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Asymptotic Constancy for Pseudo Monotone Dynamical Systems on Function Spaces

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A pseudo monotone dynamical system is a dynamical system which preserves the order relation between initial points and equilibrium points. The purpose of this paper is to present some convergence, oscillation, and order stability criteria for pseudo monotone dynamical systems on function spaces for which each constant function is an equilibrium point. Some applications to neutral functional differential equations and semilinear parabolic partial differential equations with Neumann boundary condition are given.

1. INTRODUCTION

Let $\mathbf{R}^+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$, and $\mathbf{R}^n$ denote the usual Euclidean space of dimension n.

Let M be a compact topological space or a compact n-dimensional submanifold of $\mathbf{R}^n$, and let $C^0(M) := C^0(M, \mathbf{R})$ denote the Banach space of all

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continuous mappings $u: M \to \mathbf{R}$. On the function space $C^0(M, \mathbf{R})$ we consider the following usual partial ordering

$$u \geqslant 0 \Leftrightarrow u(x) \geqslant 0 \qquad \text{on } M.$$

$u > 0$ means $u(x) \geqslant 0$ with $u \not\equiv 0$ on M, and $u \gg 0$ means $\inf_{x \in M} u(x) > 0$.

We will always assume that X is a subspace of $C^0(M)$ which has a topology making its inclusion into $C^0(M)$ continuous, so that X is a (partially) ordered function space with the ordering considered above.

Throughout this paper, we will consider a dynamical system on a given subspace $X \subseteq C^0(M, \mathbf{R})$, that is, a mapping $\phi : \mathrm{Dom}(\phi) \subseteq \mathbf{R}^+ \times X \to X$ satisfying the following continuity and determinism axioms.

(1) *Continuity*: the domain $\mathrm{Dom}(\phi)$ is an open set in $\mathbf{R}^+ \times X$ containing $\{0\} \times X$, and ϕ is continuous.

(2) *Determinism*: $\phi_t(u) = \phi(t, u)$ is such that $\phi_t : \mathrm{Dom}(\phi_t) \to X$ is a mapping with $\mathrm{Dom}(\phi_t)$ open in X. ϕ_0 is the identity mapping on X. For all $s, t \geqslant 0$, one has $\mathrm{Dom}(\phi_{t+s}) = \phi_t^{-1}(\mathrm{Dom}(\phi_s))$ and $\phi_{s+t} = \phi_s \phi_t$.

Denote $\mathrm{Dom}(\phi(\cdot, u))$ by $[0, I_u)$, the mapping from $[0, I_u)$ to X defined by $t \to \phi(t, u)$ is called the *trajectory* of u and its image is the *orbit* $\gamma^+(u)$. A subset $Y \subseteq X$ is positively invariant if $\gamma^+(u) \subseteq Y$ for all $u \in Y$. For any u the ω-limit set of $\gamma^+(u)$ is $\omega(u) = \bigcap_{s \in [0, I_u)} \mathrm{Cl} \bigcup_{t \in [s, I_u)} \gamma^+(\phi(t, u))$, and thus $y \in \omega(u)$ if and only if $y = \lim_{k \to \infty} \phi(t_k, u)$ for some sequence $t_k \to I_u$ in $[0, I_u)$. A dynamical system is also called a semiflow if $I_u = \infty$ for any $u \in X$.

It is a well-known fact that $I_u = \infty$ if the orbit $\gamma^+(u)$ is precompact. In this case $\omega(u)$ is a nonempty compact invariant connected set. The simplest case is when $\omega(u)$ is a singleton. In this case, the orbit $\gamma^+(u)$ (or the point u) is said to be convergent. A slightly more complicated case is the one when $\omega(u)$ is a subset of the set of equilibrium points, that is,

$$\omega(u) \subseteq E = \{u \in X; \phi(t, u) = u \text{ for all } t \geqslant 0\}.$$

In this case, we say that the orbit $\gamma^+(u)$ (or the point u) is quasiconvergent.

A mapping f from X into itself is monotone if $x \geqslant y$ implies $f(x) \geqslant f(y)$, and strongly monotone if $x > y$ implies $f(x) \gg f(y)$.

The semiflow ϕ is monotone (respectively strongly monotone) if ϕ_t is monotone (respectively strongly monotone) for all $t > 0$.

ϕ is eventually strongly monotone if it is monotone and there exists a constant $T > 0$ such that ϕ_t is strongly monotone for all $t \geqslant T$.

A weaker concept than that of monotone semiflow and eventually strongly monotone semiflow is the following pseudo monotone semiflow and eventually strongly pseudo monotone semiflow.

DEFINITION 2.1. The semiflow ϕ is *pseudo monotone* if for any $u \in X$ and $e \in E$ with $u \geqslant e$, we have $\phi(t, u) \geqslant e$ for all $t \geqslant 0$.

The semiflow ϕ is *eventually strongly pseudo montone* if it is pseudo monotone and if there exists a constant $T > 0$ such that for any $u \in X$ and $e \in E$ with $u > e$, we have $\phi(T, u) \gg e$.

It is clear that each eventually strongly monotone semiflow is eventually strongly pseudo monotone. In Section 3, we provide an example of an eventually strongly pseudo monotone semiflow whose strong mononicity cannot be guaranteed (see Lemma 3.1).

Strongly monotone dynamical systems on function spaces arise from various evolution equations. In [16, 17], Hirsch proved that a cooperative and irreducible ordinary differential equation generates a strongly monotone semiflow. According to Smith [24], a cooperative and irreducible retarded functional differential equation produces an eventually strongly monotone semiflow on $C^0(M)$ with $M = [-h, 0]$, where $h > 0$ is a given constant. Also, see [22] for abstract functional differential equations and reaction–diffusion systems with delay. For a similar result related to neutral functional differential equations on product spaces, we refer to [27]. Another class of strongly monotone semiflows is given by some semilinear parabolic partial differential equations with second order uniformly strongly elliptic operators with Neumann or Dirichlet boundary conditions. The strong monotonicitiy is an immediate consequence of the well-known maximum principle. For details, we refer to Amann [1–3, 16, 19, 20, 23]. By using the monotonicity and the positive semigroup theory, Hirsch [16], Matano [19, 20], and Matano and Mimura [21] sketched the proof of the strong monotonicity of the semiflow generated by certain semilinear evolution equations including some weakly coupled systems of parabolic partial differential equations where the reaction term is given by a cooperative vector field. Also, see [14] for related results.

