Nonlinear Analysis, Theory, Methods & Applications, Vol. 18, No. 6, pp. 517-542, 1992.
Printed in Great Britain.

0362-546X/92 $5.00 + .00

STEADY-STATE ANALYSIS IN A MODEL FOR POPULATION DIFFUSION IN A MULTI-PATCH ENVIRONMENT

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(*Received* 15 *October* 1990; *in revised form* 6 *March* 1991; *received for publication* 2 *May* 1991)

Key words and phrases: Boundary value problem, diffusion, homotopy, multi-patch, populations, steady state.

1. INTRODUCTION

ONE PROBLEM of recent interest in the theory of mathematical ecology is the attempt to describe the effect of the spatiotemporal structure of a heterogeneous environment on the growth and diffusion of a population. Most single-species models that have been analysed in the literature (cf. [5, 6]) have a constant diffusion coefficient and a constant carrying capacity of the environment. However, except for a controlled laboratory environment, the carrying capacity is usually varying in space and time. For details, we refer to [7-9, 11-14] and references therein.

Several methods can be utilized to describe this environmental heterogeneity and its effect on the growth and diffusion of the population by a reaction–diffusion equation. One method is to allow the diffusion coefficient and reaction term to depend on spatial position. For example, this method has recently been used by Cantrell and Cosner [1] to show the existence of a positive steady-state solution under appropriate assumptions in the case of logistic growth, where the diffusion is held constant, but the carrying capacity continually varies in the reaction term.

A second method, which is the one adopted in this paper, is to approximate the environment by a sequence of patches, in each of which the diffusion rate and carrying capacity are constants, not necessarily the same from patch to patch. This technique was utilized in [7] for the case of two patches, where the existence of a positive, monotonic, asymptotically stable steady state solution was proved constructively.

However, there are many situations arising in nature where two patches are inadequate for approximating the environmental variation. In [16] it is noted that since the opening of the Suez Canal, there has been a steady diffusion of certain plankton species into the Red Sea, which has gradually spread down its length. Since these plankton form a food basis for many fish and other populations, this has led to a spatially monotonically decreasing carrying capacity from very high to very low levels. In [10], a study of forest degradation in the Doon Valley, India, due to limestone quarries and other activities, was reported. The degradation was more severe for forest biomass closer to the quarries than for the biomass farther away due to dust

[‡]Research partially supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. NSERC A4823.
[‖]Research partially supported by the G. Gordin Kaplan Memorial Postdoctoral Fellowship of the University of Alberta.

settlement, human and cattle activity, etc. This leads to a spatially monotonically increasing carrying capacity. One also has carrying capacities flanked on both sides by higher or lower values, e.g. if forest land is interrupted along a linear transect by a stream, or if grazing land along a transect is interrupted by a burn or a change in soil conditions. In such circumstances, a minimum of three patches is needed to approximate the spatially changing environment.

In this paper we propose a reaction–diffusion model of population growth in three patches and analyse the existence, piecewise monotonicity and stability of the steady states. We will show that there exist positive steady states no matter what the patch configuration. The methods utilized in [7] do not apply here. We require a topological technique. In the case of reservoir boundary conditions, we give sufficiency criteria for these steady states to be piecewise monotonic, asymptotically stable, and indicate how they may be constructed.

In the two-patch case discussed in [7], several types of boundary conditions were considered, reservoir and no flux. In this paper we discuss only reservoir boundary conditions. This corresponds to a biological situation where spatially, the changes occur over a particular region, outside the region the environments are relatively constant over large distances. The analysis required to analyse this case is already lengthy. The case of no-flux boundary conditions, corresponding to hostile conditions outside the considered environment, is reserved for a future paper. It should also be mentioned that our theory can be easily extended to the corresponding n-patch model.

The paper is organized as follows. In Section 2, we describe the reaction–diffusion equation for a single-species population diffusing in a three-patch environment. Section 3 gives an existence proof and local asymptotic stability analysis of a steady state solution by using a topological transversality theorem, an *a priori* bound technique and the energy function method. Sections 4 and 5 are devoted to constructive proofs of piecewise monotonic solutions in the case of the middle carrying capacity being greater or less than the others, and in the case of strictly increasing or decreasing carrying capacities in patches, respectively. The proof is based on a shooting type argument and indicates a nonuniform distribution of populations accumulating in the most favourable environment. Section 6 contains a brief discussion of our results and some relevant biological implications.

2. THE MODEL

We consider a single-species population diffusing in a homogeneous three-patch environment. Assume that the ith path lies along the spacial length $L_{i-1} \le x \le L_i$, $i = 1, 2, 3$, $L_0 = 0$. Let $N_i(x, t)$ denote the population density at point x and time t in the ith patch, $D_i > 0$ the diffusion coefficient of N_i in the ith patch, and $g_i(N_i)$ the specific growth rate of N_i. Then the evolution of this population is described by the following system of autonomous reaction–diffusion equations

$$\frac{\partial}{\partial t} N_i(x, t) = D_i \frac{\partial^2}{\partial x^2} N_i(x, t) + N_i(x, t) g_i(N_i(x, t)),$$
$$t \ge 0,\ L_{i-1} \le x \le L_i,\ i = 1, 2, 3. \tag{2.1}$$

We assume g_i, $i = 1, 2, 3$, satisfies the following standard assumptions (cf. [6, Chapter 1]):

(H) $g_i \in C^1([0, \infty), R)$; $g_i(0) > 0$, $g_i'(N_i) < 0$ for all $N_i \ge 0$, and there exists $K_i > 0$ such that $g_i(K_i) = 0$.

To specify the solutions, we assume the following initial distribution

$$N_i(x, 0) = \eta_i(x), \qquad L_{i-1} \le x \le L_i \tag{2.2}$$

and the following boundary (reservoir) condition

$$N_1(0, t) = K_1, \qquad N_3(L_3, t) = K_3, \qquad t \ge 0 \tag{2.3}$$

where $\eta_i \in C([L_{i-1}, L_i], R)$ and satisfies the matching conditions $\eta_1(0) = K_1$, $\eta_3(L_3) = K_3$.

By a solution of the initial-boundary value problem (2.1)–(2.3), we mean a function $N(x, t)$, $0 \le x \le L_3$, $t \ge 0$, of the form

$$N(x, t) = N_i(x, t), \qquad L_{i-1} \le x \le L_i, \qquad i = 1, 2, 3$$

such that $N_i(x, t)$ satisfies the ith component of system (2.1) on $L_{i-1} \le x \le L_i$, the initial and boundary value conditions (2.2) and (2.3), and the following continuous flux matching condition at the interface:

$$N_1(L_1, t) = N_2(L_1, t), \qquad N_2(L_2, t) = N_3(L_2, t), \qquad t \ge 0, \tag{2.4}$$

$$D_1 \frac{\partial}{\partial x} N_1(L_1, t) = D_2 \frac{\partial}{\partial x} N_2(L_1, t), \qquad D_2 \frac{\partial}{\partial x} N_2(L_2, t) = D_3 \frac{\partial}{\partial x} N_3(L_2, t), \qquad t \ge 0. \tag{2.5}$$

In the case of no diffusion ($D_i = 0$ for $i = 1, 2, 3$), the behaviour of solutions of our models are well known [6]. At each x in the ith patch, $N_i(x, t) \to K_i$ as $t \to \infty$. Therefore, if $K_1 = K_2 = K_3$, then the unique positive equilibrium is globally asymptotically stable. If $\min_{1 \le i \le 3} K_i \ne \max_{1 \le i \le 3} K_i$, then a solution $N(x, t)$ satisfying the continuous flux matching condition (2.3) is impossible. For details, we refer to [6].

In the case of diffusion, stable patterns may or may not exist, depending on whether the environment is heterogeneous or homogeneous. For the homogeneous environment, Casten and Holland [2] and Chafee [3] have demonstrated that no stable, nonuniform steady-state solution exists. However, for a single-species population diffusing in a two-patch environment ($K_1 = K_2 \ne K_3$ or $K_1 \ne K_2 = K_3$), Freedman *et al.* [7] proved the existence of a positive, monotonic, continuous and asymptotically stable steady-state solution with continuous flux.

We are interested here in a heterogeneous environment consisting of three patches, that is, $K_1 \ne K_2 \ne K_3$. For the sake of simplifying the presentation, we distinguish the following cases:

Case 1. $K_1 < K_2 < K_3$.
Case 2. $K_3 < K_2 < K_1$.
Case 3. $\mathrm{Max}\{K_1, K_3\} < K_2$.
Case 4. $K_2 < \min\{K_1, K_3\}$.

In cases 1 and 2 the carrying capacities of successive patches are monotonically increasing or decreasing. We will prove that there exists a positive, monotonic, continuous, steady-state solution with continuous flux which is asymptotically stable. In cases 3 and 4, the monotonicities of carrying capacities of successive patches change. We will show that there exists a positive, piecewise monotonic, continuous, steady-state solution with continuous flux which is asymptotically stable.

3. EXISTENCE AND STABILITY OF STEADY-STATE SOLUTIONS

In this section, we employ the topological transversality theorem, an *a priori* bound technique and the energy function method to establish the existence and (local) stability of steady-state solutions. The main idea in proving the existence is to reformulate the existence problem of steady-state solutions as a fixed point problem of a completely continuous map. By embedding the steady-state equation in a family of equations, we obtain a series of homotopies from the aforementioned completely continuous map to a relatively simpler map which corresponds to a steady-state equation of an essentially "two-patch" problem. In this way, we reduce our "three-patch" problem to a "two-patch" problem. The same technique is also applied to reduce the "two-patch" problem to a "one-patch" problem and therefore the existence of a steady-state solution for the "three-patch" problem follows from the existence of a solution for a "one-patch" problem by the topological transversality theorem due to Dugunji and Granas (cf. [4]) described as follows: let B denote a Banach space and C a fixed closed convex subset of B. For any given pair of closed bounded subsets, $A, X \subseteq B$ such that $A \subseteq X$, we denote the class $\mathcal{K}(X, A)$ to be the class of all maps $F: X \to C$ such that F is completely continuous and is fixed point free in A, i.e. $x \neq F(x)$ for all $x \in A$. A map $H: X \times [0, 1] \to C$ is said to be a homotopy in $\mathcal{K}(X, A)$ provided that $H \in \mathcal{K}(X \times [0, 1], A \times [0, 1])$ and $H_t := H(\cdot, t) \in \mathcal{K}(X, A)$ for all $t \in [0, 1]$. Maps $F, G \in \mathcal{K}(X, A)$ are called homotopic if there exists a homotopy H in $\mathcal{K}(X, A)$ such that $H_0 \equiv F$ and $H_1 \equiv G$. A map $F \in \mathcal{K}(X, A)$ is called essential if every map $G \in \mathcal{K}(X, A)$ with $G|_A \equiv F|_A$ has a fixed point in X. The following topological transversality theorem and the simple criterion of essential maps will be repeatedly used throughout this paper.

TOPOLOGICAL TRANSVERSALITY THEOREM. Suppose that F, G are homotopic in $\mathcal{K}(X, A)$. Then F is essential iff G is essential.

LEMMA 3.1. Let U be a bounded open subset of C, $x_0 \in C$ and $x_0 \notin \partial U$. A constant map $U \to \{x_0\}$ is essential in $\mathcal{K}(U, \partial U)$ iff $x_0 \in U$.

