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Existence of periodic solutions to integro-differential equations of neutral type via limiting equations

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Abstract

In this paper, we present some results on the existence of periodic solutions to Volterra integro-differential equations of neutral type. The main idea is to show the convergence of an equibounded sequence of periodic solutions of certain limiting equations which are of finite delay. This makes it possible to apply the existing Liapunov–Razumikhin technique for neutral equations with finite delay to obtain existence of periodic solutions of Volterra neutral integro-differential equations (of infinite delay). Some comparisons between our results and the existing ideas are also provided.

1. *Introduction*

The purpose of this paper is to provide an existence theorem for periodic solutions of the following Volterra integro-differential equation of neutral type:

$$\frac{d}{dt}\left(x(t)-\int_{-\infty}^{t} C(t,s,x(s))\,ds\right)=H(t,x(t))+\int_{-\infty}^{t} G(t,s,x(s))\,ds, \tag{1·1}$$

where $H(t,x)$, $C(t,s,x)$ and $G(t,s,x)$ are $\mathbb{R}^n$-valued continuous functions and there exists a constant $T>0$ such that

$$H(t+T,x)=H(t,x),\quad C(t+T,s+T,x)=C(t,s,x)\quad\text{and}\quad G(t+T,s+T,x)=G(t,s,x)$$

for $-\infty<s\leqslant t<\infty$ and $x\in\mathbb{R}^n$.

Our main idea, motivated by [3], is to regard the following neutral equations with finite delay

$$\frac{d}{dt}\left(x(t)-\int_{t-kT}^{t} C(t,s,x(s))\,ds\right)=H(t,x(t))+\int_{t-kT}^{t} G(t,s,x(s))\,ds \tag{1·2}$$

as limiting equations of (1·1) and to demonstrate the convergence (to a T-periodic solution of (1·1)) of a certain equibounded sequence of periodic solutions $\{x_k(t)\}_{k=1}^{\infty}$ of (1·2). Applying Horn's asymptotic fixed point theorem in a standard way, we will show that the existence of such an equibounded sequence of periodic solutions follows from the boundedness and ultimate boundedness of solutions to (1·2). Consequently, we can apply the existing Liapunov–Razumikhin technique for boundedness of solutions of neutral equations with finite delay (cf. [**10**]) to obtain sufficient conditions guaranteeing the existence of periodic solutions of neutral equation with infinite delay.

Our existence theorem represents an extension of that in [**3**] from retarded equations to neutral equations. Due to the addition of the neutral term

$$\textstyle\int_{-\infty}^{t} C(t,s,x(s))\,ds,$$

our approach applies to a class of integral equations

$$x(t) = \int_{-\infty}^{t} C(t,s,x(s))\,ds + h(t) \tag{1·3}$$

as well, where h is a given T-periodic continuous $\mathbb{R}^n$-valued function. However, the addition of this term creates certain difficulties in our study of the existence of periodic solutions to (1·1). In particular, solutions of (1·1) are no longer differentiable, and consequently, the uniform continuity of solutions does not necessarily follow directly from the boundedness. Moreover, since $C \not\equiv 0$, the qualitative behaviour of solutions of (1·1) depend heavily upon that of the solutions of the associated integral equation (1·3). For details, we refer to [**4**] for neutral equations with finite delay, and to [**8**, **9**, **13–18**] for neutral equations with infinite delay.

It has been proved that for retarded functional differential equations and for some integral equations, the existence of a periodic solution is implied by the uniform boundedness and uniform ultimate boundedness of solutions in phase spaces satisfying certain well-known phase space axioms formulated in [**5**] (see [**1–3**, **7**, **19**] and references therein). One should be able to extend this result to neutral equations of type (1·1) without using the limiting equations (1·2) at all. Such an extension can, however, be applied to a concrete example only when (i) an appropriate phase space is chosen and (ii) the uniform boundedness and uniform ultimate boundedness of solutions in the phase space are verified. Even for retarded equations with infinite delay, the choice of an appropriate phase space is not a trivial task. Moreover, little has been done for the boundedness of solutions of neutral equations with infinite delay. Our present research provides an alternative for establishing the existence of periodic solutions of neutral equations with infinite delay. Namely, by going through the limiting equations (1·2), we avoid the afore-mentioned two technical tasks but still obtain the existence of periodic solutions of (1·1) from the existing theory of periodic solutions of neutral equations with finite delay, such as the Liapunov–Razumikhin technique developed in [**10**]. For details, we refer to Remark 4·5.

The rest of this paper is organized as follows. In Section 2, we present our existence results and relate the existence of periodic solutions of (1·1) to the uniform boundedness and uniform ultimate boundedness of solutions of (1·2). A relation between the boundedness of solutions of (1·2) and the qualitative behaviour of solutions of the integral equation (1·3) is given in Section 3 where some sufficient conditions for boundedness of solutions of (1·2) are provided in the spirit of the Liapunov–Razumikhin technique. Finally, in Section 4 we briefly discuss the assumptions in our major existence theorem and illustrate the difference between our major result and that in [**7**] by a simple example.

2. *Main results*

Consider the Volterra integro-differential equation of neutral type

$$\frac{d}{dt}\left(x(t)-\int_{-\infty}^{t} C(t,s,x(s))\,ds\right)=H(t,x(t))+\int_{-\infty}^{t} G(t,s,x(s))\,ds, \tag{2·1}$$

where $H(t,x)$, $C(t,s,x)$ and $G(t,s,x)$ are $\mathbb{R}^n$-valued continuous functions for $-\infty < s \leqslant t < \infty$ and $x \in \mathbb{R}^n$, and there exists a constant $T>0$ such that

$$H(t,x)=H(t+T,x), \quad C(t+T,s+T,x)=C(t,s,x), \quad \text{and} \quad G(t+T,s+T,x)=G(t,s,x)$$

for $-\infty < s \leqslant t < \infty$ and $x \in \mathbb{R}^n$.

Let BC denote the Banach space of bounded continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$ with the supremum norm $|\cdot|_{BC}$. By a solution of (2·1) through $(t_0,\phi)\in\mathbb{R}\times BC$, we mean a continuous function $x:\mathbb{R}\to\mathbb{R}^n$ such that $x_{t_0}(s):=x(t_0+s)$ is identical to $\phi(s)$ for $s\leqslant 0$, $x(t)-\int_{-\infty}^{t} C(t,s,x(s))\,ds$ is continuously differentiable and (2·1) is satisfied for $t\geqslant t_0$. For the remainder of this section, we assume that for every $(t_0,\phi)\in\mathbb{R}\times BC$ there exists a unique solution of (2·1) through (t_0,ϕ), denoted by $x(t;t_0,\phi)$. Moreover, we assume that solutions of (2·1) depend continuously on initial functions in the sense that for any $t_0\in\mathbb{R}$, $\phi\in BC$, $A>0$ and $\epsilon>0$ there exists $\delta>0$ such that $|x(t;t_0,\psi)-x(t;t_0,\phi)|<\epsilon$ for all $t\in[t_0,t_0+A]$ provided that $\psi\in BC$ and $|\psi-\phi|_{BC}<\delta$. The fundamental existence, uniqueness and continuous dependence theory for neutral equations has been developed in [**13**, **14**, **18**]. Some new results in this theory will be presented in Section 4.

