Fundamental Inequalities and Applications to Neutral Equations

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Two fundamental inequalities are derived for neutral functional differential equations with stable D-operator. These inequalities provide some exponential estimates about the relation between solution operators and D-operators, an essential characterization of neutral functional differential equations, and an effective tool for the application of Liapunov's direct method and Razumikhin techniques to boundedness, stability, and convergence of solutions. © 1991 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to establish two basic inequalities for functional differential equations (FDEs)

$$\frac{d}{dt} D(x_t) = f(t, x_t),$$

where $D: C([-r, 0], R^n) \to R^n, f: R \times C([-r, 0], R^n) \to R^n$ are continuous, and x_t is the usual notation for FDEs. These inequalities involve some exponential estimates with respect to the solution operator x_t and the corresponding D-operator. They will provide an essential characterization of neutral FDEs, and an effective tool for the application of Liapunov's direct method and Razumikhin techniques to boundedness, stability, and convergence of solutions.

The motivation of this study comes from the investigation of the neutral equation

$$\frac{d}{dt} [x(t) - cx(t - r)] = g(t, x(t), x(t - r)), \qquad (1.1)$$

where $0 \leqslant c < 1$, $g: R \times R \times R \to R$ is continuous and satisfies the order relation

$$g(t, x, y) \leqslant 0 \qquad \text{if} \quad 0 \leqslant y \leqslant x.$$

78

A prototype is

$$\frac{d}{dt}[x(t) - cx(t-r)] = -h(t, x(t)) + h(t, x(t-r)) \tag{1.2}$$

with

$$h: (t, x) \in R \times R \to h(t, x) \in R \quad \text{continuous and nondecreasing in } x.$$

Equation (1.2) can be considered as describing the motion of a compartmental system with one pipe and one compartment producing material itself (see, e.g., Győri [2]). For the retarded equation case ($c = 0$), boundedness of solutions and convergence of orbits to equilibria can be obtained via an invariance principle of Liapunov–Razumikhin type (see, e.g., Haddock and Terjéki [3]). However, for neutral equations (in the case $c \neq 0$), very little has been done along these lines. Available literature, based on the inequality

$$\|x_t\| \leqslant be^{-\alpha t}\|x_0\| + p \sup_{0 \leqslant s \leqslant t} |Dx_s| \tag{1.3}$$

(where b, p, and α are nonnegative constants), allows one to deal with the boundedness, stability, and convergence of solutions in the case where the ordinary part dominates the functional part such as

$$g(t, x, y) < 0 \quad \text{if} \quad 0 < y \leqslant x.$$

But this essentially excludes such equations as those describing compartmental systems.

To lay the foundation for a qualitative theory of NFDEs whose ordinary parts do not dominate the functional parts, we will derive two fundamental inequalities,

$$\|x_t\| \leqslant be^{-\alpha t}\|x_0\| + (p - qe^{-\gamma t}) \sup_{0 \leqslant s \leqslant t} |Dx_s| \tag{1.4}$$

and

$$\|x_t\| \leqslant be^{-\alpha t}\|x_0\| + (p - qe^{-\gamma t}) \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |Dx_s|, \tag{1.5}$$

where b, p, q and α, β, γ are nonnegative constants. They indicate, more precisely than (1.3), how properties of D reflect some general property of the solution operator to NFDEs. In these inequalities, whether q is zero or not provides an essential classification of retarded equations and neutral equations; the case where $q \neq 0$ corresponds to a neutral equation which can not be reduced to a retarded equation. The irreducibility condition

$q \neq 0$ plays an important role in qualitative thery of NFDEs. By inserting the term $-qe^{-\gamma t}$ in inequality (1.3), we are able to apply Liapunov–Razumikhin methods to such irreducible neutral equations as (1.2) to obtain some stability, boundedness, and convergence results. It is interesting to note that these results for irreducible neutral equations can not be applied to retarded equations.

At this point we find it convenient to introduce some fundamental notation. Let R^n denote the real Euclidean space of n-vectors and let $|x|$ denote the norm of the vector x in R^n. Let $r \geqslant 0$ be given, and let $C = C([-r, 0], R^n)$ denote the space of continuous functions from the interval $[-r, 0]$ into R^n. For $\phi \in C$, the norm of ϕ is defined by

$$\|\phi\| = \sup_{-r \leqslant s \leqslant 0} |\phi(s)|.$$

Suppose $x: [-r, \infty) \to R^n$ is continuous. Then for any $t \geqslant 0$, $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $-r \leqslant s \leqslant 0$. We consider the following neutral functional differential equation

$$\frac{d}{dt} D(x_t) = f(t, x_t) \tag{1.6}$$

with the initial condition

$$x_\sigma = \phi, \tag{1.7}$$

where $t \geqslant \sigma$ and $f: R \times C \to R^n$ is continuous. We assume $D: C \to R^n$ is a bounded linear operator of the form

$$D\phi = \phi(0) - \int_{-r}^{0} [d\mu(\theta)] \, \phi(\theta)$$

in which μ is an $n \times n$ matrix, $-r \leqslant \theta \leqslant 0$, whose elements are of bounded variation and

$$\operatorname*{Var}_{[-s, 0]} \mu \to 0 \qquad \text{as} \quad s \to 0.$$

Under the above assumptions, the solution, denoted by $x(\sigma, \phi)$, of initial value problem (1.6)–(1.7) exists. We further assume the uniqueness, continuous dependence, and continuation of the solution of the initial value problem (1.6)–(1.7). For details we refer to [5].

