JOURNAL OF DIFFERENTIAL EQUATIONS **107**, 395–417 (1994)

An Invariance Principle of Lyapunov–Razumikhin Type for Neutral Functional Differential Equations*

J. R. HADDOCK

Department of Mathematical Sciences, Memphis State University, Memphis, Tennessee 38152

T. KRISZTIN AND J. TERJÉKI

Bolyai Institute, Aradi Vértanúk tere 1, 6720, Szeged, Hungary

AND

J. H. WU

Department of Mathematics and Statistics, York University, North York, Ontario M3J 1P3, Canada

Received March 19, 1991; revised February 19, 1992

1. INTRODUCTION

During the past several years, invariance principles of Lyapunov–Razumikhin type have been provided for *retarded* functional differential equations (RFDES). The initial results and applications were given in [8] for equations with finite delay with extensions to infinite delay equations being supplied in [5, 9]. The purpose of this paper is to develop invariance principles along with applications to include *neutral* functional differential equations (NFDEs)

$$\frac{d}{dt}(Dx_t) = f(x_t), \tag{1.1}$$

where $r > 0$ is a given constant, $f\colon C = C([-r, 0], R^n) \to R^n$ is completely continuous, $D\colon C \to R^n$ is linear, continuous and atomic at zero (cf. [12, p. 50]). For any continuous mapping $x\colon [-r, \infty) \to R^n$ and $t \geqslant 0$, $x_t \in C$ is defined by

$$x_t(s) = x(t+s), \qquad -r \leqslant s \leqslant 0.$$

Clearly, (1.1) reduces to an RFDE if $D\varphi = \varphi(0)$.

* This work was supported in part by the National Science Foundation under Grant DMS 9002431, and the Natural Sciences and Engineering Research Council of Canada.

0022-0396/94 $6.00

To make the transition from finite to infinite delay RFDEs required new results involving (i) comparison theorems using differential inequalities, (ii) conditions for precompactness of positive orbits, and (iii) constructions of various phase spaces. (See [5, 9] for references and details.) Additional complications, which have stalled extensions of Lyapunov–Razumikhin invariance principles to NFDEs, are due to the natural relationship between (1.1) and its generalized difference equation

$$\begin{aligned} Dy_t &= h(t) \qquad (t \geqslant 0) \\ y_0 &= \varphi, \end{aligned} \tag{1.2}$$

where $h\colon [0, \infty) \to R^n$ is continuous, $\varphi \in C$.

In particular, since $D\varphi$ plays a fundamental role in results for NFDEs much as $\varphi(0)$ does for the RFDE case, Lyapunov–Razumikhin conditions have to be expressed in terms inequalities which compare between the values of $D\varphi$ and $\varphi(s)$ for $-r \leqslant s \leqslant 0$.

The invariance principles in [5, 8, and 9] have been applied to obtain information about asymptotic behavior of solutions of various RFDEs, and these applications have inspired the present work. More precisely, we develop invariance principles of Lyapunov–Razumikhin type which provides an effective tool for investigating asymptotic stability and asymptotic constanct (convergence to constants at $t \to \infty$) of solutions of NFDEs. Prototypes of equations to be studied for asymptotic constancy are

$$\frac{d}{dt}\left(x(t) - \sum_{i=1}^{n} c_i x(t - r_i)\right) = H(x(t)) + H\left(\sum_{i=1}^{m} b_i x(t - r_i)\right) \tag{1.3}$$

and

$$\frac{d}{dt}\left(x(t) - \int_0^r g(s)\, x(t-s)\, ds\right) = F\left(-ax(t) + \int_0^r k(s)\, x(t-s)\, ds\right) \tag{1.4}$$

under the assumption that each constant function is a solution.

Although an invariance principle for NFDEs is available [12, p. 297], it often is difficult to apply since it requires the construction of a functional $V\colon C \to R$ whose derivative is negative semi-definite along solutions of a given equation. Likewise, Lyapunov–Razumikhin asymptotic stability theorems, based on an important inequality of Cruz and Hale [2], already exist (see, cf. [11, 19]) but applications of these results generally require the equation at hand to have an ordinary part which "dominates" a functional (delay) part. Our results and techniques are designed to avoid this restriction. In order to accomplish this, we have been forced to generalize the Cruz–Hale inequality in a nontrivial way. In particular, we obtain a new estimate in terms of h and φ for the solution y of (1.2) (see Lemma 2.3).

The notation, preliminary results, and the above mentioned estimate with its consequences are contained in Section 2. An invariance principle is developed in Section 3, and then applied to obtain new results for asymptotic stability (Section 4) and asymptotic constancy of solutions (Section 5).

Finally, an interesting phenomenon should be emphasized. Namely, some of our results (Lemmas 3.3 and 4.1 and Theorems 4.1 and 5.1) cannot be applied to retarded equations. The fact $D\varphi \not\equiv \varphi(0)$ plays a crucial role in the proof of these results.

2. Preliminaries

Let $r>0$ be given and let $C=C([-r,0], R^n)$. For $\varphi \in C$ the norm of φ is defined by

$$\|\varphi\| = \max_{-r \leqslant s \leqslant 0} |\varphi(s)|,$$

where $|\cdot|$ denotes the Euclidean norm in R^n. For $x, y \in R^n$, $\langle x, y\rangle$ is the inner product in R^n. Suppose $x\colon [-r, \infty) \to R^n$ is continuous. Then, for any $t \geqslant 0$, $x_t \in C$ is defined by $x_t(s)=x(t+s)$, $-r\leqslant s\leqslant 0$.

Consider the NFDE

$$\frac{d}{dt}(Dx_t)=f(x_t) \qquad (t\geqslant 0) \tag{2.1}$$

with the initial condition

$$x_0=\varphi, \tag{2.2}$$

where $f\colon C\to R^n$ is completely continuous and $D\colon C\to R^n$ is linear, continuous, and atomic at zero in the sense of [12, p. 50]. Under these assumptions the solution $x(\varphi)(\cdot)$ of the initial value problem (2.1), (2.2) exists. Further, we assume the uniqueness, continuous dependence, and continuation of solutions (for details, see [12]).

By an $n\times n$ matrix measure on $[0,\infty)$ we mean an $n\times n$ matrix-valued function η whose entries are of bounded variation on $[0,\infty)$. The total variation measure $|\eta|$ of η is deduced from the $n\times n$ matrix norm.

For a given $\alpha\in R$ we introduce the weighted space of continuous functions C_α by

$$C_\alpha=\{\varphi\in C((-\infty,0], R^n)\colon \lim_{t\to-\infty} e^{\alpha t}\varphi(t)=0\}$$

with norm

$$\|\varphi\|_\alpha = \sup_{t \leqslant 0} e^{\alpha t} |\varphi(t)|.$$

For $\alpha \in R$ let M_α be the set of the $n \times n$ matrix measures η satisfying

$$\|\eta\|_\alpha = \int_0^\infty e^{\alpha t} \, d|\eta| \, (t) < \infty.$$

For $\eta \in M_\alpha$ the convolution operator $v*: C_\alpha \to C_\alpha$ is defined by

$$v * \varphi(t) = \int_0^\infty [d\eta(s)] \, \varphi(t-s) \qquad (t \leqslant 0).$$

Since the operator D in (2.1) is atomic at zero, without loss of generality we may assume that

$$D\varphi = \varphi(0) - \int_0^\infty [dv(s)] \, \varphi(-s),$$

where v is an $n \times n$ matrix measure with $v(0) = 0$ and $v(s) = v(r)$ for $s \geqslant r$, moreover $\int_0^s d|v| \to 0$ as $s \to 0+$ (see [12, p. 280]).

Throughout this paper we assume that the operator D in (2.1) is stable. That is, the zero solution of the homogeneous "generalized difference" equation

$$Dy_t = 0 \qquad (t \geqslant 0), \tag{2.3}$$

$$y_0 = \varphi, \tag{2.4}$$

is uniformly asymptotically stable, where $\varphi \in C_D = \{\varphi \in C : D\varphi = 0\}$.

It is shown in [12] that (2.3), (2.4) generates a strongly continuous semigroup of linear transformations $T_D(t): C_D \to C_D$, by $T_D(t)\varphi = y_t(\varphi)$, $t \geqslant 0$. Moreover, the following lemma is valid.

LEMMA 2.1 (Hale [12, p. 287]). *The following statements are equivalent:*

(i) *D is stable.*

(ii) *$a_D < 0$, where a_D is the order of the semigroup $T_D(t)$, defined by*

$$a_D = \inf\{a \in R: \text{there is a } K = K(a) \text{ such that } \|T_D(t)\| \leqslant Ke^{at}, \, t \geqslant 0\}.$$

(iii) *There are constants $a > 0$, $b > 0$ such that for any $\varphi \in C$ any $h \in C([0, \infty), R^n)$ the solution y of the nonhomogeneous equation*

$$Dy_t = h(t) \qquad (t \geqslant 0), \tag{2.5}$$

$$y_0 = \varphi, \tag{2.6}$$

satisfies

$$\|y_t\| \leq be^{-at}\|\varphi\| + b \sup_{0 \leq s \leq t} |h(s)| \qquad (t \geq 0). \tag{2.7}$$

Let us define the measure δ by

$$\delta(t) = \begin{cases} 0 & \text{if } t = 0 \\ I & \text{if } t > 0, \end{cases}$$

where I is the $n \times n$ identity matrix. Let $\mu = \delta - \nu$. Obviously, $\mu \in M_\alpha$ for any $\alpha \in R$.