Denote by X_T the Banach space consisting of all T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ and endowed with the supremum norm $|\cdot|_T$. We assume:

(A 1) There exists a constant $M>0$ such that for each positive integer k, the equation

$$\frac{d}{dt}\left(y(t)-\int_{t-kT}^{t} C(t,s,y(s))\,ds\right)=H(t,y)+\int_{t-kT}^{t} G(t,s,y(s))\,ds \tag{2·2$_k$}$$

has a T-periodic solution $y_k(t)$ such that $|y_k|_T\leqslant M$.

(A 2) The mapping $P:X_T\to X_T$ defined by

$$P(\phi)(t)=\phi(0)-\int_{-\infty}^{0} C(0,s,\phi(s))\,ds+\int_{-\infty}^{t} C(t,s,\phi(s))\,ds$$
$$+\int_{0}^{t} H(u,\phi(u))\,du+\int_{0}^{t}\int_{-\infty}^{u} G(u,s,\phi(s))\,ds\,du,$$

for $(t,\phi)\in[0,T]\times X_T$, is continuous.

(A 3) $\sup_{t\in[0,T],\,\phi\in X_T(M)}\left|\int_{t-kT}^{t} G(t,s,\phi(s))\,ds\right|<\infty$ and the sequence

$$\left\{\int_{t-kT}^{t} C(t,s,\phi(s))\,ds\right\}_{k=1}^{\infty}$$

is equicontinuous on $[0,T]$ for every given $\phi\in X_T(M)$, where

$$X_T(M)=\{\phi\in X_T;|\phi|_T\leqslant M\}.$$

(A 4) $$\lim_{k\to\infty}\ \sup_{t\in[0,T],\,\phi\in X_T(M)}\left(\left|\int_{-\infty}^{t-kT} G(t,s,\phi(s))\,ds\right|+\left|\int_{-\infty}^{t-kT} C(t,s,\phi(s))\,ds\right|\right)=0.$$

(A 1) is our major assumption. It will be shown below that this assumption is satisfied if solutions to $(2\cdot2)_k$ satisfy certain boundedness conditions which can be verified by the Liapunov–Razumikhin technique. (A 4) is a certain 'fading memory' assumption, in the sense of [2], which represents the reality that a system should remember its past, but that the memory should fade with time. In Section 4, we will illustrate that (A 2) and (A 3) are very weak assumptions which are satisfied if C and G satisfy certain integrability conditions.

We are now in the position to state our main result on the existence of T-periodic solutions of the equation (2·1) as a limit of T-periodic solutions to the system $(2\cdot2)_k$ with finite delay as $k\to\infty$.

THEOREM 2·1. *If assumptions* (A 1)–(A 4) *hold, then equation* (2·1) *has a T-periodic solution.*

Proof. Let $\{y_k(t)\}_{k=1}^{\infty}$ be a sequence of T-periodic solutions of equation $(2\cdot2)_k$ with $|y_k(t)|\leqslant M$ for all integers k and $t\in[0,T]$. Then (A 3) implies that

$$|(d/dt)\,(y(t)-\textstyle\int_{t-kT}^{t}C(t,s,y_k(s))\,ds)|\leqslant N$$

for some constant $N>0$ and for $t\in[0,T]$, for $k=1,2,\ldots$. Therefore for all $t_1,t_2\in[0,T]$, we have

$$\left|y_k(t_2)-y_k(t_1)+\int_{t_2-kT}^{t_2}C(t_2,s,y_k(s))\,ds-\int_{t_1-kT}^{t_1}C(t_1,s,y_k(s))\,ds\right|\leqslant N|t_1-t_2|$$

which yields

$$|y_k(t_2)-y_k(t_1)|\leqslant\left|\int_{t_2-kT}^{t_2}C(t_2,s,y_k(s))\,ds-\int_{t_1-kT}^{t_1}C(t_1,s,y_k(s))\,ds\right|+N|t_1-t_2|.$$

Note that the family of functions $\{\int_{t-kT}^{t}C(t,s,y_k(s))\,ds\}_{k=1}^{\infty}$ is equicontinuous. It follows that $\{y_k(t)\}$ is also equicontinuous. By the Ascoli–Arzela theorem, we can assume, without loss of generality, that $\{y_k(t)\}$ converges to a continuous T-periodic function $x(t)$, uniformly on $[0,T]$.

Due to the continuity of the map P, we have

$$\lim_{k\to\infty}P(y_k)\,(t)=P(x)\,(t),$$

for $t\in[0,T]$. On the other hand, by definition of P, for $t\in[0,T]$,

$$\begin{aligned}P(y_k)\,(t)&=y_k(0)-\int_{-\infty}^{0}C(0,s,y_k(s))\,ds+\int_{-\infty}^{t}C(t,s,y_k(s))\,ds+\int_0^t H(u,y_k(u))\,du\\&\quad+\int_0^t\int_{-\infty}^{u}G(u,s,y_k(s))\,ds\,du\\&=\left\{y_k(0)-\int_{-kT}^{0}C(0,s,y_k(s))\,ds+\int_{t-kT}^{t}C(t,s,y_k(s))\,ds\right.\\&\quad\left.+\int_0^t\int_{u-kT}^{u}G(u,s,y_k(s))\,ds\,du+\int_0^t H(u,y_k(u))\,du\right\}\\&\quad+\int_{-\infty}^{t-kT}C(t,s,y_k(s))\,ds-\int_{-\infty}^{-kT}C(0,s,y_k(s))\,ds\\&\quad+\int_0^t\int_{-\infty}^{u-kT}G(u,s,y_k(s))\,ds\,du.\end{aligned}$$

Notice that $y_k(t)$ satisfies equation $(2{\cdot}2)_k$, so we obtain

$$P(y_k)(t) = y_k(t) + \int_{-\infty}^{t-kT} C(t,s,y_k(s))\,ds - \int_{-\infty}^{-kT} C(0,s,y_k(s))\,ds$$
$$+ \int_0^t \int_{-\infty}^{u-kT} G(u,s,y_k(s))\,du.$$

Consequently, it follows from $(A\ 4)$ that

$$\lim_{k\to\infty} P(y_k)(t) = x(t),$$

for $t \in [0,T]$. Therefore $x(t) = P(x)(t)$ for $t \in [0,T]$. This means that $x(t)$ is a solution of equation $(2{\cdot}1)$. Clearly $x(t)$ is T-periodic. This completes the proof.

To guarantee the existence of equibounded T-periodic solutions for the family of equations $(2{\cdot}2)_k$ with finite delay, we introduce the following concepts.

Definition 2·2. Let $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ be a given continuous function. A family of functions $\{f(t)\}$ is called a δ-equicontinuous family of functions on $[a,b]$ if each function f is defined on $[a,b]$ and, for any $\epsilon > 0$ and $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\epsilon)$, we have $|f(t_2) - f(t_1)| < \epsilon$.

Clearly, if $\beta: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $0 \leqslant \beta(x) \leqslant \delta(x)$ for all $x > 0$, then a δ-equicontinuous family of functions is also a β-equicontinuous family of functions.

Definition 2·3. Let $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ be a given continuous function. A function f defined on $[a,b]$ is said to be δ-uniformly continuous on $[a,b]$ if, for any $\epsilon > 0$ and t_1, $t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\epsilon)$, we have $|f(t_1) - f(t_2)| < \epsilon$.