In order to use the topological transversality theorem to establish the existence of a steady-state solution to problems (2.1)–(2.3) subject to the matching condition (2.4)–(2.5), we consider the following family of problems

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_1(x) + u_1(x) g_1(u_1(x)) = 0, & 0 \leq x \leq L_1, \\ D_{12}^{\lambda} \dfrac{d^2}{dx^2} u_2(x) + u_2(x) g_{12}^{\lambda}(u_2(x)) = 0, & L_1 \leq x \leq L_2, \\ D_3 \dfrac{d^2}{dx^2} u_3(x) + u_3(x) g_3(u_3(x)) = 0, & L_2 \leq x \leq L_3, \\ u_1(0) = K_1,\ u_3(L_3) = K_3, & \\ u_1(L_1) = u_2(L_1),\ u_2(L_2) = u_3(L_2), & \\ D_1 \dfrac{d}{dx} u_1(L_1) = D_{12}^{\lambda} \dfrac{d}{dx} u_2(L_1),\ D_{12}^{\lambda} \dfrac{d}{dx} u_2(L_2) = D_3 \dfrac{d}{dx} u_3(L_2), & \end{cases} \tag{3.1$_\lambda$}$$

where, and in what follows, $D_{ij}^{\lambda} = (1-\lambda)D_i + \lambda D_j$, $g_{ij}^{\lambda} = (1-\lambda)g_i + \lambda g_j$ for $i, j = 1, 2, 3$, and $\lambda \in [0, 1]$.

Let $\underline{K} = \min\{K_1, K_2, K_3\}$ and $\bar{K} = \max\{K_1, K_2, K_3\}$ and choose $\varepsilon > 0$ such that $\underline{K} - \varepsilon > 0$. Then we have the following *a priori* bounds for solutions of the problem $(3.1)_\lambda$.

LEMMA 3.2. There exist constants M_{i1}^1 and M_{i2}^1, $i = 1, 2, 3$, such that if $\lambda \in [0, 1]$ is a given number and (u_1, u_2, u_3) is a solution of $(3.1)_\lambda$ with $\underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, then $\underline{K} \le u_i(x) \le \bar{K}$, $|(d/dx)u_i(x)| \le M_{i1}^1$ and $|(d^2/dx^2)u_i(x)| \le M_{i2}^1$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$.

Proof. We first prove the conclusion $\underline{K} \le u_i(x) \le \bar{K}$ for the case of $K_1 \le K_2 \le K_3$. Other cases can be treated analogously.

In the case of $K_1 \le K_2 \le K_3$, we claim that $(d/dx)u_1(0) \ge 0$. For otherwise, if $(d/dx)u_1(0) < 0$, then $u_1(x) < K_1$ for $x > 0$ and close to 0. Noting that $D_1(d^2/dx^2)u_1(x) = -u_1(x)g_1(u_1(x))$ and $g_1(u_1) > 0$ for $0 \le u_1 < K_1$, we get $D_1(d^2/dx^2)u_1(x) < 0$, and thus $(d/dx)u_1(x) \le (d/dx)u_1(0) < 0$, which implies that $u_1(x) < K_1$ on $(0, L_1]$. By the matching condition, we get $(d/dx)u_2(L_1) = (D_1/D_{12}^{\lambda})(d/dx)u_1(L_1) < 0$ and $u_2(L_1) = u_1(L_1) < K_1$. Repeating the above argument for $u_2(x)$, we can prove that $u_2(x) < K_1$ and $(d/dx)u_2(x) < 0$ for $x \in [L_1, L_2]$. Again by the matching condition $u_3(L_2) = u_2(L_2) < K_1$ and $(d/dx)u_3(L_2) = (D_{12}^{\lambda}/D_3)(d/dx)u_2(L_2) < 0$, and by repeating the above argument for $u_3(x)$, we get that $u_3(x) < K_1$ for $x \in [L_2, L_3]$ which contradicts the fact that $u_3(L_3) = K_3 \ge K_1$.

Therefore, $(d/dx)u_1(0) \ge 0$. Noting that $D_1(d^2/dx^2)u_1(x) = -u_1(x)g_1(u_1(x))$ and $g_1(u_1) < 0$ for $u \ge K_1$, we can conclude that $(d^2/dx^2)u_1(x) \ge 0$, $(d/dx)u_1(x) \ge (d/dx)u_1(0) \ge 0$ and $u_1(x) \ge K_1$ for all $x \in [0, L_1]$. Modifying the above argument, one can prove that $u_1(x) \le K_3$ for $x \in [0, L_1]$ and $K_1 \le u_i(x) \le K_3$ for $i = 2, 3$ and $x \in [L_{i-1}, L_i]$.

We now prove the conclusion concerning the derivative of $u_2(x)$. Similar estimates for $u_1(x)$ and $u_3(x)$ can be obtained analogously. By $(d^2/dx^2)u_2(x) = (D_{12}^{\lambda})^{-1}u_2(x)g_{12}^{\lambda}(u_2(x))$, we can easily verify that $|(d^2/dx^2)u_2(x)| \le M_{22}^1$ for $x \in [L_1, L_2]$, where $M_{22}^1 = \Lambda_2 \underline{D}_{12}^{-1}$, $\Lambda_2 = \max_{\underline{K}-\varepsilon \le u_2 \le \bar{K}+\varepsilon} |u_2|(|g_1(u_2)| + |g_2(u_2)|)$, and $\underline{D}_{12} = \frac{1}{2}\min\{D_1, D_2\}$. On the other hand, by the well-known mean value theorem, there exists $\xi \in (L_1, L_2)$ such that

$$\bar{K} + \varepsilon - (\underline{K} - \varepsilon) \ge |u_2(L_2) - u_2(L_1)| = \left|\frac{d}{dx}u_2(\xi)\right|(L_2 - L_1)$$

from which it follows that

$$\left|\frac{d}{dx}u_2(\xi)\right| \le \frac{\bar{K} + \varepsilon - (\underline{K} - \varepsilon)}{L_2 - L_1}.$$

Therefore, for any $x \in [L_1, L_2]$, we have

$$\begin{aligned}\left|\frac{d}{dx}u_2(x)\right| &\le \left|\frac{d}{dx}u_2(\xi)\right| + \left|\int_{\xi}^{x}\frac{d^2}{dx^2}u_2(\theta)\,d\theta\right| \\ &\le M_{21}^1 := (\bar{K} - \underline{K} + 2\varepsilon)(L_2 - L_1)^{-1} + M_{22}^1(L_2 - L_1),\end{aligned}$$

completing the proof.

Similarly, one can prove the following lemma.

LEMMA 3.3. There exist constants M_{i1}^2 and M_{i2}^2 for $i = 1, 2, 3$ such that if (u_1, u_2, u_3) is a solution of the following problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) + u_i(x) g_1(u_i(x)) = 0, & L_{i-1} \le x \le L_i,\ i = 1, 2, \\ D_{13}^{\lambda} \dfrac{d^2}{dx^2} u_3(x) + u_3(x) g_{13}^{\lambda}(u_3(x)) = 0, & L_2 \le x \le L_3 \\ u_1(0) = K_1 \min\{K_1, K_3\} \le u_3(L_3) \le \max\{K_1, K_3\} & \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), & \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \quad D_1 \dfrac{d}{dx} u_2(L_2) = D_{13}^{\lambda} \dfrac{d}{dx} u_3(L_2) & \end{cases} \tag{3.2$_\lambda$}$$

satisfying $\underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, then $\underline{K} \le u_i(x) \le \bar{K}$, $|(d/dx) u_i(x)| \le M_{i1}^2$ and $|(d^2/dx^2) u_i(x)| \le M_{i2}^2$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$.

LEMMA 3.4. There exist constants M_{i1}^3 and M_{i2}^3 for $i = 1, 2, 3$ such that if (u_1, u_2, u_3) is a solution of the following problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) + \lambda u_i(x) g_1(u_i(x)) = 0, & L_{i-1} \le x \le L_i,\ i = 1, 2, 3, \\ u_1(0) = K_1, \quad u_3(L_3) = K_1, & \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), & \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \quad \dfrac{d}{dx} u_2(L_2) = \dfrac{d}{dx} u_3(L_2) & \end{cases} \tag{3.3$_\lambda$}$$

satisfying $\underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, then $\underline{K} \le u_i(x) \le \bar{K}$, $|(d/dx) u_i(x)| \le M_{i1}^3$ and $|(d^2/dx^2) u_i(x)| \le M_{i2}^3$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$.

In what follows, for any u: $[0, L_3] \to R$, we denote $u_i = u|_{[L_{i-1}, L_i]}$ for $i = 1, 2, 3$, and write $u = (u_1, u_2, u_3)$. Let

$$Y = \{u: [0, L_3] \to R;\ u_i \in C([L_{i-1}, L_i], R),\ i = 1, 2, 3\},$$

$$B = \{u \in C([0, L_3], R);\ u_i \in C^2([L_{i-1}, L_i], R),\ i = 1, 2, 3\}.$$

Evidently, N is a Banach space with a norm

$$\|u\| = \sum_{i=1}^{3} \max_{j=0,1,2} \sup_{x \in [L_{i-1}, L_i]} \left| \frac{d^j}{dx^j} u_i(x) \right|. \tag{3.4}$$

To obtain a fixed point reformulation of the existence problem of steady state solutions to (2.1)–(2.3) subject to matching conditions (2.4)–(2.5), for any given $F = (F_1, F_2, F_3) \in Y$

and $\lambda \in [0, 1]$, we consider the following problem

$$\begin{cases} D_1 \dfrac{d^2 u_1(x)}{dx^2} = F_1(x), & 0 \le x \le L_1, \\ D_{12}^{\lambda} \dfrac{d^2 u_2(x)}{dx^2} = F_2(x), & L_1 \le x \le L_2, \\ D_3 \dfrac{d^2 u_3(x)}{dx^2} = F_3(x), & L_2 \le x \le L_3, \\ u_1(0) = K_1, \quad u_3(L_3) = K_3, \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), \\ D_1 \dfrac{d}{dx} u_1(L_1) = D_{12}^{\lambda} \dfrac{d}{dx} u_2(L_1), \quad D_{12}^{\lambda} \dfrac{d}{dx} u_2(L_2) = D_3 \dfrac{d}{dx} u_3(L_2). \end{cases} \tag{3.5}_{\lambda}$$

LEMMA 3.5. For any $F = (F_1, F_2, F_3) \in Y$, there exists one and only one solution $\mu(F) = (\mu_{1\lambda}(F), \mu_{2\lambda}(F), \mu_{3\lambda}(F)) \in B$ of $(3.5)_{\lambda}$, explicitly defined as follows:

$$\mu_{1\lambda}(F)(x) = K_1 + M_{\lambda}(F)x + \frac{1}{D_1} \int_0^x \int_0^s F_1(\theta)\, d\theta\, ds, \qquad 0 \le x \le L_1,$$

$$\mu_{2\lambda}(F)(x) = \mu_{1\lambda}(F)(L_1) + \frac{D_1}{D_{12}^{\lambda}} \frac{d}{dx} \mu_{1\lambda}(F)(L_1)(x - L_1) + \frac{1}{D_{12}^{\lambda}} \int_{L_1}^x \int_{L_1}^s F_2(\theta)\, d\theta\, ds, \qquad L_1 \le x \le L_2,$$

$$\mu_{3\lambda}(F)(x) = \mu_{2\lambda}(F)(L_2) + \frac{D_{12}^{\lambda}}{D_3} \frac{d}{dx} \mu_{2\lambda}(F)(L_2)(x - L_2) + \frac{1}{D_3} \int_{L_2}^x \int_{L_2}^s F_3(\theta)\, d\theta\, ds, \qquad L_2 \le x \le L_3,$$

$$M_{\lambda}(F) = [L_1 + D_1(L_2 - L_1)(D_{12}^{\lambda})^{-1} + D_1(L_3 - L_2)D_3^{-1}]^{-1},$$
$$\Bigg[K_3 - K_1 - \frac{1}{D_1} \int_0^{L_1} \int_0^s F_1(\theta)\, d\theta\, ds - \frac{1}{D_{12}^{\lambda}} \int_0^{L_1} F_1(\theta)\, d\theta (L_2 - L_1) - \frac{1}{D_3} \int_0^{L_1} F_1(\theta)\, d\theta (L_3 - L_2) - \frac{1}{D_{12}^{\lambda}} \int_{L_1}^{L_2} \int_{L_1}^s F_2(\theta)\, d\theta\, ds - \frac{1}{D_3} \int_{L_1}^{L_2} F_2(\theta)\, d\theta (L_3 - L_2) - \frac{1}{D_3} \int_{L_2}^{L_3} \int_{L_2}^s F_3(\theta)\, d\theta\, ds \Bigg].$$

The proof is a direct verification, and is therefore omitted.

