{R}^n \times V^H$. The projection $\pi$: $V_H \times \mathbb{R}^n \times V^H \to V_H \times \mathbb{R}^n$ onto the space $V_H \times \mathbb{R}^n$ is clearly an equivariant map with respect to the above actions and induces a smooth map $p$: $E \to M$ between smooth manifolds $E := (V_H \times \mathbb{R}^n \times V^H)/W(H)$ and $M := (V_H \times \mathbb{R}^n)/W(H)$. In fact, $p$: $E \to M$ is a smooth vector bundle with a typical fibre $V^H$. Note that an orientation of $V^H$ determines the product orientation of $V^H \times V^H$ which does not depend on the orientation of $V^H$. Thus the product $V^H \times V^H$ has a natural preferred orientation. This preferred orientation of $V^H \times V^H$ together with the standard orientation of $\mathbb{R}^n$ determine the orientation of $V_H \times \mathbb{R}^n \times V^H$.

From Lemma 3.4, we have the following proposition.

PROPOSITION 3.5. *If $W(H)$ is bi-orientable, then the manifold*

$$E = (V_H \times \mathbb{R}^n \times V^H)/W(H)$$

*is orientable. Moreover, the orientation of $E$ is determined by the choice of the orientation of $W(H)$.*

Assume now that $W(H)$ is bi-orientable. Fix an orientation of $W(H)$. If $H_1 \sim H$, that is, $H_1 = gHg^{-1}$ for some $g \in G$, then the orientation of $W(H_1)$ is uniquely determined by the natural isomorphism $t_g$: $W(H_1) \to W(H)$ defined by

$$t_g(hH_1) = g^{-1}hgH, \quad \text{for } h \in N(H_1).$$

Indeed, if $H_1 = \tilde{g}H\tilde{g}^{-1}$ for another element $\tilde{g} \in G$, then we can easily show that $t_g \circ (t_{\tilde{g}})^{-1}$ is an inner automorphism of $W(H)$ and hence preserves the fixed orientation of $W(H)$. Therefore, we can choose an orientation for every class $(H) \in OA_n(G)$.

We now consider an $\Omega$-admissible map $f$: $V \oplus \mathbb{R}^n \to V$. Since $f$ is $G$-equivariant, the restriction of $f$ to the subspace $V_H \times \mathbb{R}^n$ induces a $W(H)$-equivariant map $f_H$: $V_H \times \mathbb{R}^n \to V^H$, $f_H(x) = f(x)$, for $x \in V_H \times \mathbb{R}^n$. Clearly $f_H$ is vector bundle is of dimension $\dim V^H$ and $\dim W(H) = n$, we have

$$\dim M = \dim(V^H \times \mathbb{R}^n) - n = \dim V^H.$$

That is, the dimension of $M$ is equal to the dimension of the fibre $E$. Therefore, the intersection number of $s_{f,H}$ with respect to $\Omega_H/W(H)$ is well defined.

DEFINITION 3.6. Fix an orientation for every class $(H) \in OA_n(G)$. The *$G$-equivariant degree of $f$ with respect to $\Omega$*, denoted by $G\text{-Deg}(f, \Omega) = \{\gamma_\alpha\}$, is defined by

$$\gamma_\alpha = \begin{cases} \chi(s_{f,H}, \Omega_H/W(H)) & \text{if } \alpha = (H) \text{ and } (H) \in OA_n(G), \\ \chi_2(s_{f,H}, \Omega_H/W(H)) & \text{if } \alpha = (H) \text{ and } (H) \in OB_n(G), \\ 0 & \text{if } \alpha \neq (H). \end{cases}$$

To justify the above definition, we need the following proposition.

PROPOSITION 3.7. *The above definition does not depend on the choice of the subgroup $H$ representing the class $(H)$.*

*Proof.* Let $H_1$ be another representation of the class $(H)$. Then there exists $g \in G$ such that $H_1 = gHg^{-1}$. The mappings $\tau_g$: $V_{H_1} \times \mathbb{R}^n \to V_H \times \mathbb{R}^n$ and $\bar{\tau}_g$: $V_{H_1} \times \mathbb{R}^n \times V^{H_1} \to V_H \times \mathbb{R}^n \times V^H$ defined by

$$\tau_g(x, v) = (gxg^{-1}, v),$$
$$\bar{\tau}_g(x, v, y) = (gxg^{-1}, v, gyg^{-1}), \quad \text{for } (x, v, y) \in V_H \times \mathbb{R}^n \times V^H,$$

are equivariant (from $W(H_1)$-space to $W(H)$-space) diffeomorphisms and provide an equivariant isomorphism of the vector bundles $V_{H_1} \times \mathbb{R}^n \times V^{H_1} \to V_{H_1} \times \mathbb{R}^n$ and $V_H \times \mathbb{R}^n \times V^H \to V_H \times \mathbb{R}^n$. By passing to orbit spaces, we obtain the following isomorphisms of vector bundles

$$\begin{array}{ccc} E_1 := (V_{H_1} \times \mathbb{R}^n \times V^{H_1})/W(H_1) & \xrightarrow[\cong]{[\bar{\tau}_g]} & E \\ p_1 \downarrow & & \downarrow p \\ M_1 := (V_{H_1} \times \mathbb{R}^n)/W(H_1) & \xrightarrow[\cong]{[\tau_g]} & M \end{array}$$

Define the section $s_{f,H_1}$: $M_1 \to E_1$ of the vector bundle $p_1$: $E_1 \to M_1$ in the same way as $s_{f,H}$: $M \to E$ is defined. Then we have the commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow[\cong]{[\bar{\tau}_g]} & E \\ s_{f,H_1} \uparrow\downarrow p_1 & & p \downarrow\uparrow s_{f,H} \\ M_1 & \xrightarrow[{[\tau_g]}]{\cong} & M \end{array}$$

from which, together with the definition of intersection numbers, it follows that

$$\chi(s_{f,H}, \Omega_H/W(H)) = \chi(s_{f,H_1}, \Omega_{H_1}/W(H_1))$$

if $(H) \in OA_n(G)$, and

$$\chi_2(s_{f,H}, \Omega_H/W(H)) = \chi_2(s_{f,H_1}, \Omega_{H_1}/W(H_1))$$

if $(H) \in OB_n(G)$. This completes the proof.

