JOURNAL OF DIFFERENTIAL EQUATIONS 98, 277–298 (1992)

S^1-Degree and Global Hopf Bifurcation Theory of Functional Differential Equations

L. H. ERBE* AND W. KRAWCEWICZ*

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

K. GĘBA[†]

Institute of Mathematics, Gdańsk University, Gdańsk, Poland

AND

J. WU[‡]

Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3

Received February 2, 1990; revised November 1, 1990

The recently developed S^1-degree and bifrucation theory are applied to provide a purely topological argument of a global Hopf bifurcation theory for functional differential equations of mixed type. In the special case where the equation is of retarded type, the established result represents an analog of Alexander and Yorke's global Hopf bifurcation theorem which has been obtained by Chow, Fiedler, Mallet-Paret, and Nussbaum, using different approaches. © 1992 Academic Press, Inc.

1. INTRODUCTION

A great deal of research has been devoted to the Hopf bifurcation of functional differential equations of retarded type. The Lyapunov–Schmidt method (or generally, alternation method), center manifold theory, and integral averaging method, etc., have been used to reduce the bifurcation problem to a finite dimensional problem. Since the implicit function theorem has been applied, certain differentiability conditions must be imposed on the vector field. For details, we refer to [6, 7, 11, 23, 25] and

* Research partially supported by NSERC–Canada.
† Research done while the author visited the University of Alberta.
‡ Research partially supported by a Gordin Kaplan Award Postdoctoral Fellowship.

0022-0396/92 $5.00

references therein. Moreover, Fuller index theory, fixed point index for local condensing maps and Alexander–Yorke global bifurcation theory etc., have also been extended to retarded functional differential equations in [8, 39–41]. Their proofs were based on either the approximation of a functional differential equation by Kupka–Smale systems or a strongly proper approximation of the operator of translation along trajectories of functional differential equations with respect to a natural set of projections. Therefore, the decomposition theory of linear functional differential equations plays an essential role.

The purpose of this paper is to apply the S^1-degree and bifurcation theory recently developed in [16, 21] to provide a purely topological proof of a global Hopf bifurcation theory for functional differential equations. Our result represents an analog of the Alexander–Yorke global bifurcation theorem for periodic orbits which has been obtained by Chow, Fielder, Mallet-Paret, and Nussbaum, using different approaches (see [8, 9, 17, 18, 41]). Because of the purely topological nature of our approach, we avoid the solution operator and the sophisticated decomposition theory of linear functional differential equations. Therefore our results can be applied to functional differential equations of mixed type (with both advanced and retarded arguments).

The paper is organized as follows: in the next section we introduce the S^1-degree (see also the Appendix for additional discussion) and bifurcation theory due to [16, 21]. For the convenience of the reader, we outline briefly the geometric essence in the argument of the bifurcation theorem. A computation formula is also provided for the calculation of the degree. These general results are then applied, in Section 3, to obtain a global Hopf bifurcation theorem for functional differential equations of mixed type. Some comparison of our approach with existing older ideas is also presented in Section 3.

2. S^1-Degree and Bifurcation Theory

In this section we present an outline of S^1-equivariant bifurcation theory that was developed recently by Gęba and Marzantowicz [21].

Let $\mathbb{E}$ be a real Hilbert space. We assume that $\mathbb{E}$ is an *isometric representation* of the group $G = S^1 := \{z \in \mathbb{C}; |z| = 1\}$; i.e., we assume that there is a continuous map $\mu: G \times \mathbb{E} \to \mathbb{E}$ such that

(a) for every $g \in G$, $gx := \mu(g, x)$ defines a linear isometry on $\mathbb{E}$;

(b) $1x = x$ for all $x \in \mathbb{E}$;

(c) $g_1(g_2 x) = (g_1 g_2)x$ for all $g_1, g_2 \in G$ and $x \in \mathbb{E}$.

For $x \in \mathbb{E}$ we denote by $G_x := \{\gamma \in G; \gamma x = x\}$ the *isotropy group* of x.

The isometric representation $\mathbb{E}$ has the *isotypical direct sum decomposition*

$$\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k \oplus \cdots,$$

where $\mathbb{E}_0 = \mathbb{E}^G := \{x \in \mathbb{E};\ gx = x \text{ for all } g \in G\}$ is the subspace of G-fixed points, and for $k \geqslant 1$, $x \in \mathbb{E}_k \setminus \{0\}$ implies $G_x = \mathbb{Z}_k := \{\gamma \in G;\ \gamma^k = 1\}$. Throughout this section, we assume the following:

(A.1) For all $k = 0, 1, ...,$ the subspaces $\mathbb{E}_k$ are of finite dimensions.

All subspaces $\mathbb{E}_k$, $k \geqslant 1$, admit a natural structure of complex vector spaces. Indeed, setting $\xi_k = \exp(\pi i/2k)$, we can define a multiplication of an element $x \in \mathbb{E}_k$ by a complex number $z = a + ib$ according to the formula $z * x = ax + b\xi_k x$. It can be shown easily that an $\mathbb{R}$-linear operator $A: \mathbb{E}_k \to \mathbb{E}_k$ is G-equivariant if and only if it is $\mathbb{C}$-linear with respect to this complex structure. Therefore, by choosing a $\mathbb{C}$ basis in $\mathbb{E}_k$, $k \geqslant 1$, we can define an isomorphim between the group of all G-equivariant automorphisms of $\mathbb{E}_k$, denoted by $GL_G(\mathbb{E}_k)$, and the general linear group $GL(m_k, \mathbb{C})$, where $m_k = \dim_{\mathbb{C}} \mathbb{E}_k$.

For a topological space X, we denote by $[S^1, X]$ the set of homotopy classes of continuous maps $\alpha: S^1 \to X$. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\alpha: S^1 \to \mathbb{C}^*$ be a continuous map. The correspondence $[\alpha] \mapsto \deg_B(\alpha)$, where $\deg_B$ denotes the Brouwer degree, defines the bijection of $[S^1, \mathbb{C}^*]$ onto $\mathbb{Z}$. The following is a well-known fact:

Proposition 2.1. *There exists a canonical bijection* $\nabla: [S^1, GL(n, \mathbb{C})] \to \mathbb{Z}$ *defined by* $\nabla([\alpha]) := \deg_B(\det_{\mathbb{C}} \alpha)$, *where* $\alpha: S^1 \to GL(n, \mathbb{E})$ *and* $\det_{\mathbb{C}}: GL(n, \mathbb{C}) \to C^*$ *is the usual determinant homomorphism. Moreover, if* $\alpha_i: S^1 \to GL(n_i, \mathbb{C})$, $i = 1, 2$, *are two continuous maps, then*

$$\nabla([\alpha_1 \oplus \alpha_2]) = \nabla([\alpha_1]) \cdot \nabla([\alpha_2]),$$

where $\alpha_1 \oplus \alpha_2: S^1 \to GL(n_1 + n_2, \mathbb{C})$ *is the canonical direct sum of* α_1 *and* α_2.

Let $\mathbb{F}$ be another Hilbert isometric representation of $G = S^1$, and $L: \mathbb{E} \to \mathbb{F}$ be a given equivariant linear bounded Fredholm operator of index zero. We say that an equivariant compact operator $K: \mathbb{E} \to \mathbb{F}$ is an *equivariant compact resolvent* of L, if $L + K: \mathbb{E} \to \mathbb{F}$ is an isomorphism. We will denote by $CR_G(L)$ te set of all equivariant compact resolvents of L, and assume

(A.2) The set $CR_G(L)$ is non-empty.

In what follows, a point of the Banach space $\mathbb{E} \oplus \mathbb{R}^2$ is denoted by (x, λ), where $x \in \mathbb{E}$ and $\lambda \in \mathbb{R}^2$, and the action of G on $\mathbb{E} \oplus \mathbb{R}^2$ is defined by $g(x, \lambda) := (gx, \lambda)$, $g \in G$, $x \in \mathbb{E}$, and $\lambda \in \mathbb{R}^2$.

We consider a G-equivariant continuous map $f\colon \mathbb{E}\oplus\mathbb{R}^2\to\mathbb{F}$ such that

$$f(x,\lambda)=Lx+\varphi(x,\lambda), \qquad (x,\lambda)\in\mathbb{E}\oplus\mathbb{R}^2,$$

where $\varphi\colon \mathbb{E}\oplus\mathbb{R}^2\to\mathbb{R}$ is a completely continuous map and the following assumption is satisfied.

(A.3) There exists a 2- dimensional submanifold $N\subseteq\mathbb{E}_0\oplus\mathbb{R}^2$ such that

(i) $N\subseteq f^{-1}(0)$;

(ii) if $(x_0,\lambda_0)\in N$, then there exists an open neighbourhood U_{λ_0} of λ_0 in $\mathbb{R}^2$, an open neighbourhood U_{x_0} of x_0 in $\mathbb{E}_0$, and a C^1-map $\eta\colon U_{\lambda_0}\to\mathbb{E}_0$ such that

$$N\cap(U_{x_0}\times U_{\lambda_0})=\{(\eta(\lambda),\lambda);\lambda\in U_{\lambda_0}\}.$$

We are interested in the structure of the set of solutions to the equation

$$f(x,\lambda)=0, \qquad (x,\lambda)\in\mathbb{E}\oplus\mathbb{R}^2. \tag{2.1}$$

By (i) of (A.3), all points $(x,\lambda)\in N$ are solutions to (2.1). We call those points *trivial solutions.* All other solutions of (2.1), namely points in $f^{-1}(0)\setminus N$, will be called *nontrivial solutions.* A point $(x_0,\lambda_0)\in N$ is called a *bifurcation point* if in any neighbourhood of (x_0,λ_0), there exists a nontrivial solution for (2.1).