Thus, if $\varphi \in C$ and φ is extended to $(-\infty, 0]$ such that the extended function is in C_α for some $\alpha \in R$, then

$$D\varphi = \mu * \varphi(0).$$

Some important properties of μ as an operator in C_α are formulated as follows.

LEMMA 2.2 (Staffans [21]). *If $\alpha < -a_D$, then the operator $\mu*$ maps C_α into itself continuously, is invertible, and its inverse operator $\mu^{-1}*$ is continuous with $\mu^{-1} * \mu = \mu * \mu^{-1} = \delta$ and $\mu, \mu^{-1} \in M_\alpha$. Moreover,*

$$-a_D = \sup\left\{\alpha \in R: \int_0^\infty e^{\alpha t}\, d|\mu^{-1}|(t) < \infty\right\}.$$

Applying Lemma 2.2, the estimate (2.7) for the solution of (2.5), (2.6) can be improved significantly as follows.

LEMMA 2.3. *If $\alpha \in [0, -a_D)$, then for any $\varphi \in C$ and any $h \in C([0, \infty), R^n)$ the solution y of (2.5), (2.6) satisfies*

$$\begin{aligned} |y(t)| &\leq \|\mu\|_\alpha \|\varphi\| e^{-\alpha t} \int_t^\infty e^{\alpha s}\, d|\mu^{-1}|(s) + \int_0^t |h(t-s)|\, d|\mu^{-1}|(s) \\ &\leq \|\mu\|_\alpha \|\varphi\| e^{-\alpha t} \int_t^\infty e^{\alpha s}\, d|\mu^{-1}|(s) \\ &\quad + \int_0^t e^{\alpha s}\, d|\mu^{-1}|(s) \max_{0 \leq u \leq t} e^{-\alpha(t-u)} |h(u)| \qquad (t \geq 0). \end{aligned}$$

Proof. Let $\alpha \in [0, -a_D)$ be given. Choose $v \in C(R, R^n)$ such that $\lim_{t \to -\infty} e^{\alpha t} v(t) = 0$, $v(t) = \varphi(t)$ for $-r \leq t \leq 0$ and $e^{\alpha t}|v(t)| \leq \|\varphi\|$ for all $t \in R$.

Let $y(t) = v(t)$ for $t < -r$ and define $z(t)$ by $z(t) = y(t) - v(t)$ for all $t \in R$. Then $z(t) = 0$ for $t \leq 0$ and $Dy_t = \mu * y_t(0) = \mu * z_t(0) + \mu * v_t(0) = h(t)$ $(t \geq 0)$.

Let

$$g(t)=\begin{cases}0 & \text{if } t\leqslant 0\\ h(t)-\mu * v_t(0) & \text{if } t\geqslant 0.\end{cases}$$

From $D\varphi=\mu * v_0(0)=h(0)$ it follows that g is continuous on R and $g_t\in C_\alpha$ for all $t\in R$, where $g_t(s)=g(t+s)$, $s\leqslant 0$. Since

$$\mu * z_t(0)=g(t) \qquad (-\infty<t<\infty),$$

Lemma 2.2 implies that

$$z(t)=\mu^{-1} * g_t(0) \qquad (-\infty<t<\infty).$$

Hence,

$$y(t)=v(t)+\mu^{-1} * g_t(0) \qquad (-\infty<t<\infty).$$

Using the estimate

$$\sup_{s\in R} e^{\alpha s}\,|\mu * v_s(0)|\leqslant \|\mu\|_\alpha \sup_{s\in R} e^{\alpha s}\,|v(s)|\leqslant \|\mu\|_\alpha\,\|\varphi\|,$$

we readily obtain that

$$\begin{aligned}
|y(t)| &= \left|v(t)+\int_0^\infty [d\mu^{-1}(s)]\,g(t-s)\right|\\
&= \left|v(t)+\int_0^t [d\mu^{-1}(s)][h(t-s)-\mu * v_{(t-s)}(0)]\right|\\
&\leqslant \left|v(t)-\int_0^\infty [d\mu^{-1}(s)]\,\mu * v_{(t-s)}(0)+\int_t^\infty [d\mu^{-1}(s)]\,\mu * v_{(t-s)}(0)\right|\\
&\quad +\left|\int_0^t [d\mu^{-1}(s)]\,h(t-s)\right|\\
&\leqslant e^{-\alpha t}\int_t^\infty e^{\alpha s} d\,|\mu^{-1}|\,(s)\sup_{u\geqslant t} e^{\alpha(t-u)}\,|\mu * v_{(t-u)}(0)|\\
&\quad +\int_0^t |h(t-s)|\,d\,|\mu^{-1}|\,(s)\\
&\leqslant \|\mu\|_\alpha\,\|\varphi\|\,e^{-\alpha t}\int_t^\infty e^{\alpha s}\,d\,|\mu^{-1}|\,(s)+\int_0^t |h(t-s)|\,d\,|\mu^{-1}|\,(s)\\
&\leqslant \|\mu\|_\alpha\,\|\varphi\|\,e^{-\alpha t}\int_t^\infty e^{\alpha s}\,d\,|\mu^{-1}|\,(s)\\
&\quad +\int_0^t e^{\alpha s}\,d\,|\mu^{-1}|\,(s)\max_{0\leqslant u\leqslant t} e^{-\alpha(t-u)}\,|h(u)| \qquad (t\geqslant 0),
\end{aligned}$$

which completes the proof.

Remark 2.1. (i) If, for some $\alpha \geqslant 0$, $\int_0^r e^{\alpha t}\, d|\nu|(t) < 1$ holds, then $\int_0^\infty e^{\alpha t}\, d|\mu|(t) < \infty$ also holds, and by Lemma 2.2, $-a_D \geqslant \alpha$. In this case $\|\mu\|_\alpha$ and $\|\mu^{-1}\|_\alpha$ can be easily evaluated. Obviously,

$$\|\mu\|_\alpha \leqslant 1 + \int_0^t e^{\alpha t}\, d|\nu|(t).$$

$\mu^{-1} = (\delta - \nu)^{-1}$ can be expressed by the convergent series $\delta + \nu + \nu * \nu + \nu * \nu * \nu + \cdots$ from which it follows that

$$\|\mu^{-1}\|_\alpha = \int_0^\infty e^{\alpha t}\, d|\mu^{-1}|(t) \leqslant 1 + \sum_{k=1}^\infty \int_0^\infty e^{\alpha t}\, d|\underbrace{\nu * \nu * \cdots * \nu}_{k\text{-times}}|(t)$$
$$\leqslant 1 + \sum_{k=1}^\infty \left(\int_0^r e^{\alpha t}\, d|\nu|(t)\right)^k = \left[1 - \int_0^r e^{\alpha t}\, d|\nu|(t)\right]^{-1}.$$

(ii) It is easy to see that if $\mu \neq \delta$; i.e., $D\varphi \not\equiv \varphi(0)$, then $\|\mu\|_\alpha$ and $\|\mu^{-1}\|_\alpha$ are strictly increasing in $\alpha \in [0, -a_D)$.

(iii) It is obvious that $\lim_{\alpha \to \infty} \|\mu\|_\alpha = 1$ and $1/\|\mu\|_\alpha \leqslant \|\mu^{-1}\|_\alpha$. Therefore, the monotonicity of $\|\mu^{-1}\|_\alpha$ implies that $\|\mu^{-1}\|_0 = 1$.

Lemmas 2.4, 2.5, and 2.6, which are used frequently in the next sections, contain special consequences of Lemmas 2.2 and 2.3.

LEMMA 2.4. *If $y \in C([-r, \infty), R^n)$, $b \in R$, $t_0 \in [0, \infty)$, $\|\mu\|_0 \|y_0\| \leqslant b$ and $\max_{0 \leqslant s \leqslant t_0} |Dy_s| \leqslant b$, then $\|y_{t_0}\| \leqslant b \|\mu^{-1}\|_0$.*

Proof. Lemma 2.3 with $\alpha = 0$, $h(t) = Dy_t$, $\varphi = y_0$ implies that

$$|y(t)| \leqslant \|\mu\|_0 \|y_0\| \int_t^\infty d|\mu^{-1}|(s) + b \int_0^t d|\mu^{-1}|(s) \leqslant b \|\mu^{-1}\|_0$$

for all $0 \leqslant t \leqslant t_0$. If $-r \leqslant t \leqslant 0$, then $|y(t)| \leqslant \|y_0\| \leqslant \|\mu\|_0 \|y_0\| \leqslant b \leqslant b \|\mu^{-1}\|_0$, since $\|\mu\|_0 \geqslant 1$ and $\|\mu^{-1}\|_0 \geqslant 1$. Thus, $\|y_{t_0}\| \leqslant b \|\mu^{-1}\|_0$.