PROPOSITION 3.8. *The G-equivariant degree of f with respect to Ω satisfies the properties* (P1)–(P5).

*Proof.* The properties (P1)–(P4) can easily be verified from the construction of $G$-Deg$(f, \Omega)$. Property (P5) is a direct consequence of Theorem 2.11(v).

We conclude this section with a regular value formula under the assumption that $\bar{\Omega}$ has only one orbit type. A similar formula in more general cases will be established in § 4. Assume 0 is a regular value of $f|_{\Omega}$. For $x \in f^{-1}(0)$, let $S_x = \{y \in V^H\colon y - x \perp T_x(W(H)x)\}$ be the linear slice of $W(H)$-action at $x$. Set $D_v f(x) = Df(x)|_{S_x}\colon S_x \to V^H$. Since $x$ is a regular point for $f$, $D_v f(x)$ is a linear isomorphism. Choose an orientation of $V^H$. Orient $S_x$ in such a way that the orientation of $T_x(W(H)x)$, followed by the orientation of $S_x$, gives the product orientation of $V^H \oplus \mathbb{R}^n$. Set

$$n(x) = \begin{cases} +1 & \text{if } D_v f(x) \text{ preserves the orientation,} \\ -1 & \text{if } D_v f(x) \text{ reverses the orientation.} \end{cases}$$

Clearly $f^{-1}(0)$ is composed of a finite number of orbits, that is, $f^{-1}(0) = W(H)x_1 \cup \ldots \cup W(H)x_m$. From the definition of $G$-Deg, we obtain

PROPOSITION 3.9. *If* 0 *is a regular value for* $f|_{\Omega}$ *then*

$$\gamma_0 = \begin{cases} \sum n(x_i) & \text{if } \alpha = (H) \text{ and } (H) \in OA_n(G), \\ m \pmod 2 & \text{if } \alpha = (H) \text{ and } (H) \in OB_n(G), \\ 0 & \text{if } \alpha \neq (H), \end{cases}$$

*where* $G$-Deg$(f, \Omega) = \sum_\alpha \gamma_\alpha \cdot \alpha$ *and* $f^{-1}(0) = W(H)x_1 \cup \ldots \cup W(H)x_m$, *and* $W(H)x_i \cap W(H)x_j = 0$ *for* $i \neq j$.

3.C. *Construction of G-degree: several orbit types*

To simplify notation, we put $W := V \oplus \mathbb{R}^n$, where $V$ is a real orthogonal finite-dimensional representation of the Lie group $G$. In this section, we are going to construct $G$-Deg$(f, \Omega)$ for an $\Omega$-admissible map $f\colon V \oplus \mathbb{R}^n \to V$ in the general case where $\Omega$ may have several orbit types.

Given $\alpha \in O(G)$, it is well known that $W_\alpha := \{x \in W\colon (G_x) = \alpha\}$ is an invariant submanifold, and $\nu_\alpha\colon N^\alpha \to W_\alpha$, the normal bundle to $W_\alpha$ in $W$, is a $G$-vector bundle over $W_\alpha$.

DEFINITION 3.10. Let $D$ be an open invariant relatively compact subset of $W_\alpha$ and $\varepsilon > 0$ a positive number. We call

$$\mathcal{N}(D, \varepsilon) = \{v + w \in N^\alpha\colon v \in D, w \in N_v^\alpha, \|w\|_v < \varepsilon\}$$

an *α-normal neighbourhood of D* if each element of $\mathcal{N}(D, \varepsilon)$ can be written uniquely as $x = v + w$, where $v \in D$ and $w \in N_v^\alpha$ with $\|w\|_v < \varepsilon$.

Since $\bar{D}$ is a compact subset of $W_\alpha$, clearly, for every sufficiently small $\varepsilon > 0$, the set $\mathcal{N}(D, \varepsilon)$ is an $\alpha$-normal neighbourhood of $D$.

DEFINITION 3.11. Let $f\colon W \to V$ be an $\Omega$-admissible map and $\alpha \in O(G)$. We say that $f$ is *α-normal* (*with respect to* Ω) if there exist an open invariant relatively compact subset $D \subseteq W_\alpha$ and an $\alpha$-normal neighbourhood $\mathcal{N}(D, \varepsilon)$ such that

(i) $f^{-1}(0) \cap W_\alpha \cap \Omega \subseteq D$;

(ii) $\mathcal{N}(D, \varepsilon) \subseteq \Omega$;

(iii) $f(x) = f(v + w) = f(v) + w$ for all $x \in \mathcal{N}(D, \varepsilon)$, where $x = v + w$, $v \in D$, $w \in N_v^\alpha$.

DEFINITION 3.12. Let $h$: $W \times [0, 1] \to V$ be an $\Omega$-admissible homotopy and $\alpha \in O(G)$. We say that $h$ is *$\alpha$-normal* (*with respect to* $\Omega$) if there are an open invariant relatively compact subset $D \subseteq W_\alpha$ and an $\alpha$-normal neighbourhood $\mathcal{N}(D, \varepsilon)$ such that

(i) $h^{-1}(0) \cap (W_\alpha \cap \Omega) \times [0, 1] \subseteq D \times [0, 1]$;

(ii) $\mathcal{N}(D, \varepsilon) \subseteq \Omega$;

(iii) $h(x, t) = h(v + w, t) = h(v, t) + w$ for all $x \in \mathcal{N}(D, \varepsilon)$, $t \in [0, 1]$, where $x = v + w$, $v \in D$, $w \in N_v^\alpha$.