Recent research shows that for strongly monotone dynamical systems precompact orbits have a strong tendency to converge to the set of equilibrium points E. When this set is not connected, it often can be shown that a dense set of points has convergent orbits.

There are numerous functional differential equations and partial differential equations which arise in applications and seemingly lend themselves to this type of behavior, but which have not been investigated in the context of monotone flows. This is true particularly for equations for which each constant (function) in the phase space is an equilibrium point.

Very little has been accomplished with respect to monotone dynamical systems defined, for instance, by the following functional differential equation of neutral type

$$\frac{d}{dt}[x(t) - cx(t-r)] = f(x(t), x(t-r)),$$

where $c, r \in \mathbf{R}$ with $0 \leqslant c < 1$, $r \geqslant 0$, $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous, locally Lipschitz in the first argument, increasing in the second argument, and

$$f(x, x) = 0 \qquad \text{for any} \quad x \in \mathbf{R};$$

or the following nonlinear parabolic partial differential equation with Neumann boundary condition

$$\begin{aligned} &\frac{\partial u}{\partial t} = Au + g(x, u, \nabla u), && t > 0, x \in \bar{\Omega}, \\ &u(x, 0) = v(x), && x \in \bar{\Omega}, \\ &\frac{\partial u}{\partial \eta}(x, t) = 0, && x \in \partial\Omega, t \geqslant 0, \end{aligned}$$

where A is a second order uniformly strongly elliptic differential operator and $g: \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is smooth and satisfies

$$g(x, c, 0) = 0 \qquad \text{on } \bar{\Omega} \text{ for any constant } c \in \mathbf{R}.$$

For these systems, each constant function is a solution, the set of equilibrium points is connected, and therefore convergence to the set of equilibria says nothing about the asymptotic behavior of solutions except for boundedness.

On the other hand, many papers are available dealing with the convergence of solutions of some special cases of the above neutral functional differential equation based mainly on monotonicity techniques (see, e.g., [4–6, 8, 26]) or Liapunov–Razumikhin type invariance principle. For details, refer to a survey paper by Haddock [9] and recent papers by Haddock, Krisztin, Terjéki, and Wu [10], and Haddock, Krisztin, and Wu [11].

Let us mention here that some special cases of systems satisfying the above conditions include the neutral functional differential equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = -\sinh[x(t) - x(t-r)]$$

which arises in the study of the motion of a classically radiating electron [18], and the Burgers equation in one space-variable

$$u_t + uu_x = \varepsilon u_{xx}, \qquad \varepsilon > 0,$$

which arises in the study of gas dynamics and turbulence.

In this paper, by taking the point of view of monotone dynamical systems, we present a unified treatment of asymptotic constancy of solutions for neutral differential equations and some parabolic partial differential equations. In Section 2, we establish convergence, oscillation, and order stability results for eventually strongly pseudo monotone dynamical systems for which each constant function is an equilibrium point (Theorems 2.1–2.3). In Section 3, we apply our general results to various evolution equations (Theorems 3.1–3.2). Finally, we use an example from parabolic partial differential equation theory to show how our idea can be applied to some dynamical systems defined on noncompact manifolds (Theorem 3.3).

2. Convergence, Oscillation, and Order Stability

In this section, we prove some general convergence, oscillation, and order stability theorems for eventually strongly pseudo monotone semiflows. Throughout this section we make the following assumptions:

(1) ϕ is an eventually strongly pseudo monotone semiflow for some given $T>0$.

(2) Each constant function is an equilibrium point for the semiflow ϕ.

Since M is compact, by using the fact that every continuous real-valued function attains its maximum and minimum values at points in M, it is relatively easy to show (by contradiction for instance) that assumptions (1) and (2) imply that constant functions are the *only* equilibrium points for the semiflow ϕ.

Theorem 2.1 (Convergence Principle). *Each precompact orbit tends to a constant function.*

Proof. For each $u \in X \subseteq C^0(M)$, one has $m_0 \leqslant u(x) \leqslant M_0$, $x \in M$, where $m_0 = \min_{x \in M} u(x) > -\infty$ and $M_0 = \max_{x \in M} u(x) < \infty$.

Let

$$m_k = \min_{x \in M} \phi(kT, u)(x) \qquad \text{and} \qquad M_k = \max_{x \in M} \phi(kT, u)(x)$$

for all $k = 0, 1, \ldots$. Then $\hat{m}_k$ and $\hat{M}_k$ are equilibrium points, where throughout this paper $x \to \hat{x}$ is the inclusion mapping from $\mathbf{R}$ into X.

By definition, one has $\hat{m}_k \leqslant \phi(kT, u) \leqslant \hat{M}_k$. Therefore by pseudo monotonicity one obtains

$$\phi(T, \hat{m}_k) \leqslant \phi(T, \phi(kT, u)) \leqslant \phi(T, \hat{M}_k),$$

that is,

$$\hat{m}_k \leqslant \phi((k+1)T, u) \leqslant \hat{M}_k.$$

This implies that

$$\hat{m}_k \leqslant \hat{m}_{k+1} \leqslant \hat{M}_{k+1} \leqslant \hat{M}_k.$$

In this way, one obtains the following nested closed intervals

$$\cdots \subseteq [m_{k+1}, M_{k+1}] \subseteq [m_k, M_k] \subseteq \cdots \subseteq [m_1, M_1] \subseteq [m_0, M_0].$$

Therefore $\lim_{k \to \infty} m_k = a$ and $\lim_{k \to \infty} M_k = b$ exist.