Clearly, a family of functions consisting of all δ-uniformly continuous functions on $[a,b]$ is a δ-equicontinuous family of functions. Conversely, every function in a δ-equicontinuous family is δ-uniformly continuous.

THEOREM 2·4. *Suppose that*

(i) *for each integer k and each given initial datum, the solution of* $(2{\cdot}2)_k$ *is unique, and solutions are continuous with respect to initial data;*

(ii) *the solutions of equations* $(2{\cdot}2)_k$ *are uniformly bounded and uniformly ultimately bounded for B at* $t = 0$;

(iii) *for any* $L > 0$ *there exists* $M > 0$ *such that for all continuous functions* $x: \mathbb{R} \to \mathbb{R}^n$ *with* $\sup_{t\in\mathbb{R}} |x(t)| \leqslant L$, $\int_{-\infty}^{t} |G(t,s,x(s))\,ds| \leqslant M$ *and* $\{\int_{t-kT}^{t} C(t,s,x(s))\,ds\}$ *is a* δ-*equicontinuous family of functions on* $\mathbb{R}_+$ *for some* δ.

Then for every $k = 1, 2, \ldots,$ *equation* $(2{\cdot}2)_k$ *has a T-periodic solution* $y_k(t)$ *with* $|y_k(t)| \leqslant B$ *for* $t \in [0,T]$.

Here and in what follows, solutions of equations $(2{\cdot}2)_k$ are said to be uniformly bounded at $t = 0$ if for any $B_1 > 0$ there exists $B_2 > 0$ such that $\phi \in BC$ and $|\phi|_{BC} \leqslant B_1$ imply $|y(t,\phi,k)| \leqslant B_2$ for $t \geqslant 0$, where $y(t,\phi,k)$ denotes the solution of $(2{\cdot}2)_k$ through $(0,\phi)$. Moreover, solutions of $(2{\cdot}2)_k$ are uniformly ultimately bounded for B at $t = 0$ if for any $B_3 > 0$ there exists $T(B_3) > 0$ such that $\phi \in BC$ and $|\phi|_{BC} \leqslant B_3$ imply that $|y(t,\phi,k)| < B$ for all $t \geqslant T(B_3)$.

Proof. For any fixed k, we set

$$Y_k = \{\phi : [-kT, 0] \to \mathbb{R}^n; \phi \text{ is continuous}\} \quad \text{and} \quad |\phi|_k = \sup_{-kT \leqslant s \leqslant 0} |\phi(s)|.$$

Then $(Y_k, |\cdot|_k)$ is a Banach space.

Evidently $y(t, \phi, k)$ depends on the restriction of ϕ to $[-kT, 0]$ only. Therefore from the uniform boundedness assumption, we can find $\beta_3 \geqslant \beta_2 \geqslant \beta_1 \geqslant B$ such that

$$\begin{aligned} &\phi \in Y_k \text{ with } |\phi|_k \leqslant B \quad \text{implies} \quad |y(t, \phi, k)| < \beta_1 \text{ for } t \geqslant 0, \\ &\phi \in Y_k \text{ with } |\phi|_k \leqslant \beta_1 \quad \text{implies} \quad |y(t, \phi, k)| < \beta_2 \text{ for } t \geqslant 0, \\ &\phi \in Y_k \text{ with } |\phi|_k \leqslant \beta_2 \quad \text{implies} \quad |y(t, \phi, k)| < \beta_3 \text{ for } t \geqslant 0, \end{aligned}$$

On the other hand, from the uniform ultimate boundedness assumption, we can find a positive integer m such that

$$|y(t, \phi, k)| < B \quad \text{for } t \geqslant mT \tag{2·3}$$

if $\phi \in Y_k$ and $|\phi|_k < B_1$. Moreover, by the uniform boundedness assumption and (iii), there exists a constant $M > 0$ such that

$$\left|\frac{d}{dt}\left(y(t, \phi, k) - \int_{t-kT}^{t} C(t, s, y(s, \phi, k))\, ds\right)\right| \leqslant 2M$$

for all $\phi \in Y_k$ with $|\phi|_k < \beta_3$, which implies that $\{y(t, \phi, k)\}$ is a δ_M-equicontinuous family of functions of $t \geqslant 0$, where $\delta_M(\epsilon) = \min\{\delta(\epsilon/2), \epsilon/(4M)\}$. We now define three subsets of Y_k as follows:

$$\begin{aligned} S_0^k &= \{\phi \in Y_k; |\phi|_k \leqslant B \text{ and } \phi \text{ is } \delta_M\text{-uniformly continuous}\}; \\ S_1^k &= \{\phi \in Y_k; |\phi|_k < \beta_1 \text{ and } \phi \text{ is } \delta_M\text{-uniformly continuous}\}; \\ S_2^k &= \{\phi \in Y_k; |\phi|_k \leqslant \beta_2 \text{ and } \phi \text{ is } \delta_M\text{-uniformly continuous}). \end{aligned}$$

Then $S_0^k \subseteq S_1^k \subseteq S_2^k$ are convex subsets of Y_k, S_0^k and S_2^k are compact and S_1^k is relatively open in S_2^k.

We define the Poincaré map $f: S_2^k \to Y_k$ as follows:

$$f(\phi)(t) = y(t+T, \phi, k) \quad \text{for } t \in [-kT, 0].$$

Note that for $\phi \in S_1^k$ we have $|f(\phi)|_k < \beta_2$. Thus $f^2(\phi)$ is well-defined. We claim that $f^2(\phi)(t) = y(t+2T, \phi, k)$ for $t \in [-kT, 0]$. In fact, for $\phi \in S_1^k$, the function $f(\phi)$ is δ_M-uniformly continuous and thus $f(\phi) \in S_2^k$. Moreover, $y(t+T, \phi, k)$ and $y(t, f(\phi), k)$ are both solutions of $(2{\cdot}2)_k$ with the same initial datum $(0, f(\phi))$. So by uniqueness, $y(t+T, \phi, k) = y(t, f.f(\phi), k)$ for all $t \in [-kT, \infty)$. In particular,

$$y(t+2T, \phi, k) = y(t+T, f(\phi), k) \quad \text{for } -kT \leqslant t \leqslant 0,$$

or equivalently,

$$f^2(\phi)(t) = y(t+2T, \phi, k) \quad \text{for } -kT \leqslant t \leqslant 0.$$

Repeating the above argument, we see that $f^j(S_1^k) \subseteq S_2^k$ for all positive integers j. Similarly, $f^j(S_0^k) \subseteq S_1^k$. It is also clear from (2·3) that $f^j(S_1^k) \subseteq S_0^k$ for all $j \geqslant m$. Furthermore, f is a continuous map since solutions of $(2{\cdot}2)_k$ are continuous with respect to initial data. Applying Horn's fixed point theorem, we obtain a fixed point $\phi \in S_0^k$ of f. That is, $y(t+T, \phi, k) = \phi(t)$ for $t \in [-kT, 0]$. Again by uniqueness, $y(t, \phi, k) = y(t+T, \phi, k)$, and thus $y(t, \phi, k)$ is T-periodic. This completes the proof.