We assume that at all points $(x_0,\lambda_0)\in N$ the derivative $D_xf(x_0,\lambda_0)\colon\mathbb{E}\to\mathbb{F}$ of f with respect to x exists and is continuous on N. We say that $(x_0,\lambda_0)\in N$ is *$\mathbb{E}$-singular* if $D_xf(x_0,\lambda_0)\colon\mathbb{E}\to\mathbb{F}$ is not an isomorphism. An $\mathbb{E}$-singular point $(x_0,\lambda_0)\in N$ is called an *isolated $\mathbb{E}$-singular point* if there are no other $\mathbb{E}$-singular points in some neighbourhood of (x_0,λ_0) in N. It follows from the implicit function theorem that if (x_0,λ_0) is a bifurcation point then (x_0,λ_0) is an $\mathbb{E}$-singular point.

Suppose that $(x_0,\lambda_0)\in N$ is an isolated $\mathbb{E}$-singular point. We are interested in finding nontrivial solutions of Eq. (2.1) in a neighbourhood of (x_0,λ_0) in $\mathbb{E}\oplus\mathbb{R}^2$. First of all, we remark that (2.1) can be transformed into the equivariant fixed point problem

$$x=R_K\circ[Kx-\varphi(x,\lambda)], \qquad (x,\lambda)\in\mathbb{E}\oplus\mathbb{R}^1,$$

where $K\in cR_G(L)$ and $R_K:=(L+K)^{-1}$. Therefore Eq. (2.1) is equivalent to the equation

$$h(x,\lambda)=0, \qquad (x,\lambda)\in\mathbb{E}\oplus\mathbb{R}^2, \tag{2.2}$$

where $h: \mathbb{E} \times \mathbb{R}^2 \to \mathbb{E}$ is defined by the formula

$$h(x, \lambda) = x - R_K \circ [Kx - \varphi(x, \lambda)], \qquad (x, \lambda) \in \mathbb{E} \oplus \mathbb{R}^2.$$

The approach of finding nontrivial solutions to (2.2) in an open invariant neighbourhood $U \subseteq \mathbb{E} \times \mathbb{R}^2$ of $(x_0, \lambda_0) \in N$ is based on the idea of a *complementing function* to Eq. (2.2). This method was developed (in the non-equivariant case) by Ize [26] and recently was applied in [21] to the equivariant bifurcation problem (2.2). Such a method can be described roughly as follows. Suppose that there exists an equivariant function $\psi: \bar{U} \to \mathbb{R}$ satisfying the condition $\psi(x, \lambda) < 0$ for all $(x, \lambda) \in \bar{U} \cap N$ (called a *complementing function*). Then every solution to the system

$$\begin{aligned} h(x, \lambda) &= 0, \\ \psi(x, \lambda) &= 0, \end{aligned} \qquad (x, \lambda) \in \mathbb{E} \oplus \mathbb{R}^2 \tag{2.3}$$

is evidently a nontrivial solution to (2.1). This leads us to the equivariant map $F_\psi: \bar{U} \to \mathbb{E} \oplus \mathbb{R}$ defined by $F_\psi(x, \lambda) = (h(x, \lambda), \psi(x, \lambda))$, $(x, \lambda) \in \mathbb{E} \oplus \mathbb{R}^2$, and the problem of finding nontrivial soutions to (2.1) in U can be reduced to the problem of finding a solution to the equation $F_\psi(x, \lambda) = 0$ in U which can be solved by using the so-called *S^1-equivariant degree* as a topological invariant associated with the problem (2.3).

To describe the definition and basic properties of S^1-degree, we assume that V is an isometric Hilbert representation of $G = S^1$. If U is an open bounded invariant subset of $V \oplus \mathbb{R}$ (where S^1 acts trivially on $\mathbb{R}$) and $F: (\bar{U}, \partial U) \to (V, V \setminus \{0\})$ is an equivariant compact vector field on $\bar{U}$, then there is defined the S^1-equivariant degree of F with respect to U, which is a sequence of integers

$$\mathrm{Deg}(F, U) := \{\deg_k(F, U)\}_{k=1}^{\infty} \in \bigoplus_{k=1}^{\infty} \mathbb{Z}$$

such that only for finite number of indices k, $\deg_k(F, U) \neq 0$. The basic properties of Deg are as follows.

(i) *Existence*: If $\mathrm{Deg}(F, U) = \{\deg_k(F, U)\}_{k=1}^{\infty} \neq 0$, i.e., there exists $k \in \{1, 2, ...\}$ such that $\deg_k(F, U) \neq 0$, then $F^{-1}(0) \cap U^H \neq \emptyset$, where $H = \mathbb{Z}_k$ and

$$U^H := \{v \in U; gv = v \text{ for any } g \in H\}.$$

(ii) *Additivity*: If U_1 and U_2 are two open invariant subsets of U such that $U_1 \cap U_2 = \emptyset$ and $F^{-1}(0) \cap U \subseteq U_1 \cup U_2$, then $\mathrm{Deg}(F, U) = \mathrm{Deg}(F, U_1) + \mathrm{Deg}(F, U_2)$.

(iii) *Homotopy invariance*: If $\mathscr{H}: (\bar{U}; \partial U) \times [0, 1] \to (V, V \setminus \{0\})$ is an S^1-equivariant homotopy of compact vector fields, then $\mathrm{Deg}(\mathscr{H}_0, U) = \mathrm{Deg}(\mathscr{H}_1, U)$, where $\mathscr{H}_t(\theta) = \mathscr{H}(t, \theta)$ for $t \in [0, 1]$ and $\theta \in \bar{U}$.

(iv) *Contraction*: Suppose that W is another isometric Hilbert representation of S^1 and let Ω be an open bounded, invariant subset of W such that $0 \in \Omega$. Define $\Phi: \bar{\Omega} \times \bar{U} \to W \oplus V$ by $\Phi(y, x, t) = (y, F(x, t))$. Then $\mathrm{Deg}(\Phi, \Omega) = \mathrm{Deg}(F, U)$.

For more details of S^1-degree, we refer to [16] (see also the Appendix for a brief description of S^1-*degree* and a comparison with that introduced by Ize *et al.* [29]).

Now we return to the problem (2.3). If the mapping $F_\psi: \bar{U} \to \mathbb{E} \oplus \mathbb{R}$ has no zero on ∂U, then there is a well-defined S^1-equivariant degree of F_ψ for $V = \mathbb{E} \oplus \mathbb{R}$,

$$\mathrm{Deg}(F_\psi, U) = \{\deg_k(F_\psi, U)\}_{k=1}^{\infty}.$$

(In fact, such a degree is well defined for any equivariant function $\psi: \bar{U} \to \mathbb{R}$ such that $F_\psi(x, \lambda) \neq 0$ for all $(x, \lambda) \in \partial U$. This remark wil be used later.) The existence property of the S^1-degree guarantees that if $\mathrm{Deg}(F_\psi, U) \neq 0$, then (2.1) has a nontrivial solution in U. Therefore, in order to be able to use the above S^1-degree to our bifurcation problem, we need some facts about the construction of the open neighbourhood U and the complementing function ϕ as well as the possible computation of $\mathrm{Deg}(F_\psi, U)$.

For this purpose, we identify $\mathbb{R}^2$ with $\mathbb{C}$, and for sufficiently small $\rho > 0$, we define $\alpha: D \to N$, $D := \{z \in \mathbb{C}; |z| \leqslant 1\}$, by putting

$$\alpha(z) = (\eta(\lambda_0 + \rho z), \lambda_0 + \rho z) \in \mathbb{E}_0 \oplus \mathbb{R}^2.$$

Since we assumed that $(x_0, \lambda_0) = (\eta(\lambda_0), \lambda_0) \in N$ is an isolated $\mathbb{E}$-singular point, it is clear that we can choose sufficiently small $\rho > 0$ such that $\alpha(D)$ contains only one $\mathbb{E}$-singular point, namely (x_0, λ_0). Consequently, the formula $\Psi(z) := D_x h(\alpha(z))$, $z \in S^1 \subseteq D$, defines a continuous mapping $\Psi: S^1 \to GL_G(\mathbb{E})$ which has the decomposition $\Psi = \Psi_0 \oplus \Psi_1 \oplus \cdots \oplus \Psi_k \oplus \cdots$, where $\Psi_k: S^1 \to GL_G(\mathbb{E}_k)$ for $k = 1, 2, \ldots$ and $\Psi_0: S^1 \to GL(\mathbb{E}_0)$. We now define

$$\varepsilon = \mathrm{sgn}\ \det \Psi_0(z)$$

and

$$\gamma_k(x_0, \lambda_0) = \varepsilon \nabla([\Psi_k]), \qquad k = 1, 2, \ldots. \tag{2.4}$$

Obviously, ε does not depend on the choice of $z \in S^1$.