According to the above result, one can easily show that $u = (u_1, u_2, u_3)$ is a solution to $(3.1)_{\lambda}$ iff u is a fixed point of the map $H_{\lambda}: B \to B$ defined by

$$H_{\lambda}(u) = (\mu_{1\lambda}(-u_1 g_1(u_1)), \mu_{2\lambda}(-u_2 g_{12}^{\lambda}(u_2)), \mu_{3\lambda}(-u_3 g_3(u_3))).$$

By direct verification, we can show that there exist constants Λ_{ij}^1, $i = 1, 2, 3$, $j = 0, 1, 2$, such that if $\underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, then

$$\left|\frac{d^j}{dx^j}[\mu_{i\lambda}(-u_i g_i(u_i))(x)]\right| \le \Lambda_{ij}^1, \qquad \left|\frac{d^j}{dx^j}[\mu_{2\lambda}(-u_2 g_{12}^{\lambda}(u_2))(x)]\right| \le \Lambda_{2j}^1$$

for $j = 0, 1, 2$ and $i = 1, 3$.

Let

$$X = \left\{u \in B: \underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon, \left|\frac{d}{dx}u_i(x)\right| \le M_{i1} + 1, \left|\frac{d^2}{dx^2}u_i(x)\right| \le M_{i2} + 1 \right.$$
$$\left. \text{for } x \in [L_{i-1}, L_i] \text{ and } i = 1, 2, 3\right\}$$

and

$$C_1 = \left\{u \in B: |u_i(x)| \le \Lambda_{i0}^1, \left|\frac{d}{dx}u_i(x)\right| \le \Lambda_{i1}^1, \left|\frac{d^2}{dx^2}u_i(x)\right| \le \Lambda_{i2}^1 \right.$$
$$\left. \text{for } x \in [L_{i-1}, L_i] \text{ and } i = 1, 2, 3\right\},$$

where

$$M_{ij} = \max\{M_{ij}^1, M_{ij}^2, M_{ij}^3\}, \qquad i, j = 1, 2.$$

Then $H_\lambda(X) \subseteq C_1$ for all $\lambda \in [0, 1]$. Let $H: X \times [0, 1] \to C_1$ be defined by $H(u, \lambda) = H_\lambda(u)$. We have the following lemma.

LEMMA 3.6. The map $H: X \times [0, 1] \to C_1$ is a completely continuous homotopy between H_0 and H_1 in $\mathcal{K}(X, \partial X)$.

Proof. For any $u \in X$, we have

$$\begin{aligned}\left|\frac{d^3}{dx^3}\mu_{1\lambda}(-u_1 g_1(u_1))(x)\right| &= \frac{1}{D_1}\left|\frac{d}{dx}[u_1(x) g_1(u_1(x))]\right| \\ &\le \frac{1}{D_1}[|g_1(u_1(x))| + |u_1(x)|\,|g_1'(u_1(x))|]\,|u_1'(x)| \\ &\le \frac{1}{D_1}\max_{\underline{K}-\varepsilon \le u_1 \le \bar{K}+\varepsilon}[|g_1(u_1)| + |u_1|\,|g_1'(u_1)|](M_{11} + 1) < \infty\end{aligned}$$

for $x \in [0, L_1]$. Likewise, we can prove

$$\begin{aligned}&\left|\frac{d^3}{dx^3}\mu_{2\lambda}(-u_2 g_{12}^{\lambda}(u_2))(x)\right| \\ &\le \frac{1}{\underline{D}_{12}}\max_{\underline{K}-\varepsilon \le u_2 \le \underline{K}+\varepsilon}[|g_1(u_2)| + |g_2(u_2)| + |u_2|(|g_1'(u_2)| + |g_2'(u_2)|)](M_{21} + 1) < \infty\end{aligned}$$

for $x \in [L_1, L_2]$ and

$$\left|\frac{d^3}{dx^3}\mu_{3\lambda}(-u_3 g_3(u_3))(x)\right| \le \frac{1}{D_3}\max_{\underline{K}-\varepsilon \le u_3 \le \bar{K}+\varepsilon}[|g_3(u_3)| + |u_3|\,|g_3'(u_3)|](M_{31} + 1) < \infty$$

for $x \in [L_2, L_3]$. Therefore $H(X \times [0, 1])$ is relatively compact in C_1.

We note that for any $\lambda \in [0, 1]$, u is a fixed point of $H_\lambda: X \to C_1$ iff u is a solution to $(3.1)_\lambda$. By lemma 3.2, $|(d/dx)u_i(x)| \leq M_{i1}^1 < M_{i1} + 1$, $|(d^2/dx^2)u_i(x)| \leq M_{i2}^1 < M_{i2} + 1$ and $\underline{K} \leq u_i(x) \leq \bar{K}$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$. Therefore $u \notin \partial X$. This proves that H is a completely continuous homotopy between H_0 and H_1.

By the topological transversality theorem, H_1 is essential iff H_0 is essential. We note that if $\lambda = 0$, then $(3.1)_0$ reduces to the following essential "two-patch" problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) + u_i(x) g_1(u_i(x)) = 0, & L_{i-1} \leq x \leq L_i,\ i = 1, 2, \\ D_3 \dfrac{d^2}{dx^2} u_3(x) + u_3(x) g_3(u_3(x)) = 0, & L_2 \leq x \leq L_3, \\ u_1(0) = K_1, \quad u_3(L_3) = K_3, \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \quad D_1 \dfrac{d}{dx} u_2(L_2) = D_3 \dfrac{d}{dx} u_3(L_2). \end{cases} \tag{3.6}$$

To verify that H_0 is an essential map, we consider the following family of problems

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) + u_i(x) g_1(u_i(x)) = 0, & L_{i-1} \leq x \leq L_i,\ i = 1, 2, \\ D_{13}^\lambda \dfrac{d^2}{dx^2} u_3(x) + u_3(x) g_{13}^\lambda(u_3(x)) = 0, & L_2 \leq x \leq L_3, \\ u_1(0) = K_1, \quad u_3(L_3) = K_{3,\lambda}, \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \quad D_1 \dfrac{d}{dx} u_2(L_2) = D_{3,\lambda} \dfrac{d}{dx} u_3(L_2) \end{cases} \tag{3.7$_\lambda$}$$

where $K_{3,\lambda}$ is defined as follows: since $(d/d\theta)g_{13}^\lambda(\theta) = (1 - \lambda)(d/d\theta)g_1(\theta) + \lambda(d/d\theta)g_3(\theta) < 0$ for $\theta \geq 0$, and $g_{13}^\lambda(\min\{K_1, K_3\}) \geq 0$, $g_{13}^\lambda(\max\{K_1, K_3\}) \leq 0$, there exists a unique $K_{3\lambda}$ such that $g_{3\lambda}(K_{3\lambda}) = 0$.

As before, we associate $(3.7)_\lambda$ with the following nonhomogeneous problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) = F_i(x), & L_{i-1} \leq x \leq L_i,\ i = 1, 2, \\ D_{13}^\lambda \dfrac{d^2}{dx^2} u_3(x) = F_3(x), & L_2 \leq x \leq L_3, \\ u_1(0) = K_1, \quad u_3(L_3) = K_{3,\lambda}, \\ u_1(L_1) = u_2(L_1), \quad u_2(L_2) = u_3(L_2), \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \quad D_1 \dfrac{d}{dx} u_2(L_2) = D_{13}^\lambda \dfrac{d}{dx} u_3(L_2). \end{cases} \tag{3.8$_\lambda$}$$

The following lemma is an analogue of lemma 3.5.

LEMMA 3.7. For any $F \in Y$, there exists one and only one solution $v_\lambda(F) = (v_{1\lambda}(F), v_{2\lambda}(F), v_{3\lambda}(F)) \in B$ of $(3.8)_\lambda$, explicitly defined as follows:

$$v_{1\lambda}(F)(x) = K_1 + P_\lambda(F)x + \frac{1}{D_1}\int_0^x \int_0^s F_1(\theta)\,\mathrm{d}\theta\,\mathrm{d}s, \qquad 0 \le x \le L_1,$$

$$v_{2\lambda}(F)(x) = v_{1\lambda}(F)(L_1) + \frac{\mathrm{d}}{\mathrm{d}x} v_{1\lambda}(F)(L_1)(x - L_1) + \frac{1}{D_1}\int_{L_1}^x \int_{L_1}^s F_2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s, \qquad L_1 \le x \le L_2,$$

$$v_{3\lambda}(F)(x) = v_{2\lambda}(F)(L_2) + \frac{D_1}{D_{13}^\lambda}\frac{\mathrm{d}}{\mathrm{d}x} v_{2\lambda}(F)(L_2)(x + L_2) + \frac{1}{D_{13}^\lambda}\int_{L_2}^x \int_{L_2}^s F_3(\theta)\,\mathrm{d}\theta\,\mathrm{d}s, \qquad L_2 \le x \le L_3,$$

where

$$\begin{aligned} P_\lambda(F) = [L_2 + D_1(L_3 - L_2)(D_{13}^\lambda)^{-1}]^{-1}\Bigg[& K_{3\lambda} - K_1 - \frac{1}{D_1}\int_0^{L_1}\int_0^s F_1(\theta)\,\mathrm{d}\theta\,\mathrm{d}s \\ & - \frac{1}{D_1}\int_0^{L_1} F_1(\theta)\,\mathrm{d}\theta(L_2 - L_1) - \frac{1}{D_{13}^\lambda}\int_0^{L_1} F_1(\theta)\,\mathrm{d}\theta(L_3 - L_2) \\ & - \frac{1}{D_1}\int_0^{L_2}\int_{L_1}^s F_2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s - \frac{1}{D_{13}^\lambda}\int_{L_1}^{L_2} F_2(\theta)\,\mathrm{d}\theta(L_3 - L_2) \\ & - \frac{1}{D_{13}^\lambda}\int_{L_2}^{L_3}\int_{L_2}^s F_3(\theta)\,\mathrm{d}\theta\,\mathrm{d}s\Bigg]. \end{aligned}$$

From the above result, it follows that there exist constants Λ_{ij}^2, $i = 1, 2, 3$, $j = 0, 1, 2$ such that if $\underline{K} - \varepsilon \le u_i(x) \le \bar{K} + \varepsilon$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, then

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}x^j} v_{i\lambda}(-u_i g_1(u_i))(x)\right| \le \Lambda_{ij}^2 \qquad \text{and} \qquad \left|\frac{\mathrm{d}^j}{\mathrm{d}x^j} v_{3\lambda}(-u_3 g_{13}^\lambda(u_3))(x)\right| \le \Lambda_{3j}^2$$

for $i = 1, 2$ and $j = 0, 1, 2$.

Define the map $Q: X \times [0, 1] \to B$ by

$$Q(u, \lambda) = (v_{1\lambda}(-u_1 g_1(u_1)),\ v_{2\lambda}(-u_2 g_1(u_2)),\ v_{3\lambda}(-u_3 g_{13}^\lambda(u_3))).$$

Then $Q(X \times [0, 1]) \subseteq C_2$, where

$$C_2 = \left\{u \in B : |u_i(x)| \le \Lambda_{i0}^2,\ \left|\frac{\mathrm{d}}{\mathrm{d}x}u_i(x)\right| \le \Lambda_{i1}^2,\ \left|\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_i(x)\right| \le \Lambda_{i2}^2 \right. \\ \left. \text{for } x \in [L_{i-1}, L_i] \text{ and } i = 1, 2, 3\right\}.$$

By using lemma 3.3 and employing the argument for lemma 3.6, we can prove the following lemma.