3. *Uniform boundedness and uniform ultimate boundedness of solutions*

We now discuss the uniform boundedness and uniform ultimate boundedness of solutions to parametrized neutral functional differential equations of the following type

$$\frac{d}{dt}D_\alpha(t,x_t) = F_\alpha(t,x_t), \tag{3·1}$$

where $\alpha \in \Lambda$ for some index set Λ, $D_\alpha : \mathbb{R}\times C_\alpha \to \mathbb{R}^n$ and $F_\alpha : \mathbb{R}\times C_\alpha \to \mathbb{R}^n$ are given continuous functionals such that the standard initial value problem to (3·1) is well posed, $C_\alpha = \{\phi;\phi:[-\nu_\alpha,0]\to\mathbb{R}^n \text{ is continuous}\}$ is endowed with the supremum norm $|\cdot|_\alpha$ and $\nu_\alpha \geqslant 0$ is a constant. In particular, if $\Lambda = \{k\}_{k=1}^\infty$ and $\nu_k = kT$, then we arrive at $(2·2_k)$ with specific D_k and F_k.

Definition 3·1. A family of maps $\{D_\alpha\}_{\alpha\in\Lambda}$ is said to be uniformly bounded if there exists an unbounded non-decreasing function $S:\mathbb{R}_+\to\mathbb{R}$ such that for any $t_0\in\mathbb{R}$, $\alpha\in\Lambda$, $\phi\in C_\alpha$, $h\in C([t_0,\infty);\ \mathbb{R}^n)$, $H>0$ with $|\phi|_\alpha < H$ and $\sup_{t\geqslant t_0}|h(t)| \leqslant H$, the solution, denoted by $x^\alpha(t)$, of the equation

$$D_\alpha(t,x_t) = h(t), \quad x_{t_0} = \phi \tag{3·2}$$

satisfies $|x^\alpha(t)| \leqslant S(H)$ for all $t\geqslant t_0$.

Definition 3·2. A family of maps $\{D_\alpha\}_{\alpha\in\Lambda}$ is said to be quasi-uniformly ultimately bounded if, there exists an unbounded non-decreasing function $B:\mathbb{R}_+\to\mathbb{R}$ such that for any $M_1, M_2 > 0$ there exists $T(M_1,M_2) > 0$ so that for any $t_0\in\mathbb{R}$, $\alpha\in\Lambda$, $\phi\in C_\alpha$ and $h\in C([t_0,\infty);\ \mathbb{R}^n)$ with $\sup_{t\geqslant 0}|h(t)| \leqslant M_2$ and $\sup_{t\geqslant t_0}|x_t^\alpha|_\alpha \leqslant M_1$, the solution of (3·2) satisfies $|x^\alpha(t)| \leqslant B(M_2)$ for $t \geqslant t_0 + T(M_1,M_2)$.

A uniformly bounded and quasi-uniformly ultimately bounded family of maps is said to be uniformly ultimately bounded.

The following result gives a very simple sufficient condition ensuring the uniform ultimate boundedness of $\{D_\alpha\}_{\alpha\in\Lambda}$. For a broad class of neutral equations satisfying the conditions in the next proposition, we refer to [**4**].

PROPOSITION 3·3. *Suppose that there exist positive constants K_1, K_2 and a such that for any $t_0\in\mathbb{R}$, $\alpha\in\Lambda$, $\phi\in C_\alpha$ and $h\in C([t_0,\infty);\ \mathbb{R}^n)$, the solution $x^\alpha(t)$ of* (3·2) *through* (t_0,ϕ) *satisfies*

$$|x^\alpha(t)| \leqslant K_1 e^{-a(t-t_0)}|\phi|_\alpha + K_2 \sup_{t_0\leqslant s\leqslant t}|h(s)|$$

for $t\geqslant t_0$. Then $\{D_\alpha\}_{\alpha\in\Lambda}$ is uniformly ultimately bounded.

Proof. It is clear that we can take

$$S(r) = (K_1+K_2)\,r, \quad T(M_1,M_2) = a^{-1}\left|\frac{\ln(\beta M_2)}{K_1 M_1}\right|, \quad B(r) = (\beta+K_2)\,r,$$

where $\beta > 0$ is any given constant.

In the following two propositions, we restrict our attention to the case where $\{D_\alpha\}$ is of the form

$$D_\alpha(t,\phi) = \phi(0) - \int_{t-\alpha}^{t} C(t,s,\phi(s))\,ds$$

in which $\alpha\geqslant 0$ and $\phi\in C_\alpha := C([-\alpha,0];\mathbb{R}^n)$.

PROPOSITION 3·4. *Suppose that there exist constants $B_1 \geqslant 0$ and $\beta \in [0,1)$ and an increasing function $Q: \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\int_{t_0}^{t} |C(t,s,x(s))|\,ds \leqslant \beta \max_{t_0 \leqslant s \leqslant t} |x(s)| + B_1$$

and

$$\int_{t_0-\alpha}^{t_0} |C(t,s,x(s))|\,ds \leqslant Q(\max_{t_0-\alpha \leqslant s \leqslant t_0} |x(s)|)$$

for all $t \geqslant t_0$ and $x \in C([t_0-\alpha,t];\mathbb{R}^n)$. Then $\{D_\alpha\}$ is uniformly bounded, where $S(H) = (H+Q(H)+B_1)/(1-\beta)$ for $H \geqslant 0$.

Proof. Let (α, t_0, ϕ) be given such that $|\phi|_\alpha \leqslant H$ and $\sup_{t_0 \leqslant s \leqslant t} h(s)| \leqslant H$. Denote by $t^* > t_0$ the number such that $|x^\alpha(t^*)| = \sup_{t_0 \leqslant s \leqslant t^*} |x^\alpha(s)|$. Then

$$\begin{aligned}|x^\alpha(t^*)| &\leqslant |D_\alpha(t^*, x^\alpha_{t^*})| + \left|\int_{t^*-\alpha}^{t^*} C(t^*,s,x^\alpha(s))\,ds\right| \\ &\leqslant H + \int_{t_0-\alpha}^{t_0} |C(t^*,s,\phi(s-t_0))|\,ds + \int_{t_0}^{t^*} |C(t^*,s,x^\alpha(s))|\,ds \\ &\leqslant H + Q(H) + B_1 + \beta \max_{t_0 \leqslant s \leqslant t^*} |x^\alpha(s)|,\end{aligned}$$

from which it follows that

$$|x^\alpha(t^*)| \leqslant \frac{H+Q(H)+B_1}{1-\beta} = S(H).$$

So $|x^\alpha(t)| \leqslant S(H)$ for all $t \geqslant t_0$. Therefore $\{D_\alpha\}$ is uniformly bounded.

PROPOSITION 3·5. *Assume that there exist positive constants B_1, B_2 and $\beta \in [0,1)$ such that for any $M_1, M_2 > 0$ there exists $\Gamma(M_1,M_2) > 0$ such that for any $x \in C([t_0-\alpha,\infty);\mathbb{R}^n)$ with $|x_t|_\alpha \leqslant M_1$ and $|D_\alpha(t,x_t)| \leqslant M_2$ for $t \geqslant t_0$, we have*

$$\int_{t_0-\alpha}^{t-\Gamma(M_1,M_2)} |C(t,s,x(s))|\,ds < B_1 M_2$$

and

$$\int_{t-\Gamma(M_1,M_2)}^{t} |C(t,s,x(s))|\,ds \leqslant \beta \max_{t-\Gamma(M_1,M_2) \leqslant s \leqslant t} |x(s)| + B_2$$

for $t \geqslant t_0 + \Gamma(M_1,M_2)$. Then $\{D_\alpha\}$ is quasi-uniformly ultimately bounded with $B(M_2) = a^{-1}(B_2 + (1+B_1)M_2)$, where $0 < a < 1-\beta$ is an arbitrarily given constant.