2. FUNDAMENTAL INEQUALITIES

Throughout this paper, we always assume that the D-operator is stable. Then the zero solution of the homogeneous difference equation

$$Dy_t = 0, \qquad t \geq 0 \tag{2.1}$$

$$y_0 = \phi \in C_D := \{\phi \in C : D\phi = 0\} \tag{2.2}$$

is uniformly asymptotically stable. Therefore there are n functions $\phi_1, \phi_2, ..., \phi_n$ such that $D\Phi = I(n \times n$ identity matrix), where $\Phi = (\phi_1, \phi_2, ..., \phi_n)$ (see, e.g., [5, p. 281]). The operator defined by

$$\Psi = I - \Phi D$$

is a continuous projection $\Psi : C \to C_D$, and (2.1) defines a strongly continuous semigroup of linear transformations $T_D(t) : C_D \to C_D$, $t \geq 0$, by

$$T_D(t)\psi = y_t(\psi) \qquad \text{for} \quad t \geq 0, \psi \in C_D.$$

Again, see [5].

Now consider the nonhomogeneous equation

$$Dy_t = h(t), \qquad t \geq 0 \tag{2.3}$$

with the initial condition

$$y_0 = \phi \in C, \tag{2.4}$$

where $h \in C([0, \infty), R^n)$. According to [5, p. 281] we have the following result.

LEMMA 2.1. *There exist constants a and b so that, for any $\phi \in C$ and $h \in C([0, \infty), R^n)$, the solution to (2.3)–(2.4) satisfies*

$$\|y_t\| \leq ae^{bt}[\|\phi\| + \sup_{0 \leq s \leq t} |h(s)|].$$

Let a_D be the order of the semigroup $T_D(t)$; that is,

$$a_D = \inf\{a \in R : \text{there is a } K = K(a) \text{ such that } \|T_D(t)\| \leq Ke^{at}, t \geq 0\}.$$

It is known [1] that the stability of the D-operator is equivalent to $a_D < 0$, which turns out to be equivalent to the inequality

$$\|y_t\| \leq be^{-at}\|\phi\| + p \sup_{0 \leq s \leq t} |h(s)|$$

(see [5, p. 287]). The following result shows the above inequality can be strengthened significantly.

THEOREM 2.1. *If D is stable, then there exist constants b, p, $q \geqslant 0$ and $a > 0$ such that $p \geqslant q \geqslant 0$ and, for any $h \in C([0, \infty), R^n)$, each solution y of the nonhomogeneous system (2.3)–(2.4) satisfies*

$$\| y_t \| \leqslant be^{-at} \|\phi\| + (p - e^{-at}q) \sup_{0 \leqslant s \leqslant t} |h(s)|. \tag{2.5}$$

Moreover, $q \neq 0$ whenever $\Psi \neq 0$.

Proof. Let $y_t(a, \phi, h)(0) = y(t; a, \phi, h)$ denote the solution of (2.3) through (a, ϕ). By the superposition principle of solutions of linear systems and (2.3) with $t = 0$, we have

$$\begin{aligned} y(t; 0, \phi, h) &= y(t; 0, \Psi\phi, 0) + y(t; 0, \Phi D\phi, h) \\ &= y(t; 0, \Psi\phi, 0) + y(t; 0, \Phi h(0), h). \end{aligned}$$

Since D is stable, $a_D < 0$. This implies that there exist constants $L > 1$ and $\alpha > 0$ so that $L \|\Psi\| e^{-\alpha r} > 1$ and $\| T_D(t)\phi \| \leqslant Le^{-\alpha t} \|\phi\|$ for any $\phi \in C_D$. Therefore,

$$\| y_t(0, \Psi\phi, 0) \| \leqslant Le^{-\alpha t} \|\Psi\phi\| \leqslant L \|\Psi\| e^{-\alpha t} \|\phi\|. \tag{2.6}$$

Next find a $\sigma > r$ sufficiently large so that

$$L \|\Psi\| e^{-\alpha\sigma} e^{\alpha r} = L^* < 1.$$

For any $t \geqslant 0$, there exists a nonnegative integer j so that $j\sigma \leqslant t < (j+1)\sigma$, and, thus, by the superposition principle we have the decomposition

$$\begin{aligned} y(t; 0, \Phi h(0), h) &= y(t; j\sigma, y_{j\sigma}(0, \Phi h(0), h), h) \\ &= y(t; j\sigma, \Psi h(0), h), 0) + y(t; j\sigma, \Phi h(j\sigma), h). \end{aligned}$$

