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LERAY-SCHAUDER DEGREE FOR SEMILINEAR FREDHOLM MAPS AND PERIODIC BOUNDARY VALUE PROBLEMS OF NEUTRAL EQUATIONS

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1. INTRODUCTION

In a series of his papers (cf. [9–10, 26–29]), Mawhin developed a coincidence degree theory for perturbations of a linear Fredholm operator of index zero. This coincidence degree theory turned out to be a very powerful and useful tool in the study of the existence of solutions for nonlinear boundary value problems.

In [15], Hale and Mawhin applied the coincidence degree to the periodic boundary value problem of some neutral functional differential equations. Their method uses a continuation principle based on the homotopy invariance of the coincidence degree. Let us recall this property. Let X and Y be two real Banach spaces and let $L\colon \mathrm{Dom}(L) \subseteq X \to Y$ be a linear (possibly unbounded) Fredholm operator of index zero. Suppose that $\Omega \subseteq X$ is an open bounded subset and $H\colon \bar{\Omega} \times [0, 1] \to Y$ is a homotopy of L-compact mappings (see [27, 29] for precise definitions) such that $L(x) \neq H(x, \lambda)$ for all $x \in \partial\Omega \cap \mathrm{Dom}(L)$ and $\lambda \in [0, 1]$. Then the coincidence degree $d[(L, H(\cdot, \lambda)), \Omega]$ is a constant function with respect to $\lambda \in [0, 1]$.

In this paper we study the periodic boundary value problem of a neutral functional differential equation of the following type

$$\begin{cases} \dfrac{d}{dt}[x(t) - c(t)x(t-r)] = f(t, x_t) \\ x(0) = x(w) \end{cases} \tag{1.1}$$

where $C\colon R \to R$ is a continuous w-periodic function and $f\colon R \times C([-r, 0], R) \to R$ is a completely continuous mapping w-periodic with respect to the first variable.

In our approach we reduce the problem (1.1) to the, relatively simpler, periodic boundary value problem for the following retarded equation of the type

$$\begin{cases} \dfrac{d}{dt}x(t) = f(t, x_t) \\ x(0) = x(w). \end{cases} \tag{1.2}$$

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The simplest way for such a reduction is through the following natural deformation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - \lambda c(t)x(t-r)] = f(t, x_t), & \lambda \in [0, 1] \\ x(0) = x(w). \end{cases} \tag{1.1$_\lambda$}$$

However, since the deformation $(1.1)_\lambda$ involves considerable modifications of the linear part, representing the Fredholm operator L, the homotopy invariance property of the coincidence degree, in the above formulation, is not sufficient for our purposes. In order to use such deformations we need more general homotopy invariance, allowing "continuous" deformations of the Fredholm operator L as well as deformations of the perturbing map.

On the other hand, in [7, 8] Fitzpatrick and Pejsachowicz developed a degree theory for continuous semilinear Fredholm maps of index zero. The advantage of their approach lies in the fact that their generalizations of the Leray-Schauder degree does not depend on the representation of a semilinear Fredholm map $f(x)$ in the form $f(x) = Lx - F(x)$, where L is a continuous Fredholm operator and F is compact. Another important advantage of this degree theory consists in the homotopy invariance property which permits also for continuous deformations of the linear part. Since the coincidence degree of Mawhin is defined for a more general (unbounded) Fredholm operator of index zero it is an interesting question if it is possible to obtain a similar homotopy invariance for the coincidence degree in this case.

In the first part of this paper, we apply the Fitzpatrick–Pejsachowicz degree theory in order to obtain a variant of the homotopy invariance property, allowing deformations of the linear part, for the coincidence degree of Mawhin. In the second part, we apply this homotopy invariance property to provide a natural and unified way of extending the existence results established for the periodic boundary value problems of certain retarded equations to neutral equations.