Proof. Let $t_0 \in \mathbb{R}$, $\phi \in C_\alpha$ and $h \in C([t_0-\alpha,\infty);\mathbb{R}^n)$ be given. If $x^\alpha(t)$ satisfies $|D_\alpha(t,x^\alpha_t)| \leqslant M_2$ and $|x^\alpha_t|_\alpha \leqslant M_1$ for $t \geqslant t_0$, then for $t \geqslant t_0 + \Gamma$ we have

$$\begin{aligned}|x^\alpha(t)| &\leqslant \int_{t-\alpha}^{t} |C(t,s,x^\alpha(s))|\,ds + |D(t,x^\alpha_t)| \leqslant M_2 + \int_{t_0-\alpha}^{t} |C(t,s,x^\alpha(s))|\,ds \\ &\leqslant M_2 + \left(\int_{t_0-\alpha}^{t-\Gamma} + \int_{t-\Gamma}^{t}\right) |C(t,s,x^\alpha(s))|\,ds \leqslant M_2 + B_1M_2 + B_2 + \beta \max_{t-\Gamma \leqslant s \leqslant t} |x^\alpha(s)|.\end{aligned} \tag{3·3}$$

If $\beta = 0$, then we are finished. So we assume that $\beta \neq 0$.

Consider the intervals $I_n = [t_0 + n\Gamma, t_0 + (n+1)\Gamma]$ for $n \geqslant 2$, and choose $t_n \in I_n$ so that $|x^\alpha(t_n)| = \max_{s \in I_n} |x^\alpha(s)|$. By (3·3), we get

$$|x^\alpha(t_n)| \leqslant B_2 + (1+B_1)M_2 + \beta \max_{t_n - \Gamma \leqslant s \leqslant t_n} |x^\alpha(s)|. \tag{3·4}$$

Put $L = B_2 + (1+B_1)M_2$. We examine two possible cases.

(I) For some $t^* \in [t_n - \Gamma, t_n + n\Gamma]$ we have $|x^\alpha(t^*)| = \max_{t_n - \Gamma \leqslant s \leqslant t_n} |x^\alpha(s)|$. Then (3·4) implies that

$$|x^\alpha(t_n)| \leqslant L + \beta |x^\alpha(t_{n-1})|. \tag{3·5}$$

(II) Otherwise, (3·4) gives $|x^\alpha(t_n)| \leqslant L + \beta |x^\alpha(t_n)|$, or

$$|x^\alpha(t_n)| \leqslant \frac{L}{1-\beta}. \tag{3·6}$$

If (II) happens for some integer $K \geqslant 2$, then for all $n \geqslant K$ and in both cases,

$$|x^\alpha(t_n)| \leqslant \frac{L}{1-\beta} < \frac{L}{a} = B(M_2).$$

If there is no such integer, then for all $n \geqslant 2$, (I) happens. Thus from (3·5) it follows that

$$\begin{aligned} |x^\alpha(t_n)| &\leqslant L + \beta |x^\alpha(t_{n-1})| \leqslant L(1 + \beta + \beta^2 + \ldots + \beta^{n-3}) + \beta^{n-2} |x^\alpha(t_2)| \\ &\leqslant \frac{L}{1-\beta} + \beta^{n-2} M_1. \end{aligned} \tag{3·7}$$

Taking

$$N = \left(3 + \left|\ln L\left(\frac{1}{a} - \frac{1}{1-\beta}\right) - \ln M_1\right| / |\ln \beta|\right),$$

we obtain from (3·7) that $|x^\alpha(t_n)| \leqslant L/a = B(M_2)$ for all $n \geqslant N$.

Therefore $|x^\alpha(t_n)| \leqslant B(M_2)$ for all $n \geqslant N$, and hence $|x^\alpha(t)| \leqslant B(M_2)$ for all $t \geqslant t_0 + N\Gamma$. This completes the proof.

Example 3·6. Let $D_\alpha(t, \phi)$ be given by

$$D_\alpha(t, \phi) = \phi(0) - \int_{t-\alpha}^{t} (C_1(t, s, x(s)) + C_2(t, s, x(s)))\, ds \tag{3·8}$$

where $|C_1(t, s, x)| \leqslant K_1(t-s)|x|$, $|C_2(t, s, x)| \leqslant K_2(t-s)$ for $-\infty < s \leqslant t < \infty$, and where $\int_0^\infty K_1(u)\, du < 1$ and $\int_0^\infty K_2(u)\, du < \infty$. Then direct verification shows that $\{D_\alpha\}$ satisfies all the conditions in Propositions 3·4 and 3·5. Therefore $\{D_\alpha\}$ is uniformly ultimately bounded.

The following two results, in the spirit of Lyapunov–Razumikhin technique, give sufficient conditions guaranteeing uniform boundedness and uniform ultimate boundedness of solutions to (3·1).

THEOREM 3·7. *Suppose that $\{D_\alpha\}$ is uniformly bounded, and that there exist continuous, increasing, unbounded functions $W_i: \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, 2, 3$, and continuous functions $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$, $W: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as well as a constant $M > 0$ such that the following conditions hold:*

(i) $|D_\alpha(t,\phi)| \leqslant W_3(|\phi|_\alpha)$, *for all* $\alpha \in \Lambda$, $\phi \in C_\alpha$, $t \in \mathbb{R}$;

(ii) $W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)$, *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}^n$;

(iii) *if at some* $t \in \mathbb{R}$, $|D_\alpha(t,x_t)| \geqslant M$ *and* $V(s,x(s)) \leqslant W_2 \circ S \circ W_1^{-1}(V(t,D_\alpha(t,x_t)))$ *for* $s \leqslant t$, *then*

$$\dot V_{(3\cdot1)}(t,D_\alpha(t,x_t)) \leqslant W(t, V(t,D_\alpha(t,x_t))),$$

where S *is defined in Definition* 3·1;

(iv) *solutions of* $\dot z = W(t,z)$ *are uniformly bounded.*

Then solutions of $(3\cdot1)_\alpha$ *are uniformly bounded.*

THEOREM 3·8. *Suppose that* $\{D_\alpha\}$ *is uniformly ultimately bounded and solutions of* $(3\cdot1)_\alpha$ *are uniformly bounded. Assume also that there exist continuous, increasing and unbounded functions* $W_i:\mathbb{R}_+\to\mathbb{R}_+$ *for* $i=1,2,3$, *continuous functions* $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$, $W:\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ *and constants* $M, N>0$ *such that* (i) *and* (ii) *of Theorem* 3·7 *are satisfied and such that the following conditions hold*:

(iii) *for any* $\beta>0$ *there exist* $\delta>0$ *and* $h>0$ *with the property that whenever* t *is such that* $|x_t| \leqslant \beta$, $M \leqslant V(t,D_\alpha(t,x_t))$ *and* $V(s,x(s)) \leqslant W_2\circ B\circ W_1^{-1}(V(t,D_\alpha(t,x_t)))+\delta$ *for* $s\in[t-h,t]$, *then*

$$\dot V_{(3\cdot1)}(t,D_\alpha(t,x_t)) \leqslant -W(t, V(t,D_\alpha(t,x_t)),\delta);$$

(iv) *for any* $H>0$ *and* $t_0\in\mathbb{R}$ *there exists* $T_3(H)>0$ *such that the solution of* $\dot z = -W(t,z,\delta)$ *with* $z(t_0)\leqslant H$ *satisfies* $z(t)<N$ *for all* $t\geqslant t_0+T_3(H)$.
Then solutions of (3·1) *are uniformly ultimately bounded at* $t=0$.