According to Lemma 2.1, we can find constants $c, d > 0$ so that

$$\| y_t(j\sigma, \Phi h(j\sigma), h) \| \leqslant ce^{d(t-j\sigma)}(1 + \|\Phi\|) \sup_{j\sigma \leqslant s \leqslant t} |h(s)| \tag{2.7}$$

for all $t \geqslant j\sigma$, and c and d can be chosen so that

$$ce^{d\sigma}(1 + \|\Phi\|) > 2(1 - L^*) e^{-\ln L^*} \|\Phi\|. \tag{2.8}$$

Thus,

$$|y(t; j\sigma, \Phi h(j\sigma), h)| \leqslant ce^{d\sigma}(1 + \|\Phi\|) \sup_{j\sigma \leqslant s \leqslant t} |h(s)| \tag{2.9}$$

for $j\sigma \leqslant t \leqslant (j+1)\sigma$. On the other hand,

$$\begin{aligned} |y(t; j\sigma, \Psi y_{j\sigma}(0, \Phi h(0), h), 0)| &= |y(t - j\sigma; 0, \Psi y_{j\sigma}(0, \Phi h(0), h), 0)| \\ &\leqslant Le^{-\alpha(t-j\sigma)} \|\Psi y_{j\sigma}(0, \Phi h(0), h)\| \\ &\leqslant Le^{-\alpha(t-j\sigma)} \|\Psi\| \|y_{j\sigma}(0, \Phi h(0), h)\|. \tag{2.10} \end{aligned}$$

Therefore, by using (2.9), (2.10) as well as

$$\| y_t(0, \Phi h(0), h)\| = \sup_{-r \leqslant s \leqslant 0} | y(t+s, 0, \Phi h(0), h)|$$

and considering the cases $t + s \geqslant j\sigma$ and $t + s < j\sigma$, we obtain

$$\| y_t(0, \Phi h(0), h)\| \leqslant L e^{\alpha r} e^{-\alpha(t - j\sigma)} \|\Psi\| \, \| y_{j\sigma}(0, \Phi h(0), h)\|$$
$$+ c e^{d\sigma}(1 + \|\Phi\|) \sup_{j\sigma \leqslant s \leqslant t} |h(s)|.$$

Using the above inequality at $t = j\sigma$, $(j-1)\sigma$, ..., σ, we get

$$\| y_{j\sigma}(0, \Phi h(0), h)\|$$
$$\leqslant [L e^{\alpha r} e^{-\alpha \sigma} \|\Psi\|]^j \|\Phi\| \, |h(0)|$$
$$+ \{ |L e^{\alpha r} e^{-\alpha \sigma} \|\Psi\|]^{j-1} + \cdots + [L e^{\alpha r} e^{-\alpha \sigma} \|\Psi\|] + 1 \}$$
$$\times (1 + \|\Phi\|) \, c e^{d\sigma} \sup_{0 \leqslant s \leqslant j\sigma} |h(s)|$$
$$\leqslant [L^{*j} \|\Phi\| + c e^{d\sigma}(1 + \|\Phi\|)(1 - L^{*j})/(1 - L^*)] \sup_{0 \leqslant s \leqslant t} |h(s)|.$$

Therefore, from the previous decomposition and the estimates (2.7) and (2.10), we get

$$| y(t; 0, \Phi h(0), h)| \leqslant L \|\Psi\| \, e^{-\alpha(t - j\sigma)} [L^{*j} \|\Phi\| + c e^{d\sigma}(1 + \|\Phi\|)$$
$$\times (1 - L^{*j})/(1 - L^*)] \sup_{0 \leqslant s \leqslant t} |h(s)|$$
$$+ c e^{d(t - j\sigma)}(1 + \|\Phi\|) \sup_{j\sigma \leqslant s \leqslant t} |h(s)|$$
$$\leqslant L \|\Psi\| \, [e^{j \ln L^*} \|\Phi\| + c e^{d\sigma}(1 + \|\Phi\|)(1 - e^{j \ln L^*})/(1 - L^*)$$
$$+ c e^{d\sigma}(1 + \|\Phi\|)] \sup_{0 \leqslant s \leqslant t} |h(s)|.$$

From $j\sigma \leqslant t < (j+1)\sigma$, it follows that

$$j \leqslant \frac{t}{\sigma} \quad \text{and} \quad j > t/\sigma - 1,$$

and, thus,

$$| y(t; 0, \Phi h(0), h)| \leqslant L \|\Psi\| \, [e^{(t/\sigma) \ln L^*} e^{-\ln L^*} \|\Phi\|$$
$$+ c e^{d\sigma}(1 + \|\Phi\|)(1 - e^{(t/\sigma) \ln L^*})/(1 - L^*)$$
$$+ c e^{d\sigma}(1 + \|\Phi\|)] \sup_{0 \leqslant s \leqslant t} |h(s)|.$$