LEMMA 2.5. *If $y \in C(R, R^n)$, $\beta \in [0, -a_D)$, $b \in R$, $t_0 \in R$ and $\sup_{s \leqslant t_0} e^{-\beta(t_0 - s)} |Dy_s| \leqslant b$, then*

$$|y(t_0 - s)| \leqslant \int_0^\infty |Dy_{t_0 - s - u}|\, d|\mu^{-1}|(u) \qquad (s \geqslant 0)$$

and

$$\|y_{t_0}\| \leqslant b e^{\beta r} \|\mu^{-1}\|_\beta.$$

Proof. Since the function $z(s)=Dy_{t_0+s}$, $s\leqslant 0$, is in C_α for $\alpha\in(\beta, -a_D)$, by Lemma 2.2, the function $\mu^{-1}*z_s(0)$, $s\leqslant 0$, is well defined. From our assumption, $|z(s)|\leqslant be^{-\beta s}$, $s\leqslant 0$. So, it follows that $|\mu^{-1}*z_{(-s)}(0)|\leqslant \int_0^\infty |z(-s-u)|\,d|\mu^{-1}|(u)=\int_0^\infty |Dy_{t_0-s-u}|\,d|\mu^{-1}|(u)$, $s\geqslant 0$. In addition, $|\mu^{-1}*z_\theta(0)|\leqslant e^{-\beta\theta}b\int_0^\infty e^{\beta u}\,d|\mu^{-1}|(u)\leqslant e^{\beta r}\|\mu^{-1}\|_\beta b$, $-r\leqslant\theta\leqslant 0$. From $z(\theta)=Dy_{t_0+\theta}=\mu * y_{t_0+\theta}(0)$ and $\mu^{-1}*\mu=\delta$ it follows that $\mu^{-1}*z_\theta(0)=y(t_0+\theta)$. Thus the proof is complete.

LEMMA 2.6. *If $y\in C(R, R^n)$ and y is a bounded solution of* (2.1) *on R, then $y(t)$ and Dy_t are uniformly continuous on R.*

Proof. From $Dy_t=\mu * y_t(0)$ it follows that $|Dy_{t_2}-Dy_{t_1}|\leqslant \|\mu\|_0\,\|y_{t_2}-y_{t_1}\|$. So, the uniform continuity of $y(t)$ implies that of Dy_t. There is $K\geqslant 0$ such that $\|y_t\|\leqslant K$ for all $t\in R$. From Eq. (2.1)

$$Dy_{t_2-u}-Dy_{t_1-u}=\int_{t_1-u}^{t_2-u} f(y_s)\,ds,$$

and by $\|y_t\|\leqslant K$, $|Dy_t|\leqslant\|\mu\|_0 K$ follows. Thus Lemma 2.5 can be applied with any $t_0\in R$ to get

$$\begin{aligned}|y(t_2)-y(t_1)|&\leqslant\int_0^\infty |Dy_{t_2-u}-Dy_{t_1-u})|\,d|\mu^{-1}|(u)\\&\leqslant\int_0^\infty\int_{t_1-u}^{t_2-u}|f(y_s)|\,ds\,d|\mu^{-1}|(u)\\&\leqslant\|\mu^{-1}\|_0\,L\,|t_2-t_1|,\end{aligned}$$

where $L=\sup\{|f(\varphi)|:\varphi\in C,\ \|\varphi\|\leqslant K\}$. This completes the proof.

Let $\varphi\in C$ and suppose $x(\varphi)(\cdot)$ is defined on $[-r,\infty)$. The ω-limit set $\Omega(\varphi)$ of $x(\varphi)$ is defined by $\Omega(\varphi)=\{\psi\in C$: there is a sequence $\{t_n\}$ such that $t_n\geqslant 0$, $t_n\to\infty$ and $\|x_{t_n}(\varphi)-\psi\|\to 0$ as $n\to\infty\}$.

A set $M\subseteq C$ is said to be invariant with respect to (2.1) if for any $\varphi\in M$ there is a function $y\colon(-\infty,\infty)\to R^n$ such that $y_0=\varphi$, $y_t\in M$ for all $t\in R$ and y is a solution of (2.1) on R. For the ω-limit set of bounded orbits of (2.1) we have the following

LEMMA 2.7 (Hale, [12, p. 293]). *If $\{x_t(\varphi): t\geqslant 0\}$ is bounded, then $\Omega(\varphi)$ is nonempty, compact, connected, invariant and $x_t(\varphi)\to\Omega(\varphi)$ as $t\to\infty$.*

Our main purpose is to locate $\Omega(\varphi)$ by using a Lyapunov function, in the spirit of [18] for ordinary differential equations and [8] for RFDEs. For NFDEs it is natural to use Lyapunov functions of the form $V(D\varphi)$, where $V\colon R^n\to R$ is continuous. In order not to hide the main ideas behind technicalities, we consider only the special case $V(x) = \langle x, x \rangle$. The derivative of $V(D\varphi) = \langle D\varphi, D\varphi \rangle$ with respect to system (2.1) is defined by

$$\dot{V}_{(2.1)}(D\varphi) = 2 \langle D\varphi, f(\varphi) \rangle.$$

Obviously, if x is a solution of (2.1) on $[0, a)$, $a > 0$, then for any $t \in [0, a)$

$$\frac{d}{dt} V(Dx_t) = \dot{V}_{(2.1)}(Dx_t).$$

A set $M \subseteq C$ is said to be stable if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $\text{dist}(\varphi, M) < \delta$ implies $\text{dist}(x_t(\varphi), M) < \varepsilon$ for all $t \geqslant 0$. M is asymptotically stable if M is stable and there exists $\delta_0 > 0$ such that $\text{dist}(\varphi, M) < \delta_0$ implies $\text{dist}(x_t(\varphi), M) \to 0$ as $t \to \infty$.

For a nonnegative c, define $K(c) = \{\varphi \in C : |D\varphi| = c\}$. Let $M(c)$ be the largest subset of $K(c)$ which is invariant with respect to (2.1), that is, $M(c) = \{\varphi \in C$: there is $y \in C(R, R^n)$ so that y is a solution of (2.1) on R, $y_0 = \varphi$, $y_t \in K(c)$ for all $t \in R\}$. Let $M = \bigcup_{c \geqslant 0} M(c)$.

For $\alpha \in [0, -a_D)$ consider the conditions

$$\|\varphi\| \leqslant \|\mu^{-1}\|_\alpha \, |D\varphi| \text{ implies } \langle D\varphi, f(\varphi) \rangle \leqslant 0, \tag{A$_\alpha$}$$

$$\|\varphi\| \leqslant \|\mu^{-1}\|_\alpha \, |D\varphi| \text{ implies } \langle D\varphi, f(\varphi) \rangle < 0. \tag{B$_\alpha$}$$

For a nonempty set $H \subseteq (0, r)$ we also introduce the assumption

$$\begin{aligned} &\text{If } \|\varphi\| \leqslant \|\mu^{-1}\|_0 \, |D\varphi| \text{ and there exist } s \in H \text{ such that} \\ &|\varphi(-s)| < \|\mu^{-1}\|_0 \, |D\varphi|, \text{ then } \langle D\varphi, f(\varphi) \rangle < 0. \end{aligned} \tag{C$_H$}$$

The condition

$$\begin{aligned} 0 < |\varphi(-s)| = \|\varphi\| \leqslant \|\mu^{-1}\|_0 \, |D\varphi| \\ \text{for all } \quad s \in [0, r) \text{ implies } \langle D\varphi, f(\varphi) \rangle > 0 \end{aligned} \tag{E}$$

is also used in the next sections.

Note that in the case $\alpha = 0$, $D\varphi \equiv \varphi(0)$, condition (A$_\alpha$) is the usual Razumikhin type one used, e.g., in [3, 8, 14, 16, 20]:

$$|\varphi(0)| = \|\varphi\| \text{ implies } \langle \varphi(0), f(\varphi) \rangle \leqslant 0.$$

3. An Invariance Principle Via Lyapunov Functions

The following lemma provides the cadre for our additional results. The idea is simple and certainly not new in the literature (see, cf. [7]), but, for the sake of completeness, we prove it.

LEMMA 3.1. *If* $\{x_t(\varphi): t \geqslant 0\}$ *is bounded and there exists a stable subset* $S \subseteq C$ *such that* $\Omega(\varphi) \cap S$ *is nonempty, then* $x_t(\varphi) \to S$ *as* $t \to \infty$.