As the open neighbourhood U of $(x_0, \lambda_0) \in N$, we take the set

$$B_N(x_0, \lambda_0; r, \rho) := \{(x, \lambda) \in \mathbb{E} \oplus \mathbb{R}^2; |\lambda - \lambda_0| < \rho, \|x - \eta(\lambda)\| < r\},$$

where $r > 0$, on which we assume that

(i) $h(x, \lambda) \neq 0$ for all $(x, \lambda) \in \{(x, \lambda) \in B_N(x_0, \lambda_0; r, \rho);\ |\lambda - \lambda_0| = \rho, |x - \eta(\lambda)| \neq 0\}$;

(ii) (x_0, λ_0) is the only $\mathbb{E}$-singular point in $B_N(x_0, \lambda_0; r, \rho)$.

We call a set U a *special neighbourhood of* (x_0, λ_0) *determined by* r, ρ. The existence of a special neighbourhood follows from the implicit function theorem. Moreover, by the Gleason–Tietze G-extension theorem, there exists a continuous S^1-equivariant function θ: $\bar{U} \to \mathbb{R}$ such that

(i) $\theta(\eta(\lambda), \lambda) = -|\lambda - \lambda_0|$ for all $(\eta(\lambda), \lambda) \in \bar{U} \cap N$;

(ii) $\theta(x, \lambda) = r$ if $\|x - \eta(\lambda)\| = r$.

Such a function will be called an *auxiliary function.* Clearly, if θ is an auxiliary function, then $\theta_\delta(x, \lambda) := \theta(x, \lambda) - \delta$ will be negative on the subset of trivial solutions $U \cap N$, provided $\delta > 0$. Consequently, θ_δ is a complementing function. Now for sufficiently small $\delta > 0$, we can define $F_{\theta_\delta}: \bar{U} \to \mathbb{E} \oplus \mathbb{R}$, $F_{\theta_\delta}(x, \lambda) = (h(x, \lambda), \theta_\delta(x, \lambda))$, and the S^1-equivariant degree $\mathrm{Deg}(F_{\theta_\delta}, U)$.

By the homotopy invariance of the S^1-degree, we obtain $\mathrm{Deg}(F_{\theta_\delta}, U) = \mathrm{Deg}(F_\theta, U)$. Therefore the nontriviality of the degree $\mathrm{Deg}(F_\theta, U)$ implies the existence of nontrivial solutions of (2.1) in U.

The following result due to [21] provides an important formula for the computation of $\mathrm{Deg}(F_\theta, U)$.

PROPOSITION 2.2. *Assume that* $U = B_N(x_0, \lambda_0; \gamma, \rho)$ *is a special neighbourhood of an isolated* $\mathbb{E}$*-singular point* $(x_0, \lambda_0) \in N$, *and let* θ *be an auxiliary function. Then the* S^1*-equivalent degree* $\mathrm{Deg}(F_\theta, U)$, *where* $F_\theta(x, \lambda) = (h(x, \lambda), \theta(x, \lambda))$, $(x, \lambda) \in \bar{U}$, *is well defined and we have the formula*

$$\deg(F_\theta, U) = \gamma_k(x_0, \lambda_0), \qquad k = 1, 2, ...,$$

where $\gamma_k(x_0, \lambda_0)$ *is defined by* (2.4).

From the above result, one can easily obtain the following local Hopf bifurcation theorem of Krasnosel'skii type.

THEOREM 2.3. *Suppose that* $f: \mathbb{E} \oplus \mathbb{R}^2 \to \mathbb{F}$ *is an equivariant continuous map which is continuously differentiable with respect to* x *at points* $(x, \lambda) \in N$

and satisfies (A.1), (A.2), *and* (A.3). *If* $(x_0, \lambda_0) \in N$ *is an isolated* $\mathbb{E}$*-singular point and there exists* $k \geqslant 1$ *such that* $\gamma_k(x_0, \lambda_0) \neq 0$, *then* (x_0, λ_0) *is a bifurcation point. More precisely, there exists a sequence* $(x_n, \lambda_n) \to (x_0, \lambda_0)$ *of nontrivial solutions to* (2.1) *such that the isotropy group of* x_n *contains* $\mathbb{Z}_k$.

We remark that the above result holds when $\mathbb{R}^2$ is replaced by an open subset of $\mathbb{R}^2$.

The following result represents an analog of the Rabinowitz global Hopf bifurcation theorem in our setting.

THEOREM 2.4. *Suppose that* $f\colon \mathbb{E} \oplus \mathbb{R}^2 \to \mathbb{F}$ *is as in Theorem* 2.3 *and suppose further that* N *is complete and every* $\mathbb{E}$*-singular point in* N *is isolated in* N. *Let* $S(f)$ *denote the closure of the set of all nontrivial solutions of* (2.1). *Then for each bounded connected component* $\mathfrak{J}$ *of* $S(f)$, *the set* $\mathfrak{J} \cap N$ *is finite. Moreover, if* $\mathfrak{J} \cap N = \{(x_1, \lambda_1), ..., (x_q, \lambda_q)\}$, *then for every* $k = 1, 2, ...$

$$\gamma_k(x_1, \lambda_1) + \cdots + \gamma_k(x_q, \lambda_q) = 0.$$

The detailed proofs of Theorems 2.3 and 2.4 can be found in [21].

We will need one more technical result related to the notion of crossing numbers.

Suppose a_1, a_2, b, c, and β are given numbers with $b, c > 0$ and $a_2 > a_1$. Let $\Omega := (0, b) \times (\beta - c, \beta + c) \subseteq \mathbb{R}^2$. Assume $H\colon [a_1, a_2] \times \bar{\Omega} \to \mathbb{R}^2$ is a continuous function. For every $\alpha \in [a_1, a_2]$, we put $H_\alpha(x, y) := H(\alpha, x, y)$, $(x, y) \in \bar{\Omega}$. The following conditions on H are assumed:

(C1) $H(\alpha, x, y) \neq 0$ for all $\alpha \in [a_1, a_2]$ and $(x, y) \in \partial\Omega \setminus \{(0, y);\ y \in (\beta - c, \beta + c)\}$.

(C.2) For $(x, y) \in \Omega$, if $H_{a_i}(x, y) = 0$, $i = 1$ or 2, then $x \neq 0$.

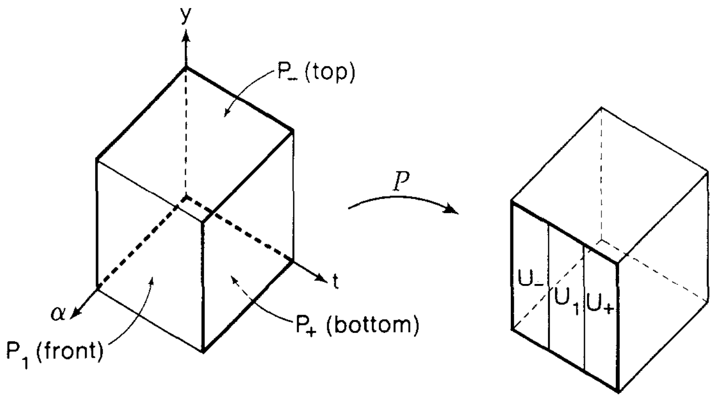
Conditions (C.1) and (C.2) imply that $H_{a_i}\colon \bar{\Omega} \to \mathbb{R}^2$ has no zero on the boundary $\partial\Omega$, $i = 1, 2$. Consequently, we can define the following integers: $d_i := \deg_B(H_{a_i})$, where $\deg_B$ denotes the Brouwer degree. We define the (*formal*) *crossing number* d for the family $\{h_\alpha\}_{\alpha \in [a_1, a_2]}$ by

$$d := d_1 - d_2.$$

LEMMA 2.5. *Suppose that* $H\colon [a_1, a_2] \times \bar{\Omega} \to \mathbb{R}^2$ *is continuous and satisfies conditions* (C.1) *and* (C.2). *Put* $\Omega_1 := (a_1, a_2) \times (\beta - c, \beta + c)$ *and define the function* $\psi_H\colon \bar{\Omega}_2 \to \mathbb{R}^2$ *by* $\psi_H((\alpha, y) = H(\alpha, 0, y)$, $\alpha \in [a_1, a_2]$, $y \in [\beta - c, \beta + c]$. *Then* $\psi_H(\alpha, y) \neq 0$ *for* $(\alpha, y) \in \partial\Omega_1$ *and* $\deg_B(\psi_H, \Omega_1) = d$.

Proof. The conclusion that $\psi_H(\alpha, y) \neq 0$ for $(\alpha, y) \in \partial\Omega_1$ follows immediately from (C.1) and (C.2). To prove that $\deg_B(\psi_H, \Omega_1) = d$, we consider the parallelepiped $P := [a_1, a_2] \times [0, b] \times [\beta -, \beta + c]$. It is easy to see that there exists an orientation preserving homeomorphism $\mathscr{P}\colon P \to P$

such that it transforms the faces $P_- := \{a_1\} \times [0, b] \times [\beta - c, \beta + c]$, $P_+ := \{a_2\} \times [0, b] \times [\beta - c, \beta + c]$, and $P_1 := [a_1, a_2] \times \{0\} \times [\beta - c, \beta + c]$ onto the face P_1. We denote $\mathscr{P}(P_+)$ by U_+, $\mathscr{P}(P_-)$ by U_-, and $\mathscr{P}(P_1)$ by U_1.