LEMMA 3.8. The map $Q: X \times [0, 1] \to C_2$ is a completely continuous homotopy in $\mathcal{K}(X, \partial X)$.

Noting that $Q_1(\cdot) \triangleq Q(\cdot, 1) = H_0(\cdot)$, it follows by the topological transversality theorem that H_0 is an essential map in $\mathcal{K}(X, \partial X)$, iff $Q_0(\cdot) \equiv Q(\cdot, 0)$ also is. On the other hand, if $\lambda = 0$, then $(3.7)_\lambda$ reduces to the following essential "one-patch" problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} u_i(x) + u_i(x) g_1(u_i(x)) = 0, \qquad L_{i-1} \le x \le L_i, \qquad i = 1, 2, 3, \\ u_1(0) = K_1, \qquad u_3(L_3) = K_1, \\ u_1(L_1) = u_2(L_1), \qquad u_2(L_2) = u_3(L_2), \\ \dfrac{d}{dx} u_1(L_1) = \dfrac{d}{dx} u_2(L_1), \qquad \dfrac{d}{dx} u_2(L_2) = \dfrac{d}{dx} u_3(L_2). \end{cases} \tag{3.9}$$

It is easy to verify that $u = (u_1, u_2, u_3)$ is a solution to (3.9) iff

$$u(x) = K_1 - \frac{x}{L_3 D_1} \int_0^{L_3} \int_0^s G(\theta)\, d\theta\, ds + \frac{1}{D_1} \int_0^x \int_0^s G(\theta)\, d\theta\, ds$$

for $x \in [0, L_3]$, where $G(x) = -u_i(x) g_1(u_i(x))$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$. Define $W: X \times [0, 1] \to B$ by

$$W(u, \lambda)(x) = K_1 - \lambda \frac{x}{L_3 D_1} \int_0^{L_3} \int_0^s G(\theta)\, d\theta + \frac{\lambda}{D_1} \int_0^x \int_0^s G(\theta)\, d\theta\, ds.$$

Noting that $|G(x)| \le \Lambda_1 := \max_{\underline{K} - \varepsilon \le u_1 \le \bar{K} + \varepsilon} |u_1 g_1(u_1)|$ for $x \in [0, L_3]$, we get

$$|W(u, \lambda)(x)| \le K_1 + 2\Lambda_1 L_3^2 D_1^{-1},$$

$$\left| \frac{d}{dx} W(u, \lambda)(x) \right| \le 2\Lambda_1 L_3 D_1^{-1},$$

$$\left| \frac{d^2}{dx^2} W(u, \lambda)(x) \right| \le \Lambda_1 D_1^{-1}$$

for $x \in [0, L_3]$ and $\lambda \in [0, 1]$. Hence, we have the following lemma.

LEMMA 3.9. $W: X \times [0, 1] \to C_3$ is a completely continuous homotopy in $\mathcal{K}(X, \partial X)$, where

$$C_3 = \left\{ u \in B: |u_i(x)| \le K_1 + 2\Lambda_1 L_3^2 D_1^{-1}, \left| \frac{d}{dx} u_i(x) \right| \le 2\Lambda_1 L_3 D_1^{-1}, \left| \frac{d^2}{dx^2} u_i(x) \right| \le \Lambda_1 D_1^{-1} \right.$$
$$\left. \text{for } x \in [L_{i-1}, L_i] \text{ and } i = 1, 2, 3 \right\}.$$

Proof. Suppose $u \in X$ is a fixed point of W_λ with $\lambda \in [0, 1]$. Then u solves the problem $(3.3)_\lambda$. By lemma 3.4, $\underline{K} \le u_i(x) \le \bar{K}$, $|(d/dx) u_i(x)| \le M_{i1}^3 < M_{i1} + 1$, and $|(d^2/dx^2) u_i(x)| \le M_{i2}^3 < M_{i2} + 1$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$. Therefore, $u \notin \partial X$. It is easy to prove that W is completely continuous. Therefore W is a homotopy in $\mathcal{K}(X, \partial X)$, proving the lemma.

Now we are in the position to state our main result of this section.

THEOREM 3.1. There exists a steady-state solution $u(x)$ such that $\underline{K} \le u(x) \le \bar{K}$ on $0 \le x \le L_3$ to the problem (2.1)–(2.3) subject to the matching conditions (2.4) and (2.5).

Proof. Let

$$C = \left\{u \in B\colon |u_i(x)| \le \Lambda_{i0},\ \left|\frac{d}{dx}u_i(x)\right| \le \Lambda_{i1},\ \left|\frac{d^2}{dx^2}u_i(x)\right| \le \Lambda_{i2} \text{ for } x \in [L_{i-1}, L_i] \text{ and } i = 1,2,3\right\},$$

where

$$\Lambda_{i0} = \max\{\Lambda_{i0}^1, \Lambda_{i0}^2, \Lambda_{i0}^3, K_1 + 2\Lambda_1 L_3^2 D_1^{-1}, \bar{K} + \varepsilon\},$$
$$\Lambda_{i1} = \max\{\Lambda_{i1}^1, \Lambda_{i1}^2, \Lambda_{i1}^3, 2\Lambda_1 L_3 D_1^{-1}, M_{i1} + 1\},$$
$$\Lambda_{i2} = \max\{\Lambda_{i2}^1, \Lambda_{i2}^2, \Lambda_{i2}^3, \Lambda_1 D_1^{-1}, M_{i2} + 1\}.$$

Define

$$U = \left\{u \in B\colon \underline{K} - \varepsilon < u_i(x) < \bar{K} + \varepsilon,\ \left|\frac{d}{dx}u_i(x)\right| < \Lambda_{i1},\ \left|\frac{d^2}{dx^2}u_i(x)\right| < \Lambda_{i2} \text{ for } x \in [L_{i-1}, L_i] \text{ and } i = 1,2,3\right\}.$$

Then $X = \bar{U}$, and by lemma 3.1, $W_0\colon X \to \{K_1\}$ is essential in $K(\bar{U}, \partial U) = K(X, \partial X)$. On the other hand, by lemmas 3.9, 3.8 and 3.6, we get the following chain of homotopies in $K(X, \partial X)$

$$W_0 \overset{W}{\simeq} W_1 = Q_0 \overset{Q}{\simeq} Q_1 = H_0 \overset{H}{\simeq} H_1.$$

By the topological transversality theorem, H_1 is an essential map. Therefore, H_1 has a fixed point in $X\backslash\partial X$. Noting that a fixed point of H_1 is a solution of $(3.1)_1$, and thus a steady-state solution to the problem (2.1)–(2.5), the proof is complete.

The following result describes the local stability of the steady state solution of the problem (2.1)–(2.3) subject to matching conditions (2.4)–(2.5).

THEOREM 3.2. Suppose that

$$\frac{d}{dx}[u_i g_i(u_i)] < 0 \qquad \text{for } u_i \in [\underline{K}, \bar{K}],\ i = 1,2,3. \tag{3.10}$$

Then the steady-state solution of the problem (2.1)–(2.3) subject to matching conditions (2.4)–(2.5) is (locally) asymptotically stable.

Proof. Let $u = (u_1, u_2, u_3)$ be the steady-state solution to the problem (2.1)–(2.3) subject to matching conditions (2.4)–(2.5). Define $n_i(x,t) = N_i(x,t) - u_i(x)$ for $x \in [L_{i-1}, L_i]$ and $i = 1,2,3$. Then the linearization of system (2.1), (2.2), (2.3) at $u = (u_1, u_2, u_3)$ leads to the following equation

$$\frac{\partial n_i}{\partial t} = D_i \frac{\partial^2}{\partial x^2} n_i + \frac{d}{du_i}[u_i g_i(u_i)] n_i$$

and the initial, boundary, and matching conditions become

$$\begin{cases} n_i(x,0) = \eta_i(x) - u_i(x), \\ n_i(0,t) = 0, \qquad n_3(L_3,t) = 0, \\ n_i(L_1,t) = n_2(L_1,t), \qquad n_2(L_2,t) = n_3(L_2,t), \\ D_1 \dfrac{\partial}{\partial x} n_1(L_1,t) = D_2 \dfrac{\partial}{\partial x} n_2(L_1,t), \qquad D_2 \dfrac{\partial}{\partial x} n_2(L_2,t) = D_3 \dfrac{\partial}{\partial x} n_3(L_2,t). \end{cases}$$

We consider the following positive definite function

$$V(t) = \int_0^{L_1} \tfrac{1}{2} n_1^2 \, \mathrm{d}x + \int_{L_1}^{L_2} \tfrac{1}{2} n_2^2 \, \mathrm{d}x + \int_{L_2}^{L_3} \tfrac{1}{2} n_3^2 \, \mathrm{d}x.$$

Then the derivative along solutions is

$$\begin{aligned} \dot{V}(t) = & \int_0^{L_1} \frac{\mathrm{d}}{\mathrm{d}u_1} [u_1 g_1(u_1)] n_1^2 \, \mathrm{d}x + \int_{L_1}^{L_2} \frac{\mathrm{d}}{\mathrm{d}u_2} [u_2 g_2(u_2)] n_2^2 \, \mathrm{d}x + \int_{L_2}^{L_3} \frac{\mathrm{d}}{\mathrm{d}u_3} [u_3 g_3(u_3)] n_3^2 \, \mathrm{d}x \\ & + \int_0^{L_1} D_1 n_1 \frac{\partial^2}{\partial x^2} n_1 \, \mathrm{d}x + \int_{L_1}^{L_2} D_2 n_2 \frac{\partial^2}{\partial x^2} n_2 \, \mathrm{d}x + \int_{L_2}^{L_3} D_3 n_3 \frac{\partial^2}{\partial x^2} n_3 \, \mathrm{d}x. \end{aligned}$$

Substituting the boundary and matching conditions and using integration by parts we get

$$\begin{aligned} \dot{V}(t) = & \int_0^{L_1} \frac{\mathrm{d}}{\mathrm{d}u_1} [u_1 g_1(u_1)] n_1^2 \, \mathrm{d}x + \int_{L_1}^{L_2} \frac{\mathrm{d}}{\mathrm{d}u_2} [u_2 g_2(u_2)] n_2^2 \, \mathrm{d}x + \int_{L_2}^{L_3} \frac{\mathrm{d}}{\mathrm{d}u_3} [u_3 g_3(u_3)] n_3^2 \, \mathrm{d}x \\ & - \int_0^{L_1} D_1 \left(\frac{\partial n_1}{\partial x}\right)^2 \mathrm{d}x - \int_{L_1}^{L_2} D_2 \left(\frac{\partial n_2}{\partial x}\right)^2 - \int_{L_1}^{L_3} D_3 \left(\frac{\partial n_3}{\partial x}\right)^2 \mathrm{d}x \\ & + D_1 n_1(L_1, t) \frac{\partial}{\partial x} n_1(L_1, t) - D_2 n_2(L_1, t) \frac{\partial}{\partial x} n_2(L_1, t) \\ & + D_2 n_2(L_2, t) \frac{\partial}{\partial x} n_2(L_2, t) - D_3 n_3(L_2, t) \frac{\partial}{\partial x} n_3(L_2, t) \\ & + D_3 n_3(L_3, t) \frac{\partial}{\partial x} n_3(L_3, t) - D_1 n_1(0, t) \frac{\partial}{\partial x} n_1(0, t) \\ \le & \int_0^{L_1} \frac{\mathrm{d}}{\mathrm{d}u_1} [u_1 g_1(u_1)] n_1^2 \, \mathrm{d}u + \int_{L_1}^{L_2} \frac{\mathrm{d}}{\mathrm{d}u_2} [u_2 g_2(u_2)] n_2^2 \, \mathrm{d}x + \int_{L_2}^{L_3} \frac{\mathrm{d}}{\mathrm{d}v_3} [u_3 g_3(u_3)] n_3^2 \, \mathrm{d}x. \end{aligned}$$

Since $\underline{K} \le u_i(x) \le \bar{K}$ for $x \in [L_{i-1}, L_i]$ and $i = 1, 2, 3$, by the assumption (3.10) $\dot{V}(t)$ is negative definite and therefore the proof is complete.