PROPOSITION 3.13. *Let $\alpha$ be a minimal element in $\mathcal{J}(\bar{\Omega}) := \{(G_x): x \in \bar{\Omega}\}$ and $f$: $W \to V$ an $\Omega$-admissible map. Then there exist an open invariant subset $\Omega_0 \subseteq \Omega$ and an $\alpha$-normal (with respect to $\Omega$) map $f_0$: $W \to V$ such that*

(i) *$f_0(x) = f(x)$ for all $x \in W_\alpha$;*

(ii) *$f_0(x) = f(x)$ for all $x \in W \backslash \Omega_0$;*

(iii) *$f^{-1}(0) \cap W_\alpha \cap \Omega \subseteq W_\alpha \cap \Omega_0$.*

*Proof.* We choose $\varepsilon > 0$ and open invariant subsets $D_0$ and $D_1$ of $W_\alpha$ such that

$$f^{-1}(0) \cap W_\alpha \cap \Omega \subseteq D_0 \subseteq \bar{D}_0 \subseteq D_1 \subseteq \bar{D}_1 \subseteq \Omega,$$
$$\Omega_0 := \mathcal{N}(D_1, 2\varepsilon) \subseteq \Omega.$$

We can choose an invariant continuous function $\gamma$: $W \to [0, 1]$ such that $\gamma(x) = 1$ for $x \in \mathcal{N}(D_0, \varepsilon)$ and $\gamma(x) = 0$ for $x \in W \backslash \Omega_0$. Then the equivariant map $f_0$: $W \to V$, defined by

$$f_0(x) = \begin{cases} f(x) & \text{for } x \in W \backslash \Omega_0, \\ \gamma(x)[f(v) + w] + [1 - \gamma(x)]f(x) & \text{for } x = v + w \in \Omega_0, \end{cases}$$

satisfies all the required properties, where $x = v + w$, $v \in D_1$, $w \in N_v^\alpha$ and $\|w\|_v < 2\varepsilon$.

Using the same argument, we can obtain the following proposition.

PROPOSITION 3.14. *Let $\alpha$ be a minimal element in $\mathcal{J}(\bar{\Omega})$. If $h$: $W \times [0, 1] \to V$ is an $\Omega$-admissible homotopy, then there exist an open invariant subset $\Omega_0 \subseteq \Omega$ and an $\alpha$-normal homotopy $h_0$: $W \times [0, 1] \to V$ such that*

(i) *$h_0(x, t) = h(x, t)$ for $x \in W_\alpha$ and $t \in [0, 1]$;*

(ii) *$h_0(x, t) = h(x, t)$ for $x \in W \backslash \Omega_0$ and $t \in [0, 1]$;*

(iii) *$h^{-1}(0) \cap W_\alpha \times [0, 1] \cap \Omega \times [0, 1] \subseteq W_\alpha \times [0, 1] \cap \Omega_0 \times [0, 1]$.*

Let $f$: $V \oplus \mathbb{R}^n \to V$ be an $\Omega$-admissible map. Our construction of $G$-equivariant degree $G$-Deg$(f, \Omega)$ will be conducted by induction over the orbit type in $\bar{\Omega}$. Let $\#\Omega$ denote the number of elements of $\mathcal{J}(\bar{\Omega})$. Throughout our inductive procedure, we will use the following versions of the Additivity Property, the Equivariant Homotopy Invariance Property, and the Product Formula.

$(\mathrm{A})_k$ Assume that $\#\Omega \leqslant k$. If $\Omega_1$ and $\Omega_2$ are two disjoint open invariant subsets of $\Omega$ and $f$: $W \to V$ is an $\Omega$-admissible map such that $f(x) \neq 0$ for $x \in \overline{\Omega \backslash (\Omega_1 \cup \Omega_2)}$, then

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(f, \Omega_1) + G\text{-Deg}(f, \Omega_2).$$

$(\mathrm{H})_k$ Assume that $\#\Omega \leqslant k$. If $h$: $W \times [0, 1] \to V$ is an $\Omega$-admissible homotopy, then

$$G\text{-Deg}(h_0, \Omega) = G\text{-Deg}(h_1, \Omega).$$

$(\mathrm{P})_k$ Assume that $\#\Omega \leqslant k$. If $U$ is another orthogonal representation of $G$, $\mathscr{U}$ is an open invariant subset of $U$ such that $0 \in \mathscr{U}$, and $g$: $V \oplus \mathbb{R}^n \times U \to V \times U$ is defined by $g(x, y) = (f(x), y)$ for $(x, y) \in V \oplus \mathbb{R}^n \times U$, then $G\text{-Deg}(f, \Omega) = G\text{-Deg}(g, \Omega \times \mathscr{U})$.

The proofs of $(\mathrm{A})_k$, $(\mathrm{H})_k$, and $(\mathrm{P})_k$ will be carried in parallel with the construction of $G$-Deg$(f, \Omega)$ which is described as follows.

*Step* 1. In the case where $\#\Omega = 1$, the $G$-Deg$(f, \Omega)$ has been constructed in the previous section, and it has been verified that $(\mathrm{A})_1$, $(\mathrm{H})_1$, and $(\mathrm{P})_1$ are satisfied by the constructed $G$-Deg$(f, \Omega)$.

*Step* 2. Assume that in the case where $\#\Omega \leqslant k$, $G$-Deg$(f, \Omega)$ has been constructed for all $\Omega$-admissible maps $f$: $W \to V$ and satisfies $(\mathrm{A})_k$, $(\mathrm{H})_k$, and $(\mathrm{P})_k$. We now consider the case where $\#\Omega \leqslant k+1$ and $f$: $W \to V$ is $\Omega$-admissible. First of all, we choose a minimal element $\alpha \in \mathscr{J}(\bar{\Omega})$. By Proposition 3.13, there exist an open invariant subset $\Omega_1 \subseteq \Omega$ and an $\alpha$-normal map $f_0$: $W \to V$ such that

$$\bar{\Omega}_1 \subseteq \Omega \backslash W^\alpha \quad \text{and} \quad f_0^{-1}(0) \cap \Omega \subseteq W^\alpha \cup \Omega_1.$$

Notice that $\#\Omega_1 \leqslant k$. By the inductive assumption, $G$-Deg$(f_0, \Omega_1)$ is well defined. Choose $x \in \Omega \cap W_\alpha$ and set $H = G_x$. From Step 1, we know that $N(H)$-Deg$(f_H, \Omega_H)$ has been constructed. Because $\Omega_H$ conains only one orbit type $(H)$ (with respect to the $N(H)$-action), we may identify the degree $N(H)$-Deg$(f_H, \Omega_H)$ with an element in $AB_n[G]$. Consequently, we may put

$$G\text{-Deg}(f, \Omega) := G\text{-Deg}(f_0, \Omega_1) + N(H)\text{-Deg}(f_H, \Omega_H),$$

where $+$ denotes the addition in the group $AB_n[G]$.