Next, we consider the composition

$$\tilde{H} := H \circ \mathscr{P}^{-1}: P \to \mathbb{R}^2.$$

By the construction, for each $t \in [0, b]$, the mapping $\tilde{H}_t: P_1 \to \mathbb{R}^2$ defined by $\tilde{H}_t(a, 0, y) = \tilde{H}(a, t, y)$ for $(a, t, y) \in P$ has no zero in ∂P_1. Therefore, $\tilde{H}$: $[0, b] \times P_1 \to \mathbb{R}^2$ may be viewed as a homotopy $\tilde{H}_t$ from $\tilde{H}_0: P_1 \to \mathbb{R}^2$ to $\tilde{H}_b: P_1 \to \mathbb{R}^2$. Moreover, by condition (C.1), $\tilde{H}_b$ has no zero in P_1, consequently, $\deg_B(\tilde{H}_b, P_1) = 0$. This implies $\deg(\tilde{H}_0, P_1) = 0$.

On the other hand, since $\mathscr{P}^{-1}$ is a homeomorphism, in view of orientation preserving and reversing properties of $\mathscr{P}^{-1}$ on the boundary ∂P, we have

$$\deg_B(\tilde{H}|_{\bar{U}_+}, U_+) = \deg_B(H_{\alpha_2}, \Omega),$$

$$\deg_B(\tilde{H}|_{\bar{U}_-}, U_-) = 1 \deg_B(H_{\alpha_1}, \Omega),$$

$$\deg_B(\tilde{H}|_{\bar{U}_1}, U_1) = \deg_B(\psi_H, \Omega_1).$$

By applying the additivity and excision properties of the Brouwer degree we obtain

$$\begin{aligned} 0 &= \deg_B(\tilde{H}_0, P_1) \\ &= \deg_B(\tilde{H}|_{\bar{U}_+}, U_+) + \deg_B(\tilde{H}|_{\bar{U}_1}, U_1) + \deg_B(\tilde{H}|_{\bar{U}_-}, U_-) \\ &= \deg_B(H_{a_2}, \Omega) + \deg_B(\psi_H, \Omega_1) - \deg_B(H_{a_1}, \Omega) \\ &= \deg_B(\psi_H, \Omega_1) + d_2 - d_1. \end{aligned}$$

This completes the proof. ∎

3. Local and Global Bifurcations of Functional Differential Equations

We denote by $C(\mathbb{R}, \mathbb{R}^n)$ the Banach space of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$ equipped with the usual supremum norm $\|\phi\| = \sup_{\theta \in \mathbb{R}} |\phi(\theta)|$ for $\phi \in C(\mathbb{R}; \mathbb{R}^n)$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. If $\phi \in C(\mathbb{R}; \mathbb{R}^n)$ and $t \in \mathbb{R}$, then $\phi_t \in C(\mathbb{R}; \mathbb{R}^n)$ is defined by $\phi_t = \phi(t+\theta)$ for $\theta \in \mathbb{R}$.

Consider the one parameter family of functional differential equations

$$\dot{x}(t) = F(x_t, \alpha), \tag{3.1}$$

where $F\colon C(\mathbb{R}; \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is a continuous map and satisfies the following assumption.

(H.1) There exists a number $s > 0$ such that the map F has a continuous factorization $\hat{F}\colon C((-\infty, s]; \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$; i.e., we have the following commutative diagram (where $r\colon C(\mathbb{R}; \mathbb{R}^n) \to C((-\infty, s]; \mathbb{R})$ denotes the restriction operator).

$$\begin{array}{ccc} C(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R} & \xrightarrow{\quad F \quad} & \mathbb{R}^n \\ {\scriptstyle r \times \mathrm{id}} \big\downarrow & \nearrow {\scriptstyle \hat{\mathcal{F}}} & \\ C((-\infty, s]; \mathbb{R}^n) \times \mathbb{R} & & \end{array}$$

The existence of such a factorization $\hat{F}$ means that in Eq. (3.1) we admit functionals with both unbounded *delayed* argument and bounded *advanced* argument. That is, we will consider a class of functional differential equations of *mixed type*.

In what follows, we will assume that $\hat{F}$ is a C^1-map. We should remark that, because of the nature of our approach, this assumption can be weakened.

For $x_0 \in \mathbb{R}^n$, we will use the same symbol to denote the constant function $x_0(\theta) = x_0$, for all $\theta \in \mathbb{R}$. Since $(x_0)_t = x_0$ for all $t \in \mathbb{R}$, if $F(x_0, \alpha_0) = 0$, then x_0 is a solution of (3.1) with $\alpha = \alpha_0$. We will call such points $(x_0, \alpha_0) \in \mathbb{R}^n \times \mathbb{R}$ *stationary points* of (3.1). Moreover, we say that a stationary point (x_0, α_0) is *nonsingular* if the restriction of F to the space $\mathbb{R}^n \times \mathbb{R} \subseteq C(\mathbb{R}; \mathbb{R}^n) \times \mathbb{R}$, still denoted by F, has the derivative $D_x F(x_0, \alpha_0)\colon \mathbb{R}^n \to \mathbb{R}^n$ (with respect to $x \in \mathbb{R}^n$), which is an isomorphism.

Let (x_0, α_0) be a stationary point of (3.1). The linearization of Eq. (3.1) at (x_0, α_0) leads to the following *characteristic equation* for the stationary point (x_0, α_0),

$$\det[\lambda I - T(x_0, \alpha_0)(e^{\lambda \cdot} \cdot)] = 0, \tag{3.2}$$

where $T(x_0, \alpha_0) := D_\phi \hat{F}(x_0, \alpha_0)$: $C((-\infty, s]; \mathbb{C}^n) \to \mathbb{C}^n$, $(e^{\lambda \cdot} \cdot)(\theta, x) = e^{\lambda\theta} x$, for $(\theta, x) \in \mathbb{R} \times \mathbb{C}^n$, and

$$\Delta_{(x_0, \alpha_0)}(\lambda) := \lambda I - T(x_0, \alpha_0)(e^{\lambda \cdot} \cdot): \mathbb{C}^n \to \mathbb{C}^n$$

is a complex $n \times n$ matrix. By our assumption on the factorization $\hat{F}$, formula (3.2) is well defined for all complex numbers λ such that $\operatorname{Re} \lambda \geqslant 0$.

A solution λ_0 to Eq. (3.2) will be called a *characteristic value* of the stationary point (x_0, α_0). It is clear that (x_0, α_0) is a nonsingular stationary point if and only if 0 is not a characteristic value of stationary point (x_0, α_0).

In what follows, we say that a nonsingular stationary point (x_0, α_0) is a *center* if it has a purely imaginary characteristic value, and we will call (x_0, α_0) an *isolated center* if it is the only center in some neighbourhood of (x_0, α_0) in $\mathbb{R}^n \times \mathbb{R}$.

For a local Hopf bifurcation, we assume the following

(H.2) There exists $(x_0, \alpha_0) \in \mathbb{R}^n \times \mathbb{R}$ *such that* (x_0, α_0) is an isolated center of (3.1).

Since (x_0, α_0) is a nonsingular stationary point, i.e., $D_x F(x_0, \alpha_0)$: $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, there is a continuously differentiable function $x(\alpha)$ for α near α_0 such that $(x(\alpha), \alpha)$ is a stationary point for each α. To simplify the notation, we put

$$T(\alpha) = T(x(\alpha), \alpha) := D_\phi F(x(\alpha), \alpha): C((-\infty, s]; \mathbb{C}^n) \to \mathbb{C}^n$$

and

$$\Delta_\alpha(\lambda) := \lambda I - T(\alpha)(e^{\lambda \cdot} \cdot),$$

where $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, for sufficiently small $\delta > 0$.

By assumption (H.2) there is $\beta_0 > 0$ such that $\det \Delta_{\alpha_0}(i\beta_0) = 0$ and if $0 < |\alpha - \alpha_0| < \delta$, then $iR \cap \{\lambda \in \mathbb{C};\ \operatorname{Re} \lambda \geqslant 0,\ \det \Delta_\alpha(\lambda) = 0\} = \varnothing$.

Choose constants $b = b(\alpha_0, \beta_0) > 0$ and $c = c(\alpha_0, \beta_0) > 0$ such that the closure of $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c) \subseteq \mathbb{R}^2 \simeq \mathbb{C}$ contains no other zero of $\det \Delta_{\alpha_0}(\lambda)$. It is well known that $\det \Delta_\alpha(\lambda)$, $0 < |\alpha - \alpha_0| \leqslant \delta$, is analytic in $\lambda \in \Omega$ and continuous in α. Since for sufficiently small $\delta > 0$, there exists no zero of $\det \Delta_{\alpha_0 \pm \delta}(\lambda)$ in $\partial\Omega$, the notation

$$\gamma_\pm(x_0, \alpha_0, \beta_0) := \deg_B(\det \Delta_{\alpha_0 \pm \delta}(\cdot), \Omega)$$

is well defined. We now can introduce the following important concept:

DEFINITION 3.1. The *crossing number* of (x_0, α_0, β_0) is defined as

$$\gamma(x_0, \alpha_0, \beta_0) := \gamma_-(x_0, \alpha_0, \beta_0) - \gamma_+(\chi_0, \alpha_0, \beta_0).$$

We are now in a position to state the local Hopf bifurcation theorem.

THEOREM 3.2. *Suppose that hypotheses* (H.1) *and* (H.2) *are satisfied. If* $\gamma(x_0, \alpha_0, \beta_0) \neq 0$, *then there exists a bifurcation of nonconstant periodic solutions from* (x_0, α_0). *More precisely, there exists a sequence* $\{(x_n(t), \alpha_n, \beta_n)\}$ *such that* $\alpha_n \to \alpha_0$, $\beta_n \to \beta_0$, $x_n(t) \to x_0$ for all $t \in \mathbb{R}$ *as* $n \to \infty$, *and* $x_n(t)$ *is a nonconstant periodic solution with a period* $2\pi/\beta_n$ *of Eq.* (3.1) *with* $\alpha = \alpha_n$.