The ideas of the proof of the last two results are similar to those in [**10**]. For details, we refer to [**16, 17**].

We conclude this section with the following simple example:

$$\frac{d}{dt}\left(x(t)-\int_{t-\alpha}^{t} C(t,s,x(s))\,ds\right) = Ax(t)+\int_{t-\alpha}^{t} G(t,s,x(s))\,ds+f(t) \tag{3·9}$$

where $x\in\mathbb{R}^n$, A is an $n\times n$ stable matrix, $f:\mathbb{R}\to\mathbb{R}^n$ is continuous with

$$|f(t)|\leqslant L<\infty,\quad |C(t,s,x)|\leqslant K(t-s)|x| \quad\text{and}\quad |G(t,s,x)|\leqslant U(t-s)|x|$$

for $s\leqslant t$ and $x\in\mathbb{R}^n$, where K and U are non-negative functions such that $\int_0^\infty K(u)\,du = m<1$ and $\int_0^\infty U(t)\,dt<\infty$.

Since A is stable, there exist a unique $n\times n$ positive definite and symmetric matrix B and constants a and $b>0$ such that

$$A^{\mathrm T}B+BA=-I,\quad a^2|x|^2\leqslant x^{\mathrm T}Bx\leqslant b^2|x|^2.$$

PROPOSITION 3·9. *If*

$$l = 1-\frac{b^4}{a^4}\left(2m\left(\frac{1+m}{1-m}\right)^2+m^2\left(\frac{1+m}{1-m}\right)^2+2|A^{\mathrm T}B|\,m\left(\frac{1+m}{1-m}\right)^2 + \frac{2a^2}{b^2}\left(\frac{1+m}{1-m}\right)|B|\int_0^\infty U(t)\,dt\right)>0,$$

then solutions of (3·9) *are uniformly bounded.*

Proof. First of all, we know from Example 3·6 that $\{D_\alpha\}$ is uniformly bounded, and $S(H) = ((1+m)/(1-m))H$ for $H \geqslant 0$. Let

$$V(t,x) = x^{\mathrm{T}}Bx, \quad W_1(r) = a^2r^2, \quad W_2(r) = b^2r^2 \quad \text{and} \quad W_3(r) = (1+m)r$$

for $r \geqslant 0$. It can be easily proved that $V(s,x(s)) \leqslant W_2 \circ S \circ W_1^{-1}(V(t, D_\alpha(t,x_t))$ for $s \leqslant t$ implies that

$$|x(s)|^2 \leqslant \frac{b^4}{a^2}\left(\frac{1+m}{1-m}\right)^2 |D_\alpha(t,x_t)|^2 \quad \text{for} \quad s \leqslant t, \tag{3·10}$$

and thus

$$\dot{V}_{(3\cdot 9)}(t, D_\alpha(t,x_t)) \leqslant -l|D_\alpha(t,x_t)|^2 + 2L|B|\,|D_\alpha(t,x_t)|.$$

Therefore by Theorem 3·7, solutions of (3·9) are uniformly bounded.

PROPOSITION 3·10. *If conditions of Proposition* 3·9 *are satisfied, and there exist a constant* $\gamma > 0$ *such that* $m+\gamma < 1$ *and*

$$l^* = 1 - \frac{b^4}{a^4}\left(2m\left(\frac{2-\gamma}{\gamma}\right)^2 + m^2\left(\frac{2-\gamma}{\gamma}\right)^2 + 2\,|A^{\mathrm{T}}B|m\left(\frac{2-\gamma}{\gamma}\right)^2 + \frac{2a^2(2-\gamma)}{b^2\gamma}|B|\int_0^\infty U(t)\,dt\right) > 0,$$

then solutions of (3·9) *are uniformly ultimately bounded.*

Proof. First of all, we know from Proposition 3·5 and Example 3·6 that $\{D_\alpha\}$ is uniformly ultimately bounded with $B(M_2) = ((2-\gamma)/\gamma)M_2$ for $M_2 \geqslant 0$. Using all the functions employed in the argument of Proposition 3·9, we can show that condition (iii) in Theorem 3·8 is satisfied with $W(t,z,\delta) = l^*/(2b^2)z$ (cf. [**16**] and [**17**] for details). Therefore, by Theorem 3·8, solutions of (3·9) are uniformly ultimately bounded.

4. *Discussion*

For convenience of later reference, we start this section by listing some growth, Lipschitz and uniform continuity conditions on the kernel functions C and G of equation (2·1).

(H 1) For any $M > 0$ there exists a continuous function $Q_M: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^\infty Q_M(t) < \infty \quad \text{and} \quad |C(t,s,x)| + |G(t,s,x)| \leqslant Q_M(t-s)$$

for $0 \leqslant t, s \leqslant T$ and $x \in R^n(M) := \{x \in \mathbb{R}^n; |x| \leqslant M\}$.

(H 2) For any $M > 0$ there exists a continuous function $U_M: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty U_M(s)\,ds < \infty$ and

$$|C(t,s,x) - C(t,s,y)| + |G(t,s,x) - G(t,s,y)| \leqslant U_M(t-s)\,|x-y|$$

for $0 \leqslant s,t \leqslant T$ and $x, y \in \mathbb{R}^N(M)$.

(H 3) For any ϵ, $M > 0$ there exists $\delta > 0$ such that

$$\int_{-\infty}^t |C(t+\Delta, s, x) - C(t,s,x)|\,ds < \epsilon$$

for any $t \in \mathbb{R}$, $x \in \mathbb{R}^n(M)$ and $\Delta \in (0,\delta)$.

Remark 4·1. The local existence and uniqueness of solutions to (2·1) through $(t_0, \phi) \in \mathbb{R}_+ \times BC$ has been considered in [**13**, **18**]. A standard application of the fixed point theorem for set-condensing mappings shows that if $H(t, x)$ is local Lipschitz in $x \in \mathbb{R}^n$ and if $(H\ 1)$ and $(H\ 2)$ are satisfied, then for any $(t_0, \phi) \in \mathbb{R}_+ \times BC$, (2·1) has a unique solution through (t_0, ϕ).

Remark 4·2. Under the assumptions $(H\ 1)$ and $(H\ 3)$, solutions of (2·1) depend continuously on the initial data. In fact, if the conclusion is not true, then we can find a constant $\epsilon_0 \in (0, 1)$ and sequences $\{\psi_k\} \subseteq BC$ and $\{t_k\} \subseteq (t_0, t_0 + A]$ such that $|\psi_k - \phi|_{BC} < k^{-1}$ and $|x(t; t_0, \psi_k) - x(t; t_0, \phi)| < \epsilon_0$ for $t \in [t_0, t_k)$ and

$$|x(t_k; t_0, \psi_k) - x(t_k; t_0, \phi)| = \epsilon_0.$$

For simplicity, let $x_k(t) = x(t; t_0, \psi_k)$ and $x(t) = x(t; t_0, \phi)$. Without loss of generality, we may assume that $\{t_k\}$ is an increasing sequence and $t_k \to t_0 + \alpha$ as $k \to \infty$ for some $\alpha \in (0, A]$.