This paper is organized as follows. In Section 2, we present a brief description of the Fitzpatrick–Pejsachowicz degree theory for semilinear continuous Fredholm maps of index zero and next we indicate how its homotopy invariance implies the forementioned homotopy invariance of the coincidence degree of Mawhin. A continuous deformation of a closed unbounded Fredholm operators of index zero is a continuous deformations of their graphs with respect to an appropriate topology. Presentation of a parametrized family of unbounded Fredholm operators as a morphism from certain vector bundle modelled on a Banach space into a Banach space permits us to simplify the presentation and, with the use of some homotopy arguments, to describe the direct link of coincidence degree with the setting of Fitzpatrick–Pejsachowicz. The proof of this extension is surprisingly simple but nevertheless, as it is presented in the Section 3, this result can produce some interesting application. Namely, the existence of *a priori* bounds for the deformation $(1.1)_\lambda$ implies, by the homotopy invariance, that the coincidence degree for the both ends of this deformation are, up to sign, the same. Consequently, the existence results for the neutral equation (1.1) follow from their counterparts for the retarded equation (1.2). In the Section 4, we develop the method of guiding functions and the Liapunov–Razumikhin technique which are consequently applied to establish the existence of *a priori* bounds. The obtained results extend those due to Krasnosel'skii [18, 19], Gaines and Mawhin [9] for ordinary differential equations, and to Gustafson and Schmitt [11], Hetzer [16, 17] and Mawhin [26, 29] for retarded equations to neutral equations.

Some examples are given to illustrate possible applications of our results.

2. FITZPATRICK-PEJSACHOWICZ DEGREE THEORY AND HOMOTOPY INVARIANCE OF COINCIDENCE DEGREE

We start with a brief description of the Fitzpatrick-Pejsachowicz degree theory (cf. [7, 8, 32]).

A continuous map $f: X \to Y$, from a Banach space X into a Banach space Y, is called a *semilinear Fredholm map* of index zero if f can be represented in the form $f(x) = L(x) - F(x)$, $x \in X$, where L is a bounded linear Fredholm operator of index zero and F is a completely continuous map. For given Banach spaces X and Y, let $\mathcal{L}(X, Y)$ denote the Banach space of bounded linear operators from X into Y, $\mathcal{K}(X, Y)$ and $GL(X, Y)$ denote the subsets of $\mathcal{L}(X, Y)$ consisting of all compact operators and all isomorphisms, respectively. $GL_c(X)$ will denote the group of all isomorphisms of the form $Id - K$ with $K \in \mathcal{K}(X, X)$. An orientation is a function $\varepsilon: GL(X, Y) \to \{-1, 1\}$ satisfying the following properties:

(i) If $M_1, M_2 \in GL(X, Y)$ with $M_1 - M_2 \in \mathcal{K}(X, Y)$, then $\varepsilon(M_1) = \varepsilon(M_2)$ if and only if $M_1^{-1}M_2 \in GL_c^+(X)$, where $GL_c^+ = \{\Phi \in GL_c X); \deg_{L.S.}(\Phi, B, 0) = 1\}$ and $\deg_{L.S.}(\Phi, B, 0)$ is the Leray-Schauder degree of Φ with respect to a ball B, centred at the origin and of positive radius;

(ii) If $X = Y$, then $\varepsilon(Id) = 1$.

Such an orientation can be constructed in the following way: the group $GL_c(X)$ acts on $GL(X, Y)$ by multiplication on the right. By choosing a representation M_α in each orbit α we obtain the orientation function $\varepsilon: GL(X, Y) \to \{-1, 1\} = Z_2$ by

$$\varepsilon(M) = \deg_{L.S.}(M_\alpha^{-1}M, B, 0) \qquad \text{if } M \in \alpha,$$

where the right side is the Leray-Schauder degree of $M_\alpha^{-1}M \in GL_c(X)$ with respect to any ball of positive radius.

It is well known that for a bounded Fredholm operator of index zero $L: X \to Y$ there exists a $K \in \mathcal{K}(X, Y)$ such that $L + K \in GL(X, Y)$. Therefore, a semilinear Fredholm map of index zero having the representation $f(x) = L(x) - F(x)$ can also be represented by $f(x) = (L + K)(x) - (F + K)(x)$, and the problem of solving the equation $f(x) = 0$ is equivalent to finding a zero of the compact vector field $Id - (L + K)^{-1}(F + K)$.