Let $v \in \omega(u)$, then we can find a sequence $t_n \to \infty$ such that $\phi(t_n, u) \to v$ in X as $n \to \infty$. Obviously, there exist a nonnegative integer sequence $\{p_n\}$ and a nonnegative real number sequence $\{q_n\}$ such that $t_n = p_n T + q_n$ and $q_n \in [0, T]$. Owing to the compactness of $[0, T]$ and the precompactness of the orbit, we may assume, without loss of generality, that $\lim_{n \to \infty} q_n = q \in [0, T]$ and $\lim_{n \to \infty} \phi(p_n T, u) = w \in X$. Then by the semigroup property and continuity of ϕ we have $\phi(q, w) = v$.

On the other hand, we can find $y_n, z_n \in M$ such that

$$m_{p_n} = \phi(p_n T, u)(y_n) \qquad \text{and} \qquad M_{p_n} = \phi(p_n T, u)(z_n).$$

Without loss of generality, we may assume that $y_n \to y_0 \in M$ and $z_n \to z_0 \in M$ as $n \to \infty$. Again by continuity of ϕ and the fact that X is continuously imbedded into $C^0(M)$, we have $w(y_0) = a$ and $w(z_0) = b$.

Summarizing the above discussion, we can assert that for any $v \in \omega(u)$, there exist $q \in [0, T]$, $w \in \omega(u)$ such that $v = \phi(q, w)$ and

$$a = \min_{x \in M} w(x) \leqslant \max_{x \in M} w(x) = b.$$

Recalling that $\hat{a}, \hat{b} \in E$, one obtains $\hat{a} \leqslant w \leqslant \hat{b}$, and thus, by pseudo monotonicity one has $\hat{a} \leqslant v \leqslant \hat{b}$. Since v is an arbitrary element in $\omega(u)$, we have that

$$\hat{a} \leqslant v \leqslant \hat{b} \qquad \text{for any} \quad v \in \omega(u).$$

Let $\bar{v}$ be a given element in $\omega(u)$ and $\bar{w}$ be associated with $\bar{v}$ as above. Then $\phi(q, \bar{w}) = \bar{v}$ and there exist $\bar{y}_0, \bar{z}_0 \in M$ such that

$$\bar{w}(\bar{y}_0) = a \qquad \text{and} \qquad \bar{w}(\bar{z}_0) = b_0.$$

By invariance of the limit set $\omega(u)$, one can find an element $\bar{z} \in \omega(u)$ such that $\bar{w} = \phi(T + q, \bar{z})$. Since $\bar{z} \in \omega(u)$, we have $\bar{z} \geqslant \hat{a}$. We want to show that actually $\bar{z} = \hat{a}$.

For that purpose, suppose $\bar{z} \neq \hat{a}$, then $\bar{z} > \hat{a}$, and thus by strong pseudo monotonicity, one has

$$\phi(T, \bar{z}) \gg \phi(T, \hat{a}) = \hat{a},$$

that is,

$$c = \inf_{x \in M} \phi(T, \bar{z})(x) \gg a.$$

Therefore by pseudo monotonicity one obtains

$$\bar{w} = \phi(q, \phi(T, \bar{z})) \geqslant \phi(q, \hat{c}) = \hat{c} \gg \hat{a},$$

which is a contradction to $\bar{w}(\bar{y}_0) = a$.

Thus, $\bar{z} = \hat{a}$ on M, that is $\bar{w} = \phi(T+q, \hat{a}) = \hat{a}$, and $\bar{v} = \phi(q, \bar{w}) = \phi(q, \hat{a}) = \hat{a}$. Likewise, by using a similar argument, it is easily shown that $\bar{v} = \hat{b}$. Hence $\bar{v} = \hat{a} = \hat{b}$. This completes the proof.

Therefore, for any point $u \in X$ with precompact orbit $\gamma^+(u)$ there exists a constant $c = c(u) \in \mathbf{R}$ such that $\lim_{t \to \infty} \phi(t, u) = \hat{c}$. Note that the constant $c = c(u)$ is constructed in the proof of Theorem 2.1 as the unique limit of the sequences (m_k) and (M_k).

The following theorem shows that $\phi(t, u)$ oscillates about $\hat{c}$.

THEOREM 2.2 (Oscillation Principle). *Suppose $u \in X$ is a given point such that $\gamma^+(u)$ is precompact. Let $c = c(u)$ denote the unique limit $\lim_{t \to \infty} \phi(t, u)$. If u is not a constant function, then either there exists $\tau > 0$ such that*

$$\phi(t, u) = \hat{c} \qquad \textit{for all} \quad t \geqslant \tau,$$

or, for any $t \geqslant 0$, there exist $y, z \in M$ such that

$$\phi(t, u)(y) > c \qquad \textit{and} \qquad \phi(t, u)(z) < c.$$

Proof. Obviously $u = \phi(0, u) \neq \hat{c}$ since u is not a constant function.

Therefore, for a given $\tau \geqslant 0$, if $\phi(\tau, u) = \hat{c}$, then necessarily $\tau > 0$, and $\phi(t, u) = \hat{c}$ for all $t \geqslant \tau$ by the semigroup property of the semiflow ϕ and the assumption that each constant function is an equilibrium point. If $\phi(\tau, u) \neq \hat{c}$, then it is impossible that $\phi(\tau, u) > \hat{c}$ (the proof for the case $\phi(\tau, u) < \hat{c}$ is similar). Otherwise by strong pseudo monotonicity of ϕ one has $\phi(\tau + T, u) \gg \hat{c}$. Let $\inf_{x \in M} \phi(\tau + T, u)(x) = c^*$, then $\hat{c}^* \gg \hat{c}$, and $\phi(\tau + T, u) \geqslant \hat{c}^*$. Therefore, by pseudo monotonicity

$$\phi(t, u) \geqslant \hat{c}^* \gg \hat{c} \qquad \text{for all} \quad t \geqslant \tau + T.$$

This contradicts the fact that $\lim_{t \to \infty} \phi(t, u) = \hat{c}$ in $C^0(M)$-topology also. The proof is complete.