By (2.8), we obtain

$$|y(t; 0, \Phi h(0), h)| \leqslant L \|\Psi\| [ce^{d\sigma}(1 + \|\Phi\|)/(1 - L^*) + ce^{d\sigma}(1 + \|\Phi\|)$$
$$- ce^{d\sigma}(1 + \|\Phi\|)/_{2(1 - L^*)} e^{(t/\sigma)\ln L^*}] \sup_{0 \leqslant s \leqslant t} |h(s)|$$
$$\leqslant [K_1 - K_2 e^{-\gamma t}] \sup_{0 \leqslant s \leqslant t} |h(s)|,$$

where

$$K_1 = ce^{d\sigma}L \|\Psi\| (1 + \|\Phi\|)/(1 - L^*) + ce^{d\sigma}(1 + \|\Phi\|),$$
$$K_2 = ce^{d\sigma}L \|\Psi\| (1 + \|\Psi\|)/2(1 - L^*),$$

and

$$\gamma = \frac{-\ln L^*}{\sigma} = \alpha - \frac{\ln(L \|\Psi\| e^{\alpha r})}{\sigma} < \alpha.$$

It follows that

$$|y(t; 0, \phi, h)| \leqslant Le^{-\alpha t} \|\Psi\| \|\phi\| + [K_1 - K_2 e^{-\gamma t}] \sup_{0 \leqslant s \leqslant t} |h(s)|$$
$$\leqslant be^{-at} \|\phi\| + (p - qe^{-at}) \sup_{0 \leqslant s \leqslant t} |h(s)|,$$

where $b = L \|\Psi\| e^{\alpha r}$, $p = K_1$, $q = K_2$, and $a = \alpha$. The proof is completed.

THEOREM 2.2.   *If D is stable, then there exist constants $B, P, Q \geqslant 0$ and $\alpha, \beta, \gamma > 0$ such that $\alpha > \gamma$ and for any $h \in C([0, \infty), R^n)$, any solution y of the nonhomogeneous system* (2.3)–(2.4) *satisfies*

$$\|y_t\| \leqslant Be^{-\alpha t} \|\phi\| + (P - e^{-\gamma t}Q) \sup_{0 \leqslant s \leqslant t} e^{-\beta(t - s)} |h(s)|. \qquad (2.11)$$

Moreover, $Q > 0$ whenever $\Psi \neq 0$.

Proof.   In the argument of Theorem 2.1, we have proved that there exist constants $L > 1$ and $\alpha > 0$ so that $L \|\Psi\| e^{-\alpha r} > 1$ and

$$\|y_t(0, \Psi\phi, 0)\| \leqslant Le^{-\alpha t} \|\Psi\phi\| \leqslant L \|\Psi\| e^{-\alpha t} \|\phi\|.$$

Evidently, we can find a constant $\sigma > r$ sufficiently large so that

$$L \|\Psi\| e^{-\alpha\sigma} e^{\alpha r} = M < 1.$$

For this given M, choose $\beta > 0$ so that $Me^{\beta\sigma} < 1$. Let $z(t) = y(t; 0, \Phi h(0), h)$. For any $t \geqslant 0$, there exists a nonnegative integer j so that $j\sigma \leqslant t < (j + 1)\sigma$, and thus

$$z(t) = y(t; j\sigma, \Psi z_{j\sigma}, 0) + y(t; j\sigma, \Phi h(j\sigma), h). \qquad (2.12)$$

Acording to Lemma 2.1, we can find constants $c, d > 0$ so that

$$\| y_t(j\sigma, \Psi h(j\sigma), h) \| \leqslant c e^{d(t - j\sigma)} (1 + \|\Phi\|) \sup_{j\sigma \leqslant s \leqslant t} |h(s)|$$

for all $t \geqslant j\sigma$, and c and d can be chosen so that

$$N e^{2\beta\sigma} > (1 - M e^{\beta\sigma}) \|\Psi\| e^{\beta\sigma},$$

where

$$N = c e^{d\sigma} (1 + \|\Phi\|).$$

Therefore,

$$|y(t; j\sigma, \Phi h(j\sigma), h)| \leqslant N \sup_{j\sigma \leqslant s \leqslant t} |h(s)| \tag{2.13}$$

for $j\sigma \leqslant t \leqslant (j+1)\sigma$. On the other hand,

$$|y(t; j\sigma, \Psi z^{j\sigma}, 0)| \leqslant L e^{-\alpha(t - j\sigma)} \|\Psi\| \|z_{j\sigma}\|. \tag{2.14}$$

Hence, by decomposition (2.12) and estimates (2.13) and (2.14) we get, for $j\sigma \leqslant t \leqslant (j+1)\sigma$,

$$\|z_t\| \leqslant L e^{\alpha r} e^{-\alpha(t - j\sigma)} \|\Psi\| \|z_{j\sigma}\| + N \sup_{j\sigma \leqslant s \leqslant t} |h(s)|. \tag{2.15}$$

Specifically, we have

$$\|z_{(j+1)\sigma}\| \leqslant M \|z_{j\sigma}\| + N \sup_{j\sigma \leqslant s \leqslant (j+1)\sigma} |h(s)|.$$