Now we are able to present the definition of the Fitzpatrick-Pejsachowicz degree. Suppose that $f: X \to Y$ is a semilinear Fredholm map of index zero having the representation $f(x) = L(x) - F(x)$, $\Omega \subseteq X$ is an open bounded subset and $L(x) \neq F(x)$ for all $x \in \partial\Omega$. Let ε be an orientation function on $GL(X, Y)$ and let $K \in \mathcal{K}(X, Y)$ be such that $L + K \in GL(X, Y)$. Then the *Fitzpatrick-Pejsachowicz* degree of f on Ω is defined by

$$\deg_{F.P.}(f, \Omega, 0) := \varepsilon(L + K) \deg_{L.S.}(Id - (L + K)^{-1}(F + K), \Omega, 0). \tag{2.1}$$

It was shown in [7, 8] that $\deg_{F.P.}(f, \Omega, 0)$ is a well-defined function of f and that the additivity, excision and normalization axioms of degree theory are satisfied.

In order to present the homotopy invariance property, the following concept of semilinear Fredholm homotopy is introduced. A *semilinear Fredholm homotopy* is a map $H: [0, 1] \times X \to Y$ having a representation $H(\lambda, x) = L(\lambda, x) - F(\lambda, x)$, $\lambda \in [0, 1]$, where the map $\lambda \to L(\lambda, \cdot)$ is continuous from $[0, 1]$ to $\mathcal{L}(X, Y)$, $L(\lambda, \cdot)$ is a bounded Fredholm operator of index zero for all $\lambda \in [0, 1]$, and $F: [0, 1] \times X \to Y$ is completely continuous. When $\Omega \subseteq X$ is an open bounded subset, H is said to be *admissible* with respect to Ω if $L(\lambda, x) \neq F(\lambda, x)$ for all $(\lambda, x) \in [0, 1] \times \partial\Omega$. The *homotopy invariance property* of the Fitzpatrick-Pejsachowicz degree is the following.

THEOREM 2.1. Suppose that $\Omega \subseteq X$ is an open bounded set and that $H: [0, 1] \times X \to Y$ is a semilinear homotopy which is admissible with respect to Ω. Then

$$\deg_{F.P.}(H_1, \Omega, 0) = \varepsilon(L_0 + K_0)\varepsilon(L_1 + K_1)\deg_{F.P.}(H_0, \Omega, 0)$$

where $H_\lambda := H(\lambda, \cdot)$ has the semilinear representation $H_\lambda = L_\lambda - F_\lambda$ with L_λ being a bounded Fredholm operator of index zero for all $\lambda \in [0, 1]$, F_λ being completely continuous, $K_\lambda \in \mathcal{K}(X, Y)$ and $L_\lambda + K_\lambda \in GL(X, Y)$ for all $\lambda \in [0, 1]$. The number $\delta(H) = \varepsilon(L_0 + K_0)\varepsilon(L_1 + K_1) \in Z_2$ depends only on the homotopy H and on ε and is independent of the semilinear representation $H_\lambda = L_\lambda - F_\lambda$.

Let us present now the coincidence degree of Mawhin. Let $L: \mathrm{Dom}(L) \subseteq X \to Y$ be a Fredholm operator (possibly unbounded) of index zero. A *resolvent* of L is an $K \in \mathcal{K}(X, Y)$ such that $L + K$ is a one-to-one mapping onto Y. The set of all resolvents of L is denoted by $CR(L)$. Let us fix a resolvent $K \in CR(L)$ and let $\Omega \subseteq X$ be an open bounded subset. A map $F: \bar{\Omega} \to Y$ is called *L-compact* if $(L + K)^{-1}F: \bar{\Omega} \to X$ is compact. Let $F: \bar{\Omega} \to Y$ be an L-compact map such that $L(x) \neq F(x)$ for all $x \in \mathrm{Dom}(L) \cap \partial\Omega$, then the *coincidence degree* of L with F on Ω is defined by

$$d[(L, F), \Omega] := \deg_{L.S.}(Id - (L + K)^{-1}(F + K), \Omega, 0).$$

If L is a bounded Fredholm operator of index zero, then L-compact maps are exactly compact maps and it follows from (2.1) that

$$d[(L, F), \Omega] = \pm\deg_{F.P.}(f, \Omega, 0) \tag{2.2}$$

where $f(x) = L(x) - F(x)$, $x \in \bar{\Omega}$. By theorem 2.1 and (2.2), we have the following *homotopy invariance property* for the coincidence degree.