In the next section, we will show that for a strongly pseudo monotone dynamical system generated by a functional differential equation, the above oscillation principle implies a very strong oscillation property for solutions (see Theorem 3.1). A similar remark is valid for parabolic partial differential equation (see Theorems 3.2–3.3).

The following result shows order stability of equilibrium points. Even though its proof is elementary, we will call it a theorem for sake of consistency.

An equilibrium point $\hat{c}$ is *order stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $u \in X$ with $\hat{c} - \hat{\delta} \leqslant u \leqslant \hat{c} + \hat{\delta}$ one has $\hat{c} - \hat{\varepsilon} \leqslant \phi(t, u) \leqslant \hat{c} + \hat{\varepsilon}$ for all $t > 0$.

THEOREM 2.3 (Order Stability Principle). *Each constant function is order stable.*

Proof. Let $c \in \mathbf{R}$ be (arbitrarily) given. For every $\varepsilon > 0$, choose $\delta = \varepsilon > 0$. Then for all $u \in X$ with

$$\hat{c} - \hat{\delta} \leqslant u \leqslant \hat{c} + \hat{\delta},$$

we have, by pseudo monotonicity and the fact that $\hat{c} - \hat{\delta} \in E$ and $\hat{c} + \hat{\delta} \in E$,

$$\hat{c} - \hat{\varepsilon} \leqslant \phi(t, u) \leqslant \hat{c} + \hat{\varepsilon} \qquad \text{for all} \quad t \geqslant 0.$$

This shows order stability of $\hat{c}$ with $\delta = \varepsilon$, and the proof is complete.

Note that order stability is a very weak stability notion if X is a space of smooth functions.

3. Applications to Functional and Partial Differential Equations

In this section, we apply our general convergence, oscillation, and order stability results to neutral functional differential equations and second order parabolic partial differential equations. For simplicity, we concentrate on two systems.

The first example is the following neutral functional differential equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = f(x(t), x(t-r)) \tag{3.1}$$

(which models active compartmental systems with pipes, the motion of a classically radiating electron, the spread of epidemics, population growth, and the growth of capital stocks [7, 8, 12, 13, 18]), where c, r are real numbers with $0 \leqslant c < 1$, $r \geqslant 0$, and $f\colon \mathbf{R}^2 \to \mathbf{R}$ is continuous, f is locally

Lipschitz in the first argument (see, e.g., [18, p. 54] for definition), f is increasing in the second argument,

$$f(x, x) = 0 \quad \text{for} \quad x \in \mathbf{R},$$

and f satisfies the following growth condition: for any bounded set $W \subseteq \mathbf{R}^2$, there exists a constant $L = L(f, W) > 0$ such that $f(x, y) \geqslant -L\,|x - y|$ for all $x, y \in W$.

Note that (3.1) reduces to a retarded functional differential equation for the special case $c = 0$.

Obviously, the above general conditions are satisfied, in particular, by the neutral functional differential equation (see, e.g., [18])

$$\frac{d}{dt}[x(t) - cx(t-r)] = -\sinh[x(t) - x(t-r)].$$

The basic existence, boundedness, precompactness, and pseudo monotonicity results are stated and proved below.

LEMMA 3.1. *Let $C = C([-r, 0), \mathbf{R})$. Then*

(1) *for any $\phi \in C$ there exists a unique solution, denoted by $x(\cdot, \phi)$, of (3.1) through $(0, \phi)$. That is, there exists a unique continuous function $x \in C([-r, \infty), \mathbf{R})$ such that $x_0 = \phi$, $x(t) - cx(t-r)$ is differentiable and (3.1) holds for all $t \geqslant 0$.*

(2) *let $P = \{(a, \phi) \in \mathbf{R} \times C; a = \phi(0) - c\phi(-r)\}$. Then the solution of (3.1) generates an eventually strongly pseudo monotone semiflow $u: [0, \infty) \times P \to P$, defined by $u(t, D(\phi), \phi) = (D(x_t(\phi)), x_t(\phi))$, for which each orbit is precompact. Here $D(\phi) = \phi(0) - c\phi(-r)$, $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for all $s \in [-r, 0]$, and the partial ordering in P is defined by*

$$(a, \phi) \leqslant (b, \psi) \Leftrightarrow a \leqslant b \quad \text{and} \quad \phi(s) \leqslant \psi(s) \quad \text{for } s \in [-r, 0].$$

Proof. The local existence–uniqueness of solutions is guaranteed by the general theory of neutral equations with atomic D-operator at zero. For details, we refer to Hale [13, Chap. 12]. By using a standard Liapunov–Razumikhin argument, like in [26], one obtains the inequality

$$m(\phi) \leqslant x(t, \phi) \leqslant M(\phi) \qquad \text{for all} \quad t \geqslant 0,$$

where

$$m(\phi) = \min\left\{\frac{\phi(0) - c\phi(-r)}{1 - c},\ \min_{s \in [-r, 0]} \phi(s)\right\}$$

and

$$M(\phi)=\max\left\{\frac{\phi(0)-c\phi(-r)}{1-c},\ \max_{s\in[-r,0]}\phi(s)\right\}.$$

This implies boundedness of solutions.

Since the D-operator is stable for $0\leqslant c<1$, the precompactness of the orbit $\{x_t(\phi);\, t\geqslant 0\}$ follows from the fact that boundedness implies precompactness of orbits for a neutral equation with stable D-operator (see, e.g., [13]). Therefore Eq. (3.1) generates a dynamical system $u\colon [0,\infty)\times P\to P$ on P, and each orbit of this semiflow is precompact.