Proof. We first introduce the period as an additional parameter so that we can work in a space of functions of fixed period 2π. By making a change of variable

$$x(t) = z(\beta t) \qquad \text{or} \qquad z(t) = x\left(\frac{1}{\beta} t\right)$$

we obtain

$$\dot{z}(t) = \frac{1}{\beta} F(z_{t,\beta}, \alpha), \tag{3.3}$$

where $z_{t,\beta}(\theta) = z(t + \beta\theta)$ for $\theta \in \mathbb{R}$. Evidently, $z(t)$ is a 2π-periodic solution of (3.3) if and only if $x(t)$ is a $2\pi/\beta$-periodic solution of (3.1).

We put $S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$, $\mathbb{E} = H^1(S^1; \mathbb{R}^n)$, $\mathbb{F} = L^2(S^1; \mathbb{R}^n)$, $\mathbb{D}(\alpha_0, \beta_0) := (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$ and define $L: \mathbb{E} \to \mathbb{F}$, $\mathcal{F}: \mathbb{E} \times \mathbb{D}(\alpha_0, \beta_0) \to \mathbb{F}$ as

$$Lz(t) = \dot{z}(t), \qquad \mathcal{F}(z, \alpha, \beta)(t) = \frac{1}{\beta} F(z_{t,\beta}, \alpha),$$

where $z \in \mathbb{E}$, $(\alpha, \beta) \in \mathbb{D}(\alpha_0, \beta_0)$, $t \in \mathbb{R}$. The spaces $\mathbb{E}$ and $\mathbb{F}$ are isometric Hilbert representations of the group $G = S^1$, where S^1 acts simply by shifting the argument. Evidently, L is a bounded equivariant Fredholm operator of index zero such that $\operatorname{Ker}(L) = \mathbb{R}^n \subseteq \mathbb{E}$, the subspace of constant functions. It is useful to consider L as an unbounded self-adjoint operator from $\operatorname{Dom}(L) = H^1(S^1; \mathbb{R}^n) \subset L^2(S^1; \mathbb{R}^n)$ into $L^2(S^1; \mathbb{R}^n)$. Since we have the orthogonal direct decomposition $L^2 := \operatorname{Ker}(L) \oplus \operatorname{Im}(L)$, we can define an equivariant resolvent $K: H^1(S^1; \mathbb{R}^n) \to L^2(S^1; \mathbb{R}^n)$ of the operator L just by taking L^2-orthogonal projection P onto the subspace of constant functions $\mathbb{R}^n = \operatorname{Ker}(L)$, i.e., for every $x \in H^1(S^1; \mathbb{R}^n)$, $Kx := (1/2\pi) \int_0^{2\pi} x(t)\, dt$. We also note that the inclusion $\mathbb{E} \subsetneq C(S^1; \mathbb{R}^n)$ is a compact operator and the mapping $\hat{\mathscr{F}}: C(S^1; \mathbb{R}^n) \times \mathbb{D}(\alpha_0, \beta_0) \to L^2(S^1; \mathbb{R}^n)$ defined by

$$\hat{\mathscr{F}}(z, \alpha, \beta)(t) = \frac{1}{\beta} F(z_{t,\beta}, \alpha)$$

is continuous. Thus it follows from the following commutative diagram that $\mathscr{F}$ is a completely continuous map.

$$\begin{array}{ccc} H^1(S^1; \mathbb{R}^n) \times \mathscr{D}(\alpha_0, \beta_0) & \xrightarrow{\mathscr{F}} & L^2(S^1; \mathbb{R}^n) \\ \downarrow j & \nearrow \hat{\mathscr{F}} & \\ C(S^1; \mathbb{R}^n) \times \mathscr{D}(\alpha_0, \beta_0) & & \end{array}$$

Define $f: \mathbb{E} \times \mathbb{D}(\alpha_0, \beta_0) \to \mathbb{F}$ by

$$f(z, \alpha, \beta)(t) = \dot{z}(t) - \frac{1}{\beta} F(z_{t,\beta}, \alpha),$$

that is,

$$f(z, \alpha, \beta) = Lz - \mathscr{F}(z, \alpha, \beta).$$

Then finding a 2π-periodic solution of Eq. (3.3) is equivalent to finding a solution of the problem

$$f(z, \alpha, \beta) = 0 \tag{3.4}$$

in $\mathbb{E} \times \mathbb{D}(\alpha_0, \beta_0)$. Evidently, f is S^1-equivariant, and so its derivative

$$D_z f(x(\alpha), \alpha, \beta) z(t) = \dot{z}(t) - \frac{1}{\beta} T(\alpha)(z_{t,\beta}).$$

By assumption (H.2), we can define the 2-dimensional submanifold $N \subseteq \mathbb{E}_0 \oplus \mathbb{R}^2$ by taking

$$N := \{(x(\alpha), \alpha, \beta); \alpha \in (\alpha_0 - \delta, \alpha_0 + \delta), \beta \in (\beta_0 - c, \beta_0 + c)\}.$$

It is an easy observation that N satisfies assumption (A.3) of the previous section, and (x_0, α_0, β_0) is the only $\mathbb{E}$-singular point in N.

It is well known that $\mathbb{E}$ has the following isotypical direct sum decomposition $\mathbb{E} = \bigoplus_{k=0}^{\infty} \mathbb{E}_k$, where $\mathbb{E}_k$ is the subspace spanned by $\cos(kt)\varepsilon_j$ and $\sin(kt)\varepsilon_j$ for $j = 1, 2, ..., n$, where $\{\varepsilon_1, ..., \varepsilon_k\}$ denotes the standard basis of $\mathbb{R}^n$. In what follows, we will identify $\mathbb{E}_k$, $k > 0$, with the linear space over the complex numbers spanned by $\exp(kt)\varepsilon_j$, $j = 1, ..., n$.

Put $\Psi(\alpha, \beta) = (L+K)^{-1} D_z f(x(\alpha), \alpha, \beta)\colon \mathbb{E} \to \mathbb{E}$ for $(\alpha, \beta) \in \mathbb{D}(\alpha_0, \beta_0)$. Since $\Psi(\alpha, \beta)$ is S^1-equivariant, $\Psi(\alpha, \beta)(\mathbb{E}_k) \subseteq \mathbb{E}_k$ for all $k = 0, 1, ..,$ and therefore we obtain the maps $\Psi_k\colon \mathbb{D}(\alpha_0, \beta_0) \to \mathscr{L}(\mathbb{E}_k, \mathbb{E}_k)$ by

$$\Psi_k(\alpha, \beta) := \Psi(\alpha, \beta)|_{\mathbb{E}_k}.$$

By direct verification, we have, for $k = 1, 2, ...,$ that

$$\Psi_k(\alpha, \beta) z = z - \frac{1}{\beta} L^{-1} T(\alpha)(z_{t,\beta}), \qquad z \in \mathbb{E}_k,$$

thus

$$\begin{aligned}\Psi_k(\alpha, \beta)(e^{ikt}\varepsilon_j) &= e^{ikt}\varepsilon_j - \frac{1}{\beta} L^{-1} T(\alpha)(e^{ikt} e^{ik\beta} \cdot \varepsilon_j)\\ &= e^{ikt}\varepsilon_j - \frac{1}{\beta} L^{-1}(e^{ikt})\, T(\alpha)(e^{ik\beta} \cdot \varepsilon_j)\\ &= e^{ikt}\varepsilon_j - \frac{1}{\beta ik}\, T(\alpha)(e^{ik\beta} \cdot \varepsilon_j).\end{aligned}$$

Therefore the matrix representation of $\Psi_k(\alpha, \beta)$ with respect to the ordered basis $(e^{ikt}\varepsilon_1, ..., e^{ikt}\varepsilon_n)$ is $(1/ik\beta)\Delta_\alpha(ik\beta)$.

Now, we can use the construction of $\gamma_1(x_0, \alpha_0, \beta_0)$ explained in the previous section. First of all, we put $\Omega_1 := (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$. We identify $\partial\Omega_1$ with S^1 and define $Q\colon \bar{\Omega}_1 \to \mathbb{R}^2 \simeq \mathbb{C}$ by $Q(\alpha, \beta) = \det_{\mathbb{C}} \Delta_\alpha(i\beta)$. By Proposition 2.2, we obtain

$$\gamma_1(x_0, \alpha_0, \beta_0) = \varepsilon \deg_{\mathrm{B}}(Q, \Omega_1),$$

where $\varepsilon = \operatorname{sign} \det \Psi_0(\alpha, \beta)$, $(\alpha, \beta) \in \partial\Omega_1$. Then we define the function $H\colon [\alpha_0 - \delta, \alpha_0 + \delta] \times \bar{\Omega} \to \mathbb{R}^2 \simeq \mathbb{C}$, where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c)$, by

$$H(\alpha, u, v) = \det_{\mathbb{C}} \Delta_\alpha(u + iv).$$

The choice of b, c, and δ guarantees that all conditions in Lemma 2.5 are satisfied, consequently

$$\deg_{\mathrm{B}}(Q, \Omega_1) = \gamma(x_0, \alpha_0, \beta_0).$$

By the assumption that $\gamma(x_0, \alpha_0, \beta_0) \neq 0$, we have

$$\gamma_1(x_0, \alpha_0, \beta_0) = \varepsilon\gamma(x_0, \alpha_0, \beta_0) \neq 0.$$

Therefore, by Theorem 2.3, (x_0, α_0, β_0) is a bifurcation point. More precisely, there exists a sequence $\{(x_n(t), \alpha_n, \beta_n)\}$ such that $\alpha_n \to \alpha_0$,

$\beta_n \to \beta_0$ as $n \to \infty$, and $x_n(t)$ is a nontrivial periodic solution of the equation $\dot{x}(t) = F(x_t, \alpha_n)$ with a period $2\pi/\beta_n$ such that $x_n \to x_0$ in the H^1-norm as $n \to \infty$. This completes the proof. ▌

We now consider the global bifurcation problem. In what follows, we assume the following conditions are satisfied:

(H.3) For any bounded set $W \subseteq \mathbb{E} \times \mathbb{R}$, there exists a constant $L > 0$ such that $|F(\phi, \alpha) - F(\psi, \alpha)| \leqslant L \|\phi - \psi\|$ for all (ϕ, α), $(\psi, \alpha) \in W$.