Proof. It suffices to prove that for any $\varepsilon > 0$ there is $T = T(\varepsilon) \geqslant 0$ such that $\text{dist}(x_t(\varphi), S) < \varepsilon$ for all $t \geqslant T$. Let $\varepsilon > 0$ be fixed. Since S is stable, there exists $\delta > 0$ such that $\lambda \in C$, $\text{dist}(\lambda, S) < \delta$ imply $\text{dist}(x_t(\lambda), S) < \varepsilon$ for all $t \geqslant 0$. Let $\psi \in \Omega(\varphi) \cap S$. By the definition of $\Omega(\varphi)$, one can find $T \geqslant 0$ such that $\|x_T(\varphi) - \psi\| < \delta$. Then $\text{dist}(x_T(\varphi), S) < \delta$ and, thus, $\text{dist}(x_t(\varphi), S) = \text{dist}(x_{t-T}(x_T(\varphi)), S) < \varepsilon$ for all $t \geqslant T$. The proof is complete.

In this paper, Lemma 3.1 will be used in the case $S = M(c)$ for some $c \geqslant 0$. In order to do this we have to guarantee that

(i) $\{x_t(\varphi): t \geqslant 0\}$ is bounded;

(ii) $\Omega(\varphi) \cap M(c) \neq \varnothing$;

(iii) $M(c)$ is stable.

It turns out that sufficient conditions for (i), (ii), and (iii) can be given and $M(c)$ can be located by using the Lyapunov function $V(D\varphi) = \langle D\varphi, D\varphi \rangle$.

Boundedness of solutions and stability of the zero solution of (2.1) are guaranteed by the following lemma

LEMMA 3.2. *If* (A_0) *holds and* $\varphi \in C$, *then*

$$\|x_t(\varphi)\| \leqslant \|\mu\|_0 \, \|\mu^{-1}\|_0 \, \|\varphi\| \qquad (t \geqslant 0).$$

Proof. Let $x = x(\varphi)$ and

$$W(t) = \max\{V(Dx_t), \|\mu\|_0^2 \, \|\varphi\|^2\} \qquad (t \geqslant 0).$$

We will show that the upper right hand Dini derivative $D^+W(t) \leqslant 0$ for all $t \geqslant 0$. This will imply $W(t) \leqslant W(0)$, i.e.,

$$|Dx_t| \leqslant \max\{|D\varphi|, \|\mu\|_0 \, \|\varphi\|\} \leqslant \|\mu\|_0 \, \|\varphi\| \qquad (t \geqslant 0),$$

since $|D\varphi| \leqslant \|\mu\|_0 \, \|\varphi\|$. Hence, by applying Lemma 2.4 with $b = \|\mu\|_0 \, \|\varphi\|$, $y(t) = x(t)$, we obtain that $\|x_t\| \leqslant \|\mu\|_0 \, \|\mu^{-1}\|_0 \, \|\varphi\|$, $t \geqslant 0$.

If $D^+W(t) \leqslant 0$ is false, then there is $\tau \geqslant 0$ with $W(s) \leqslant W(\tau)$, $0 \leqslant s \leqslant \tau$, and $D^+W(\tau) > 0$. Since $V(Dx_\tau) < W(\tau)$ implies $D^+W(\tau) = 0$, one gets $W(\tau) = |Dx_\tau|^2 = \max_{0 \leqslant s \leqslant \tau} |Dx_s|^2 \geqslant \|\mu\|_0^2 \, \|\varphi\|^2$. Consequently, there exists a sequence $t_n \to 0+$ such that $W(\tau + t_n) = |Dx_{\tau + t_n}|^2 > |Dx_\tau|^2 = W(\tau)$ and

$$0 < D^+W(\tau) = \lim_{n \to \infty} t_n^{-1}(W(\tau - t_n) - W(\tau)) = \dot{V}_{(2.1)}(Dx_\tau) = 2\langle Dx_\tau, f(x_\tau)\rangle.$$

On the other hand, from $\max_{0 \leqslant s \leqslant \tau} |Dx_s| = |Dx_\tau| \geqslant \|\mu\|_0 \|\varphi\|$, and Lemma 2.4, $\|x_\tau\| \leqslant \|\mu^{-1}\|_0 |Dx_\tau|$ follows. So, (A_0) implies $\langle Dx_\tau, f(x_\tau)\rangle \leqslant 0$, a contradiction, which completes the proof.

Remark 3.1. In case $D\varphi \equiv \varphi(0)$, Lemma 3.2 gives the following well known result for retarded equations [3, 4, 8, 14, 16, 20]:

If, for any $\varphi \in C$, $|\varphi(0)| = \|\varphi\|$ implies $\langle \varphi(0), f(\varphi)\rangle \leqslant 0$, then for any $\varphi \in C$ we have $\|x_t(\varphi)\| \leqslant \|\varphi\|$, $t \geqslant 0$.

Now, we turn to the problem of $\Omega(\varphi) \cap M(c) \neq \varnothing$ for some $c \geqslant 0$.

LEMMA 3.3. *If* $\mu \neq \delta$ *(i.e.,* $D\varphi \not\equiv \varphi(0)$*) and* (A_α) *holds for some* $\alpha \in (0, -a_D)$, *then* $\Omega(\varphi) \cap M$ *is nonempty.*

Proof. Let $\varphi \in C$ be given. Lemma 3.2 implies $((\mathrm{A}_\alpha) \Rightarrow (\mathrm{A}_0))$ that $\{x_t(\varphi) : t \geqslant 0\}$ is bounded. By Lemma 2.7, $\Omega(\varphi) \neq \varnothing$. Let $\psi \in \Omega(\varphi)$. From the invariance property of $\Omega(\varphi)$ there is a bounded $y \in C(R, R^n)$ such that $y_0 = \psi$, y is a solution of (2.1) on R and $y_t \in \Omega(\varphi)$ for all $t \in R$. By (ii) of Remark 2.1 there is $\beta \in (0, \alpha)$ such that $e^{\beta r} \|\mu^{-1}\|_\beta \leqslant \|\mu^{-1}\|_\alpha$. From Lemma 2.5 with $b = |Dy_t|$, it follows that if $\sup_{s \leqslant t} e^{-\beta(t-s)} |Dy_s| = |Dy_t|$ then $\|y_t\| \leqslant e^{\beta r} \|\mu^{-1}\|_\beta |Dy_t| \leqslant \|\mu^{-1}\|_\alpha |Dy_t|$ and, by (A_α), $(d/dt)\, V(Dy_t) \leqslant 0$. Thus, a standard Razumikhin type argument (see, cf. [6, 10, 22]) can be employed to show that $\sup_{s \leqslant t} e^{-\beta(t-s)} |Dy_s|$ is a nonincreasing function of t on R. Let

$$c = \lim_{t \to \infty} \left(\sup_{s \leqslant t} e^{-\beta(t-s)} |Dy_s|\right).$$

Claim. $\lim_{t \to \infty} |Dy_t| = c$.

If not, then $c > 0$ and there exist $\varepsilon > 0$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $|Dy_{t_n}| \leqslant c - 2\varepsilon$. Since Dy_t is uniformly continuous on R by Lemma 2.6, there is $\tau > 0$ such that $|Dy_{t_n + \theta}| \leqslant c - \varepsilon$ for all $-\tau \leqslant \theta \leqslant 0$. Therefore, we obtain

$$\begin{aligned} c &= \lim_{n \to \infty} \left(\sup_{s \leqslant t_n} e^{-\beta(t_n - s)} |Dy_s|\right) \\ &\leqslant \lim_{n \to \infty} \max\left\{e^{-\beta\tau} \sup_{s \leqslant t_n - \tau} e^{-\beta(t_n - \tau - s)} |Dy_s|, c - \varepsilon\right\} \\ &\leqslant \max\left\{-e^{\beta\tau} c, c - \varepsilon\right\} < c, \end{aligned}$$

a contradiction.

From the claim, $\Omega(\psi) \subseteq K(c)$ follows. Since $\Omega(\psi)$ is invariant, $\Omega(\psi) \subseteq M(c)$, thereby we complete the proof.

Now, we show that $\Omega(\varphi) \cap M \neq \varnothing$ can be obtained by assuming (A_0) and an additional natural condition instead of (A_α), $0 < \alpha < -a_D$. In order to do that, for a nonempty set $H \subseteq (0, r]$, let us define $\mathscr{H} = \{s \in R$: there is a positive integer k and $r_1, r_2, ..., r_k \in H$ such that $s = r_1 + r_2 + \cdots + r_k\}$. We need the following simple, technical lemma, which essentially was shown in [8, p. 108].

LEMMA 3.4. *Assume that $H \subseteq (0, r]$ is a nonempty set such that either there exist $r_1, r_2 \in H$ so that r_1/r_2 is irrational, or the set H is infinite. Then, for any $\varepsilon > 0$, there exists $T = T(\varepsilon) \geqslant 0$ such that if $t \geqslant T$, then $|t - s| < \varepsilon$ for some $s \in \mathscr{H}$.*

Proof. Let $\varepsilon > 0$ be given. We can choose positive integers p, q and suitable $r_1, r_2 \in H$ such that $0 < qr_2 - pr_1 \leqslant \varepsilon$. Indeed, if H is an infinite set, then in a neighborhood of every accumulation point of H there are suitable r_1, r_2 with $p = q = 1$. If $r_1, r_2 \in H$ and r_1/r_2 is irrational, then, from Dirichlet's theorem in number theory, for every natural number m, there exists a rational number $p/q \geqslant 0$ such that $|r_2/r_1 - p/q| < 1/mq$, from which one gets the desired numbers.