Define

$$y_k(t) = \begin{cases} x_k(t) & \text{for } t \in [t_0, t_k] \\ x_k(t_k) & \text{for } t \in (t_k, t_0 + \alpha]. \end{cases}$$

We now claim that $\{y_k(t)\}$ is equicontinuous on $[t_0, t_0 + \alpha]$. Indeed, if not, then there exist $\epsilon_1 > 0$ and sequences $\{s_k\} \subseteq (t_0, t_k]$ and $\{\Delta_k\} \subseteq (0, \infty)$ such that $\lim_{k \to \infty} \Delta_k = 0$, $s_k - \Delta_k \geqslant t_0$ and $|y_k(s_k) - y_k(s_k - \Delta_k)| \geqslant \epsilon_1$ for $k = 1, 2, \ldots$.

Let

$$\tau_k = \inf\{s \in [t_0, t_k]; |y_k(s) - y_k(s - \Delta_k)| \geqslant \epsilon_1\}.$$

Clearly $|y_k(\tau_k) - y_k(\tau_k - \Delta_k)| \geqslant \epsilon_1$. On the other hand, it follows from equation (2·1) that

$$\begin{aligned} |y_k(\tau_k) - y_k(\tau_k - \Delta_k)| &= |x_k(\tau_k) - x_k(\tau_k - \Delta_k)| \\ &= \left| \int_{-\infty}^{\tau_k} C(\tau_k, s, x_k(s))\, ds - \int_{-\infty}^{\tau_k - \Delta_k} C(\tau_k - \Delta_k, s, x_k(s))\, ds \right. \\ &\quad \left. + \int_{\tau_k - \Delta_k}^{\tau_k} H(s, x_k(s))\, ds + \int_{\tau_k - \Delta_k}^{\tau_k} \int_{-\infty}^{u} G(u, s, x_k(s))\, ds\, du \right| \\ &\leqslant \int_{-\infty}^{\tau_k - \Delta_k} |C(\tau_k, s, x_k(s)) - C(\tau_k - \Delta_k, s, x_k(s))|\, ds \\ &\quad + \int_{\tau_k - \Delta_k}^{\tau_k} |C(\tau_k, s, x_k(s))|\, ds + \int_{\tau_k - \Delta_k}^{\tau_k} |H(s, x_k(s))|\, ds \\ &\quad + \int_{\tau_k - \Delta_k}^{\tau_k} \left| \int_{-\infty}^{u} G(u, s, x_k(s))\, ds \right| du. \end{aligned}$$

Consequently, by the assumption $(H\ 1)$ we have

$$\begin{aligned} |y_k(\tau_k) - y_k(\tau_k - \Delta_k)| &\leqslant \int_{-\infty}^{\tau_k - \Delta_k} |C(\tau_k, s, x_k(s)) - C(\tau_k - \Delta_k, s, x_k(s))|\, ds \\ &\quad + C^* \Delta_k + H^* \Delta_k + \int_{\tau_k - \Delta_k}^{\tau_k} \int_{-\infty}^{u} Q_M(u - s)\, ds\, du, \end{aligned}$$

where

$$M = 1 + \sup_{-\infty < s \leqslant t_0+\alpha} |x(s)|,$$

$$C^* = \sup\{|C(t,s,z)|\,; t_0 \leqslant s \leqslant t \leqslant t_0+\alpha, |z| \leqslant M\},$$

$$H^* = \sup\{|H(s,z)|\,; t_0 \leqslant s \leqslant t_0+\alpha,\ |z| \leqslant M\}.$$

By the assumption (H 3), we can choose K so large that for all $k \geqslant K$, we have

$$\int_{-\infty}^{\tau_k-\Delta_k} |C(\tau_k, s, x_k(s)) - C(\tau_k-\Delta_k, s, x_k(s))|\,ds < \tfrac{1}{4}\epsilon_1,$$

and
$$C^*\Delta_k + H^*\Delta_k + \int_0^\infty Q_M(s)\,ds\,\Delta_k < \tfrac{1}{4}\epsilon_1.$$

Therefore $|y_k(\tau_k) - y_k(\tau_k-\Delta_k)| < \frac{1}{4}\epsilon_1 + \frac{1}{4}\epsilon_1 = \frac{1}{2}\epsilon_1$, which contradicts the fact that $|y_k(\tau_k) - y_k(\tau_k-\Delta_k)| \geqslant \epsilon_1$. This proves the equicontinuity of $\{y_k(t)\}$ on $[t_0, t_0+\alpha]$.

By the well-known Ascoli–Arzela theorem, we may now assume that $\{y_k(t)\}$ is uniformly convergent to $y(t)$ on $[t_0, t_0+\alpha]$. Moreover, from equation (2·1), we have

$$y_k(t) = y_k(t_0) - \int_{-\infty}^{t_0} C(t_0, s, y_k(s))\,ds + \int_{-\infty}^{t} C(t, s, y_k(s))\,ds + \int_{t_0}^{t} H(s, y_k(s))\,ds + \int_{t_0}^{t}\int_{-\infty}^{u} G(u, s, y_k(s))\,ds\,du$$

for $t \in [t_0, t_0+\alpha]$. Taking $k \to \infty$ in the above equality and applying the Lebesgue dominated convergence theorem, we obtain

$$y(t) = y(t_0) - \int_{-\infty}^{t_0} C(t_0, s, y(s))\,ds + \int_{-\infty}^{t} C(t, s, y(s))\,ds + \int_{t_0}^{t} H(s, y(s))\,ds + \int_{t_0}^{t}\int_{-\infty}^{u} G(u, s, y(s))\,ds\,du$$

for $t \in [t_0, t_0+\alpha]$. Obviously $y_{t_0} = \phi$. So y is a solution of (2·1) through (t_0, ϕ), and thus, by uniqueness, $x(t) = y(t)$ on $[t_0, t_0+\alpha]$ and

$$|x_k(t_k) - x(t_k)| = |y_k(t_k) - y(t_k)| \to 0 \quad \text{as } k \to \infty.$$

This contradicts the fact that $|x_k(t_k) - x(t_k)| = \epsilon_0$ for $k = 1, 2, \ldots$, completing the proof.