(H.4) All stationary points of (3.1) are nonsingular and all centers of (3.1) are isolated.

In a way similar to that in the proof of Theorem 3.2, it is possible to reformulate the problem (3.1) as a problem of finding a 2π-periodic solution to the equation

$$\dot{y}(t) = pF(y_{t, 1/p}, \alpha), \tag{3.5}$$

where $y_{t,1/p}(\theta) = y(t + \theta/p)$ for $\theta \in \mathbb{R}$. Here we replace $\beta > 0$ by $1/p$ in Eq. (3.3). We are looking for the solution $(y(t), \alpha, p)$ in the so-called *Fuller space* (cf. [19]) $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ := \{p \in R; p > 0\}$ and $\mathbb{E} := H^1(S^1; \mathbb{R}^n)$. This problem can again be reformulated as the coincidence problem

$$Ly = \mathscr{F}(\alpha, p)(y), \qquad y \in \mathbb{E},$$

where $L\colon \mathbb{E} \to \mathbb{F} := L^2(S^1; \mathbb{R}^n)$, $Ly = \dot{y}$, and $\mathscr{F}\colon \mathbb{E} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{F}$, $\mathscr{F}(\alpha, \beta)(y)(t) = pF(y_{t,1/p}, \alpha)$.

Under assumption (H.4), zero is a regular value of the restriction $\mathscr{F}_0 := \mathscr{F}|_{\mathbb{E}_0 \times \mathbb{R} \times \mathbb{R}_+}\colon \mathbb{E}_0 \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{F}_0$, where $\mathbb{E}_0 = \mathbb{E}^G$ and $\mathbb{F}_0 = \mathbb{F}^G$. Consequently, $\mathscr{F}^{-1}(0) := N$ is a 2-dimensional complete submanifold of $\mathbb{E}_0 \times \mathbb{R}^2$ such that $N \subseteq f^{-1}(0)$, where $f := L - \mathscr{F}$. It is easy to verify that N satisfies the assumption (4.3). We will call N the set of *trivial* (*stationary*) solutions of $(3.1)_\alpha$.

Let $\Xi(f)$ denote the closure of the set of all nontrivial periodic solutions of (3.5) in $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$ and let $\mathfrak{I}(y_0, \alpha_0, p_0)$ denote the connected component of a bifurcation point $(y_0, \alpha_0, p_0) \in \Xi(f)$.

We point out that the closure of the set $\mathfrak{I}(y_0, \alpha_0, p_0)$ in the space $\mathbb{E} \times \mathbb{R} \times \mathbb{R}$ does not contain any point of type $(y, \alpha, 0)$. Indeed, by way of contradiction, if there exists $(y^*, \alpha^*, 0) \in \mathfrak{I}(y_0, \alpha_0, p_0)$, then there is a sequence $\{(y_n, \alpha_n, p_n)\}_{n=1}^{\infty} \subseteq \mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$ such that y_n is a nontrivial 2π-periodic solution of (3.5) with $\alpha = \alpha_n$, $p = p_n$, and $(y_n, \alpha_n, p_n) \to (y^*, \alpha^*, 0)$ in $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$ as $n \to \infty$. Therefore, $W := \bigcup_{t \in \mathbb{R}} \bigcup_{n=1}^{\infty} \{(y_{nt}, \alpha_n)\}$ is a bounded set of $C((-\infty, s]; \mathbb{R}^n) \times \mathbb{R}$. By the argument of Lasota and Yorke [34], we can show that if condition (H.3) is satisfied, then there exists a constant $\varepsilon_0 > 0$ such that $p_n \geqslant \varepsilon_0$ for all $n = 1, 2, ...,$ which contradicts $p_n \to 0$ as $n \to \infty$.

Therefore, without loss of generality we can assume that the problem (3.5) is posed on the whole space $\mathbb{E} \times \mathbb{R}^2$. In fact, F is bounded on bounded subsets, thus Eq. (3.5) is well defined for all values of p. Therefore, we obtain the following global Hopf bifurcation theorem.

THEOREM 3.3. *Under assumptions* (H.1), (H.3), *and* (H.4), *if* $(y_0, \alpha_0, p_0) \in N$ *is a bifurcation point of* (3.6), *then either the connected component* $\mathfrak{J}(y_0, \alpha_0, p_0)$ *of* (y_0, α_0, p_0) *in* $\Xi(f)$ *is unbounded, or the number of bifurcation points in* $\mathfrak{J}(y_0, \alpha_0, p_0)$ *is finite, i.e.,*

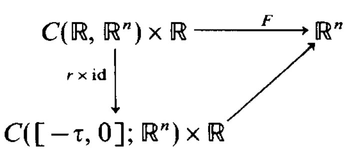
$$\mathfrak{J}(y_0, \alpha_0, p_0) \cap N = \{(y_0, \alpha_0, p_0), (y_1, \alpha_1, p_1), ..., (y_q, \alpha_q, p_q)\}.$$

Moreover, we have the equality

$$\gamma\left(y_0, \alpha_0, \frac{1}{p_0}\right) + \gamma\left(y_1, \alpha_1, \frac{1}{p_1}\right) + \cdots + \gamma\left(y_q, \alpha_q, \frac{1}{p_q}\right) = 0.$$

Proof. This is an immediate consequence of Theorem 2.4, since as we have shown in the argument of Theorem 3.2, $\gamma_1(x_i, \alpha_i, 1/p_i) = \varepsilon\gamma(x_i, \alpha_i, 1/p_i)$ and ε is independent of $(x_i, \alpha_i, 1/p_i)$ for $i = 0, 1, ..., q$. ∎

W conclude this paper by comparing the above results with some existing older ideas. The crucial concept, that of the crossing number $\gamma(x_0, \alpha_0, \beta_0)$, involved in our Hopf bifurcation theory concerns a net change of stability along trivial solutions. The close relation of the crossing number $\gamma(x_0, \alpha_0, \beta_0)$ with the *center index* $\oplus\,(x_0, \alpha_0)$ introduced in Mallet-Paret and Yorke [37], the *Hopf index* $H(f)$ of (3.1) defined by Fiedler [17], and the *index* $r(x_0, \alpha_0, \beta_0)$ with respect to (3.1) discussed in Nussbaum [41] can be demonstrated as follows: suppose that F has the continuous factorization $\hat{F}\colon C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$, i.e., we have the commutative diagram



where $r\colon C(\mathbb{R}, \mathbb{R}^n) \to C([-\tau, 0]; \mathbb{R}^n)$ denotes the restriction operator and $\tau > 0$ is a given real number. In other words, we suppose that the system (3.1) is a one parameter family of *retarded* functional differential equations with *finite* delay. If (x_0, α_0) is an isolated center and $\beta_0 > 0$ is a constant such that $\det \Delta_{\alpha_0}(i\beta_0) = 0$, then we can cover the element $i\beta_0$ by a closed disc $D(i\beta_0)$ which contains only the characteristic value $i\beta_0$ of the stationary point (x_0, α_0). Define

$$\mathrm{Mult}_\alpha(i\beta_0) = \{z \in D(\beta_0); \det \Delta_\alpha(z) = 0\}.$$

If there is an open interval Λ_{α_0} containing α_0 such that the real part of $z \in \mathrm{Mult}_\alpha(i\beta_0)$ is nonzero for $\alpha \in \Lambda_{\alpha_0} \setminus \{\alpha_0\}$, then for α sufficiently close to α_0, the number of elements in $\mathrm{Mult}_\alpha(i\beta_0)$ (counted with algebraic multiplicity) is the same as the algebraic multiplicity of $i\beta_0$ as a zero of $\det \Delta_{\alpha_0}(\lambda) = 0$. It can be verified, by direct computation of the Brouwer degree $\deg_B(\Delta_{\alpha_0 \pm \delta(\cdot)}, \Omega)$ of the analytic function $\Delta_{\alpha_0 \pm \delta(\cdot)}$, that $\gamma_\pm(x_0, \alpha_0, \beta_0)$ is the number of elements of $\mathrm{Mult}_\alpha(i\beta_0)$ (counted with algebraic multiplicity) whose real part is positive for $\alpha > \alpha_0$, or $\alpha < \alpha_0$, respectively. Therefore, $r(x_0, \alpha_0, \beta_0)$ and $\text{\c{c}}\!\!\!\!\square\ (x_0, \alpha_0)$ can be expressed by our crossing numbers according to the formulas

$$\begin{aligned} r(x_0, \alpha_0, \beta_0) &= \sum_{m \in N_1} [\gamma_+(x_0, \alpha_0, m\beta_0) - \gamma_-(x_0, \alpha_0, m\beta_0)] \\ &= -\sum_{m \in N_1} \gamma(x_0, \alpha_0, m\beta_0), \end{aligned}$$

and

$$⊕\ (x_0, \alpha_0) = -\frac{\varepsilon}{2} \sum_{\beta \in N_2} \gamma(x_0, {}_0, \beta_0).$$