4. CONSTRUCTION OF PIECEWISE MONOTONIC, STEADY-STATE SOLUTIONS IN THE CASES OF ALTERNATING CARRYING CAPACITIES

In this section, we use a shooting type argument to give a constructive proof for the existence of piecewise monotonic steady-state solutions to the problem (2.1)–(2.3) subject to matching conditions (2.4)–(2.5) in cases 3 and 4. We prove our results for case 3. Without loss of generality, assume that $K_3 \le K_1 < K_2$. All other cases can be treated analogously.

We start with the following family of initial value problems of second order equations

$$D_2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_2 + v_2 g_2(v_2) = 0, \qquad v_2(\theta) = \alpha, \qquad \frac{\mathrm{d}}{\mathrm{d}x} v_2(\theta) = 0, \tag{4.1}$$

where $\theta \in [L_1, L_2]$ and $0 \le \alpha \le K_2$. Denoting the unique solution of the problem (4.1) by $v_2^{\theta,\alpha}$, we have the following lemma.

LEMMA 4.1. Assume

$$\frac{\mathrm{d}}{\mathrm{d}v_2}[v_2 g_2(v_2)] < 0 \qquad \text{for } v_2 \in [K_3, K_2]. \tag{4.2}$$

Then for any fixed $\theta \in [L_1, L_2]$, there exists $\alpha_0(\theta) \in (K_3, K_2)$ such that

(i) $v_2^{\theta,\alpha_0(\theta)}(L_2) = K_3$;

(ii) $v_2^{\theta,\alpha}(x) \in (K_3, K_2]$ for all $x \in [\theta, L_2]$ and $\alpha \in (\alpha_0(\theta), K_2]$;

(iii) $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,K_2}(L_2) = 0$;

(iv) $v_2^{\theta,\alpha}(L_2)$ and $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(L_2)$ are continuous and increasing in $\alpha \in [\alpha_0(\theta), K_2]$.

Proof. As the first step, we prove that $v_2^{\theta,\alpha}(x) < K_2$ for all $\alpha \in [K_3, K_2)$ and $x \in [\theta, L_2]$. Suppose, to the contrary, there exist $\alpha \in [K_3, K_2)$ and $\tau \in (\theta, L_2]$ such that $v_2^{\theta,\alpha}(\tau) = K_2$ and $v_2^{\theta,\alpha}(x) < K_2$ on $[\theta, \tau)$. Then $D_2(\mathrm{d}^2/\mathrm{d}x^2)v_2^{\theta,\alpha}(x) = -v_2^{\theta,\alpha}g_2(v_2^{\theta,\alpha}(x)) < 0$, and thus $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(x)$ is decreasing for $x \in [0, \tau)$. Therefore $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(x) < (\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(\theta) = 0$ for $x \in (\theta, \tau)$, from which we obtain $v_2^{\theta,\alpha}(x) \le v_2^{\theta,\alpha}(\theta) = \alpha < K_2$ for all $x \in [\theta, \tau]$, a contradiction to $v_2^{\theta,\alpha}(\tau) = K_2$.

Since for any given $\alpha \in [K_3, K_2)$, $v_2^{\theta,\alpha}(x) < K_2$ for $x \in [\theta, L_2]$, we have $(\mathrm{d}^2/\mathrm{d}x^2)v_2^{\theta,\alpha}(x) < 0$, $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(x) < 0$ and $v_2^{\theta,\alpha}(x)$ is decreasing on $[\theta, L_2]$ by the same argument as above. Therefore, $v_2^{\theta,\alpha}(L_2) < v_2^{\theta,\alpha}(\theta) = \alpha$ for all $\alpha \in [K_3, K_2]$. As a consequence, $v_2^{\theta,K_3}(L_2) < v_2^{\theta,K_3}(\theta) = K_3$. Evidently, $v_2^{\theta,K_2}(L_2) = K_2$. Hence by the continuity of $v_2^{\theta,\alpha}$ in $\alpha \in [K_3, K_2]$, one can find a constant $\alpha_0 = \alpha_0(\theta) \in (K_3, K_2)$ such that $v_2^{\theta,\alpha}(L_2) \in (K_3, K_2]$ for $\alpha \in (\alpha_0, K_2]$ and $v_2^{\theta,\alpha_0}(L_2) = K_3$.

Evidently, (iii) holds.

To prove (iv), we claim that for any $\alpha, \bar{\alpha} \in [\alpha_0, K_2)$ with $\alpha < \bar{\alpha}$, $v_2^{\theta,\alpha}(x) < v_2^{\theta,\bar{\alpha}}(x)$ for all $x \in [\theta, L_2]$. Suppose not, then there exists $t \in (\theta, L_2]$ such that $v_2^{\theta,\alpha}(t) = v_2^{\theta,\bar{\alpha}}(t)$ and $v_2^{\theta,\alpha}(x) < v_2^{\theta,\bar{\alpha}}(x)$ on $[\theta, t)$. Then from equation (4.1) it follows that

$$D_2\frac{\mathrm{d}^2}{\mathrm{d}x^2}[v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x)] = -[v_2^{\theta,\bar{\alpha}}(x)g_2(v_2^{\theta,\bar{\alpha}}(x)) - v_2^{\theta,\alpha}(x)g_2(v_2^{\theta,\alpha}(x))].$$

Therefore if $(\mathrm{d}/\mathrm{d}v_2)[v_2 g_2(v_2)) < 0$ for $v_2 \in [K_3, K_2]$, then $(\mathrm{d}^2/\mathrm{d}x^2)[v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x)] > 0$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}x}[v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x)] > \frac{\mathrm{d}}{\mathrm{d}x}[v_2^{\theta,\bar{\alpha}}(\theta) - v_2^{\theta,\alpha}(\theta)] = 0$$

for $x \in (\theta, t)$. So

$$v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x) \ge v_2^{\theta,\bar{\alpha}}(\theta) - v_2^{\theta,\alpha}(\theta) = \bar{\alpha} - \alpha > 0$$

for $x \in [\theta, t]$ which is contrary to $v_2^{\theta,\bar{\alpha}}(t) = v_2^{\theta,\alpha}(t)$. Hence $v_2^{\theta,\bar{\alpha}}(x) > v_2^{\theta,\alpha}(x)$ for all $x \in [\theta, L_2]$. Consequently, $v_2^{\theta,\alpha}(L_2)$ is increasing in $\alpha \in [\alpha_0, K_2]$.

Using the same argument as above, we can verify that $v_2^{\theta,\alpha}(x) < v_2^{\theta,\bar{\alpha}}(x)$ for all $x \in [\theta, L_2]$ implies that $(\mathrm{d}^2/\mathrm{d}x^2)[v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x)] > 0$ for $x \in [\theta, L_2]$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}[v_2^{\theta,\bar{\alpha}}(x) - v_2^{\theta,\alpha}(x)] > \frac{\mathrm{d}}{\mathrm{d}x}[v_2^{\theta,\bar{\alpha}}(\theta) - v_2^{\theta,\alpha}(\theta)] = 0$$

at $x = L_2$ which implies that $(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(L_2)$ is increasing in $\alpha \in [\alpha_0, K_2]$, completing the proof.

LEMMA 4.2. Suppose that

$$\frac{\mathrm{d}}{\mathrm{d}v_3}[v_3 g_3(v_3)] < 0 \qquad \text{for } v_3 \in [K_3, K_2]. \tag{4.3}$$

Then there exists a constant $\beta_0 < 0$ such that the solution, denoted by v_3^{β}, on $[L_2, L_3]$ to the following initial value problem

$$D_3 \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_3 + v_3 g_3(v_3) = 0, \qquad v_3(L_3) = K_3, \qquad \frac{\mathrm{d}}{\mathrm{d}x} v_3(L_3) = \beta \leq 0 \tag{4.4}$$

satisfies

(i) $v_3^{\beta}(x) \in [K_3, K_2]$ for all $x \in [L_2, L_3]$ and $\beta \in [\beta_0, 0]$;
(ii) $v_3^{\beta_0}(L_2) = K_2$;
(iii) $v_3^{\beta}(L_2)$ is continuous and decreasing in $\beta \in [\beta_0, 0]$;
(iv) $(\mathrm{d}/\mathrm{d}x)v_3^{\beta}(L_2)$ is continuous and increasing in $\beta \in [\beta_0, 0]$.

Proof. Using a similar argument to the first step in the proof of lemma 4.1, we can prove that for any $\beta < 0$, $(\mathrm{d}/\mathrm{d}x)v_3^{\beta}(x) \leq \beta < 0$ and $v_3^{\beta}(x) \geq K_3$ for all $x \in [L_2, L_3]$. Therefore, $v_3^{\beta}(L_2) > v_3^{\beta}(L_3) - \beta(L_3 - L_2) \to \infty$ as $\beta \to -\infty$. Moreover, $v_3^{0}(L_2) = K_3$. Therefore by the continuity in β of $v_3^{\beta}(L_2)$, there exists $\beta_0 < 0$ such that $v_3^{\beta_0}(L_2) = K_2$ and $v_3^{\beta}(x) \in [K_3, K_2]$ for all $x \in [L_2, L_3]$ and $\beta \in [\beta_0, 0]$.

For any given β, $\bar\beta$ with $\beta_0 \leq \bar\beta < \beta < 0$ we claim that $v_3^{\beta}(x) < v_3^{\bar\beta}(x)$ for all $x \in [L_2, L_3)$. Suppose not, then there exists $t \in [L_2, L_3)$ such that $v_3^{\beta}(x) < v_3^{\bar\beta}(x)$ on (t, L_3), $v_3^{\beta}(t) = v_3^{\bar\beta}(t)$. On (t, L_3), one has

$$D_3 \frac{\mathrm{d}^2}{\mathrm{d}x^2}[v_3^{\bar\beta}(x) - v_3^{\beta}(x)] = -[v_3^{\bar\beta}(x) g_3(v_3^{\bar\beta}(x)) - v_3^{\beta}(x) g_3(v_3^{\beta}(x))].$$

Therefore if $(\mathrm{d}/\mathrm{d}v_3)[v_3 g_3(v_3)] < 0$ for $v_3 \in [K_3, K_2]$, then $(\mathrm{d}^2/\mathrm{d}x^2)[v_3^{\bar\beta}(x) - v_3^{\beta}(x)] > 0$ on $[t, L_3)$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}[v_3^{\bar\beta}(x) - v_3^{\beta}(x)] < \frac{\mathrm{d}}{\mathrm{d}x}[v_3^{\bar\beta}(L_3) - v_3^{\beta}(L_3)] = \bar\beta - \beta < 0$$

which implies that

$$v_3^{\bar\beta}(t) - v_3^{\beta}(t) > v_3^{\bar\beta}(L_3) - v_3^{\beta}(L_3) = K_3 - K_3 = 0,$$

a contradiction to $v_3^{\bar\beta}(t) = v_3^{\beta}(t)$. Therefore $v_3^{\beta}(x) < v_3^{\bar\beta}(x)$ for all $x \in (L_2, L_3)$ and all $\bar\beta$, β with $\beta_0 \leq \bar\beta < \beta < 0$. Consequently, $v_3^{\beta}(L_2)$ is decreasing in $\beta \in [\beta_0, 0]$.