Let N be an integer such that $N \geqslant pr_1/(qr_2 - pr_1)$ and $T = T(\varepsilon) = Npr_1$. For any $t \geqslant T$ there is a positive integer $k \geqslant N$ such that $t \in [kpr_1, kqr_2]$, because $l \geqslant N$ implies $lqr_2 \geqslant (l+1)pr_1$. Since the numbers kpr_1, $kpr_1 + (qr_2 - pr_1)$, $kpr_1 + 2(qr_2 - pr_1)$, ..., $kpr_1 + (k-1)(qr_2 - pr_1)$, kqr_2 are in $\mathscr{H}$ and form an arithmetical sequence with difference $qr_2 - pr_1 \leqslant \varepsilon$, the proof is complete.

LEMMA 3.5. *Let $H \subseteq (0, r]$ be a set such that either there exist $r_1, r_2 \in H$ so that r_1/r_2 is irrational, or H is infinite. If* (A_0) *and* (C_H) *hold, then $\Omega(\varphi) \cap M$ is nonempty.*

Proof. Let $\varphi \in C$ be given. By Lemmas 2.7 and 3.2, $\Omega(\varphi)$ is nonempty, compact and invariant. Let $\psi \in \Omega(\varphi)$. There is a bounded solution y of (2.1) on R such that $y_0 = \psi$ and $y_t \in \Omega(\varphi)$ for all $t \in R$.

Lemma 2.5 implies that if $\sup_{s \leqslant t} |Dy_s| = |Dy_t|$ for some $t \in R$, then $\|y_t\| \leqslant \|\mu^{-1}\|_0 \, |Dy_t|$. Then, from ($A_0$), $(d/dt)\, V(Dy_t) \leqslant 0$ follows. Thus, $\sup_{s \leqslant t} |Dy_s|$ is a nonincreasing function of t on R; consequently, it is constant. Let $c = \sup_{s \leqslant t} |Dy_s|$. There is a sequence $\{t_n\}$ such that $t_n \to -\infty$ and $|Dy_{t_n}| \to c$ as $n \to \infty$. By using the diagonalization procedure, there is a subsequence, again denoted by $\{t_n\}$, such that

$$y(t_n + s) \to z(s) \qquad (n \to \infty)$$

uniformly in s on any compact subset of R. Then, z will be a bounded solution of (2.1) on R, $z_t \in \Omega(\varphi)$ for all $t \in R$, $|Dz_0| = c$ and $|Dz_t| \leqslant c$ for all $t \in R$.

Claim. $\lim_{t\to-\infty}|Dz_t| = c$.

If not, then $c > 0$ and there exist $\gamma > 0$ and a sequence $\{t_n\}$ such that $t_n \to -\infty$ and $|Dz_{t_n}| \leq c - 2\gamma$. By Lemma 2.6, Dz_t is uniformly continuous on R. So, one can choose $\varepsilon > 0$ such that $|Dz_{t_n+\theta}| \leq c - \gamma$ for all $\theta \in (-\varepsilon, \varepsilon)$. Since $V(Dz_t)$ attains its maximum c_2 at $t = 0$, $(d/dt)\, V(Dz_t)|_{t=0} = 2\langle Dz_0, f(z_0)\rangle = 0$, assumption (C_H) gives that $|z(-s)| = \|\mu^{-1}\|_0\, c$ for all $s \in H$. On the other hand, from Lemma 2.5, it follows that $|Dz_0| = c$ implies $|z(t_0 - s)| \leq \int_0^\infty |Dz_{-s-u}|\, d\,|\mu^{-1}|\,(u)$, $s \geq 0$. Thus, $|z(-s)| = \|\mu^{-1}\|_0\, c$ implies $|Dz_{-s}| = c$, because 0 is in the support of the measure $|\mu^{-1}|$. So, from $|Dz_0| = c$ and $|z(-s)| = \|\mu^{-1}\|_0\, c$, $s \in H$, one obtains $|Dz_{-s}| = c$ for all $s \in H$. Since from $|Dz_{-s}| = c$, $\langle Dz_{-s}, f(z_{-s})\rangle = 0$ follows, the above argument can be iterated to get

$$|Dz_{-s}| = c \qquad (s \in \mathscr{H}).$$

Let $T = T(\varepsilon)$ be the number given in Lemma 3.4. Let n be so large that $-t_n \geq T$. By Lemma 3.4 there exists $s \in \mathscr{H}$ such that $t_n - \varepsilon < -s < t_n + \varepsilon$. Then, $|Dz_{-s}| \leq c - \gamma$, a contradiction.

Again, by the diagonalization procedure, one can find a sequence $\{t_n\}$ such that $t_n \to -\infty$ and

$$z(t_n + s) \to v(s) \qquad (n \to \infty)$$

uniformly in s on any compact subset of R. Then, v is a solution of (2.1) on R, $v_t \in \Omega(\varphi)$ for all $t \in R$ and $|Dv_t| = c$ for all $t \in R$. So, $v_0 \in \Omega(\varphi) \cap M(c)$, and the proof is complete.

Remark 3.2. Note that in the proof of Lemma 3.5 we used only the fact that 0 is in the support of the measure $|\mu^{-1}|$, denoted by U. Thus, the condition on H in Lemma 3.5 can be weakened. Let $H + U = \{s \in R : s = h + u, h \in H, u \in U\}$. It is not difficult to see that it is enough to assume that either there are $r_1, r_2 \in H + U$ such that r_1/r_2 is irrational or the set $(H + U) \cap (0, a)$ is infinite for some $a > 0$.

We conclude this section with a brief discussion concerning stability of $M(c)$. From Lemma 2.5 it follows that $M(0) = \{0\}$. Thus, by Lemma 3.2, $M(0)$ is stable under condition (A_0). This will be useful to prove asymptotic stability of the zero solution of (2.1) in Section 4. The stability of $M(c)$ for $c > 0$ will be considered in Section 5 to get asymptotic constancy of the solutions of (2.1).

4. Applications to Asymptotic Stability

Asymptotic stability of the zero solution of (2.1) can be obtained from the results of Section 3 by guaranteeing $\Omega(\varphi) \cap M(0) \neq \emptyset$; i.e., $0 \in \Omega(\varphi)$.

From the next two lemmas we see that $M(c) = \emptyset$ for $c > 0$.

Lemma 4.1. *If* $\mu \neq \delta$ *(i.e.,* $D\varphi \not\equiv \varphi(0)$*),* (B_α) *holds with some* $\alpha \in (0, -a_D)$*, then* $M(c)$ *is empty for all* $c > 0$.

Proof. Since $\mu \neq \delta$, $\beta \in (0, \alpha)$ can be found such that $e^{\beta r} \|\mu^{-1}\|_\beta < \|\mu^{-1}\|_\alpha$. If $M(c) \neq \emptyset$ for some $c > 0$, then there is a bounded $y \in C(R, R^n)$, which is a solution of (2.1) on R and $|Dy_t| = c$ for all $t \in R$. Then, by Lemma 2.5, $\|y_t\| \leqslant ce^{\beta r} \|\mu^{-1}\|_\beta < \|\mu^{-1}\|_\alpha |Dy_t|$. So, (B_α) gives that $\langle Dy_t, f(y_t)\rangle < 0$. This is a contradiction, because $2\langle Dy_t, f(y_t)\rangle = (d/dt) V(Dy_t) = (d/dt) |Dy_t|^2 = (d/dt) c^2 = 0$.

Lemma 4.2. *If* $H \subseteq (0, r)$ *is a nonempty set and* (C_H), (E) *hold, then* $M(c)$ *is empty for all* $c > 0$.

Proof. Let $c > 0$ and $\varphi \in M(c)$. There is a bounded solution y of (2.1) on R such that $y_0 = \varphi$ and $|Dy_t| = c$ for all $t \in R$. Then, $\langle Dy_t, f(y_t)\rangle = 0$ for all $t \in R$. Since, by Lemma 2.5, we have $\|y_t\| \leqslant \|\mu^{-1}\|_0 |Dy_t|$ for all $t \in R$, (C_H) implies that $|y(t-s) = \|\mu^{-1}\|_0 |Dy_t| = \|\mu^{-1}\|_0 c$ for all $t \in R$ and $s \in H$. Consequently, $|y(t)| = \|y_t\| = \|\mu^{-1}\|_0 |Dy_t| = \|\mu^{-1}\| \mu_0 c > 0$ for all $t \in R$. Thus (E) gives the contradiction $\langle Dy_t, f(y_t)\rangle < 0$.

Now, we are in the position to state the following asymptotic stability results.

Theorem 4.1. *Assume that* $\mu \neq \delta$ *(i.e.,* $D\varphi \not\equiv \varphi(0)$*),* $f(0) = 0$ *and* (B_α) *holds for some* $\alpha \in (0, -a_D)$*. Then the zero solution of* (2.1) *is asymptotically stable.*

Proof. Observing that (B_α) implies (A_α), $0 < \alpha < -a_D$, and $M(0) = \{0\}$, we readily obtain our statement by combining Lemmas 3.1, 3.2, 3.3, and 4.1.