This inequality implies, by induction on j,

$$\|z_{j\sigma}\| \leqslant M^j \|z_0\| + N \sum_{k=0}^{j-1} M^k \sup_{(j-1-k)\sigma \leqslant s \leqslant (j-k)\sigma} |h(s)|$$

$$\leqslant M^j \|\Phi\| |h(0)| + N e^{\beta[t - (j-1)\sigma]} \sum_{k=0}^{j-1} (M e^{\beta\sigma})^k \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|$$

$$\leqslant M^j \|\Phi\| |h(0)| + N e^{2\beta\sigma} \frac{1 - (M e^{\beta\sigma})^j}{1 - M e^{\beta\sigma}} \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|$$

$$\leqslant \left[M^j \|\Phi\| e^{\beta t} + N e^{2\beta\sigma} \frac{1 - (M e^{\beta\sigma})^j}{1 - M e^{\beta\sigma}} \right] \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|$$

$$\leqslant \left[(M e^{\beta\sigma})^j \|\Phi\| e^{\beta\sigma} + N e^{2\beta\sigma} \frac{1 - (M e^{\beta\sigma})^j}{1 - M e^{\beta\sigma}} \right] \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|$$

$$\leqslant [U - V e^{(\ln(M e^{\beta\sigma})/\sigma) t}] \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|,$$

where $U = Ne^{2\beta\sigma}/(1 - Me^{\beta\sigma})$ and $V = Ne^{2\beta\sigma}/(1 - Me^{\beta\sigma}) - \|\Phi\|\, e^{\beta\sigma}$. Therefore,

$$\|z_t\| \leqslant Le^{\alpha r} \|\Psi\| \, [U - Ve^{(\ln(Me^{\beta\sigma})/\sigma)t}] \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)| + N \sup_{j\sigma \leqslant s \leqslant t} |h(s)|$$

$$\leqslant [P - Qe^{-\gamma t}] \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|,$$

with

$$P = Le^{\alpha r} \|\Psi\| \, U + Ne^{\beta\sigma},$$

$$Q = Le^{\alpha r} \|\Psi\| \, V,$$

and

$$\gamma = -\frac{\ln(Me^{\beta\sigma})}{\sigma} < \alpha.$$

Hence,

$$\|y_t\| \leqslant L \|\Psi\| e^{-\alpha t} \|\phi\| + (P - Qe^{-\gamma t}) \sup_{0 \leqslant s \leqslant t} e^{-\beta(t-s)} |h(s)|.$$

This completes the proof.

Evidently, inequality (2.11) provides more information than inequality (2.5) regarding the dependence of the solution y_t to Eq. (2.3) on the non-homogeneous term $h(t)$. As we will see in next section, qualitative results can be obtained by employing (2.11) to infinite delay retarded equations; whereas, it is difficult, if not impossible, to obtain the same results by using (2.5). However, many qualitative results depend on the value of P in inequality (2.11) which usually is larger than p in inequality (2.5), and therefore qualitative results based on (2.5) often are more accurate than those based on (2.11).

3. REMARKS AND APPLICATIONS

As we will find, constants p in inequality (2.5) and P in inequality (2.11) are of paramount importance for qualitative analysis of the neutral equation (1.6). This particularly is true for applications of Liapunov–Razumikhin techniques to stability, boundedness, and invariance principles. In some simple cases, the calculations can be accomplished specifically. For example, considering the D-operator defined by $D\phi = \phi(0) - C\phi(-r)$, where C is an $n \times n$ constant matrix, we have the following two results.

CLAIM 3.1. *If $|C| < 1$, then $p = 1/(1 - |C|)$.*

CLAIM 3.2. *For any positive $\alpha < -\ln|C|/r$, (2.11) holds with $P = 1/(1 - |C|\,e^{\alpha r})$. Moreover, $Q = |C|\,e^{\alpha r}/(1 - |C|\,e^{\alpha r})$, and $Q, \beta > 0$, $\alpha > \gamma$ if $C \neq 0$.*

Proof of Claim 3.1. Let $(j-1)r \leqslant t < jr$ for some integer r. Repeating the equality
$$x(t) - Cx(t - r) = h(t)$$
at $t, t - r, \dots, t - (j-1)r$, we get