Since on $[L_2, L_3]$, $v_3^{\beta}(x) < v_3^{\bar\beta}(x)$ and $v_3^{\beta}(x), v_3^{\bar\beta}(x) \in [K_3, K_2]$ for all $\bar\beta$, $\beta \in [\beta_0, 0]$ with $\bar\beta < \beta$, by assumption (4.3) one has

$$D_3 \frac{\mathrm{d}^2}{\mathrm{d}x^2}[v_3^{\bar\beta}(x) - v_3^{\beta}(x)] = -[v_3^{\bar\beta}(x) g_3(v_3^{\bar\beta}(x)) - v_3^{\beta}(x) g_3(v_3^{\beta}(x))] > 0,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}x}[v_3^{\bar\beta}(x) - v_3^{\beta}(x)] < \frac{\mathrm{d}}{\mathrm{d}x}[v_3^{\bar\beta}(L_3) - v_3^{\beta}(L_3)] = \bar\beta - \beta < 0.$$

Therefore, $(\mathrm{d}/\mathrm{d}x)v_3^{\beta}(L_2)$ is increasing in $\beta \in [\beta_0, 0]$. This completes the proof.

Remark 4.1. Since the inverse of a continuous decreasing function is continuous and decreasing, lemma 4.2 can be reformulated as follows. If (4.3) holds, then there exist $\beta_0 > 0$ and a continuous, decreasing function $\omega: [K_3, K_2] \to [\beta_0, 0]$ such that $\omega(K_2) = \beta_0$, $\omega(K_3) = 0$, and $v_3^{\omega(\xi)}(L_2) = \xi$ for any $\xi \in [K_3, K_2]$. Moreover, $(d/dx)v_3^{\omega(\xi)}(L_2)$ is decreasing in $\xi \in [K_3, K_2]$.

Likewise, we can prove the following lemma.

LEMMA 4.3. Assume that

$$\frac{d}{dv_1}[v_1 g_1(v_1)] < 0 \qquad \text{for } v_1 \in [K_3, K_2]. \tag{4.5}$$

Then there exists a constant $\underline{K}_1 \in (K_3, K_1)$ and a continuous function $\xi: [\underline{K}_1, K_2] \to R$ such that

(i) $\xi(K_1) = 0$;

(ii) for any $\zeta \in [\underline{K}_1, K_2]$, the unique solution, denoted by v_1^ζ, to the following intial value problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} v_1 + v_1 g_1(v_1) = 0, & 0 \le x \le L_1, \\ v_1(0) = K_1, & \dfrac{d}{dx} v_1(0) = \xi(\zeta) \end{cases}$$

satisfies $v_1(L_1) = \zeta$;

(iii) $(d/dx)v_1^\zeta(L_1) < 0$ if $\zeta \in (\underline{K}_1, K_1]$; and $(d/dx)v_1^\zeta(L_1) > 0$ if $\zeta \in (K_1, K_2]$.

LEMMA 4.4. Suppose (4.2) and (4.3) hold. Then there exists a continuous function

$$\gamma: [L_1, L_2] \to [K_3, K_2]$$

such that for any $\theta \in [L_1, L_2]$ there exists a unique solution, denoted by $(v_{2,\theta}, v_{3,\theta})$, to the following problem

$$D_2 \frac{d^2 v_2}{dx^2} + v_2 g_2(v_2) = 0, \qquad x \in [L_1, L_2] \tag{4.6}$$

$$v_2(\theta) = \gamma(\theta), \qquad \frac{d}{dx} v_2(\theta) = 0, \tag{4.7}$$

$$D_3 \frac{d^2 v_3}{dx^2} + v_3 g_3(v_3) = 0, \qquad x \in [L_2, L_3] \tag{4.8}$$

$$v_3(L_3) = K_3, \tag{4.9}$$

$$v_3(L_2) = v_2(L_2), \qquad D_3 \frac{d}{dx} v_3(L_2) = D_2 \frac{d}{dx} v_2(L_2). \tag{4.10}$$

Proof. For any given $\theta \in [L_1, L_2]$ and $\alpha \in [\alpha_0(\theta), K_2]$, let $\beta \triangleq \beta(\theta, \alpha)$ be given such that $v_3^{\beta(\theta,\alpha)}(L_2) = v_2^{\theta,\alpha}(L_2)$. By lemmas 4.1 and 4.2, such a β exists and $\beta(\theta, \alpha)$ is continuous and decreasing in $\alpha \in [\alpha_0(\theta), K_2]$ when θ is fixed. Consider the function

$$F(\theta, \alpha) = D_2 \frac{d}{dx} v_2^{\theta,\alpha}(L_2) - D_3 \frac{d}{dx} v_3^{\beta(\theta,\alpha)}(L_2)$$

for $\theta \in [L_1, L_2]$ and $\alpha \in [\alpha_0(\theta), K_2]$. By (iv) of lemmas 4.1 and 4.2, $F(\theta, \alpha)$ is continuous and increasing in $\alpha \in [\alpha_0(\theta), K_2]$. Moreover,

$$F(\theta, \alpha_0(\theta)) = D_2 \frac{\mathrm{d}}{\mathrm{d}x} v_2^{\theta, \alpha_0(\theta)}(L_2) < 0$$

and

$$F(\theta, K_2) = -D_3 \frac{\mathrm{d}}{\mathrm{d}x} v_3^{\beta_0}(L_2) > 0.$$

Therefore there exists a unique $\gamma(\theta) \in [\alpha_0(\theta), K_2]$ such that $F(\theta, \gamma(\theta)) = 0$. This verifies the existence of the function $\gamma\colon [L_1, L_2] \to [K_3, K_2]$ such that (4.6)–(4.10) has a unique solution.

It remains to prove the continuity of γ. Suppose $\theta_0 \in [L_1, L_2]$ and there exists a sequence $\{\theta_n\} \subseteq [L_1, L_2]$ such that $\theta_n \to \theta_0$ as $n \to \infty$. Since $\gamma(\theta_n) \in [K_3, K_2]$, without loss of generality, we may assume that $\gamma(\theta_n) \to \gamma_0 \in [K_3, K_2]$ as $n \to \infty$. By the continuity of $v_2^{\theta,\alpha}$ and v_3^{β} in (θ, α) and β, one can find a constant $M > 0$ such that $|v_2^{\theta,\alpha}(x)| + |(\mathrm{d}/\mathrm{d}x)v_2^{\theta,\alpha}(x)| < M$ for $x \in [L_1, L_2]$, $\theta \in [L_1, L_2]$ and $\alpha \in [K_3, K_2]$; and $|v_3^{\beta}(x)| + |(\mathrm{d}/\mathrm{d}x)v_3^{\beta}(x)| < M$ for $x \in [L_2, L_3]$ and $\beta \in [\beta_0, 0]$. Therefore

$$\left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_2^{\theta,\alpha}(x) \right| \le \frac{1}{D_2} \max_{|v_2| \le M} |v_2 g_2(v_2)| < \infty, \qquad x \in [L_1, L_2]$$

and

$$\left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_3^{\beta}(x) \right| \le \frac{1}{D_3} \max_{|v_3| \le M} |v_3 g_3(v_3)| < \infty, \qquad x \in [L_2, L_3].$$

By the Ascoli–Arzela theorem, without loss of generality, we may assume that $v_2^{\theta_n,\gamma(\theta_n)} \to \bar{v}_2$ and $v_3^{\beta(\theta_n,\gamma(\theta_n))} \to \bar{v}_3$ in C^1-topology, and $(\mathrm{d}^2/\mathrm{d}x^2)v_2^{\theta_n,\gamma(\theta_n)}(x) \to (\mathrm{d}^2/\mathrm{d}x^2)\bar{v}_2(x)$ for $x \in [L_1, L_2]$, $(\mathrm{d}^2/\mathrm{d}x^2)v_3^{\beta(\theta_n,\gamma(\theta_n))}(x) \to (\mathrm{d}^2/\mathrm{d}x^2)\bar{v}_3(x)$ for $x \in [L_2, L_3]$ as $n \to \infty$. Taking the limit as $n \to \infty$ in (4.6)–(4.10) with $\theta = \theta_n$, we verify that $(\bar{v}_2, \bar{v}_3)$ satisfies (4.6)–(4.10) with $v_2(\theta_0) = \gamma_0$. By the uniqueness of α such that $F(\theta_0, \alpha) = 0$, we get $\gamma_0 = \gamma(\theta_0)$, and therefore $\gamma(\theta_n) \to \gamma(\theta_0)$ as $n \to \infty$. This completes the proof.

We are now in the position to state our main result of this section.

THEOREM 4.1. Assume (4.2), (4.3) and (4.5) hold, and

$$v_{2,L_1}(L_1) > K_1, \tag{4.11}$$

where v_{2,L_1} is defined in lemma 4.4. Then there exists a positive, piecewise monotonic continuous steady-state solution to the problem (2.1)–(2.3) subject to matching conditions (2.4) and (2.5). Moreover, this steady-state solution is (locally) asymptotically stable.

Proof. For $\theta \in [L_1, L_2]$, we can use a similar argument to that used in lemma 4.1 in order to prove that $(\mathrm{d}/\mathrm{d}x)v_{2,\theta}(x)$ is decreasing and positive, $v_{2,\theta}(x)$ is increasing for all $x \in [L_1, \theta)$ such that $v_{2,\theta}(x) > 0$. Therefore under the assumption that $K_3 < K_1$, it follows that if $v_{2,L_2}(x) \ge 0$ for all $x \in [L_1, L_2]$, then $v_{2,L_2}(L_1) < v_{2,L_2}(L_2) = K_3 \le K_1$. On the other hand, since $v_{2,L_1}(L_1) > K_1$, by the continuity of $v_{2,\theta}(L_1)$ in $\theta \in [L_1, L_2]$, if $v_{2,L_2}(x)$ does not remain positive on $[L_1, L_2]$, then there must be $\theta_0 \in [L_1, L_2]$ such that $v_{2,\theta_0}(x) > 0$ for all $x \in [L_1, L_2]$ and $\underline{K}_1 < v_{2,\theta_0}(L_1) < K_1$. In summary, we can always find $\bar{\theta} \in [L_1, L_2]$ such that $v_{2,\bar{\theta}}(L_1) \in (\underline{K}_1, K_1)$. Without loss of generality, we may assume that $v_{2,\theta}(L_1) > v_{2,\bar{\theta}}(L_1)$ for all $\theta \in [L_1, \bar{\theta})$.

Consider the function $G: [L_1, \bar{\theta}] \to R$ defined by

$$G(\theta) = D_1 \frac{d}{dx} v_1^{\zeta}(L_1) - D_2 \frac{d}{dx} v_{2,\theta}(L_1), \qquad \theta \in [L_1, \bar{\theta}],$$

where $\zeta = \zeta(\theta) = v_{2,\theta}(L_1)$. By assumptions, $\zeta(L_1) = v_{2,L_1}(L_1) > K_1$. Therefore by lemma 4.3, $G(L_1) = D_1(d/dx)v_1^{\zeta(L_1)}(L_1) > 0$. When $\theta = \bar{\theta}$, $\underline{K}_1 < \zeta = v_{2,\bar{\theta}}(L_1) < K_1$. Hence by lemma 4.3, $D_1(d/dx)v_1^{\zeta}(L_1) < 0$. Moreover, as we have shown, $(d/dx)v_{2,\bar{\theta}}(L_1)$ is positive. Therefore, $G(\bar{\theta}) < 0$. By the continuity of $v_{2,\theta}(L_1)$ in θ and $v_1^{\zeta}(L_1)$ in ζ, we can claim that there exists $\theta^* \in [L_1, \bar{\theta}]$ such that $D_1(d/dx)v_1^{\zeta^*}(L_1) = D_2(d/dx)v_{2,\theta^*}(L_1)$, where $\zeta^* = v_{2,\theta^*}(L_1)$. Therefore

$$v(x) = \begin{cases} v_1^{\zeta^*}(x), & 0 \le x \le L_1, \\ v_{2,\theta^*}(x), & L_1 \le x \le L_2, \\ v_{3,\theta^*}(x), & L_2 \le x \le L_3 \end{cases}$$

is a steady-state solution of the problem (2.1)–(2.5). Evidently, $v(x)$ is increasing in $[0, \theta^*)$ and decreasing in $(\theta^*, L_3]$. The local asymptotic stability of $v(x)$ can be verified by an argument similar to the proof of theorem 3.2.