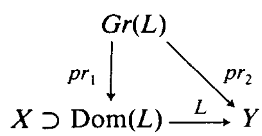
COROLLARY 2.1. Suppose that L_λ, $\lambda \in [0, 1]$, is a continuous family of bounded Fredholm operators of index zero. Let $\Omega \subseteq X$ be an open bounded set and let $H: [0, 1] \times \bar{\Omega} \to Y$ be a compact map such that $L_\lambda(x) \neq H(\lambda, x)$ for all $(\lambda, x) \in [0, 1] \times \partial\Omega$. Then

$$d[(L_0, H_0), \Omega] = \pm d[(L_1, H_1), \Omega]$$

where $H_\lambda := H(\lambda, \cdot)$ for $\lambda \in [0, 1]$.

The advantage of the Fitzpatrick–Pejsachowicz degree lies in the fact that it does not depend on the representation of the map $f(x) = L(x) - F(x)$. However, in spite of the fact that in the case of semilinear Fredholm maps, the coincidence degree is expressed by the same number (up to the sign) as the Fitzpatrick–Pejsachowicz degree, it encompasses the more general situation of unbounded Fredholm operators. The aim of this section is to present the homotopy invariance property (admitting deformations of the operator L) for the coincidence degree of Mawhin in this general case.

Let $L: \mathrm{Dom}(L) \subset X \to Y$ be a closed Fredholm operator of index zero. We denote by $Gr(L) = \{(x, y) \in X \times Y;\ x \in \mathrm{Dom}(L),\ L(x) = y\}$ the graph of L. By assumption, $Gr(L)$ is a closed subspace of $X \times Y$ and we have the following commutative diagram:



where pr_1 and pr_2 are (continuous) projections on the first and the second component respectively.

LEMMA 2.1. Let $L\colon \mathrm{Dom}(L) \subset X \to Y$ be a closed linear operator. L is a Fredholm operator of index zero if and only if there exist two closed subspaces $X^0 \subset X$ and $Y_0 \subset Y$ such that codim $X^0 = \dim Y_0 < \infty$ and $Gr(L) \oplus (X^0 \times Y_0) = X \times Y$.

Proof. Suppose that L is a Fredholm operator of index zero. Then $Im(L)$ is a closed subspace of Y and codim $Im(L) = \dim \ker(L) < \infty$. There exists $X^0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\ker L \oplus X^0 = X$ and $Im(L) \oplus Y_0 = Y$. It is easy to prove that $Gr(L) \cap (X^0 \times Y_0) = \{(0, 0)\}$. Let $(x, y) \in X \times Y$, then $(x, y) = (x_0 + x', L(x_0 + x')) + (x^0 - x', y^0)$, where $x = x_0 + x^0 \in \ker L \oplus X^0$, $y = y^0 + y_0 \in Im(L) \oplus Y_0$ and $x' \in x^0 \cap \mathrm{Dom}(L)$ is such that $L(x') = y^0$. Consequently $Gr(L) \oplus (X^0 \times Y_0) = X \times Y$. Conversely, if L is a closed operator such that $Gr(L) \oplus (X^0 \times Y_0) = X \times Y$, where $\dim Y_0 = \operatorname{codim} X^0 < \infty$, then $\ker L \cap X^0 = \{0\}$ and $\ker L + X^0 = X$, thus $\ker L \oplus X^0 = X$. On the other hand it can be verified that $Im(L) \oplus Y_0 = Y$ and thus $Im(L)$ is a closed subspace of finite codimension. Consequently L is a Fredholm operator of index zero. ■