$$
\begin{aligned}
|x(t)| &\leqslant |C|\,|x(t-r)| + |h(t)| \\
&\leqslant |C|^2\,|x(t-2r)| + |C|\,|h(t-r)| + |h(t)| \\
&\leqslant |C|^3\,|x(t-3r)| + |C|^2\,|h(t-2r)| + |C|\,|h(t-r)| + |h(t)| \\
&\quad \cdots \\
&\leqslant |C|^j\,|x(t-jr)| + |C|^{j-1}\,|h(t-(j-1)r)| + |C|^{j-2}\,|h(t-(j-2)r)| \\
&\quad + \cdots + |C|\,|h(t-r)| + |h(t)| \\
&\leqslant |C|^j\,\|\phi\| + [\,|C|^{j-1} + |C|^{j-2} + \cdots + 1\,]\ \sup_{0 \leqslant s \leqslant t}\ |h(s)| \\
&\leqslant |C|^{t/r}\,\|\phi\| + \frac{1 - |C|^j}{1 - |C|}\ \sup_{0 \leqslant s \leqslant t}\ |h(s)| \\
&\leqslant e^{-t[-\ln|C|/r]}\,\|\phi\| + \left[\frac{1}{1 - |C|} - \frac{|C|}{1 - |C|^{[-\ln|C|/r]t}}\right]\ \sup_{0 \leqslant s \leqslant t}\ |h(s)|.
\end{aligned}
$$

Hence,
$$
b = 1, \qquad a = \frac{-\ln|C|}{r}, \qquad p = \frac{1}{1 - |C|}, \qquad q = \frac{|C|}{1 - |C|}.
$$

This completes the proof.

Proof of Claim 3.2. Let $(j-1)r \leqslant t < jr$ for an integer. Then we have

$$
\begin{aligned}
|x(t)| &\leqslant |C|^j\,|x(t-jr)| + \sum_{k=0}^{j-1} |C|^k\,|h(t-kr)| \\
&\leqslant |C|^{t/r}\,\|\phi\| + \sum_{k=0}^{j-1} (|C|\,e^{\alpha r})^k\ \sup_{0 \leqslant s \leqslant t}\ e^{-\alpha(t-s)}|h(s)| \\
&\leqslant e^{-(\ln|C|/r)t}\,\|\phi\| + \frac{1 - (|C|\,e^{\alpha r})^j}{1 - |C|\,e^{\alpha r}}\ \sup_{0 \leqslant s \leqslant t}\ e^{-\alpha(t-s)}|h(s)| \\
&\leqslant e^{-(\ln|C|/r)t}\,\|\phi\| + \frac{1 - (|C|\,e^{\alpha r}(|C|\,e^{\alpha r}))^{t/r}}{1 - |C|\,e^{\alpha r}}\ \sup_{0 \leqslant s \leqslant t}\ e^{-\alpha(t-s)}|h(s)|.
\end{aligned}
$$

This completes the proof.

Remark 3.1.   The above argument reveals an interesting fact about the classification of functional differential equations. Whether q is zero or not provides an essential characterization of retarded equations and neutral equations. In fact, if (1.6) is a retarded equation, then the exponential estimate (2.5) in Theorem 2.1 holds true only if $q = 0$ and (2.11) in Theorem 2.2 holds true only if $Q = 0$.

Motivated by this observation, we introduce the following concepts

DEFINITION 3.1.   The D-operator (or neutral equation (1.6)) is *irreducible* if (2.5) holds with $q > 0$.

DEFINITION 3.2.   The D-operator (or neutral equation (1.6)) is *strongly irreducible* if (2.11) holds with $Q > 0$.

In the qualitative analysis of neutral equations, the irreducibility assumption ($q \neq 0$ or $Q \neq 0$) plays an important role. To illustrate the significance of this irreducibility assumption, we present the following boundedness and stability result. For boundedness and stability definitions, we refer to [5].

THEOREM 3.1.   *Suppose D is irreducible and $\langle D\phi, f(t, \phi)\rangle \leqslant 0$ for all $(t, \phi) \in R \times C$ with $\|\phi\| \leqslant p \,|D\phi|$. Then solutions of (1.6) are uniformly bounded. Moreover, if $f(t, 0) = 0$, then the zero solution is uniformly stable.*

Proof.   Let $x(t) = x(t; \sigma, \phi)$ be a solution of (1.6), and set

$$M = \max\{\|D\| \,\|\phi\|, b \,\|\phi\|/q\}$$

and

$$W(t) = \max\{V(Dx_t), M^2\},$$

where $\|D\|$ is the norm of bounded linear operator $D: C \to C$ and $V(x) = |x|^2$ for all $x \in R^n$. Then we assert

$$|Dx_t| \leqslant M$$

for all $t \geqslant \sigma$. To prove this assertion, it suffices to prove $W(t) \leqslant W(\sigma)$ for all $t \geqslant \sigma$. By way of contradiction, if this is false, then by a standard comparison principle we can find a real number τ so that $W(s) \leqslant W(\tau)$ for all $\sigma \leqslant s \leqslant \tau$ with $D^+ W(\tau) > 0$. For this given τ, we claim that $V(Dx_\tau) \geqslant M^2$. Otherwise, $W(t) = M$ for all $t \in [\tau, \tau + h]$, where $h > 0$ is sufficiently small, and, thus, $D^+ W(\tau) = 0$, which is contrary to $D^+ W(\tau) > 0$. So $|Dx_\tau| \geqslant M$