To prove the pseudo monotonicity, choose an element $\phi\in C$ and a constant $e\in\mathbf{R}$. Let $x(t)=x(t,\phi)$ and $z(t)=x(t)-e$. Then

$$\frac{d}{dt}D(z_t)=f(x(t),x(t-r))=F(z(t),z(t-r)),$$

where $F:\mathbf{R}^2\to\mathbf{R}$ is defined by

$$F(x,y)=f(x+e,y+e).$$

Obviously F is continuous, locally Lipschitz in the first argument, increasing in the second argument, $F(x,x)=0$ for $x\in\mathbf{R}$, and F satisfies the following growth condition: for any bounded set $W\subseteq\mathbf{R}^2$ there exists a constant $L>0$ such that $F(x,y)\geqslant -L\,|x-y|$ for all $x,y\in W$.

Introducing the transformation

$$w(t)=z(t)-cz(t-r),$$

one has

$$z(t)=w(t)+\sum_{j=1}^{[t/r]}c^jw(t-jr)+c^{[t/r]+1}z\left(t-\left(\left[\frac{t}{r}\right]+1\right)r\right)$$

and

$$z(t-r)=\sum_{j=1}^{[t/r]}c^{j-1}w(t-jr)+c^{[t/r]}z\left(t-\left(\left[\frac{t}{r}\right]+1\right)r\right),$$

where $[t/r]$ is the greatest integer less than or equal to t/r.

Therefore $w(t)$ is a solution to the following retarded equation

$$\dot{w}(t)=F\left[w(t)+\sum_{j=1}^{[t/r]}c^jw(t-jr)+c^{[t/r]+1}z\left(t-\left(\left[\frac{t}{r}\right]+1\right)r\right),\right.$$
$$\left.\sum_{j=1}^{[t/r]}c^{j-1}w(t-jr)+c^{[t/r]}z\left(t-\left(\left[\frac{t}{r}\right]+1\right)r\right)\right]$$

for all $t\geqslant 0$.

Let

$$v(t) = \min\{\min_{0 \leqslant s \leqslant t} w(s), m\},$$

where

$$m = \min\{\min_{-r \leqslant s \leqslant 0} (1-c)[\phi(s) - e], D\phi - (1-c)e\}.$$

If $w(t) > v(t)$, then evidently $D^+ v(t) = 0$. Here D^+ stands for the Dini derivative.

If $w(t) = v(t)$, then

$$D^+ v(t) \geqslant \min\{0, \dot{w}(t)\}$$

and

$$\min\{\min_{-r \leqslant \theta \leqslant 0} (1-c)[\phi(\theta) - e], w(s)\} \geqslant w(t)$$

for all $0 \leqslant s \leqslant t$.

Hence, it follows that

$$\begin{aligned} &\sum_{j=1}^{[t/r]} c^{j-1}(1-c)\, w(t-jr) + c^{[t/r]}(1-c)\, z\left(t - \left(\left[\frac{t}{r}\right] + 1\right) r\right) \\ &\qquad \geqslant \sum_{j=1}^{[t/r]} c^{j-1} w(t)(1-c) + c^{[t/r]} \min_{-r \leqslant s \leqslant 0} (1-c)[\phi(s) - e] \\ &\qquad \geqslant (1 - c^{[t/r]})\, w(t) + c^{[t/r]} w(t) = w(t), \end{aligned}$$

that is,

$$\begin{aligned} &w(t) + \sum_{j=1}^{[t/r]} c^j w(t-jr) + c^{[t/r]+1} z\left(t - \left(\left[\frac{t}{r}\right] + 1\right) r\right) \\ &\qquad \leqslant \sum_{j=1}^{[t/r]} c^{j-1} w(t-jr) + c^{[t/r]} z\left(t - \left(\left[\frac{t}{r}\right] + 1\right) r\right). \end{aligned}$$

By monotonicity of F, this implies that $\dot{w}(t) \geqslant 0$, so that $D^+ v(t) \geqslant 0$.

In any case, one has $D^+ v(t) \geqslant 0$. Consequently, $v(t) \geqslant v(0) = m$ which implies that $w(t) \geqslant m$, that is,

$$D(x_t - \hat{e}) \geqslant \min\{\min_{-r \leqslant s \leqslant 0} (1-c)[\phi(s) - e], D(\phi) - D(\hat{e})\}. \tag{3.2}$$

Therefore, $D(x_t) \geqslant D(\hat{e})$ provided $\psi \geqslant \phi$ and $D(\psi) \geqslant D(\phi)$.

To prove the inequality $x_t \geqslant \hat{e}$ for all $t \geqslant 0$, we first claim that

$$\inf_{t \in [0,T]} x(t) > 0 \qquad \text{provided} \inf_{t \in [0,T]} [x(t) - cx(t-r)] > 0$$

for any continuous function $x : [-r, T] \to \mathbf{R}$ $(T > 0)$ with $x_0 \geqslant 0$.

Indeed it is clear that if there exists a first $\tau \geqslant 0$ for which $x(\tau) = 0$, then $x(\tau) - cx(\tau - r) = -cx(\tau - r) \leqslant 0$ leads to a contradiction.

Now for the initial function ϕ given above, define $\phi_m \in C$ by $\phi_m(s) = \phi(s) + 1/m$ for all $s \in [r, 0]$, where m is any positive integer. Then

$$\min_{-r \leqslant s \leqslant 0} (1-c)[\phi_m(s) - e] \geqslant \frac{1-c}{m} > 0$$

and

$$D(\phi_m) - D(\hat{e}) = D(\phi) - D(\hat{e}) + (1-c)\frac{1}{m} > 0.$$

According to (3.2) one obtains

$$D(x_t^m - \hat{e}) > 0 \tag{3.3}$$

for all $t \geqslant 0$, where x_t^m is the solution of (3.1) through $(0, \phi_m)$. So, by the above claim, one has the inequality

$$x_t^m > \hat{e}$$

for all $t \geqslant 0$. Hence, by continuous dependence of solutions on initial functions, one obtains $x_t \geqslant \hat{e}$ for all $t \geqslant 0$. This completes the proof about the pseudo monotonicity of the semiflow $\{(Dx_t, x_t)\}_{t \geqslant 0}$ on the product space P.