Moreover, $H(f)$ can be expressed as

$$H(f) = -\frac{\varepsilon}{2} \sum_{(x_0, \alpha_0, \beta_0) \in N_3} \gamma(x_0, \alpha_0, \beta_0),$$

where

$$\begin{aligned} N_1 &= \{m \in \mathbb{N}; \det \Delta_{\alpha_0}(im\beta_0) = 0\} \\ N_2 &= \{\beta > 0; \det \Delta_{\alpha_0}(i\beta) = 0\} \\ N_3 &= \{(x_0, \alpha_0, \beta_0) \in \mathbb{R}^n \times \mathbb{R} \times (0, \infty); \det \Delta_{\alpha_0}(i\beta_0) = 0\} \end{aligned}$$

and $\varepsilon = (-1)^{E(\alpha_0)}$ and $E(\alpha_0)$ is the sum of the numbers (counted with algebraic multiplicity) of the characteristic values of (3.1) whose real parts are positive. Our global Hopf bifurcation theorem implies the following conclusions:

(i) If the crossing number $\gamma(x_0, \alpha_0, \beta_0) \neq 0$ and $\mathfrak{I}(x_0, \alpha_0, \beta_0)$ is bounded, then $\mathfrak{I}(x_0, \alpha_0, \beta_0)$ must contain another bifurcation point. This is an analogy of the well-known Alexander–Yorke theorem on global bifurcation of periodic orbits (cf. [1]) whose extension to functional differential equations with finite delay was obtained by Chow and Mallet-Paret [8], using Fuller's index for periodic solutions of autonomous systems.

(ii) If the set of centers is bounded and $H(f) \neq 0$, then there exists $(x_0, \alpha_0, \beta_0) \in \mathbb{R}^n \times \mathbb{R} \times (0, \infty)$ such that $\mathfrak{I}(x_0, \alpha_0, \beta_0)$ is bounded. This represents an analog of the global bifurcation theorem obtained by Mallet-Paret [36] and Fiedler [17], using generic approximation techniques.

Moreover, we note that if $r(x_0, \alpha_0, \beta_0) \neq 0$, then there must be an integer k such that $\gamma(x_0, \alpha_0, k\beta_0) \neq 0$. On the other hand, by the same argument as that for Theorem 3.2, we can prove that $\gamma_k(x_0, \alpha_0, \beta_0) = \varepsilon\gamma(x_0, \alpha_0, k\beta_0)$ with $\varepsilon = \pm 1$ independent of the choice of (x_0, α_0, β_0). Therefore, the argument of Theorem 3.3 can be applied to obtain the following result:

(iii) If $r(x_0, \alpha_0, \beta_0) \neq 0$ and $\mathfrak{I}(x_0, \alpha_0, \beta_0)$ is bounded, then $\mathfrak{I}(x_0, \alpha_0, \beta_0)$ must contain another bifurcation point. This result was essentially discovered by Nussbaum [41], using a strongly proper approximation of the operator of translation along trajectories of functional differential equations with respect to a natural set of projections.

Finally, it should be pointed out that the approaches in the papers mentioned above rely on a certain Kupka–Smale or generic approximation of functional differential equations (cf. [1, 8, 36, 41]) and on certain local analysis as well as pertubation arguments in the neighbourhood of a single periodic orbit (cf. [17, 41]) which, to the best of our knowledge, have not been developed to functional differential equations of mixed type (with both advanced and delayed argument). It is due to the purely topological nature of our argument that our global Hopf bifurcation theory can be applied to this very general class of functional differential equations.

APPENDIX: S^1-DEGREE

To describe the finite dimensional version of S^1-degree constructed in [16], we assume that V is a finite dimensional orthogonal representation of $G = S^1$.

If U is an open bounded invariant subset of $V \oplus \mathbb{R}$ (where S^1 acts trivally on $\mathbb{R}$) and $\varphi: (U, \partial U) \to (V, V \setminus \{0\})$ is an equivariant continuous map, then there is defined the S^1-degree of φ with respect to U,

$$\mathrm{Deg}(\varphi, U) := \{\deg_H(\varphi, U)\},$$

where H runs through the family of all closed subgroups of S^1,

$\deg_H(\varphi, U) \in \mathbb{Z}_2$ for $H = S^1$ and $\deg_H(\varphi, U) \in \mathbb{Z}$ for $H \neq S^1$. Note that if H is a closed subgroup of S^1 then either $H = S^1$ or $H = \mathbb{Z}_k = \{\gamma \in S^1; \gamma^k = 1\}$ (the cyclic group of order k). Accordingly, we sometimes use the following notation: $\deg(\varphi, U) = \deg_G(\varphi, U) \in \mathbb{Z}_2$; $\deg_k(\varphi, U) = \deg_{\mathbb{Z}_k}(\varphi, U) \in \mathbb{Z}$, $k \in \mathcal{N}$. The basic property of Deg is that $\text{Deg}(\varphi, U) \neq 0$ implies that the equation $\varphi(x, \lambda) = 0$ has a solution in U; more precisely, $\deg_H(\varphi, U) \neq 0$ implies $\varphi^{-1}(0) \cap U^H \neq \emptyset$. Further, the S^1-equivariant degree has the standard properties referred to in Section 2, namely, the additivity, homotopy invariance, and the contraction property; cf. [16] for details.

Moreover, an immediate consequence of the definition of S^1-degree is

(∗) If for a closed subgroup $H \subset G$ there is no $x \in U$ such that $G_x = H$ then

$$\deg_H(\varphi, U) = 0.$$

The construction of S^1-degree given in [16] uses only the definition and basic properties of the classical topological degree.

Another more general construction of S^1-degree was given in [30]. Suppose that V and W are two real orthogonal finite dimensional representations of G. Put $M = \dim V$, $N = \dim W$. Let $S(\mathbb{R} \oplus V)$ (resp., $S(\mathbb{R} \oplus W)$) denote the unit sphere in $\mathbb{R} \oplus V$ (resp., $\mathbb{R} \oplus W$)), where G acts trivially on $\mathbb{R}$. Suppose U is an open bounded invariant subset of $\mathbb{R} \oplus V$ and $\varphi: (\bar{U}, \partial U) \to (W, W \setminus \{0\})$ is an equivariant map. Since $S(\mathbb{R} \oplus V)$ is in a natural way equivariantly homeomorphic to one-point compactification of V, one can assing to φ an equivariant continuous map $\hat{\varphi}: S(\mathbb{R} \oplus V) \to S(\mathbb{R} \oplus W)$. Set

$$\widehat{\text{Deg}}(\varphi, U) := [\varphi] \in \pi_V(S(\mathbb{R} \oplus W)),$$

where $\pi_V(S(\mathbb{R} \oplus W))$ stands for an equivariant homotopy group denoted by $\pi_M(S^N)$ in [30]. Note that in the non-equivariant case the $\widehat{\text{Deg}}(\varphi, U)$ was introduced in [20]. The definition of $\widehat{\text{Deg}}(\varphi, U)$ is patterned after the definition of fixed-point index introduced by A. Dold in [15]. Obviously the computation of $\pi_V(S(\mathbb{R} \oplus W))$ is crucial from the point of view of applications. Before we formulate the results from [30] we need some notation.

For $n \in \mathbb{N}$, $\gamma \in G$, $z \in \mathbb{C}$ the formula $\gamma z = \gamma^n \cdot z$, where $\cdot$ denotes the standard multiplication of complex numbers, defines a real 2-dimensional representation of G, which we will denote by R_n. Without loss of generality we may assume that

$$V = \mathbb{R}^k \oplus R_{m_1} \oplus \cdots \oplus R_{m_q}, \qquad m_1 \leqslant m_2 \leqslant \cdots \leqslant m_q;$$

$$W = \mathbb{R}^l \oplus R_{n_1} \oplus \cdots \oplus R_{n_r}, \qquad n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r;$$

where the action of G on $\mathbb{R}^k$ and $\mathbb{R}^l$ is trivial. Assume that

(α) $M-N=1$;

(β) $q \leqslant r$;

(γ) $m_j = k_j n_j$ for $j=1, ..., r$;

(δ) n_j are multiples of m_s for $s=r+1, ..., q$.

Set $p=q-r$. Then (Theorem 2 in [30])

(A) if $p>1$ then $\pi_V(S(\mathbb{R} \oplus W)) = \mathbb{Z}$;

(B) if $p=1$ then $\pi_V(S(\mathbb{R} \oplus W))$ is a cyclic group, whose order is explicitly computed in [30];

(C) if $p=0$ then $\pi_V(S(\mathbb{R} \oplus W)) = \pi_k(S^l) \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (where there is one $\mathbb{Z}$ for each $H \in \mathscr{H}$, where $\mathscr{H}$ denotes the family of all finite subgroups $H \subset G$ such that $\dim V^H = 1 + \dim W^H$ and there exists $x \in V$ with $G_x = H$).