and thus $W(\tau) = V(Dx_\tau)$. On the other hand, $W(s) \leqslant W(\tau)$ for $\sigma \leqslant s \leqslant \tau$ implies $|Dx_s| \leqslant |Dx_\tau|$ for $\sigma \leqslant s \leqslant \tau$. Now, find a sequence $t_n \to 0^+$ so that

$$D^+ W(\tau) = \lim_{n \to \infty} \frac{W(\tau + t_n) - W(\tau)}{t_n}$$

$$= \lim_{n \to \infty} \frac{W(\tau + t_n) - V(Dx_\tau)}{t_n}.$$

If there is a subsequence $\{t_{n_k}\} \subseteq t_n\}$ so that $W(\tau + t_{n_k}) = M$, then $|Dx_\tau| \geqslant M$ implies

$$D^+ W(\tau) = \lim_{n_k \to \infty} \frac{M - V(Dx_\tau)}{t_{n_k}} \leqslant \lim_{n_k \to \infty} \frac{M - M}{t_{n_k}} = 0$$

which is contrary to $D^+ W(\tau) > 0$. Therefore

$$W(\tau + t_n) = V(Dx_{\tau + t_n})$$

and

$$D^+ W(t) = \lim_{n \to \infty} \frac{V(Dx_{\tau + t_n}) - V(Dx_\tau)}{t_n} = \dot{V}_{(1.6)}(Dx_\tau).$$

This implies the existence of a $\tau \geqslant \sigma$ so that $|Dx_s| \leqslant |Dx_\tau| \leqslant M$ for $\sigma \leqslant s \leqslant \tau$, and $\dot{V}_{(1.6)}(Dx_\tau) > 0$. However, by (2.5), we have

$$\|x_\tau\| \leqslant b e^{-a(\tau - \sigma)} \|\phi\| + (p - q e^{-a(\tau - \sigma)}) |Dx_\tau| \leqslant p \|Dx_\tau\|. \qquad (3.1)$$

By our assumption, this implies $\dot{V}_{(1.6)}(Dx_\tau) \leqslant 0$, which yields a contradiction. Therefore, $|Dx_t| \leqslant M$ for all $t \geqslant \sigma$. From (2.5) it follows

$$\|x_t\| \leqslant b \|\phi\| + pM.$$

This completes the proof.

The next result can be proved in a similar fashion.

THEOREM 3.2. *Suppose that D is strongly irreducible and $\langle D\phi, f(t, \phi) \rangle \leqslant 0$ for all $(t, \phi) \in R \times C$ with $\|\phi\| \leqslant P |D\phi|$. Then all solutions of (1.6) are bounded. Moreover, the zero solution is uniformly stable if $f(t, 0) = 0$.*

Remark 3.2. In the argument of Theorem 3.1, a key step is the verification of (3.1) in which the irreducibility assumption $q \neq 0$ is crucial. Therefore, it is interesting to note that the argument can not be applied to retarded equations.

Remark 3.3.   Stability conditions in Theorem 3.1 and Theorem 3.2 depend on the values of p and P. As we noticed in Claim 3.1 and Claim 3.2, P usually is larger than p, and therefore stability results based on (2.5) are more precise than those on inequality (2.11).

As an illustration, consider the scalar equation

$$\frac{d}{dt}\left[x(t) - cx(t-r)\right] = -g(t, x(t), x(t-r)),\tag{3.2}$$

where $0 < c < 1$, $g: R \times R \times R \to R$ is continuous, and

   (i) $g(t, x, y) \leqslant 0$ for all x, y, $t \in R$ with $x \leqslant y$,

   (ii) $g(t, x, y) \geqslant 0$ for all x, y, $t \in R$ with $x \geqslant y$.

Obviously,

$$\langle D\phi, g(t, \phi(0), \phi(-r))\rangle = -\left[\phi(0) - c\phi(-r)\right]g(t, \phi(0), \phi(-r)).$$

According to Claim 3.1, $D\phi$ is irreducible with $p = 1/(1-c)$. Consider any given $\phi \in C$ with

$$|\phi(-r)| \leqslant \frac{|\phi(0) - c\phi(-r)|}{1-c}.$$

Four cases can occur:

Case 1. $\phi(-r) \geqslant 0$ and $\phi(0) - c\phi(-r) \geqslant 0$.
   In this case, $(1-c)\phi(-r) \leqslant \phi(0) - c\phi(-r)$ and thus $\phi(-r) \leqslant \phi(0)$. This implies $g(t, \phi(0), \phi(-r)) \geqslant 0$ and, thus,

$$\left[\phi(0) - c\phi(-r)\right]g(t, \phi(0), \phi(-r)) \geqslant 0.$$

Case 2. $\phi(-r) \geqslant 0$ and $\phi(0) - c\phi(-r) \leqslant 0$.
   In this case, $\phi(0) \leqslant c\phi(-r) \leqslant \phi(-r)$. This shows that $g(t, \phi(0), \phi(-r)) \leqslant 0$ and, thus,

$$\left[\phi(0) - c\phi(-r)\right]g(t, \phi(0), \phi(-r)) \geqslant 0.$$

Case 3. $\phi(-r) \leqslant 0$ and $\phi(0) - c\phi(-r) \geqslant 0$.
   In this case, $\phi(0) \geqslant c\phi(-r) \geqslant \phi(-r)$. This shows that $g(t, \phi(0), \phi(-r)) \geqslant 0$ and thus

$$\left[\phi(0) - c\phi(-r)\right]g(t, \phi(0), \phi(-r)) \geqslant 0.$$

Case 4. $\phi(-r) \leqslant 0$ and $\phi(0) - c\phi(-r) \leqslant 0$.