To prove strong pseudo monotonicity, we choose $\phi \in C$ and $e \in \mathbf{R}$ such that $\phi \geqslant \hat{e}$, $D(\phi) \geqslant D(\hat{e})$, and either $D(\phi) > D(\hat{e})$ or there exists a $\theta_0 \in [-r, 0]$ such that $\phi(\theta_0) > e$. Let $x(t) = x(t, \phi)$ and $z(t) = x(t) - e$. Then one has $z(t) \geqslant 0$ and $z(t) - cz(t-r) \geqslant 0$ for all $t \geqslant 0$.

If there exists a $\theta_0 \in [-r, 0]$ such that

$$\phi(\theta_0) > e \qquad \text{(that is, } z(\theta_0) > 0)$$

and

$$z(t) - cz(t-r) = 0 \qquad at \quad t = \theta_0 + r,$$

then at this point, one has

$$\begin{aligned} \frac{d}{dt}[z(t) - cz(t-r)] &= F(z(t), z(t-r)) \\ &= F(cz(t-r), z(t-r)) > 0. \end{aligned}$$

This shows that, in any case, one can always find a $\tau \in [0, r]$ such that

$$z(\tau) - cz(\tau - r) > 0.$$

For any given constant $M \geqslant r$, let L be the growth condition constant of $F(x, y)$ on the bounded set $W = \{(x, y) \in \mathbf{R}^2; |x|, |y| \leqslant N\}$, where

$$N = \max_{t \in [-r, 0]} [|e| + |x(t)|].$$

Then

$$\begin{aligned}\frac{d}{dt}[z(t) - cz(t-r)] &= F(z(t), z(t-r)) \\ &\geqslant F(z(t), cz(t-r)) \\ &\geqslant -L[z(t) - cz(t-r)]\end{aligned}$$

for all $t \in [\tau, M]$.

This implies that

$$z(t) - cz(t-r) \geqslant e^{-L(t-\tau)}[z(\tau) - cz(\tau - r)] > 0$$

for all $t \in [\tau, M]$. Once more, by the above claim, one obtains $z(t) > 0$ for all $t \in [\tau, M]$. M is any constant chosen in a way such that it is not less than r.

Therefore the semiflow $\{(Dx_t, x_t)\}_{t \geqslant 0}$ is eventually strongly pseudo monotone with at least $T = r$. The proof is complete.

By using our general convergence, oscillation, and order stability principles, we obtain the following result for Eq. (3.1).

THEOREM 3.1. (1) *Each constant function is stable.*

(2) *For each $\phi \in C$, the solution of* (3.1) *through* $(0, \phi)$ *has a finite limit $s(\phi)$ which is a constant function on* $[-r, 0]$.

(3) *If $\phi \in C$ is not a constant function, then $x(t, \phi)$ approaches $s(\phi)$ in the following strongly oscillatory manner: Let I be any interval in $[-r, \infty)$ with length greater than r, then there exist $t_1, t_2 \in I$ such that*

$$x(t_1, \phi) \geqslant s(\phi) \qquad \textit{and} \qquad x(t_2, \phi) \leqslant s(\phi).$$

Furthermore, if there exists $t_1 \in I$ such that $x(t_1, \phi) > s(\phi)$, then there exists $t_2 \in I$ such that $x(t_2, \phi) < s(\phi)$. (The role of t_1 and t_2 may be reversed.)

Remark 3.1. Strictly speaking, the space P is not a space of the type $C^0(M)$ considered in Section 2, and so we cannot directly apply

Theorems 2.1–2.3 to the neutral equation (3.1). The space P can be considered to be a subset of $C^0(M)$ where $M = [-r, 0] \cup \{1\}$ and, for $\phi \in C^0(M)$, $\phi^* = \phi$ on $[-r, 0]$ and $\phi^*(1) = \phi(0) - c\phi(-r)$. Furthermore, we can make a slight modification in the proof of Theorem 2.1 as follows.

Replace

$$m_0 \leqslant u(x) \leqslant M_0, \qquad x \in M$$

by

$$((1-c)\, m_\phi, \hat{m}_\phi) \leqslant (D(\phi), \phi) \leqslant ((1-c)\, M_\phi, \hat{M}_\phi),$$

where

$$m_\phi = \min \left\{ \frac{\phi(0) - c\phi(-r)}{1-c}, \min_{s \in [-r, 0]} \phi(s) \right\}$$

and

$$M_\phi = \max \left\{ \frac{\phi(0) - c\phi(-r)}{1-c}, \max_{s \in [-r, 0]} \phi(s) \right\},$$

and do the same thing for m_k and M_k.

With this modification, the argument in the proof of Theorem 2.1 still applies to the neutral equation (3.1).

To provide one more example, we consider a semilinear second order parabolic partial differential equation.

Let Ω be the interior of a smooth closed bounded n-dimensional submanifold $\bar{\Omega} \subseteq \mathbf{R}^n$ with boundary $\partial\Omega$. Consider the following semilinear parabolic initial-boundary value problem with Neumann boundary condition.

$$\frac{\partial u}{\partial t} = Au + f(x, u, \nabla u), \qquad t > 0, x \in \bar{\Omega}, \tag{3.4a}$$

$$\frac{\partial u}{\partial \eta} = 0, \qquad x \in \partial\Omega, t \geqslant 0, \tag{3.4b}$$

$$u(x, 0) = v(x), \qquad x \in \bar{\Omega}, \tag{3.4c}$$

where A is a second order uniformly strongly elliptic differential operator of the form

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\, D_i\, D_j + \sum_{k=1}^{n} b_k(x)\, D_k, \tag{3.5}$$

$D_j = \partial/\partial x_j$, $a_{ij} = a_{ji}$, $b_k : \bar{\Omega} \to \mathbf{R}$ are smooth, $f : \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz such that $f \in C^{1+\alpha}(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$ with $0 < \alpha < 1$, $\nabla u = (D_1 u, ..., D_n u)$ is the spatial gradient, $\eta : \partial\Omega \to \mathbf{R}^n$ is a smooth outward pointing nowhere tangent vector field on $\partial\Omega$.