We notice that (4.11) is equivalent to the assumption that $w(L_2) < K_3$, where w is the solution to the following initial value problem

$$\begin{cases} D_2 \dfrac{d^2}{dx^2} w + w g_2(w) = 0, & x \in [L_1, L_2], \\ w(L_1) = K_1, \quad \dfrac{d}{dx} w(L_1) = 0. \end{cases}$$

From this we have the following corollary.

COROLLARY 4.1. Assume (4.2), (4.3) and (4.5) hold, and

$$K_1 - K_3 < \frac{\alpha}{2D_2}(L_2 - L_1)^2 \tag{4.12}$$

where $\alpha = \min\{v_2 g_2(v_2) : K_3 \le v_2 \le K_1\}$. Then there exists a positive, piecewise monotonic, continuous, asymptotically stable, steady-state solution to the problem (2.1)–(2.3) subject to matching conditions (2.4) and (2.5).

Proof. According to theorem 4.1 and the remark immediately following it, it suffices to prove that $w(L_2) < K_3$. Suppose not; then by lemma 4.1, $K_3 \le w(x) \le K_1$ for all $x \in [L_1, L_2]$, and, therefore, $D_2(d^2/dx^2)w = -wg_2(w) \le -\alpha$, from which we get that $(d/dx)w(x) \le -(\alpha/D_2)(x - L_1)$ and $w(x) \le -(\alpha/2D_2)(x - L_1)^2 + K_1$ for $x \in [L_1, L_2]$. Therefore

$$w(L_2) \le \frac{\alpha}{2D_2}(L_2 - L_1)^2 + K_1 < K_3$$

which is contrary to $w(x) \ge K_3$ for $x \in [L_1, L_2]$. This completes the proof.

Remark 4.1. Biologically, there are various controls on the system which will cause (4.12) to be satisfied; for example, increasing the length of the middle patch, decreasing the diffusivity of the middle patch, or decreasing the difference of the carrying capacities of the first and the third patches.

For instance, in the case of alternating carrying capacities, where $K_1 - K_3$ may not be too large, then the length of the middle patch could also be small. However, if the carrying capacities are monotonic so that $K_1 - K_3$ may be large, then either the length of the middle patch must also be large or the diffusivity D_2 small (or a combination thereof).

5. CONSTRUCTION OF MONOTONIC STEADY-STATE SOLUTIONS IN THE CASE OF MONOTONIC CARRYING CAPACITIES

In this section, we present a constructive proof for the existence of a monotonic steady-state solution of the problem (2.1)–(2.5) in the case where the carrying capacities of successive patches are increasing or decreasing. We consider only the case that $K_1 < K_2 < K_3$, the other case $K_3 < K_2 < K_1$ can be treated analogously. We start with the following family of two-patch problems.

$$D_1 \frac{d^2}{dx^2} u_1(x) + u_1(x) g_1(u_1(x)) = 0, \qquad 0 < x < L_1, \tag{5.1}$$

$$D_2 \frac{d^2}{dx^2} u_2(x) + u_2(x) g_2(u_2(x)) = 0, \qquad L_1 < x < \beta, \tag{5.2}$$

$$u_1(0) = K_1, \qquad u_2(\beta) = K_2, \tag{5.3}$$

$$u_1(L_1) = u_2(L_1), \qquad D_1 \frac{d}{dx} u_1(L_1) = D_2 \frac{d}{dx} u_2(L_1). \tag{5.4}$$

Let (u_1^β, u_2^β) denote the solution to the above problem for $\beta \in [L_1, L_2]$.

LEMMA 5.1. (i) There exists a unique, continuous, monotonic function

$$u^\beta(x) = \begin{cases} u_1^\beta(x), & 0 \le x \le L_1, \\ u_2^\beta(x), & L_1 \le x \le \beta \end{cases}$$

satisfying (5.1)–(5.4);

(ii) there exists a constant $M_1 > 0$ independent of $\beta \in [L_1, L_2]$ such that $|(d/dx)u_1^\beta(x)| \le M_1$ for $x \in [0, L_1]$;

(iii) $\lim_{\beta \to L_1^+} (d/dx)u_2^\beta(\beta) = (D_1/D_2)(d/dx)v_1(L_1)$, where $v_1(x)$ is the unique solution of the following two-point boundary value problem

$$\begin{cases} D_1 \dfrac{d^2}{dx^2} v_1(x) + v_1(x) g_1(v_1(x)) = 0, & 0 \le x \le L_1, \\ v_1(0) = K_1, \qquad v_1(L_1) = K_2. \end{cases}$$

Proof. The first part was proved in [7]. To prove the second part, we recall a result from [7] which claims that for the following initial value problem

$$\begin{cases} D_1 \dfrac{d^2 u_1}{dx^2} + u_1 g_1(u_1) = 0, & 0 \le x \le L_1, \\ u_1(0) = K_1, \quad \dfrac{d}{dx} u_1(0) = \alpha, \end{cases}$$

if $\alpha > 0$, then $(d/dx)u_1(x) > \alpha$ on $(0, L_1]$. Therefore, to guarantee $u_1(L_1) \in [K_1, K_2]$, α must satisfy

$$0 \le \alpha \le \frac{K_2 - K_1}{L_1} \tag{5.5}$$

from which it follows that

$$0 \le \frac{d}{dx} u_1^\beta(0) \le \frac{K_2 - K_1}{L_1} \quad \text{for any } \beta \in [L_1, L_2]. \tag{5.6}$$

Therefore for any $x \in (0, L_1]$, one has

$$\begin{aligned} \left| \frac{d}{dx} u_1^\beta(x) \right| &\le \left| \frac{d}{dx} u_1^\beta(0) \right| + \left| \frac{d}{dx} u_1^\beta(x) - \frac{d}{dx} u_1^\beta(0) \right| \le \frac{K_2 - K_1}{L_1} + \left| \frac{d^2}{dx^2} u_1^\beta(\theta) \right| x \\ &= \frac{K_2 - K_1}{L_1} + \frac{1}{D_1} \left| - u_1^\beta(\theta) g_1(u_1^\beta(\theta)) \right| L_1 \le \frac{K_2 - K_1}{L_1} + \frac{1}{D_1} L_1 P_1, \qquad \theta \in (0, x), \end{aligned}$$

where

$$P_1 = \max_{K_1 \le x \le K_2} |x g_1(x)|.$$

Therefore $|(d/dx)v_1^\beta(x)| \le M_1$ with $M_1 = ((K_2 - K_1)/L_1) + (L_1 P_1/D_1)$.

To prove the third part, we notice that

$$\frac{d}{dx} u_2^\beta(\beta) - \frac{d}{dx} u_2^\beta(L_1) = \frac{d^2}{dx^2} u_2^\beta(\theta)(\beta - L_1), \qquad \theta \in (L_1, \beta),$$

and

$$\left| \frac{d^2}{dx^2} u_2^\beta(\theta) \right| = \frac{1}{D_2} \left| -u_2^\beta(\theta) g_2(u_2^\beta(\theta)) \right| \le P_2 D_2^{-1}$$

where

$$P_2 \triangleq \max_{K_1 \le x \le K_2} |x g_2(x)|.$$

Therefore

$$\lim_{\beta \to L_1^+} \left[\frac{d}{dx} u_2^\beta(\beta) - \frac{d}{dx} u_2^\beta(L_1) \right] = 0,$$

from which and by the continuous flux matching condition (5.4), we get

$$\lim_{\beta \to L_1^+} \left[\frac{d}{dx} u_2^\beta(\beta) - \frac{D_1}{D_2} \frac{d}{dx} u_1^\beta(L_1) \right] = 0.$$

If $\lim_{\beta\to L_1^+}(d/dx)u_1^\beta(L_1) \neq (d/dx)v_1(L_1)$, then there exists a sequence $\beta_n \to L_1^+$ as $n \to \infty$ and a constant $\delta > 0$ such that

$$\left|\frac{d}{dx}u_1^{\beta_n}(L_1) - \frac{d}{dx}v_1(L_1)\right| \geq \delta \qquad n = 1, 2, \ldots. \tag{5.7}$$

From the conclusion (ii), we have $|(d/dx)u_1^{\beta_n}(x)| \leq M_1$ and

$$D_1\left|\frac{d^2}{dx^2}u_1^{\beta_n}(x)\right| = |u_1^{\beta_n}(x)g_1(u_1^{\beta_n}(x))| \leq P_1$$

for all $x \in [0, L_1]$. Therefore by the Ascoli-Arzela theorem, we may assume that $\lim_{n\to\infty} u_1^{\beta_n}(x) = u^*(x)$, $\lim_{n\to\infty}(d/dx)u_1^{\beta_n}(x) = (d/dx)u^*(x)$ and $\lim_{n\to\infty}(d^2/dx^2)u_1^{\beta_n}(x) = (d^2/dx^2)u^*(x)$ on $[0, L_1]$. Moreover, from the equality

$$D_1\frac{d^2}{dx^2}u_1^{\beta_n}(x) + u_1^{\beta_n}(x)g_1(u_1^{\beta_n}(x)) = 0$$

we get

$$D_1\frac{d^2}{dx^2}u^*(x) + u^*(x)g_1(u^*(x)) = 0, \qquad 0 \leq x \leq L_1.$$

Noting that

$$\left|\frac{d}{dx}u_1^{\beta_n}(L_1)\right| \leq M_1,$$

$$\left|\frac{d}{dx}u_2^{\beta_n}(L_1)\right| = \left|\frac{D_1}{D_2}\frac{d}{dx}u_1^{\beta_n}(L_1)\right| \leq \frac{D_1}{D_2}M_1,$$

and

$$D_2\left|\frac{d^2}{dx^2}u_2^{\beta_n}(x)\right| \leq P_2 \qquad \text{for } x \in [L_1, L_2],$$

we get

$$\left|\frac{d}{dx}u_2^{\beta_n}(x)\right| \leq \frac{D_1}{D_2}M_1 + P_2(L_2 - L_1)D_2^{-1}.$$

Therefore

$$\lim_{n\to\infty}|u_2^{\beta_n}(L_1) - u_2^{\beta_n}(\beta_n)| = \lim_{n\to\infty}\left|\frac{d}{dx}u_2^{\beta_2}(\theta)\right|(\beta_n - L_1) = 0, \qquad \theta \in (L_1, \beta_n).$$

This, together with the continuous flux matching condition (5.3), implies that

$$\lim_{n\to\infty} u_1^{\beta_n}(L_1) = \lim_{n\to\infty} u_2^{\beta_n}(L_1) = \lim_{n\to\infty} u_2^{\beta_n}(\beta_n) = K_2.$$

Therefore u^* satisfies the following two-point boundary value problem

$$\begin{cases} D_1\dfrac{d^2}{dx^2}u^*(x) + u^*(x)g_1(u^*(x)) = 0, & 0 \leq x \leq L_1, \\ u^*(0) = K_1, \quad u^*(L_1) = K_2. \end{cases}$$

That is $u^*(x) = v_1(x)$. This leads to a contradiction to (5.7) and

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} u_1^{\beta_n}(L_1) = \frac{\mathrm{d}}{\mathrm{d}x} u^*(L_1).$$

Therefore,

$$\lim_{\beta_n \to L_1^+} \frac{\mathrm{d}}{\mathrm{d}x} u_2^{\beta}(L_1) = \frac{D_1}{D_2} \frac{\mathrm{d}}{\mathrm{d}x} v_1(L_1).$$

The proof is then complete.

Likewise, for the following family of two-patch problems

$$D_2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_2 + v_2 g_2(v_2) = 0, \qquad \beta \le x \le L_2, \tag{5.8}$$

$$D_3 \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_3 + v_3 g_3(v_3) = 0, \qquad L_2 \le x \le L_3, \tag{5.9}$$

$$v_2(\beta) = K_2, \qquad v_3(L_3) = K_3, \tag{5.10}$$

$$v_2(L_2) = v_3(L_2), \qquad D_2 \frac{\mathrm{d}}{\mathrm{d}x} v_2(L_2) = D_3 \frac{\mathrm{d}}{\mathrm{d}x} v_3(L_2). \tag{5.11}$$

Denoting the solution by $(v_{2\beta}, v_{3\beta})$, we get the following lemma.