In this case, $-\phi(-r) \leqslant -[\phi(0) - c\phi(-r)]/(1-c)$; that is, $(1-c)\phi(-r)$ $\geqslant \phi(0) - c\phi(-r)$. This implies $\phi(-r) \geqslant \phi(0)$, and thus $g(t, \phi(0), \phi(-r)) \leqslant 0$ which shows

$$[\phi(0) - c\phi(-r)] \, g(t, \phi(0), \phi(-r)) \geqslant 0.$$

In summary, for any $\phi \in C$ with $\|\phi\| \leqslant |\phi(0) - c\phi(-r)|/(1-c)$, we have $\langle D\phi, g(t, \phi(0), \phi(-r)) \rangle \leqslant 0$. Therefore all solutions are uniformly bounded, and each constant solution (that is, the zero solution of the family of equations $(d/dt)[x(t) - cx(t-r)] = -g(t, e + x(t), e + x(t-r))$ for any $e \in R$) is uniformly stable.

Remark 3.4. For the above example, the constant P in (2.11) is $1/(1 - ce^{\alpha r})$ for any given constant $\alpha > 0$ (see Claim 3.2). Therefore $\|\phi\| \leqslant P\,|D\phi|$ does not imply necessarily $[\phi(0) - c\phi(-r)]\,g(t, \phi(0), \phi(-r)) \geqslant 0$. This indicates that certain qualitative results based on (2.11) are not as accurate as those based on inequality (2.5). However, inequality (2.11) does have certain merits over inequality (2.5). Particularly, some qualitative results like an invariance principle and convergence theorems can be established in a similar way to retarded equations by employing (2.11). The details will be given in a forthcoming paper [8].

Remark 3.5. Based on inequality (1.3) one can also derive some stability theorems of Liapunov–Razumikhin type as Lopes did. These theorems essentially require the nonnegative property of $\langle D\phi, f(t, \phi) \rangle \leqslant 0$ for all $\phi \in C$ subject to $\|\phi\| \leqslant (\alpha/(1-c))\,|D\phi|$ ($\alpha > 1$ given). Unfortunately, $\|\phi\| \leqslant (\alpha/(1-c))\,|D\phi|$ does not necessarily imply $\langle D\phi, f(t, \phi) \rangle \leqslant 0$. Therefore these stability theorems do not apply to such neutral equations as (1.2) in which the ordinary part does not dominate the functional part.

To conclude this paper, we state two asymptotic stability theorems of Liapunov–Razumikhin type. The basic idea for the proof is contained in Haddock and Wu [4] and Wu [7].

THEOREM 3.3. *Suppose $f(t, 0) = 0$, the D-operator is irreducible (strongly irreducible), and, for each pair (u, v) with $0 < u \leqslant v \leqslant \infty$, one can find positive constants $\varepsilon = \varepsilon(u, v)$ and $w = w(u, v)$ such that*

$$\langle D\phi, f(t, \phi) \rangle \leqslant -\varepsilon$$

whenever $(t, \phi) \in R \times C$, $u \leqslant |D(t, \phi)| \leqslant v$, and $\|\phi\| \leqslant p\,|D(\phi)| + \omega$ (or $\|\phi\| \leqslant P\,|D(\phi)| + w$). Then the zero solution is uniformly asymptotically stable.

As a simple consequence of this theorem, we get the following result which is a generalization of Liapunov–Razumikhin type asymptotic stability to neutral equations and a reformulation of Lopes' theorem [6].

THEOREM 3.4. *Let $f(t, 0) = 0$, the D-operator be irreducible (or strong irreducible). Suppose that there exists a function $F: [0, \infty) \to [0, \infty)$ such that $F(s) > s$ for all $s > 0$, and that*

$$\langle D\phi, f(t, \phi) \rangle \leq -\omega(|D(\phi)|)$$

whenever $\|\phi\| \leq F(p |D(\phi)|)$ (or $\|\phi\| \leq F(P | D(\phi)|)$), where ω is an increasing continuous function on $[0, \infty)$ with $\omega(0) = 0$. Then the zero solution is uniformly asymptotically stable.

Applying Theorem 3.3 to Eq. (3.2), we get the following sufficient condition to guarantee uniform asymptotic stability:

$$\inf \left\{ (x - cy) \, g(t, x, y); u \leq |x - cy| \leq v, \max\{|x|, |y|\} \leq \frac{|x - cy|}{1 - c} \right\} > 0$$

for any $v \geq u > 0$.

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