We assume that

$$f(x, c, 0) = 0 \qquad \text{on } \bar{\Omega}$$

for any given $c \in \mathbf{R}$, so that each constant function is a solution to (3.4a)–(3.4b).

Note that, without loss of generality, one may incorporate lower order terms of (3.5) into the nonlinearity f in order to reduce A to a linear symmetric operator, if necessary.

Obviously, the above general conditons are satisfied, in particular, by the Burgers equation in one space-variable

$$u_t + uu_x = \varepsilon u_{xx}, \qquad \varepsilon > 0.$$

A solution flow for (3.4) is a dynamical system (ϕ, X), where X is a space of real valued functions on $\bar{\Omega}$, such that for each $v \in X$, a solution to Eqs. (3.4a)–(3.4c) is obtained by

$$u(x, t) = (\phi_t)(x).$$

Let $C^1_B(\bar{\Omega})$ denote the Banach subspace of $C^1(\bar{\Omega})$-functions v such that

$$\frac{\partial v}{\partial \eta}(x) = 0, \qquad x \in \partial\Omega.$$

Let X be a linear subspace of $C^1_B(\bar{\Omega})$ with a topology that makes the inclusion $X \subseteq C^1_B(\bar{\Omega})$ continuous. Then X is an ordered space with function ordering. According to [15, Theorem 4.3] the dynamical system (ϕ, X) is strongly monotone and compact. Bounded solutions to Eq. (3.4) are defined for all $t \geqslant 0$.

To ensure precompactness of bounded orbits for Eq. (3.4), we suppose that there exists a function $d : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$|f(x, u, w)| \leqslant d(\rho)(1 + |w|^2)$$

for every $\rho \geqslant 0$ and $(x, u, w) \in \bar{\Omega} \times [-\rho, \rho] \times \mathbf{R}^n$. This is the so called Nagumo type growth condition in the spatial gradient. This condition and imbeddings imply boundedness of the (partial) derivatives of the solution to (3.4a)–(3.4c) if the solution itself is bounded (see, e.g., [1]).

Boundedness of the solution to (3.4a)–(3.4c) follows from the method of upper and lower solutions, where the upper and lower bounds are respectively given by the maximum and minimum values of the initial condition v in (3.4c). (See, e.g., [1, Sect. 2] and Refs. therein.)

Therefore, by our general convergence and oscillation theorems, one has the following result.

THEOREM 3.2. *Let $v \in C^{2+\alpha}(\bar{\Omega})$ be such that*

$$\frac{\partial v}{\partial \eta}(x) = 0, \qquad x \in \partial\Omega.$$

Then the solution $u(x, t)$ of (3.4a)–(3.4c) converges to a constant function as $t \to \infty$, that is, there exists a constant $c = c(v) \in \mathbf{R}$ such that $\lim_{t\to\infty} u(x, t) = c$ uniformly on $\bar{\Omega}$.

Furthermore, if v is not a constant function, then for any $t \geq 0$ there exist $y, z \in \bar{\Omega}$ such that

$$u(y, t) > c \qquad \text{and} \qquad u(z, t) < c.$$

The oscillation part follows from Theorem 2.2 and a contradiction argument which uses backwards uniqueness for the initial-boundary value problem for parabolic partial differential Eqs. (3.4a)–(3.4b), where the associated initial condition would be considered at the *first* time $\tau > 0$ such that $u(x, \tau) = c$ for all $x \in \bar{\Omega}$, if it is assumed that such a τ exists from Theorem 2.2 (see, e.g., [25, pp. 167–170] and Refs. therein.)

Our general convergence principle cannot be applied to the case when M is not compact (for instance $M = \mathbf{R}^n$) since the proof of Theorem 2.1 requires the compactness of the considered space M so that the supremum and infimum of a function $u \in C^0(M)$ can be attained at points in M, and each sequence in M has a convergent subsequence. However, the idea included in the argument of the proof of that theorem can also be adapted for general cases.

To show this, we consider the convergence problem for solutions of a parabolic partial differential equation defined on $\mathbf{R}^n$. We follow [15, Sect. 5] for notations and preliminary results. We consider a strongly monotone semiflow in spaces of almost periodic functions.

Denote by $UC^1(\mathbf{R}^n)$ the Banach space of C^1-mappings $u : \mathbf{R}^n \to \mathbf{R}$ such that u and its first partial derivatives are bounded and uniformly continuous with the usual C^1-norm denoted by $\|\cdot\|_1$. Obviously, $UC^1(\mathbf{R}^n)$ is ordered with pointwise ordering. We define $u \gg 0$ if and only if there exists a $c \in \mathbf{R}$ such that $u(x) \geq c > 0$ for all $x \in \mathbf{R}^n$. The subspace of almost periodic C^1-functions

$$AP^1(\mathbf{R}^n) \subseteq UC^1(\mathbf{R}^n)$$

is the set of those $v : \mathbf{R}^n \to \mathbf{R}$ whose orbit under the group of translations G of $\mathbf{R}^n$ has compact closure in $UC^1(\mathbf{R}^n)$.

Now, consider a second order parabolic initial value problem

$$\frac{\partial u}{\partial t} = Au + f(x, u, \nabla u), \qquad t > 0, x \in \mathbf{R}^n, \tag{3.6a}$$

$$u(x, 0) = v(x), \qquad x \in \mathbf{R}^n, \tag{3.6b}$$

where $v \in UC^1$,

$$A = \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j + \sum_{k=1}^{n} b_k(x) D_k$$

and

$$f : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$$

satisfy the following conditions.

(1) The coefficients $a_{ij} = a_{ji}$, $b_i : \mathbf{R}^n \to \mathbf{R}$ are in $UC^2(\mathbf{R}^n)$ and their partial derivatives satisfy a uniform Hölder condition for some exponent in the range $0 < \alpha < 1$.