Theorem 4.2. *Assume that* $f(0) = 0$, (A_0), (E), *and* (C_H) *hold with a set* $H \subseteq (0, r]$ *which is either infinite or contains* r_1, r_2 *such that* r_1/r_2 *is irrational. Then the zero solution of* (2.1) *is asymptotically stable.*

Proof. Apply Lemmas 3.1, 3.2, 3.5, and 4.2.

EXAMPLE 4.1. Consider the system

$$\frac{d}{dt}(x(t)-Cx(t-r))=F(x(t)-Bx(t-r)), \tag{4.1}$$

where $r>0$, B, C are $n\times n$ matrices $|B-C|<1-|C|$, $|C|\neq 0$, $F: R^n\to R^n$ is continuous, $F(0)=0$, and $\langle u,v\rangle>0$ implies $\langle u,F(v)\rangle<0$ for all $u,v\in R^n$. Then $\dot V(D\varphi)=2\langle D\varphi, F(\varphi(0)-B\varphi(-r))\rangle=2\langle D\varphi, F(D\varphi+(C-B)\,\varphi(-r))\rangle$. Choose $\alpha>0$ such that $|B-C|<1-|C|\,e^{\alpha r}$. Then, by Remark 2.1, $\|\mu^{-1}\|_\alpha\,|D\varphi|\leqslant(1-|C|\,e^{\alpha r})^{-1}\,|D\varphi|$. Therefore, under the condition $\|\varphi\|<\|\mu^{-1}\|_\alpha\,|D\varphi|$,

$$\begin{aligned}\langle D\varphi, D\varphi+(C-B)\,\varphi(-r)\rangle &\geqslant |D\varphi|^2-|\langle D\varphi,(C-B)\,\varphi(-r)\rangle|\\ &\geqslant |D\varphi|^2-|D\varphi|\,|B-C|\,(1-|C|\,e^{\alpha r})^{-1}\,|D\varphi|\\ &=|D\varphi|^2\,(1-|B-C|\,(1-|C|\,e^{\alpha r})^{-1})>0.\end{aligned}$$

Thus, Theorem 4.1 implies that the zero solution of (4.1) is asymptotically stable.

EXAMPLE 4.2. Consider the scalar equation

$$\begin{aligned}&\frac{d}{dt}(x(t)-c_1x(t-r_1)-c_2x(t-r_2))\\ &\quad=-ax(t)+b_1x(t-r_1)+b_2x(t-r_2)\\ &\quad\quad-(x(t)-c_1x(t-r_1)-c_2x(t-r_2))(x(t)-\frac{1}{2}x(t-r_3))^2, \end{aligned}\tag{4.2}$$

where $r_i>0$ $(i=1,2,3)$, $r_1\neq r_2$, $0<|c_1|+|c_2|<1$ and $|ac_1-b_1|+|ac_2-b_2|\leqslant a(1-|c_1|-|c_2|)$. Here $(D\varphi)\,f(\varphi)=-(D\varphi)(a\,D\varphi+(ac_1-b_1)\,\varphi(-r_1)+(ac_2-b_2)\,\varphi(-r_2))-(D\varphi)^2\,(\varphi(0)-\frac{1}{2}\varphi(-r_3))^2$. Clearly, $\|\mu^{-1}\|_0\leqslant(1-|c_1|-|c_2|)^{-1}$ in this case. In the same way as in Example 4.1, by applying Theorem 4.1, it can be obtained that under the condition

$$|ac_1-b_1|+|ac_2-b_2|<a(1-|c_1|-|c_2|),$$

the zero solution of (4.2) is asymptotically stable.

Assume that

$$ac_1\neq b_1,\qquad ac_2\neq b_2,\qquad |ac_1-b_1|+|ac_2-b_2|=a(1-|c_1|-|c_2|)>0.$$

Then $\|\varphi\| \leqslant \|\mu^{-1}\|_0 \, |D\varphi|$ implies that $(D\varphi) f(\varphi) \leqslant 0$. In addition, if either $|\varphi(-r_1)| < \|\varphi\|$ or $|\varphi(-r_2)| < \|\varphi\|$, then $(D\varphi) f(\varphi) < 0$. So, choosing $H = \{r_1, r_2\}$, (C_H) holds. Moreover, $0 < |\varphi(-s)| = \|\varphi\| \leqslant \|\mu^{-1}\|_0 \, |D\varphi|$, $0 \leqslant s \leqslant r$, implies $(D\varphi) f(\varphi) < 0$. Thus (E) also holds. Therefore, if r_1/r_2 is irrational and $|ac_1 - b_1| + |ac_2 - b_2| = a(1 - |c_1| - |c_2|) > 0$, then the zero solution of (4.2) is asymptotically stable.

5. Applications to Asymptotic Constancy

The asymptotic constancy problem will be considered merely for one-dimensional equations. This restriction is done because the sets $M(c)$, $c > 0$, in general, contain more functions than constants in the higher dimensional case. If fact, in the higher dimensional case there is no general theory of the problem of asymptotic constancy even for retarded functional differential equations.

Before giving asymptotic constancy results, we prove two lemmas for the n-dimensional case. The first one is very important in the study of NFDEs. It states that the asymptotic constanct if Dy_t is equivalent to that of $y(t)$.

Lemma 5.1. *If $y \in C(R, R^n)$ is bounded, the limits (provided they exist)*

$$\lim_{t \to \infty} y(t) = y(\infty) \qquad \textit{and} \qquad \lim_{t \to \infty} Dy_t = Dy(\infty)$$

are equivalent; moreover,

$$y(\infty) = \int_0^\infty [d\mu^{-1}(s)] \, Dy(\infty) \qquad \textit{and} \qquad Dy(\infty) = \int_0^r [d\mu(s)] \, y(\infty).$$

Proof. Choose $\alpha \in (0, -a_D)$. Then $y_t \in C_\alpha$ for all $t \in R$, where $y_t(s) = y(t+s)$, $s \leqslant 0$. Thus $Dy_t = \mu * y_t(0)$ and by Lemma 2.2, $y(t) = \mu^{-1} * z_t(0)$, where $z(t) = Dy_t$. Let $K \geqslant 0$ be defined so that $|y(t)| \leqslant K$, $|Dy_t| \leqslant K$ for all $t \in R$. Assume $\lim_{t \to \infty} Dy_t = Dy(\infty)$. Let $\varepsilon > 0$ and find $T_1 \geqslant 0$ such that $|Dy_t - Dy(\infty)| < \varepsilon$ for $t \geqslant T_1$. Then, for $t \geqslant T_1$,

$$\begin{aligned} \left| y(t) - \int_0^\infty [d\mu^{-1}(s)] \, Dy(\infty) \right| &= \left| \int_0^\infty [d\mu^{-1}(s)] \, (Dy_{t-s} - Dy(\infty)) \right| \\ &\leqslant \int_0^\infty |Dy_{t-s} - Dy(\infty)| \, d\,|\mu^{-1}|\,(s) \\ &\leqslant \int_0^{t-T_1} \varepsilon \, d|\mu^{-1}|\,(s) + \int_{t-T_1}^\infty 2K \, d\,|\mu^{-1}|\,(s). \end{aligned}$$

So, there exists $T_2 > 0$ such that $|y(t) - \int_0^\infty [d\mu^{-1}(s)]\, Dy(\infty)| < 2\varepsilon \, \|\mu^{-1}\|_0$ for all $t > T_2$. Therefore $y(t) \to y(\infty) = \int_0^\infty [d\mu^{-1}(s)]\, Dy(\infty)$. The other direction of the proof is analogous, so it is omitted.

Let x be a solution of (2.1) and define $y(t) = x(t) - k$ for a $k \in R^n$ satisfying $f(k) = 0$, where, and in what follows, k denotes either a vector in R^n or a constant function in C with value k. Then y satisfies

$$\frac{d}{dt}(Dy_t) = f(y_t + k). \tag{5.1}$$

Thus, the solution of the constant solution $x = k$ of (2.1) is equivalent to the stability of the zero solution of (5.1). Evidently, Lemma 3.2 implies

LEMMA 5.2. *Assume that $f(k) = 0$ for a constant k in C, and for any $\varphi \in C$*

$$\|\varphi\| \leqslant \|\mu^{-1}\|_0 \, |D\varphi| \text{ implies } \langle D\varphi, f(\varphi + k)\rangle \leqslant 0 \qquad (A_0^k)$$

Then, for any $\varphi \in C$, the solution $x(\varphi)$ of (2.1) satisfies

$$\|x_t(\varphi) - k\| \leqslant \|\mu\|_0 \, \|\mu^{-1}\|_0 \, \|\varphi - k\| \qquad (t \geqslant 0).$$

Now, we can state the following asymptotic constancy results for scalar NFDEs.