Remark 4·3. The assumption (A 2) is satisfied if C and G satisfy the Lipschitz condition (H 2). Indeed, let $\phi \in X_T$ and $\epsilon > 0$ be given. From the continuity of $H(t,s)$, we can find $\delta_1 \in (0,1)$ such that $|H(s,\psi(s)) - H(s,\phi(s))| < \epsilon/(4T)$ for $s \in [0,T]$ and $\psi \in X_T$ with $|\psi - \phi|_T < \delta_1$. Let U_M be a continuous function defined as in the assumption (H 2) with $M = |\phi|_T$, and put

$$\delta_2 = \frac{\epsilon}{2\int_0^\infty U_M(s)\,ds(2+T)}, \quad \delta = \min(\delta_1, \delta_2, \tfrac{1}{4}\epsilon).$$

Then for $(t,\psi)\in[0,T]\times X_T$ with $|\psi-\phi|_T<\delta$, we have

$$\begin{aligned}
|P(\psi)(t)-P(\phi)(t)| &= \left|\psi(0)-\phi(0)-\int_{-\infty}^{0}[C(0,s,\psi(s))-C(0,s,\phi(s))]\,ds\right. \\
&\quad+\int_{-\infty}^{t}[C(t,s,\psi(s))-C(t,s,\phi(s))]\,ds+\int_{0}^{t}[H(u,\psi(u))-H(u,\phi(u))]\,du \\
&\quad\left.+\int_{0}^{t}\int_{-\infty}^{u}[G(u,s,\psi(s))-G(u,s,\phi(s))]\,ds\right| \\
&\leqslant |\psi(0)-\phi(0)|+\int_{-\infty}^{0}U_M(-s)\,|\psi(s)-\phi(s)|\,ds \\
&\quad+\int_{-\infty}^{t}U_M(t-s)\,|\psi(s)-\phi(s)|\,ds+\int_{0}^{t}\int_{-\infty}^{u}U_M(u-s)\,|\psi(s)-\phi(s)|\,ds\,du \\
&\quad+\int_{0}^{T}|H(u,\psi(u))-H(u,\phi(u))|\,du \\
&\leqslant \tfrac{1}{4}\epsilon+\left(2\int_{0}^{\infty}U_M(s)\,ds+T\int_{0}^{\infty}U_M(s)\,ds\right)|\psi-\phi|_T+T\cdot\frac{\epsilon}{4T}
\end{aligned}$$

This completes the proof.

Remark 4·4. Assumption (*A* 3) is implied by the growth condition (*H* 1) and the uniform continuity condition (*H* 3). To prove this, we fix $\phi\in X_T$ with $|\phi|_T\leqslant M$. Let $t_1,t_2\in[0,T]$ with $t_2>t_1$. Then it follows from (*H* 1) and (*H* 3) that for any given $\epsilon>0$ there exists $\delta=\delta(M,\epsilon)>0$ such that for $k=0,1,2,\ldots,$

$$\int_{kT}^{kT+\delta}Q_M(s)\,ds<\tfrac{1}{3}\epsilon\quad\text{and}\quad\int_{-\infty}^{t_1}|C(t_2,s,\phi(s))-C(t_1,s,\phi(s))|\,ds<\tfrac{1}{3}\epsilon,$$

whenever $|t_1-t_2|<\delta$. Therefore, if $|t_1-t_2|<\delta$, then

$$\begin{aligned}
&\left|\int_{t_2-kT}^{t_2}C(t_2,s,\phi(s))\,ds-\int_{t_1-kT}^{t_1}C(t_1,s,\phi(s))\,ds\right| \\
&\qquad\leqslant\left|\int_{t_2-kT}^{t_2}C(t_2,s,\phi(s))\,ds-\int_{t_1-kT}^{t_1}C(t_2,s,\phi(s))\,ds\right| \\
&\qquad\quad+\int_{t_1-kT}^{t_1}|C(t_2,s,\phi(s))-C(t_1,s,\phi(s))|\,ds \\
&\qquad\leqslant\int_{t_1}^{t_2}|C(t_2,s,\phi(s))|\,ds+\int_{t_1-kT}^{t_2-kT}|C(t_2,s,\phi(s))|\,ds \\
&\qquad\quad+\int_{-\infty}^{t_1}|C(t_2,s,\phi(s))-C(t_1,s,\phi(s))|\,ds \\
&\qquad\leqslant\int_{0}^{t_2-t_1}Q_M(s)\,ds+\int_{kT}^{t_2-t_1+kT}Q_M(s)\,ds+\tfrac{1}{3}\epsilon \\
&\qquad<\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon=\epsilon.
\end{aligned}$$

This proves the equicontinuity of the sequence of functions

$$\int_{t-kT}^{t} C(t,s,\phi(s))\,ds,$$

for $k = 1, 2, \ldots$, on $[0, T]$.

Remark 4·5. Theorems 2·1 and 2·4 can also be applied to integral equation of the type

$$x(t) = f(t) + \int_{-\infty}^{t} C(t,s,x(s))\,ds, \tag{4·1}$$

where $f\colon \mathbb{R} \to \mathbb{R}^n$ is continuous and $f(t+T) = f(t)$ for $t \in \mathbb{R}$. In [7], Islam proved that (4·1) has a T-periodic solution if solutions of (4·1) are g-uniformly bounded and g-uniformly ultimately bounded, where $g\colon(-\infty, 0] \to [1,\infty)$ is a continuous and decreasing function with $g(0) = 1$ and $\lim_{r\to-\infty} g(r) = \infty$. Islam did not use limiting equations at all, but his result holds only when (i) solutions of (4·1) are g-uniformly bounded and g-uniformly ultimately bounded, and (ii) $\int_{-\infty}^{t} C(t,s,\phi(s))\,ds$ satisfies certain Lipschitz condition for every continuous function $\phi\colon(-\infty,0] \to \mathbb{R}^n$ such that $\sup_{s\leqslant 0}|\phi(s)|/g(s) < \infty$. Our assumptions for the existence of T-periodic solutions of (4·1) are slightly weaker than those of [7] in the following sense: first of all, g-uniform (ultimate) boundedness implies uniform (ultimate) boundedness of solutions of (4·1); secondly, we avoid the problem of choosing the weight function g and the phase space. In particular, we require certain Lipschitz conditions on $\int_{-\infty}^{t} C(t,s,\phi(s))\,ds$ only for bounded continuous functions $\phi\colon(-\infty,0] \to \mathbb{R}^n$. To illustrate this, we take as an example the simple scalar integral equation

$$x(t) = f(t) + \int_{-\infty}^{t} a(t-s)\,h(s,x(s))\,ds, \tag{4·2}$$

where $a\colon \mathbb{R}_+ \to \mathbb{R}$ and $h\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous, and $h(t+T,x) = h(t,x)$ for $(t,x) \in \mathbb{R}^2$. Applying Propositions 3·4, 3·5 and a version of Theorems 3·1, 3·2 for integral equations, we conclude that (4·2) has a T-periodic solution if the following conditions are satisfied:

(C 1) $h(t,x)$ is local Lipschitz in $x \in \mathbb{R}$ and there exists a constant $\alpha > 0$ such that $|h(t,x)| \leqslant \alpha|x|$ for all $t \in [0,T]$ and $x \in \mathbb{R}$;
(C 2) $\alpha\int_0^\infty |a(t)|\,dt < 1$.

On the other hand, under the assumption (C 2), there exists a function $g\colon(-\infty,0] \to [1,\infty)$ with $g(0) = 1$, $\lim_{r\to-\infty} g(r) = \infty$ and $\alpha\int_0^\infty |a(t)|\,g(-t)\,dt < 1$ (cf. [1, **18**]). In order to apply Islam's result to obtain a T-periodic solution of (4·2), in addition to (C 1) and (C 2), we have to assume that there exists a constant $L > 0$ such that

(C 3*) $\int_{-\infty}^{0} |a(t-s) - a(\tau-s)|\,\sqrt{(g(s))}\,ds \leqslant L\,|t-\tau|$,

(C 4*) $|\int_t^\tau a(s)\,ds| \leqslant L\,|t-\tau|$,
(C 5*) $|f(t) - f(\tau)| \leqslant L\,|t-\tau|$,

for t, $\tau \in [0,T]$. Note that (C 4*) implies that a is bounded, but we do not need this boundedness. Moreover, the verification of (C 3*) is not a trivial matter if g is not explicitly constructed.

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