Using the argument for Proposition 3.7, we can show that $N(H)$-Deg$(f_H, \Omega_H)$ does not depend on the choice of $H$ such that $(H) = \alpha$. Moreover, it follows from the Additivity Property $(\mathrm{A})_k$ that $G$-Deg$(f_0, \Omega_1)$ does not depend on the choice of $\Omega_1$, nor on the minimal element $\alpha \in \mathscr{J}(\bar{\Omega})$.

We now claim that $G$-Deg$(f_0, \Omega_1)$ does not depend on the choice of $f_0$. Indeed, assume that $f_1$ is another $\alpha$-normal map satisfying all properties specified for $f_0$ in Proposition 3.13. Then the map $h$: $W \times [0, 1] \to V$ defined by $h(x, t) = tf_0(x) + (1-t)f_1(x)$, with $(x, t) \in W \times [0, 1]$, is an $\Omega$-admissible homotopy between $f_0$ and $f_1$, and we can find an open invariant subset $\Omega_2 \subseteq \Omega$ such that

$$\Omega_1 \subseteq \Omega_2 \subseteq \bar{\Omega}_2 \subseteq \Omega \backslash W_\alpha,$$
$$h^{-1}(0) \cap \Omega \times [0, 1] \subseteq (W_\alpha \cup \Omega_2) \times [0, 1].$$

Then by the inductive assumption $(A)_k$ we have

$$G\text{-Deg}(f_i, \Omega_1) = G\text{-Deg}(f_i, \Omega_2), \quad \text{for } i = 0, 1,$$

and by the inductive assumption $(H)_k$ we have

$$G\text{-Deg}(f_0, \Omega_2) = G\text{-Deg}(f_1, \Omega_2).$$

Consequently, $G\text{-Deg}(f_0, \Omega_1) = G\text{-Deg}(f_1, \Omega_1)$. So $G\text{-Deg}(f, \Omega)$ is well defined.

It is straightforward to verify that $G\text{-Deg}(f, \Omega)$ satisfies $(A)_{k+1}$.

To prove that $G\text{-Deg}(f, \Omega)$ satisfies $(H)_{k+1}$, we assume that $h\colon W \times [0, 1] \to V$ is an $\Omega$-admissible homotopy. Choose a minimal element $\alpha \in \mathscr{J}(\bar{\Omega})$. By Proposition 3.12 there exist an $\alpha$-normal homotopy $\kappa\colon W \times [0, 1] \to V$ and an open invariant subset $\Omega_3$ of $W$ such that $\bar{\Omega}_3 \subseteq \Omega \backslash W_\alpha$, $\kappa^{-1}(0) \cap (\Omega \times [0, 1]) \subseteq (W_\alpha \cup \Omega_3) \times [0, 1]$, and $\kappa(x, t) = h(x, t)$ for $(x, t) \in W_\alpha \times [0, 1]$. Therefore, $\kappa$ is an $\Omega_3$-admissible homotopy. By the inductive assumption $(H)_k$, we have $G\text{-Deg}(\kappa_0, \Omega_3) = G\text{-Deg}(\kappa_1, \Omega_3)$. On the other hand, by Proposition 3.8 we ensure that $N(H)\text{-Deg}(h_{0H}, \Omega_H) = N(H)\text{-Deg}(h_{1H}, \Omega_H)$. Consequently, $G\text{-Deg}(h_0, \Omega) = G\text{-Deg}(h_1, \Omega)$.

It remains to verify $(P)_{k+1}$. For this purpose, we choose a minimal element $\alpha \in \mathscr{J}(\bar{\Omega})$, an open invariant subset $\Omega_1 \subseteq \Omega$, and an $\alpha$-normal map $f_0\colon W \to V$, such that $\bar{\Omega}_1 \subseteq \Omega \backslash W^\alpha$ and $f_0^{-1}(0) \cap \Omega \subseteq W_\alpha \cup \Omega_1$. Notice that $g_0\colon V \oplus \mathbb{R}^n \times \mathscr{U} \to V \times \mathscr{U}$, defined by $g_0(x, y) = (f_0(x), y)$, is still $\alpha$-normal. We find an open invariant set $\Omega_4$ containing $f_0^{-1}(0) \cap W_\alpha$, such that $\Omega_1 \cap \Omega_4 = \varnothing$. By the additivity property, $G\text{-Deg}(f_0, \Omega) = G\text{-Deg}(f_0, \Omega_1) + G\text{-Deg}(f_0, \Omega_4)$. On the other hand, we have

$$G\text{-Deg}(g_0, \Omega \times \mathscr{U}) = G\text{-Deg}(g_0, \Omega_1 \times \mathscr{U}) + G\text{-Deg}(g_0, \Omega_4 \times \mathscr{U}).$$

Moreover, by the inductive assumption $(P)_k$, we have that

$$G\text{-Deg}(g_0, \Omega_1 \times \mathscr{U}) = G\text{-Deg}(f_0, \Omega_1)$$

and

$$G\text{-Deg}(g_0, \Omega_4 \times \mathscr{U}) = G\text{-Deg}(f_0, \Omega_4).$$

Consequently,

$$G\text{-Deg}(g, \Omega \times \mathscr{U}) = G\text{-Deg}(g_0, \Omega \times \mathscr{U}) = G\text{-Deg}(f_0, \Omega) = G\text{-Deg}(f, \Omega).$$

This proves $(P)_{k+1}$ and completes our inductive proof for Theorem 3.3.

The rest of this section is devoted to the study of the multiplicity property of $G$-degree for the case where $n = 0$.