If we let $k=l+1$ (hence $p=0$ and $q=r$) and $k_j=1$ for $j=1, ..., q$ then $V = W \oplus \mathbb{R}$. In this case $\widehat{\mathrm{Deg}}(\varphi, U) = \{\alpha_H\}$, where either $H=G$ and $\alpha_H \in \pi_k(S^l)$ or $H \in \mathscr{H}$ and $\alpha_H \in \mathbb{Z}$. Then $\deg_G(\varphi, U)$ = the double suspension of α_G and $\deg_H(\varphi, U) = \alpha_H$ for $H \in \mathscr{H}$. Thus, in view of property (*), $\mathrm{Deg}(\varphi, U)$ is completely determined by $\widehat{\mathrm{Deg}}(\varphi, U)$.

Acknowledgments

The authors are indebted to the referees for detailed comments and suggestions concerning an earlier version which have contributed to an improved presentation in a more general setting. We are also indebted to Mr. Huaxing Xia for providing additional references.

References

1. J. C. Alexander and J. A. Yorke, Global bifurcation of periodic orbits, *Amer. J. Math.* **100** (1978), 263–292.
2. K. T. Alligood, J. Mallet-Paret, and J. A. Yorke, Families of periodic orbits: Local continuability does not imply global continuability, *J. Differential Geom.* **16** (1981), 483–492.
3. K. T. Alligood and J. A. Yorke, Cascades of period-doubling bifurcations: A prerequisite for horseshoes, *Bull. Amer. Math. Soc.* **9** (1983), 319–322.
4. K. T. Alligood, J. Mallet-Paret, and J. A. Yorke, An index for the global continuations of relatively isolated sets of periodic orbits, *in* "Lecture Notes Mathematics, Vol. 1007" (J. Palis, Jr., Ed.), pp. 1–21, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.

5. N. CHAFEE, A bifurcation problem for a functional differential equation of finitely retarded type, *J. Math. Anal. Appl.* **35** (1971), 312–348.
6. S. N. CHOW AND J. K. HALE, "Methods of Bifurcation Theory," Springer-Verlag, New York, 1982.
7. S. N. CHOW AND J. MALLET-PARET, Integral averaging and bifurcation, *J. Differential Equations* **26** (1977), 112–159.
8. S. N. CHOW AND J. MALLET-PARET, The Fuller index and global Hopf bifurcation, *J. Differential Equations* **29** (1978), 66–85.
9. S. N. CHOW, J. MALLET-PARET, AND J. A. YORKE, Global Hopf bifurcation from a multiple eigenvalue, *Nonlinear Anal.* **2** (1978), 753–763.
10. S. N. CHOW, J. MALLET-PARET, AND J. A. YORK, A periodic orbit index which is a bifurcation invariant, *in* "Lecture Notes in Mathematics, Vol. 1007" (J. Palis, Jr., Ed.), pp. 109–131, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.
11. J. R. CLAEYSSEN, The integral-averaging bifurcation method and the general one-delay equation, *J. Math. Anal. Appl.* **78** (1980), 428–439.
12. M. G. CRANDALL AND P. H. RABINOWITZ, Bifurcation from simple eigenvalues, *J. Funct. Anal.* **8** (1971), 321–340.
13. M. G. CRANDALL AND P. H. RABINOWITZ, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1974), 161–180.
14. M. G. CRANDALL AND P. H. RABINOWITZ, The Hopf bifurcation theorem in infinite dimensions, *Arch. Rational Mech. Anal.* **67** (1977/1978), 53–72.
15. A. DOLD, Fixed point index and fixed point theorem for euclidean neighbourhood retracts, *Topology* **4** (1965), 1–8.
16. G. DYLAWERSKI, K. GĘBA, J. JODEL, AND W. MARZANTOWICZ, S^1-equivariant degree and the Fuller index, *Ann. Polon. Math.* **52** (1991), 243–280.
17. B. FIEDLER, An index for global Hopf bifurcation in parabolic systems, *J. Reine Angew. Math.* **359** (1985), 1–36.
18. B. FIEDLER, "Global Bifurcation of Periodic Solutions with Symmetry," Lecture Notes in Mathematics, Vol. 1309, Springer-Verlag, Berlin/New York, 1988.
19. F. B. FULLER, An index of fixed point type for periodic orbits, *Amer. J. Math.* **89** (1967), 133–148.
20. K. GĘBA, I. MASSABO, AND A. VIGNOLI, Generalized topological degree and bifurcation, *in* "Nonlinear Functional Analysis and Its Applications" (Proceedings, NATO Advanced Study Institute, Italy, 1985), pp. 54–73, Math. and Phys. Sci., Vol. 173, Reidel, Dordrecht, 1986.
21. K. GĘBA AND W. MARZANTOWICZ, Global bifurcation of periodic solutions, preprint.
22. J. K. HALE, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
23. J. K. HALE, Nonlinear oscillations in equations with delays, *in* "Nonlinear Oscillations in Biology," pp. 157–189, Lectures in Applied Mathematics, Vol. 17, Amer. Math. Soc., Providence, RI, 1978.
24. J. K. HALE AND J. C. F. DE OLIVEIRA, Hopf bifurcation for functional equations, *J. Math. Anal. Appl.* **74** (1980), 41–59.
25. B. HASSARD, N. D. KAZARINOFF, AND Y. WAN, "Theory and Applications of Hopf Bifurcation," Cambridge Univ. Press, Cambridge, 1981.
26. J. IZE, Bifurcation theory for Fredholm operators, *Mem. Amer. Math. Soc.* **174** (1976).
27. J. IZE, Global bifurcation of periodic orbits, *Comm. Tecnicas CIMAS* **5** (1974). [Spanish]
28. J. IZE, Obstruction theory and multiparameter Hopf bifurcation, *Trans. Amer. Math. Soc.* **209** (1985), 757–792.
29. J. IZE, I. MASSABÓ, AND V. VIGNOLI, Degree theory for equivariant maps, 1, *Trans. Amer. Math. Soc.* **315** (1989), 433–510.

30. J. IZE, I. MASSABO, AND A. VIGNOLI, Degree theory for equivariant maps, II. The general S^1-action, preprint.
31. N. D. KAZARINOFF, P. VAN DEN DRISSCHE, AND Y. H. WAN, Hopf bifurcation and stability of periodic solutions of differential-difference and integro-differential equations, *J. Inst. Math. Appl.* **21** (1978), 461–477.
32. M. A. KRASONSEL'SKII, "Topological Methods in the Theory of Nonlinear Integral Equations," Pergamon, Elmsford, NY, 1965.
33. H. KIELHÖFER, Hopf bifurcation at multiple eigenvalues, *Arch. Rational Mech. Anal.* **69** (1979), 53–83.
34. A. LASOLTA AND J. A. YORKE, Bounds for periodic solutions of differential equations in Banach spaces, *J. Differential Equations* **10** (1971), 83–91.
35. T. Y. LI, Bounds for the periodic solutions of differential delay equations, *J. Math. Anal. Appl.* **49** (1975), 124–129.
36. J. MALLET-PARET, Generic periodic solutions of functional differential equations, *J. Differential Equations* **25** (1977), 163–183.
37. J. MALLET-PARET AND J. A. YORKE, Snakes: Oriented families of periodic orbits, their sources, sinks and continuation, *J. Differential Equations* **43** (1982), 419–450.
38. J. E. MARSDEN AND M. MCCRACKEN, "The Hopf Bifurcation and Its Applications," Springer-Verlag, Berlin/Heidelberg/New York, 1976.
39. R. D. NUSSBAUM, A global bifurcation theorem with application to functional differential equations, *J. Funct. Anal.* **19** (1975), 319–338.
40. R. D. NUSSBAUM, Global bifurcation of periodic solutions of some autonomous functional differential equations, *J. Math. Anal. Appl.* **55** (1976), 699–725.
41. R. D. NUSSBAUM, A Hopf global bifurcation theorem for retarded functional differential equations, *Trans. Amer. Math. Soc.* **238** (1976), 139–164.
42. J. C. F. DE OLIVEIRA, Hopf bifurcation for functional differential equations, *Nonlinear Anal.* **4** (1980), 217–229.
43. P. RABINOWITZ, Some global results for nonlinear eigenvalues problems, *J. Funct. Anal.* **7** (1971), 487–513.
44. A. RUSTICHINI, Hopf bifurcation for functional differential equations of mixed type, *J. Dynamics Differential Equations* **1** (1989), 145–177.
45. D. H. SATTINGER, "Topics in Stability and Bifurcation Theory," Springer-Verlag, New York, 1973.
46. H. C. SIMPSON, Stability of periodic solutions of nonlinear integrodifferential systems, *SIAM J. Appl. Math.* **38** (1980), 341–363.
47. O. J. STAFFANS, Hopf bifurcation for functional and functional differential equations with infinite delay, *J. Differential Equations* **70** (1987), 114–151.
48. H. W. STECH, The Hopf bifurcation: A stability result and application, *J. Math. Anal. Appl.* **71** (1979), 525–546.
49. H. W. STECH, Hopf bifurcation for functional differential equations, *J. Math. Anal. Appl.* **109** (1985), 472–491.
50. H. W. STECH, Nongeneric Hopf bifurcation in functional differential equations, *SIAM J. Math. Anal* **16** (1985), 1134–1151.
51. J. A. YORKE, Periods of periodic solutions and Lipschitz constant, *Proc. Amer. Math. Soc.* **22** (1969), 509–512.