(2) There is a constant $c \in \mathbf{R}$ such that

$$\det[a_{ij}] \geqslant c > 0, \qquad x \in \mathbb{R}^n$$

and

$$\langle a(x)\xi, \xi \rangle \geqslant c \langle \xi, \xi \rangle, \qquad x, \xi \in \mathbf{R}^n$$

where $a(x) = [a_{ij}(x)]$ and $\langle \cdot, \cdot \rangle$ is the usual inner product.

(3) The mapping f is locally Lipschitz and f is locally of class $C^{1+\alpha}$.

Under the above conditions, Mora [23] proved that the initial value problem (3.6) has a monotone solution flow in $UC^1(\mathbf{R}^n)$.

In addition to (1)–(3), suppose

(4) there is a closed subgroup (of translations of $\mathbf{R}^n$) $H \subseteq G$ with compact quotient G/H, such that the coefficients of A are invariant under composition with elements of H, and also

$$f(x, y, \xi) = f(hx, y, \xi)$$

for all $(x, y, \xi) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, $h \in H$.

Hirsch [15, Theorem 5.4] proved that the solution flow to (3.6) leaves $AP^1(\mathbf{R}^n)$ invariant and restricts to a strongly monotone flow in $AP^1(\mathbf{R}^n)$, that is, restricted to $AP^1(\mathbf{R}^n)$, the solution flow defines a strongly monotone dynamical system.

As above we assume that

$$f(x, y, 0) = 0 \qquad \text{for} \quad (x, y) \in \mathbf{R}^n \times \mathbf{R}$$

so that each constant function is a solution to Eq. (3.6a).

Once more, by using the characterization of supremum and infimum, the strong monotonicity of the semiflow on $AP^1(\mathbf{R}^n)$, and the fact that each constant function is an equilibrium point for the semiflow, it is relatively easy to show (by contradiction for instance) that constant functions are the *only* equilibrium points for the semiflow.

Even though Theorem 2.1 cannot be directly applied to this case because of the noncompactness of $\mathbf{R}^n$. We modify the method contained in the proof of that theorem as follows.

For any $v \in AP^1(\mathbf{R}^n)$, $x \in \mathbf{R}^n$, and $t \in \mathbf{R}$, let $\phi(t, v)(x) = u(t, v)(x)$, where $u(\cdot, v)$ is the solution of Eq. (3.6). Suppose $\{\phi_t\}_{t \geqslant 0}$ *is a precompact orbit.* Let

$$m(t) = \inf_{x \in \mathbf{R}^n} \phi(t, v)(x) \qquad \text{and} \qquad M(t) = \sup_{x \in \mathbf{R}^n} \phi(t, v)(x).$$

Then by monotonicity of $\phi, m(t)$ is nondecreasing and $M(t)$ is nonincreasing, and

$$\inf_{x \in \mathbf{R}^n} v(x) \leqslant m(t) \leqslant M(t) \leqslant \sup_{x \in \mathbf{R}^n} v(x).$$

Therefore

$$\lim_{t \to \infty} m(t) = a \qquad \text{and} \qquad \lim_{t \to \infty} M(t) = b \text{ exist.}$$

For any $\psi \in \omega(v)$, we can find a sequence $t_k \to \infty$ such that $\phi(t_k, v)(x) \to \psi(x)$ in $UC^1(\mathbf{R}^n)$-topology. Hence,

$$a \leqslant \psi(x) \leqslant b \qquad \text{for all} \quad x \in \mathbf{R}^n.$$

Evidently $\omega(v) \subseteq AP^1(\mathbf{R}^n)$ since G acts isometrically on $UC^1(\mathbf{R}^n)$, and thus $AP^1(\mathbf{R}^n)$ is closed in $UC^1(\mathbf{R}^n)$.

By the invariance property of $\omega(v)$ there exists $v \in \omega(v)$ such that $\psi = \phi(1, v)$. If $\psi(x) \neq a$ on $\mathbf{R}^n$, then $v \neq a$ on $\mathbf{R}^n$, and thus $v > a$.

By strong monotonicity

$$\psi = \phi(1, v) \gg \hat{a},$$

that is

$$\psi(x) \geqslant a + c$$

for all $x \in \mathbf{R}^n$, where $c > 0$ is a constant.

On the other hand, $\lim_{k\to\infty} \inf_{x\in\mathbf{R}^n} \phi(t_k, v)(x) = a$ uniformly on $\mathbf{R}^n$. Therefore there exists a sequence $\{x_k\}$ in $\mathbf{R}^n$ such that

$$\lim_{k\to\infty} \phi(t_k, v)(x_k) = a.$$

This is a contradiction to the fact that $\lim_{k\to\infty} \phi(t_k, v)(x) = \psi(x)$ in $UC^1(\mathbb{R}^n)$-topology and $\inf_{x\in\mathbf{R}^n} \psi(x) > a$. Therefore $\psi(x) = a$ on $\mathbf{R}^n$; that is, $\lim_{t\to\infty} \phi(t, v)(x) = a$ uniformly on $\mathbf{R}^n$. By using a similar argument, it is easily shown that $\psi(x) = b$ on $\mathbf{R}^n$. The proof is complete.

Likewise, the argument in the proof of Theorem 2.2 can be accordingly modified. Thus, we have proved the following result.

THEOREM 3.3. *Let $v \in AP^1(\mathbf{R}^n)$ be such that v has a precompact orbit. Then the solution $u(x, t)$ of* (3.6a)–(3.6b) *converges to a constant function as $t \to \infty$, that is, there exists $c \in \mathbf{R}$ such that $\lim_{t\to\infty} u(x, t) = c$ uniformly on $\mathbf{R}^n$.*

Furthermore, if v is not a constant function, then for any $t \geq 0$ there exist $y, z \in \mathbf{R}^n$ such that

$$u(y, t) > c \qquad \text{and} \qquad u(z, t) < c.$$

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