DEFINITION 3.15. *The Burnside Ring.* Two finite $G$-complexes $X$ and $Y$ are said to be *equivalent* (denoted by $X \sim Y$) if for all subgroups $H \subset G$ the spaces $X^H$ and $Y^H$ have the same Euler characteristic. We denote by $A(G)$ the set of all equivalence classes of this relation, and by $[X] \in A(G)$ the class of $X$. Disjoint union and Cartesian product of $G$-complexes are compatible with this equivalence relation and induce *addition* and *multiplication* operations on $A(G)$. The set $A(G)$, together with these two operations, is a commutative ring with identity, which is called the *Burnside Ring* of $G$.

Let $\Phi(G)$ denote the set of conjugacy classes $(H)$ such that $N(H)/H$ is finite. It is well known (see [6]) that $A(G)$ is the free abelian group on $[G/H]$, $(H) \in \Phi(G)$, and for each $G$-complex $X$, the following relation holds:

$$[X] = \sum_{(H) \in \Phi(G)} \chi_c(X_{(H)})[G/H], \tag{3.16}$$

where $\chi_c$ denotes the Euler characteristic using homology with compact support. The multiplication table of the generators $[G/H]$ is given by the relation

$$[G/H] \cdot [G/K] = \sum_{(L) \in \Phi(G)} n_L[G/L], \tag{3.17}$$

where $n_L = \chi_c((G/H \times G/K)_{(L)}/G)$. In other words, we can say that the number $n_L$, standing in the formula (3.17), represents the number of elements in $(G/H \times G/K)_{(L)}/G$, that is, it is the *number of $G$-orbits in $G/H \times G/K$ of the orbit type $(L)$*.

It is clear that, as an abelian group, $A(G)$ is naturally isomorphic to $AB_0[G] = A_0[G]$, by a transformation which identifies a generator $(H)$ of $A_0[G]$ with $[G/H] \in A(G)$. Consequently, in the case where $n = 0$, the $G$-degree coincides with the equivariant degree associated with the equivariant fixed point index studied by H. Ulrich and others (see [**27**, **30**]). In this case, $G$-degree takes values in $A(G)$, which has an additional multiplicative structure. The following property of $G$-degree corresponds to the well-known *multiplicativity property* of the fixed point index (cf. [**30**, III.1.12, p. 73]).

THEOREM 3.18. *The following property holds.*

(P6) (*Multiplicativity Property*) *Let $V$, $U$ be two orthogonal representations of $G$, and $\Omega \subset V$ and $\mathcal{U} \subset U$ two invariant open bounded subsets. Assume that $f$: $V \to V$ is an $\Omega$-admissible map and $g$: $U \to U$ a $\mathcal{U}$-admissible map. Then the map $F$: $V \times U \to V \times U$, defined by $F(x, y) = (f(x), g(y))$, where $(x, y) \in V \times U$, is $\Omega \times \mathcal{U}$-admissible and we have*

$$G\text{-Deg}(F, \Omega \times \mathcal{U}) = G\text{-Deg}(f, \Omega) \cdot G\text{-Deg}(g, \mathcal{U}),$$

*where the product is taken in $A(G)$.*

*Proof.* The proof of the Multiplicativity Property can be found in [**30**]. However, our construction of the equivariant degree is slightly different from the construction of the fixed point index in [**30**]. Therefore, for the sake of completeness we present a sketch of the proof of the Multiplicativity Property.

It follows from the excision and homotopy properties, as well as the standard argument using the induction over orbit types, that we can assume without loss of generality that $f^{-1}(0) \cap \Omega \subset \Omega_{(H)}$ and $g^{-1}(0) \cap \mathcal{U} \subset \mathcal{U}_{(K)}$, where $(H)$ and $(K)$ are the minimal orbit types in $\Omega$ and $\mathcal{U}$ respectively. Moreover, we can also assume that $f$ is $(H)$-normal and $g$ is $(K)$-normal. By using the same argument as in [**30**] (see I.3.2, Step 1), we can assume that $f^H$: $\Omega_H \to V^H$ and $g^K$: $\mathcal{U}_K \to U^K$ come transversally to zero. Next, by using the excision property, we can assume that both the sets $f^{-1}(0) \cap \Omega_H$ and $g^{-1}(0) \cap \mathcal{U}_K$ consist of single orbits.

Under the above assumptions, $F^{-1}(0) \cap \Omega \times \mathcal{U}$ contains exactly $n_L$ orbits of type $(L) \in A_0[G]$. Let $a := (g_1H, g_2K)$ be an element in $G/H \times G/K$. Then the isotropy group $G_a$ consists of all $h \in G$ such that $g_1^{-1}hg_1 \in H$ and $g_2^{-1}hg_2 \in K$, that is, $G_a = g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$. This implies that the orbit types of $F^{-1}(0) \cap \Omega \times \mathcal{U}$

are minimal in $\Omega \times \mathcal{U}$. Moreover, for every $(L)$ such that there is an orbit of zeros of $F$ of the orbit type $(L)$, the map $F$ is $(L)$-normal near this orbit. It is now clear that the Multiplicativity Property follows from the formula (3.17) and the definition of the $G$-degree.

REMARK 3.19. We will use the Multiplicativity Property later only in the case where $G$ is an abelian group. In this case the map $F$ satisfies the assumptions of the transversality theorem of Hauschild (see [**30**, Theorem I.3.2]), and moreover the formula (3.17) can be simplified to

$$[G/H] \cdot [G/K] = n_{H \cap K}[G/(H \cap K)],$$

where $n_{H \cap K}$ is equal to the number of all $(H \cap K)$-orbits in $G/H \times G/K$.

Suppose that $G$ is an abelian group; then $AB_n[G] = A_n[G]$ and we can define an $A(G)$-module structure on $A_n[G]$ as follows. For every $(H) \in O_n(G)$ and $(K) \in \Phi(G)$, the $G$-space $G/H \times G/K$ has only a finite number of $G$-orbits. Indeed, by assumption, the group $G/(H \cap K)$ has dimension $n$, and it acts freely on the manifold $G/H \times G/K$ of dimension $n$. Thus the space $(G/H \times G/K)/G = (G/H \times G/K)/(G/(H \cap K))$ has dimension 0, that is, is finite. We define the action $A(G) \times A_n[G] \to A_n[G]$ by

$$[G/K] \cdot (H) = n_{H \cap K}(H \cap K),$$

where $n_{H \cap K}$ is equal to the number of $G$-orbits in $G/H \times G/K$.