THEOREM 5.1. *Assume that $n = 1$, $\mu \neq \delta$, (A_α) holds for some $0 < \alpha < -a_D$, and (A_0^k) is satisfied for any $k \in C$ that $f(k) = 0$. Then any solution of (2.1) tends to a constant as $t \to \infty$.*

Proof. Let $\varphi \in C$ and $x = x(\varphi)$. By Lemma 3.3 there is $c \geqslant 0$ such that $\psi \in \Omega(\varphi) \cap M(c)$ for some $\psi \in C$. Then there is a solution y of (2.1) on R such that $y_0 = \psi$ and $|Dy_t| = c$ for all $t \in R$. By continuity $Dy_t = \bar{c}$ for all $t \in R$, where $\bar{c} = 0$ if $c = 0$, and $\bar{c}$ is either c or $-c$ if $c > 0$. Lemma 2.2 implies that $y(t) = \bar{c} \int_0^\infty d\mu^{-1}$ for all $t \in R$. So, ψ is constant. Moreover,

$$0 = \frac{d}{dt} V(Dy_t) = \langle \bar{c}, f(y_t)\rangle = \langle \bar{c}, f(\psi)\rangle.$$

Hence, $f(\psi) = 0$, whenever $\bar{c} \neq 0$. Thus, in the case $\bar{c} \neq 0$, ψ is stable by Lemma 5.2. The stability of the zero solution (the case $\bar{c} = 0$) follows from (A_α) by Lemma 3.2. Therefore, from Lemma 3.1 it follows that $x_t(\varphi) \to \psi$ as $t \to \infty$.

THEOREM 5.2. *Assume that $n=1$, (A_0) holds, and (A_0^k) is satisfied for any $k \in C$ such that $f(k)=0$. Moreover, assume (C_H) is valid with $H \subseteq (0, r]$ such that either H is infinite or there exist $r_1, r_2 \in H$ so that r_1/r_2 is irrational. Then any solution of* (2.1) *tends to a constant as $t \to \infty$.*

Proof. Let $\varphi \in C$ and $x = x(\varphi)$. By Lemma 3.5, there are $c \geqslant 0$ and $\psi \in C$ such that $\psi \in \Omega(\varphi) \cap M(c)$. In the same way as in the proof of Theorem 5.1, it follows that ψ is constant and a stable solution of (2.1). Then Lemma 3.1 implies our statement

EXAMPLE 5.1. Consider the scalar equation

$$\frac{d}{dt}\left(x(t) - \int_0^r g(s)\, x(t-s)\, ds\right) = F\left(-ax(t) + \int_0^r h(s)\, x(t-s)\, ds\right), \qquad (5.2)$$

where $g, h\colon [0, r] \to R$ and $F\colon R \to R$ are continuous, and

(i) $xF(x) > 0$ if $x \neq 0$, $F(0) = 0$,

(ii) $\int_0^r |g(s)|\, ds < 1$,

(iii) $h(s) \geqslant ag(s)$ for all $s \in [0, r]$, $h(\cdot) - ag(\cdot) \not\equiv 0$,

(iv) $\int_0^r h(s)\, ds = a$,

(v) $\int_0^r (h(s) - ag(s))\, ds \leqslant a(1 - \int_0^r |g(s)|\, ds)$.

For this example, by Remark 2.1, $\|\mu^{-1}\|_0 \leqslant (1 - \int_0^r |g(s)|\, ds)^{-1}$. Assuming

$$\|\varphi\| \leqslant \|\mu^{-1}\|_0\, |D\varphi|,$$

we have

$$\begin{aligned}
&(D\varphi)\left(-a\varphi(0) + \int_0^r h(s)\, \varphi(-s)\, ds\right)\\
&\quad = (D\varphi)\left(-a\, D\varphi + \int_0^r (h(s) - ag(s))\, \varphi(-s)\, ds\right)\\
&\quad \leqslant -a(D\varphi)^2 + |D\varphi| \int_0^r (h(s) - ag(s))\, \|\varphi\|\, ds\\
&\quad \leqslant -a(D\varphi)^2 + |D\varphi|^2 \int_0^r (h(s) - ag(s))\, ds \left(1 - \int_0^r |g(s)|\, ds\right)^{-1} \leqslant 0.
\end{aligned}$$

If $H = \{s \in (0, r]\colon h(s) - ag(s) > 0\}$ and $|\varphi(-s)| < \|\mu^{-1}\|_0\, |D\varphi|$ for some $s \in H$, then $(D\varphi)(-a\varphi(0) + \int_0^r h(s)\, \varphi(-s)\, ds) < 0$. It is obvious that H is infinite. So, (A_0) and (C_H) hold with an infinite H, by (i). Any constant solution is stable, since for any $k \in R$, $F(-a\varphi(0)+\int_0^r h(s)\,\varphi(-s)\,ds) = F(-a(\varphi(0)+k)+\int_0^r h(s)(\varphi(-s)+k)\,ds)$. Thus, Theorem 5.2 can be applied to assert that any solution of (5.2) tends to a constant as $t \to \infty$.

EXAMPLE 5.2. Consider the scalar equation

$$\frac{d}{dt}\left(x(t)-\sum_{i=1}^{m} c_i x(t-r_i)\right) = -h(x(t))+h\left(\sum_{i=1}^{m} b_i x(t-r_i)\right), \qquad (5.3)$$

where $m \geqslant 2$, $c_i > 0$, $r_i > 0$, $i = 1, ..., m$, $\sum_{i=1}^m c_i \leqslant 1$, r_1/r_2 is irrational, $b_i = c_i/\sum_{j=1}^m c_j$, $i = 1, ..., m$, $h: R \to R$ is continuous and strictly increasing.

Let $H = \{r_i : i = 1, ..., m\}$. From Remark 2.1, $\|\mu^{-1}\|_0 \leqslant (1-\sum_{i=1}^m c_i)^{-1}$ follows. Let $k \in R$ and assume that

$$\|\varphi\| \leqslant \|\mu^{-1}\|_0\, |D\varphi|.$$

If, in addition, $\varphi(0) \leqslant \sum_{i=1}^m c_i \varphi(-r_i)$, then we have

$$-\sum_{i=1}^{m} b_i \varphi(-r_i) \leqslant \|\varphi\| \leqslant \left(1-\sum_{i=1}^{m} c_i\right)^{-1}\left(\sum_{i=1}^{m} c_i \varphi(-r_i)-\varphi(0)\right)$$

and thus $k+\varphi(0) \leqslant k+\sum_{i=1}^m b_i \varphi(-r_i)$, that is, $(D\varphi)\, f(\varphi+k) \leqslant 0$. If $\varphi(0) \geqslant \sum_{i=1}^m c_i \varphi(-r_i)$, then

$$\sum_{i=1}^{m} b_i \varphi(-r_i) \leqslant \|\varphi\| \leqslant \left(1-\sum_{i=1}^{m} c_i\right)^{-1}\left(\varphi(0)-\sum_{i=1}^{m} c_i \varphi(-r_i)\right),$$

from which $k+\varphi(0) \geqslant k+\sum_{i=1}^k b_i \varphi(-r_i)$ follows; that is, $(D\varphi)\, f(\varphi+k) \leqslant 0$.

If we also have $|\varphi(-r_i)| < \|\mu^{-1}\|_0\, |D\varphi|$ for some $r_i \in H$, then $D\varphi \neq 0$ and $(D\varphi)\, f(\varphi) < 0$ from the above inequalities. Therefore, Theorem 5.2 can be applied to conclude that all solutions of (5.3) are asymptotically constant.

THEOREM 5.3. *Assume that in the one-dimensional case, for $\mu = \delta - \nu$, f and a nonempty set $H \subseteq (0, r]$, the following properties hold*:

(i) *ν is nondecreasing and $\int_0^r d\nu < 1$*;

(ii) *for any $\varphi \in C$ and any $k \in \{0\} \cup \{\psi \in C : \psi$ is constant, $f(\psi) = 0\}$ we have*

$$\max_{-r \leqslant s \leqslant 0} \varphi(s) \leqslant \|\mu^{-1}\|_0\, D\varphi \text{ implies } f(\varphi+k) \leqslant 0,$$

$$\min_{-r \leqslant s \leqslant 0} \varphi(s) \geqslant \|\mu^{-1}\|_0\, D\varphi \text{ implies } f(\varphi+k) \geqslant 0;$$

(iii) *if* $f(\varphi)=0$ *and either* $\max_{-r\leq s\leq 0}\varphi(s)\leq \|\mu^{-1}\|_0\, D\varphi$ *or* $\min_{-r\leq s\leq 0}\varphi(s)\geq \|\mu^{-1}\|_0\, D\varphi$ *is satisfied, then* $\varphi(-s)=\|\mu^{-1}\|_0\, D\varphi$ *for all* $s\in H$;

(iv) $H=\{-p_1r^*, -p_2r^*, ..., -p_mr^*\}$, *where* $r^*>0$, $0<p_1<\cdots<p_m\leq r/r^*$, p_i *is an integer for each* $i=1, ..., m$ *and the maximal common factor* $(p_1, ..., p_m)=1$;

(v) *for each* $\varepsilon>0$, $\alpha, \beta\in R$ *and continuous function* $u\colon R\to R$ *such that* $\alpha\leq u(t)\leq\beta$, $t\in R$, *there is a natural number* N *so that*

$$\frac{1}{N+1}\sum_{j=0}^{N} f(u_{t_0+jr^*})<\varepsilon$$

for all $t_0\in R$.