LEMMA 5.2. (i) There exists a unique, continuous monotonic function

$$v_\beta(x) = \begin{cases} v_{2\beta}(x), & \beta \le x \le L_2, \\ v_{3\beta}(x), & L_1 \le x \le L_3 \end{cases}$$

satisfying (5.8)–(5.11).

(ii) There exists a constant $M_2 > 0$ independent of $\beta \in [L_1, L_2]$ such that

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} v_{3\beta}(x) \right| \le M_2 \qquad \text{for } x \in [L_2, L_3].$$

(iii) $\lim_{\beta \to L_2^-} (\mathrm{d}/\mathrm{d}x) v_{2\beta}(\beta) = (D_3/D_2)(\mathrm{d}/\mathrm{d}x) v_3(L_2)$, where $v_3(x)$ is the unique solution of the following two-point boundary value problem

$$\begin{cases} D_3 \dfrac{\mathrm{d}^2 v_3}{\mathrm{d}x^2} + v_3 g_3(v_3) = 0, & L_2 \le x \le L_3, \\ v_3(L_2) = K_2, \qquad v_3(L_3) = K_3. \end{cases}$$

Employing a similar argument to that of (iii) of lemma 5.1, we can prove the following lemma.

LEMMA 5.3. $(\mathrm{d}/\mathrm{d}x) u_2^{\beta}(\beta)$ is continuous in $\beta \in [L_1, L_2]$ and $(\mathrm{d}/\mathrm{d}x) v_{2\beta}(\beta)$ is continuous in $\beta \in [L_1, L_2)$.

We now state the main result in this section.

THEOREM 5.1. If

$$\left[\frac{D_1}{D_2}\frac{\mathrm{d}}{\mathrm{d}x}v_1(L_1)-\frac{\mathrm{d}}{\mathrm{d}x}v_{2L_1}(L_1)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x}u_2^{L_2}(L_2)-\frac{D_3}{D_2}\frac{\mathrm{d}}{\mathrm{d}x}v_3(L_2)\right]<0, \tag{5.12}$$

then there exists a positive, monotonic continuous asymptotically stable, steady-state solution to the problem (2.1)–(2.3) subject to matching conditions (2.4)–(2.5).

Proof. To prove the existence of a steady-state solution to (2.1)–(2.5), it suffices to verify the existence of $\beta \in (L_1, L_2)$ such that $(\mathrm{d}/\mathrm{d}x)u_2^\beta(\beta) = (\mathrm{d}/\mathrm{d}x)v_{2\beta}(\beta)$ for some $\beta \in (L_1, L_2)$. For this purpose, we consider the function

$$f(\beta)=\frac{\mathrm{d}}{\mathrm{d}x}v_{2\beta}(\beta)-\frac{\mathrm{d}}{\mathrm{d}x}u_2^\beta(\beta), \qquad \beta\in(L_1,L_2).$$

By lemma 5.3, $f(\beta)$ is continuous on (L_1, L_2). By lemma 5.1,

$$\lim_{\beta\to L_1^+}\frac{\mathrm{d}}{\mathrm{d}x}u_2^\beta(\beta)=\frac{D_1}{D_2}\frac{\mathrm{d}}{\mathrm{d}x}v_1(L_1).$$

By lemma 5.2,

$$\lim_{\beta\to L_1^+}\frac{\mathrm{d}}{\mathrm{d}x}v_{2\beta}(\beta)=\frac{\mathrm{d}}{\mathrm{d}x}v_{2L_1}(L_1).$$

Therefore

$$\lim_{\beta\to L_1^+}f(\beta)=\frac{D_1}{D_2}\frac{\mathrm{d}}{\mathrm{d}x}v_1(L_1)-\frac{\mathrm{d}}{\mathrm{d}x}v_{2L_1}(L_1).$$

Likewise,

$$\lim_{\beta\to L_2^-}f_2(\beta)=\frac{\mathrm{d}}{\mathrm{d}x}u_2^{L_2}(L_2)-\frac{D_3}{D_2}\frac{\mathrm{d}}{\mathrm{d}x}v_3(L_2).$$

Therefore if (5.12) is satisfied, then by the well-known mean value theorem of continuous functions, there exists $\beta \in (L_1, L_2)$ such that $(\mathrm{d}/\mathrm{d}x)u_2^\beta(\beta) = (\mathrm{d}/\mathrm{d}x)u_{2\beta}(\beta)$ holds.

The monotonicity of the obtained steady-state solution is obvious, and the local asymptotic stability can be proved by using a similar argument to that for theorem 3.2.

The following result provides a rough but simple sufficient condition to guarantee (5.12).

COROLLARY 5.1. If

$$\frac{D_1}{D_2}\sqrt{\left(\frac{K_2-K_1}{L_1}\right)^2+\frac{2}{D_1}\int_{K_1}^{K_2}\zeta g_1(\zeta)\,\mathrm{d}\zeta}\ge\frac{K_3-K_2}{L_2-L_1} \tag{5.13}$$

and

$$\frac{D_3}{D_2}\sqrt{\left(\frac{K_3-K_2}{L_3-L_2}\right)^2-\frac{2}{D_3}\int_{K_2}^{K_3}\zeta g_3(\zeta)\,\mathrm{d}\zeta}\ge\frac{K_2-K_1}{L_2-L_1}, \tag{5.14}$$

then (5.12) holds.

Proof. Since $v_1(x) \in [K_1, K_2]$ for $x \in [0, L_1]$, $(\mathrm{d}^2/\mathrm{d}x^2)v_1(x) = -(1/D_1)v_1(x)g_1(v_1(x)) > 0$ for $x \in (0, L_1]$, and thus $(\mathrm{d}/\mathrm{d}x)v_1(x)$ is increasing. Therefore $(\mathrm{d}/\mathrm{d}x)v_1(x) > (\mathrm{d}/\mathrm{d}x)v_1(0) > 0$ on $(0, L_1]$. Integrating the equality $D_1(\mathrm{d}^2/\mathrm{d}x^2)v_1 + v_1g_1(v_1) = 0$, we get

$$D_1\left[\frac{\mathrm{d}v_1(L_1)}{\mathrm{d}x}\right]^2=D_1\left[\frac{\mathrm{d}}{\mathrm{d}x}v_1(0)\right]^2-2\int_{K_1}^{K_2}\zeta g_1(\zeta)\,\mathrm{d}\zeta.$$

Therefore

$$\frac{d}{dx} v_1(x) \le \sqrt{\left[\frac{dv_1(0)}{dx}\right]^2 - \frac{2}{D_1}\int_{K_1}^{K_2} \zeta g_1(\zeta)\, d\zeta}$$

from which it follows

$$v_1(x) \le K_1 + \left[\sqrt{\left[\frac{dv_1(0)}{dx}\right]^2 - \frac{2}{D_1}\int_{K_1}^{K_2} \zeta g_1(\zeta)\, d\zeta}\,\right] x \qquad \text{for } x \in [0, L_1].$$

Hence, since $v_1(L_1) = K_2$, we must have

$$\frac{d}{dx} v_1(L_1) \ge \frac{d}{dx} v_1(0) \ge \sqrt{\left(\frac{K_2 - K_1}{L_1}\right)^2 + \frac{2}{D_1}\int_{K_1}^{K_2} \zeta g_1(\zeta)\, d\zeta}. \tag{5.15}$$

On the other hand, $v_{2L_1}(x) \in (K_2, K_3)$ for $x \in (L_1, L_2)$, hence

$$\frac{d^2}{dx^2} v_{2L_1}(x) = -\frac{1}{D_2} v_{2L_1}(x) g_2(v_{2L_1}(x)) > 0.$$

This implies that $(d/dx)v_{2L_1}(x)$ is increasing, and thus

$$\frac{d}{dx} v_{2L_1}(x) > \frac{d}{dx} v_{2L_1}(L_1)$$

for all $x \in (L_1, L_2)$. Therefore

$$v_{2L_1}(x) > K_2 + \left[\frac{d}{dx} v_{2L_1}(L_1)\right](x - L_1).$$

Noting that $v_{2L_1}(L_2) \in [K_2, K_3]$, we get

$$\frac{d}{dx} v_{2L_1}(L_1) < \frac{K_3 - K_2}{L_2 - L_1}.$$

Therefore

$$\frac{D_1}{D_2}\frac{d}{dx} v_1(L_1) - \frac{d}{dx} v_{2L_1}(L_1) > \frac{D_1}{D_2}\sqrt{\left(\frac{K_2 - K_1}{L_1}\right)^2 + \frac{2}{D_1}\int_{K_1}^{K_2} \zeta g_1(\zeta)\, d\zeta} - \frac{K_3 - K_2}{L_2 - L_1} \ge 0.$$

Likewise, one can prove

$$\frac{D_3}{D_2}\frac{d}{dx} v_3(L_2) - \frac{d}{dx} v_2^{L_2}(L_2) > \frac{D_3}{D_2}\sqrt{\left(\frac{K_3 - K_2}{L_3 - L_2}\right)^2 - \frac{2}{D_3}\int_{K_2}^{K_3} \zeta g_3(\zeta)\, d\zeta} - \frac{K_2 - K_1}{L_2 - L_1} \ge 0.$$

Therefore (5.12) holds.

Remark 5.1. Note that if D_1/D_2 and D_3/D_2 are sufficiently large, then (5.13) and (5.14) must hold, leading to a stable steady state, that is to say that the diffusivity in the end patches are large compared to the diffusivity in the middle patch. One could easily visualize an environment (such as a lake) where the rate of flow in a stream entering and a stream leaving is high compared to the flow in the lake.

6. DISCUSSION

The focus of this paper is steady-state analysis of a system of reaction–diffusion equations describing the growth of a population which diffuses in an environment consisting of patches along a linear transect.

By using a topological transversality theorem due to Granas and an *a priori* bound technique, we have shown the existence of a positive steady state in the case of reservoir boundary conditions and continuous flux matching conditions at the interface described in (2.3), (2.4) and (2.5). In a future paper, we plan to modify our argument to the cases of no-flux boundary conditions (closed patches), zero reservoir boundary conditions (extremely hostile environment) and more general continuous flux matching conditions.

In the case of only two patches, a positive, monotonic steady state has been constructed in [7]. However, in the case of more than two patches, we have only constructed a positive steady state under further restrictive (but reasonable) hypotheses. Moreover, in more than two patches, the carrying capacities may alternate in size, therefore the positive steady state may be piecewise monotonic rather than monotonic.

Biologically, the above-mentioned additional constraints are reasonable. Constraint (4.12) is satisfied for example if $|K_1 - K_3|$ is small (e.g. forestry environment which is similar on both sides of a trench) or $|L_2 - L_1|$ is large (the trench is wide as compared to the forest on one side, as for example the Shakwak Trench in the Kluane Mountains of the Yukon Territories, Canada), or if the diffusion across the second patch is small.

Conditions (5.13) and (5.14) are also satisfied if the diffusion across the middle patch is relatively small. It is also satisfied if the ratio of the difference in carrying capacities to the difference in patch lengths is small between the outer patches and the middle patch.

Clearly some constraints are necessary for our techniques to work since they depend on utilizing constructive solutions to the two-patch problem. If, for example $L_2 - L_1$ is small and $K_2 - K_1$ is small, but $L_3 - L_2$ is large and $K_3 - K_2$ is large and $K_1 < K_2 < K_3$, the stable steady state will miss K_2 on the range $L_2 - L_1$ altogether, in which case our technique cannot work.

Possible modifications leading to future work are to consider a model with general space dependent carrying capacities and diffusion coefficients, as well as spatial diffusion in higher dimensions, rather than along a linear transect. These lead to certain complications, which we hope to address in the future.

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