By using arguments similar to those in the proof of the Multiplicativity Property, we can generalize the Product Formula (P5) as follows.

PROPOSITION 3.20. *The following property holds.*

(P5)′ (*Product Formula*) *Let $G$ be a compact abelian group, $V$ and $U$ be two orthogonal representations of $G$, $\Omega \subset V \oplus \mathbb{R}^n$ and $\mathcal{U} \subset U$ be two open bounded invariant subsets, $g$: $V \oplus \mathbb{R}^n \to V$ be an $\Omega$-admissible map, and $f$: $U \to U$ be a $\mathcal{U}$-admissible map. Then $F$: $U \times V \oplus \mathbb{R}^n \to U \times V$, where $F(x, y) = (f(x), g(y))$ with $(x, y) \in U \times V \oplus \mathbb{R}^n$, is a $\mathcal{U} \times \Omega$-admissible map and we have*

$$G\text{-Deg}(F, \mathcal{U} \times \Omega) = G\text{-Deg}(f, \mathcal{U}) \cdot G\text{-Deg}(g, \Omega).$$

## 4. *Some computational formulas*

### 4.A. *A reduction formula*

Suppose $G_0$ is a normal closed subgroup of the compact Lie group $G$ such that $G/G_0$ is a discrete group. Let $\Omega_0$ be an open bounded $G_0$-invariant subset of $V \times \mathbb{R}^n$ such that

$$g\bar{\Omega}_0 \cap \bar{\Omega}_0 = \varnothing \quad \text{for all } g \in G \backslash G_0.$$

Put $\Omega = G\Omega_0$. Assume that $f$: $V \oplus \mathbb{R}^n \to V$ is an $\Omega$-admissible $G$-equivariant mapping. Clearly, $f$ is also $G_0$-equivariant and $\Omega_0$-admissible. So the equivariant degrees $G$-Deg$(f, \Omega)$ and $G_0$-Deg$(f, \Omega_0)$ are both well defined.

The purpose of this section is to show that the computation of $G$-Deg$(f, \Omega)$

can be reduced to that of $G_0$-Deg$(f, \Omega_0)$. Such a reduction will be important in developing a global bifurcation theory for some non-linear equations in the presence of symmetry in our subsequent papers.

Let $x_0 \in \Omega_0$. Since $gx_0 \notin \Omega_0$ for every $g \in G \backslash G_0$, we have $G_{x_0} \subseteq G_0$. Furthermore, the normality of $G_0$ implies that $gG_{x_0}g^{-1} \subseteq gG_0g^{-1} = G_0$ for every $g \in G$. Therefore, $G_x \subseteq G_0$ for every $x \in \Omega$. This shows, by the property (P1), that only those orbit types $(H)$ with $H \subseteq G_0$ can contribute to the degree $G$-Deg$(f, \Omega)$.

For every $H \subseteq G_0$, we denote by $N_0(H)$ the normalizer of $H$ in $G_0$ and by $W_0(H)$ the Weyl group $N_0(H)/H$. Clearly, $N_0(H) = N(H) \cap G_0$. On the other hand, since $G/G_0$ is a discrete group, we have $\dim G = \dim G_0$. So, $\dim N_0(H) = \dim N(H)$ from which it follows that $\dim W_0(H) = \dim W(H)$. Consequently, we get

$$O_n(G_0) = \{(H)_0\colon\ H \subseteq G_0 \text{ and } (H) \in O_n(G)\}, \tag{4.1}$$

where $(H)_0$ denotes the orbit type with respect to $G_0$ and $(H)$ is the orbit type with respect to $G$. We will denote the fact (4.1) by $O_n(G_0) = O_n(G) \cap O(G_0)$.

Denote by $OA_n(G) \cap O(G_0)$ the set of all $(H)_0 \in O_n(G_0)$ such that $W(H)$ is bi-orientable. For every $(H)_0 \in OA_n(G) \cap O(G_0)$, the inclusion $W_0(H) \subseteq W(H)$ and the equality $\dim W_0(H) = \dim W(H)$ imply that $W_0(H)$ is also bi-orientable. Therefore, we have

$$OA_n(G) \cap O(G_0) \subseteq OA_n(G_0). \tag{4.2}$$

We should remark that the inverse inclusion is not necessarily true in general. For example, if $H = \mathbb{Z}_1$, $G_0 = S^1$, and $G = O(2)$, then $W(H) = O(2)$ is not bi-orientable, but $W_0(H) = S^1$ is bi-orientable.

Now, we are in a position to define a homomorphism $\Upsilon\colon AB_n[G_0] \to AB_n[G]$ by

$$\Upsilon\left(\sum \gamma_{(H)_0} \cdot (H)_0\right) = \sum n_{(H)} \cdot (H),$$

where

$$n_{(H)} = \begin{cases} \gamma_{(H)_0} & \text{if } (H) \in OA_n(G), \\ \gamma_{(H)_0} \pmod 2 & \text{if } (H) \in OB_n(G). \end{cases} \tag{4.3}$$

THEOREM 4.4 (Reduction Formula). *Suppose that $G_0$ is a normal closed subgroup of $G$ such that $G/G_0$ is discrete. Let $\Omega_0$ be an open bounded $G_0$-invariant subset of $V \oplus \mathbb{R}^n$ such that $g\bar{\Omega}_0 \cap \bar{\Omega}_0 = \varnothing$ for all $g \in G \backslash G_0$, and let $\Omega = G\Omega_0$. If $f\colon V \oplus \mathbb{R}^n \to V$ is an $\Omega$-admissible $G$-equivariant mapping, then*

$$G\text{-Deg}(f, \Omega) = \Upsilon(G_0\text{-Deg}(f, \Omega_0)).$$

*Proof.* We first consider the case where $\Omega$ has only one orbit type $\alpha = (H)$. We can assume that $H = G_{x_0}$ for some $x_0 \in \Omega_0$. Since $G_0$ is a normal subgroup of $G$, $W_0(H)$ is a normal subgroup of $W(H)$, and hence the quotient group $K := W(H)/W_0(H)$ is finite and $(\Omega_H/W_0(H))/K = \Omega_H/W(H)$. On the other hand, as $g\bar{\Omega}_0 \cap \bar{\Omega}_0 = \varnothing$ for all $g \in G \backslash G_0$, we have

$$k(\bar{\Omega}_0)_H/W_0(H) \cap (\bar{\Omega}_0)_H/W_0(H) = \varnothing$$

for every non-trivial element $k$ of $K$. So, we can identify $\Omega_H/W(H)$ with