Then any solution of (2.1) *is asymptotically constant.*

Proof. Assumption (ii) implies (A_0^k), so any solution of (2.1) is bounded and any constant solution is stable by Lemma 3.2. Let $\varphi\in C$. There is a bounded solution y of (2.1) on R such that $y_t\in\Omega(\varphi)$ for all $t\in R$. From condition (i) it follows that $\mu^{-1}=(\delta-v)^{-1}=\delta+v+v*v+...$, and μ^{-1} is also nondecreasing on $[0, \infty)$. So

$$\|\mu^{-1}\|_0=\int_0^\infty d\mu^{-1}.$$

From the equality

$$y(t+s)=\int_0^\infty Dy_{t+s-u}\, d\mu^{-1}(u) \tag{5.4}$$

and the nondecreasing property of μ^{-1}, one has that

$$\sup_{s\leq t} Dy_s\leq Dy_t \text{ implies } \max_{-r\leq s\leq 0} y(t+s)\leq\|\mu^{-1}\|_0\, Dy_t$$

and

$$\inf_{s\leq t} Dy_s\geq Dy_t \text{ implies } \min_{-r\leq s\leq 0} y(t+s)\geq\|\mu^{-1}\|_0\, Dy_t.$$

Thus, by assumption (ii) with $k=0$, $\inf_{s\leq t} Dy_s$ and $\sup_{s\leq t} Dy_s$ are constants, denoted by c_1 and c_2, respectively. By using the diagonalization procedure, as in the proof of Lemma 3.5, we can find a function z, which is a solution (2.1) on R, such that $z_t\in\Omega(\varphi)$, $Dz_0=c_2$, and $c_1\leq Dz_t\leq c_2$ for $t\in R$. Then, we must have $f(z_0)=0$ and $\max_{-r\leq s\leq 0} z(s)\leq\|\mu^{-1}\|_0\, c_2=\|\mu^{-1}\|_0\, Dz_0$. Now, (iii) implies $z(-s)=\|\mu^{-1}\|_0\, c_2$ for all $s\in H$, From (5.4) and $0 \in \operatorname{supp} \mu^{-1}$ it follows that $Dz_{-s} = c_2$ for all $s \in H$. Repeating the above argument as in the proof of Lemma 3.5, we obtain that $Dz_{-s} = c_2$ for all $s \in H$.

It is also true that $\inf_{s \leqslant t} Dz_s$ is a constant, denoted by c_3. Clearly, $c_3 \leqslant c_2$. Applying the diagonalization procedure again, there is a function $v: R \to R$, which is a solution of (2.1) on R, such that $c_3 \leqslant Dv_t \leqslant c_2$ for all $t \in R$, $Dv_0 = c_3$ and there is $t' \in [-r, 0]$ such that $Dv_{t'} = c_2$. In the same way as above we conclude $Dv_{t'-s} = c_2$, $Dv_{-s} = c_3$ for all $s \in H$. There is a natural number M (see [5, p. 114]) such that $\{Mr^*, (M+1)r^*, (M+2)r^*, \ldots\} \subseteq H$. Thus, there exist $t^*, t^{**} \in R$ such that $t^* \leqslant t^{**} \leqslant t^* + r$,

$$Dv_{t^* - jr^*} = c_3, \; Dv_{t^{**} - jr^*} = c_2 \qquad (j = 0, 1, 2, \ldots).$$

Assume $c_3 < c_2$. By assumption (v) there is a natural number N such that

$$\frac{1}{N+1} \sum_{j=0}^{N} f(v_{t+jr^*}) < \frac{c_2 - c_3}{2r} \qquad (t \in R).$$

Then, from the equality

$$\begin{aligned} c_2 - c_3 &= Dv_{t^{**} - jr^*} - Dv_{t^* - jr^*} \\ &= \int_{t^* - kr^*}^{t^{**} - jr^*} f(v_s)\, ds \\ &= \int_{t^* - Nr^*}^{t^{**} - Nr^*} f(v_{t + (N-j)r^*})\, dt, \end{aligned}$$

one gets

$$\begin{aligned} (N+1)(c_2 - c_3) &= \sum_{j=0}^{N} (c_2 - c_3) \\ &= \int_{t^* - Nr^*}^{t^{**} - Nr^*} \sum_{j=0}^{N} f(v_{t+jr^*})\, dt \\ &\leqslant (N+1) \frac{c_2 - c_3}{2}, \end{aligned}$$

a contradiction. So, $c_2 = c_3$. Then, by Lemma 5.1, v is constant. We certainly have $f(v_0) = 0$ and $v_0 \in \Omega(\varphi)$. Thus, Lemma 3.1 gives that $x_t(\varphi) \to v_0$ as $t \to \infty$, and the proof is complete.

EXAMPLE 5.3. The scalar equation

$$\frac{d}{dt}(x(t) - cx(t-r)) = -h(x(t)) + h(x(t-r)), \tag{5.5}$$

under the conditions $0 \leqslant c < 1$, $r \geqslant 0$, $h: R \to R$ being strictly increasing, satisfies the conditions of Theorem 5.3. Thus, solutions of (5.5) are asymptotically constant. It should be mentioned that this result was previously obtained in the case where $c = 0$ (see, cf. [1, 13, 15]).

We conclude this paper by emphasizing that measures of the delays in the D operators and the right-hand side of the equations are assumed to be the same throughout this paper. As will be indicated in a subsequent paper, different measures of the delays can change the qualitative behavior of solutions. For instance, the linear homogeneous equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = -x(t) + x(t-1)$$

possesses a non-constant periodic solution for certain constants c and r.

References

1. K. Cooke and J. Yorke, Some equations modelling growth processes and gonorrhea epidemics, *Math. Biosci.* **16** (1973), 75–101.
2. M. Cruz and J. Hale, Stability of functional differential equations of neutral type, *J. Differential Equations* **7** (1970), 334–355.
3. R. Driver, "Ordinary and Delay Differential Equations," Springer-Verlag, New York, 1977.
4. J. Haddock, Functional differential equations for which each constant is a solution: A narrative, *in* "Proc. of 11th ICNO, Budapest," 1987, pp. 86–93.
5. J. Haddock, T. Krisztin, and J. Terjéki, Invariance principles for autonomous functional differential equations, *J. Integral Equations* **10** (1985), 123–136.
6. J. Haddock, T. Krisztin, and J. Wu, Asymptotic equivalence of neutral equations and infinite delay equations, *Nonlinear Anal. TMA* **14** (1990), 369–377.
7. J. Haddock, N. Nkashama, and J. Wu, Asymptotic constancy of linear neutral Volterra integrodifferential equations, *Tohoku Math. J.* **49** (1989), 689–710.
8. J. Haddock and J. Terjki, Liapunov–Razumikhin functions and an invariance principle for funtional differential equations, *J. Differential Equations* **48** (1983), 95–122.
9. J. Haddock and J. Terjéki, On the location of positive limit sets for functional differential equations with infinite delay, *J. Differential Equations* **86** (1990), 1–32.
10. J. Haddock and J. Wu, Comparison theorems of Liapunov–Razumikhin type for neutral functional differential equations with infinite delay, preprint.
11. J. Hale, Sufficient conditions for stability and instability of autonomous functional differential equations, *J. Differential Equations* **1** (1965), 452–458.
12. J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
13. J. Kaplan, M. Sorg, and J. Yorke, Solutions of $x'(t) = f(x(t), x(t-L)$ have limits when f is an order relation, *Nonlinear Anal. TMA* **3** (1979), 53–58.
14. N. Krasovskii, "Stability of Motion," Stanford Univ. Press, Stanford, Ca, 1959.
15. T. Krisztin, Convergence of solutions of a nonlinear integrodifferential equation arising in compartmental systems, *Acta Sci. Math.* **47** (1984), 471–485.
16. V. Lakshmakantham and S. Leela, "Differential and Integral Inequalities II," Academic Press, New York, 1969.

17. J. LaSalle, Stability theory for ordinary differential equations, *J. Differential Equations* **4** (1968), 57–65.
18. J. LaSalle, "The Stability of Dynamical Systems," Regional Conference Series in Applied Mathematics, SIAM, Philidelphia, 1976.
19. O. Lopes, Forced oscillations in nonlinear neutral differential equations, *SIAM J. Appl. Math.* **29** (1975), 105–207.
20. B. Razumikhin, On stability systems with delay, *Prikl. Mat. Mekh.* **20** (1956), 500–512.
21. O. Staffans, A neutral functional differential equations with stable D-operator is retarded, *J. Differential Equations* **49** (1983), 208–217.
22. J. Wu, The stability of neutral functional differential equations with infinite delay, *Funkcial. Ekvac.* **29** (1986), 131–138.