$(\Omega_0)_H/W_0(H)$. Suppose that the $(H)$-component of $G$-Deg$(f, \Omega)$ and the $(H)_0$-component of $G$-Deg$(f, \Omega_0)$ are given by sections

$$s_{f,H}\colon\ V_H \times \mathbb{R}^n/W(H) \to V_H \times \mathbb{R}^n \times V^H/W(H)$$

and

$$s^0_{f,H}\colon\ V_H \times \mathbb{R}^n/W_0(H) \to V_H \times \mathbb{R}^n \times V^H/W_0(H),$$

respectively. It can easily be shown that with the identification between $\Omega_H/W(H)$ and $(\Omega_0)_H/W_0(H)$,

$$s_{f,H}\big|_{\Omega_H/W(H)} = s^0_{f,H}\big|_{(\Omega_0)_H/W_0(H)}.$$

So, the reduction formula follows.

In the case where $\Omega$ has several orbit types, the reduction formula can easily be verified by induction over orbit types.

4.B. *Regular value formula*

Throughout this subsection, we assume that $G$ is an abelian compact Lie group, $V$ is a finite-dimensional orthogonal representation of $G$, $\Omega \subseteq V \oplus \mathbb{R}^n$ is an open bounded $G$-invariant subset, and $f\colon V \oplus \mathbb{R}^n \to V$ is a $G$-equivariant $\Omega$-admissible $C^1$-mapping such that

(a) $0$ is a regular value of $f|_\Omega$;

(b) $f^{-1}(0) \cap \Omega = Gx_0$ for some $x_0 \in \Omega$;

(c) $\dim G/G_{x_0} = n$.

Let $H_0 = G_{x_0}$. Denote by $S$ the linear orthogonal slice of the orbit $Gx_0$ at $x_0$, that is, $S = \{x \in V \oplus \mathbb{R}^n\colon x - x_0 \perp T_{x_0}(Gx_0)\}$. Then $S$ is an orthogonal representation of $H_0$.

Suppose that for every $(H) \in O_n(G)$ there has been chosen an invariant orientation of $G/H$. Since for two subgroups $H_1$ and $H_2$ of $G$ such that $(H_1)$, $(H_2) \in O_n(G)$ and $H_1 \subset H_2$, the natural homomorphism $\varphi\colon G/H_1 \to G/H_2$ is a local diffeomorphism, we can assume without loss of generality that the chosen orientations in $\mathcal{J}(\bar{\Omega}) \cap O_n(G)$ are such that they are preserved by $\varphi$.

The purpose of this subsection is to provide an explicit formula for $G$-Deg$(f, \Omega)$. Let $(H) \in O_n(G)$; then there exists a natural homomorphism of abelian groups

$$\Im\colon\ A(H) \to A_n[G]$$

which can be defined on the generators $[H/K]$ as follows:

$$\Im([H/K]) = (K),$$

where $[H/K] \in \Phi(H)$. Since $H/K$ is a finite group, $G/K$ has dimension $n$.

Let $A := Df(x_0)|_S\colon S \to V$. By assumption (a), $A$ is an $H_0$-equivariant isomorphism. The spaces $S$ and $V$ can be decomposed into the following $H_0$-invariant orthogonal direct sums:

$$S = S^{H_0} \oplus T \quad \text{and} \quad V = V^{H_0} \oplus T'.$$

As $A$ is an $H_0$-equivariant isomorphism, $A(S^{H_0}) = V^{H_0}$ and $A(T) = T'$. Moreover, since $\{0\} \times \mathbb{R}^n \subset S^{H_0}$, this implies that $T \subset V \times \{0\}$ and therefore $T$ and $T'$ are the same subspace of $V$, under the identification of $V$ with $V \times \{0\}$, which we will denote simply by $T$.

Let $T_\alpha$ denote the direct sum of all irreducible subrepresentations of $T$ which are equivalent to a fixed irreducible representation of $H_0$ which has $K_\alpha$ as the isotropy group of its non-zero elements. The subspace $T_\alpha$ is called an *isotypical component* of $T$. It is clear that $A(T_\alpha) = T_\alpha$. We can write

$$T = \bigoplus_\alpha T_\alpha. \tag{3}$$

This is called the *isotypical decomposition* of $T$.

We fix an orientation of $V^{H_0}$. The orientation of $G/H_0$ followed by the orientation of $V^{H_0}$ determine the unique orientation of $S^{H_0}$, and we can define the integer $\eta_0$ by

$$\eta_0 := \begin{cases} 1 & \text{if } Df(x_0)|S^{H_0} \text{ preserves the orientations,} \\ -1 & \text{otherwise.} \end{cases}$$

DEFINITION 4.5. Let $T_\alpha$ be an isotypical component of $T$. We define the *local $\alpha$-index of $f$ on the orbit $Gx_0$* as the following element of $A(H_0)$:

$$\alpha\text{-index}(f, Gx_0) = H_0\text{-Deg}(A|_{T_\alpha}, B \cap T_\alpha),$$

where $B$ denotes the unit ball in $T$.

Now we can formulate the main result of this section, the *regular value formula for an isolated orbit of zeros*:

THEOREM 4.6. *Under the above assumptions we have*

$$G\text{-Deg}(f, \Omega) = \eta_0 \Im\Big(\prod_\alpha \alpha\text{-index}(f, Gx_0)\Big).$$

*Proof.* Let $D = \{x \in S\colon \|x\| < \delta\}$. It follows from the excision property, that for a sufficiently small $\delta > 0$, we may assume without loss of generality, that $\Omega \simeq_G G \times_{H_0} D := (G \times D)/H_0$, where $H_0$ acts on $G \times D$ by $h(g, v) = (gh^{-1}, hv)$, for $g \in G$, $v \in D$, $h \in H_0$. Such a set $\Omega$ is called a *tube* around the orbit $Gx_0$. Define $\tilde{f}\colon \bar{D} \to V$ by $\tilde{f}(x) := f(x + x_0)$. Clearly, $\tilde{f}$ is an $H_0$-equivariant $D$-admissible map. As $S$ and $V$ contain the same non-trivial $H_0$-isotypical components (under the identification of $V$ with $V \times \{0\}$), we can identify the $H_0$-representation $S$ with the $H_0$-representation $V$. Therefore, the equivariant degree $H_0\text{-Deg}(\tilde{f}, D) \in A(H_0)$ is well defined. It follows directly from the definition of the $G$-equivariant degree and the assumption on the orientations of $G/K_\alpha$, $(K_\alpha) \in O_n(G) \cap \mathscr{I}(\bar{\Omega})$, that $G\text{-Deg}(f, \Omega) = \Im(H_0\text{-Deg}(\tilde{f}, D))$.

Since 0 is a regular value of $\tilde{f}$ such that $\tilde{f}^{-1}(0) = \{0\}$, by taking $\delta$ sufficiently small, we may assume that $\tilde{f}$ is $D$-homotopic to $A$. Therefore, $H_0\text{-Deg}(A, D) = G\text{-Deg}(f, \Omega)$. Since $A|_T = \bigoplus A_\alpha\colon \bigoplus T_\alpha \to \bigoplus T_\alpha$, where $A_\alpha = A|_{T_\alpha}\colon T_\alpha \to T_\alpha$, it follows from the Multiplicativity Property (P6) that

$$\begin{aligned} H_0\text{-Deg}(A, D) &= \eta_0 \prod H_0\text{-Deg}(A_\alpha, B \cap T_\alpha) \\ &= \eta_0 \prod \alpha\text{-index}(f, Gx_0). \end{aligned}$$

This completes the proof.

We put

$$\mathscr{F}_{2,H_0} := \{\alpha\colon H_0/K_\alpha \simeq \mathbb{Z}_2\},$$

and we define the $H_0$-equivariant automorphism $A_\alpha$: $T_\alpha \to T_\alpha = \tilde{T}_\alpha$ by $A_\alpha := A|_{T_\alpha}$. We also define the following element of the Burnside Ring $A(H_0)$:

$$\eta_\alpha(A) := \begin{cases} 0 & \text{if } \det A_\alpha > 0, \\ -[H_0/K_\alpha] & \text{if } \det A_\alpha < 0. \end{cases}$$

Then we have the following:

PROPOSITION 4.7. *Under the above assumptions we have*

$$\alpha\text{-index}(f, Gx_0) = \begin{cases} \eta_\alpha(A) & \text{if } \alpha \in \mathcal{F}_{2,H_0}, \\ 1 & \text{otherwise,} \end{cases}$$

*where* 1 *in* $A(H_0)$ *denotes the element* $[H_0/H_0]$.

*Proof.* Suppose that $T_\alpha$ is such that $\alpha \notin \mathcal{F}_{2,H_2}$. Then $T_\alpha$ can be naturally equipped with a complex structure such that an automorphism of $T_\alpha$ is $H_0$-equivariant if and only if it is a complex automorphism. Therefore, by connectedness of the groups $GL(n, \mathbb{C})$, $A_\alpha$ can be connected by a continuous path in the space of $H_0$-equivariant linear automorphisms of $T_\alpha$ to Id. If we use this path as a $B \cap T_\alpha$-admissible homotopy between $A_\alpha$ and Id, we obtain $\alpha$-index$(f, Gx_0) = 1$.

Assume therefore that $\alpha \in \mathcal{F}_{2,H_0}$. Since $H_0/K_\alpha \simeq \mathbb{Z}_2$, every $\mathbb{R}$-linear automorphism of $T_\alpha$ is $H_0$-equivariant. The linear group $GL(T_\alpha)$ has two connected components. If $\det A_\alpha > 0$ then $A_\alpha$ can be connected by a continuous path in $GL(T_\alpha)$ to Id, and consequently, by the same argument as in the previous case, $\alpha$-index$(f, Gx_0) = 1$. Assume therefore that $\det A_\alpha < 0$. Then $A_\alpha$ can be connected by a path in $GL(T_\alpha)$ to an operator $\tilde{A}_\alpha$ which has the following representation with respect to a certain basis in $T_\alpha$:

$$\tilde{A}_\alpha(t_1, t_2, \ldots, t_m) = (-t_1, t_2, \ldots, t_m) \in T_\alpha.$$

We define a (non-linear) map $B_\alpha$: $T_\alpha \to T_\alpha$ by

$$B_\alpha(t_1, t_2, \ldots, t_m) = (-t_1(t_1 - \tfrac{1}{2}\delta)(t_1 + \tfrac{1}{2}\delta), t_2, \ldots, t_m).$$

Clearly, $B_\alpha$ is $(D \cap T_\alpha)$-homotopic to $A_\alpha$, and the equation $B_\alpha(t) = 0$ has the following types of solutions:

(i) the zero solution $t = 0$ of the orbit type $(H_0)$;

(ii) one orbit of solutions $t = (\pm\tfrac{1}{2}\delta, 0, \ldots, 0)$ of the orbit type $(K_\alpha)$ such that $\det DB_\alpha(t) < 0$.

Consequently, we obtain that

$$\alpha\text{-index}(f, Gx_0) = [H_0/H_0] - [H_0/K_\alpha],$$

and the proof is complete.

COROLLARY 4.8. *Under the above assumptions we have*

$$G\text{-Deg}(f, \Omega) = \eta_0 \Im\Big(\prod_{\alpha \in \mathcal{F}_{2,H_0}} \eta_\alpha(A)\Big).$$

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