Remark 2.1. Let $L\colon \mathrm{Dom}(L) \subset X \to Y$ be a closed Fredholm operator of index zero. Then $pr_2\colon Gr(L) \to Y$ is a bounded Fredholm operator of index zero. Since $pr_1\colon Gr(L) \to \mathrm{Dom}(L)$ is one-to-one and onto, the subspace $\mathrm{Dom}(L)$ can be equipped with a new norm $\|\cdot\|_L$, called the *graph norm*, as follows

$$\|x\|_L := \|(pr_1)^{-1}(x)\| = \|(x, Lx)\| = \|x\| + \|Lx\|.$$

The space $\mathrm{Dom}(L)$ equipped with $\|\cdot\|_L$ will be denoted by X_L. It is clear that X_L is a Banach space and $L\colon X_L \to Y$ is a continuous Fredholm operator of index zero.

We denote by $\mathrm{Sub}(X \times Y)$ the set of all closed subspaces of $X \times Y$. $\mathrm{Sub}(X \times Y)$ can be equipped with the following metric function:

$$\mathrm{dist}(V_1, V_2) = d(B(V_1), B(V_2)),\ V_1, V_2 \in \mathrm{Sub}(X \times Y)$$

where $B(V_1)$, $B(V_2)$ denote the closed unit balls in V_1 and V_2 respectively and $d(\cdot, \cdot)$ is the Hausdorff metric on bounded subsets of $X \times Y$. Let $O_p(X \times Y) \subseteq \mathrm{Sub}(X \times Y)$ denote the subspace of all graphs of closed linear operators from X into Y.

Definition 2.1. Let P be a topological space. A family $\{L_\lambda\}_{\lambda \in P}$ of closed linear operators from X into Y is called *continuous family of operators* parametrized by P if the mapping $\varphi\colon P \to O_p(X \times Y)$, $\varphi(\lambda) = Gr(L_\lambda)$ is continuous.

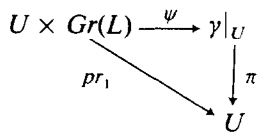
Let $\mathfrak{F}_0(X \times Y)$ be the subset of $O_p(X \times Y)$ of all graphs of closed Fredholm operators of index zero. It follows from lemma 2.1 that $\mathfrak{F}_0(X \times Y)$ is open in $O_p(X \times Y)$. Let $\{L_\lambda\}_{\lambda \in P}$ be a continuous family of Fredholm operators of index zero parametrized by P and let $\varphi\colon P \to \mathfrak{F}_0(X, Y)$ be defined by $\varphi(\lambda) = Gr(L_\lambda)$. By definition, φ is continuous. We define

$$\gamma := \{(\lambda, x, y) \in P \times X \times Y;\ (x, y) \in \varphi(\lambda)\}$$
$$= \{(\lambda, x, y) \in P \times X \times Y;\ x \in \mathrm{Dom}(L_\lambda),\ y = L_\lambda(x)\}$$

and put $\pi\colon \gamma \to P$, $\pi(\lambda, x, y) = \lambda$.

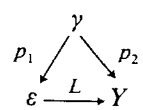
LEMMA 2.2. Under the above assumption, the map $\pi: \gamma \to P$ is a locally trivial Banach vector bundle. For every $\lambda \in P$, the fiber over λ is given by $Gr(L_\lambda) = \varphi(\lambda)$.

Proof. Let $\lambda_0 \in P$ and let $Gr(L) = \varphi(\lambda_0)$ be the fiber over λ_0. By lemma 2.1 there exist subspaces $X^0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\operatorname{codim} X^0 = \dim Y_0 < \infty$ and $Gr(L) \oplus (X^0 \times Y_0) = X \times Y$. By continuity of $\varphi: P \to \mathfrak{F}_0(X \times Y)$, there is a neighborhood U of λ_0 such that $Gr(L_\lambda) \oplus (X^0 \times Y_0) = X \times Y$ for all $\lambda \in U$, and thus $P_\lambda := P|_{Gr(L_\lambda)}: Gr(L_\lambda) \to Gr(L)$, where P is the projection onto $Gr(L)$ associated with the direct decomposition $Gr(L) \oplus (X^0 \times Y_0) = X \times Y$, is an isomorphism. Therefore we have the following commutative diagram:



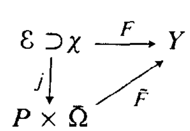
where $\psi(\lambda, (x, y)) = (\lambda, P_\lambda^{-1}(x, y))$ for $(\lambda, (x, y)) \in U \times Gr(L)$. The map ψ defines the required local trivilization of the bundle γ over U. ■

Let $\mathcal{E} := \{(\lambda, x) \in P \times X;\ x \in \operatorname{Dom}(L_\lambda)\}$ and let $p_1: \gamma \to \mathcal{E}$ be given by, $p_1(\lambda, x, y) = (\lambda, x)$ for all $(\lambda, x, y) \in \gamma$. By remark 2.1, for every $\lambda \in P$, $pr_1: Gr(L_\lambda) \to X_{L_\lambda} =: X_\lambda$ is an isometry, therefore the mapping $p_1: \gamma \to \mathcal{E}$ gives us the identification of the bundle γ with $\mathcal{E}$. Under this identification, $\pi: \mathcal{E} \to P$, $\pi(\lambda, x) = \lambda$, is a Banach vector bundle whose fiber over $\lambda \in P$ is exactly the space $X_\lambda := X_{L_\lambda}$. On the other hand, by the same remark 2.1, the projection $pr_2: Gr(L_\lambda) \to Y$ induces the vector bundle morphism $p_2: \gamma \to Y$, $p_2(\lambda, x, y) = y$ for all $(\lambda, x, y) \in \gamma$. We put $L := p_2 \circ (p_1)^{-1}$, i.e. we have the following commutative diagram of vector bundle morphisms



where $L(\lambda, x) = L_\lambda(x)$ for all $(\lambda, x) \in \mathcal{E}$. This implies that the restriction of L to the fibre X_λ over $\lambda \in P$ is exactly the bounded Fredholm operator of index zero $L_\lambda: X_\lambda \to Y$. Assume that P is a compact connected space. It is well known that if P is contractible or if $\mathcal{E}$ is modelled on Kuiper space X_0 (see [31] for examples and properties of Kuiper spaces), then there is a vector bundle isomorphism $\Psi: P \times X_0 \to \mathcal{E}$, where X_0 is the typical fibre of $\mathcal{E}$. The composition $\tilde{L} = L\Psi: P \times X_0 \to Y$ is exactly the continuous family of Fredholm operators of index zero considered in the setting of the Fitzpatrick–Pejsachowicz degree theory.

A continuous map $F: \mathcal{E} \to Y$ can be considered as a parametrized continuous family $\{F_\lambda\}_{\lambda \in P}$ of perturbations of L. The map F does not need to be defined on the whole space $\mathcal{E}$. Let $\Omega \subseteq X$ be an open and bounded set and let $\tilde{F}: P \times \bar{\Omega} \to Y$ be a mapping. The vector bundle morphism $j: \mathcal{E} \to P \times X$ induced by the imbeddings $j_\lambda: X_\lambda \to X$, i.e. $j(\lambda, x) = (\lambda, j_\lambda(x))$, is a continuous map, thus $\chi = j^{-1}(P \times \bar{\Omega})$ is a closed set and we have the following diagram:



If the mapping $\tilde{F}$ is continuous, then F is also continuous. It is clear that for every $\lambda \in P$, F_λ is compact if and only if $\tilde{F}_\lambda$ is L_λ-compact. Therefore in the case where $j_\lambda: X_\lambda \to X$ is compact for all $\lambda \in P$, the continuity of $\tilde{F}$ implies also the compactness of F.

Definition 2.2. Let P be a compact space and let $j: \mathcal{E} \to P \times X$ be as above. A mapping $\tilde{F}: P \times \bar{\Omega} \to Y$ is called *L-compact* if the composition $F := \tilde{F} \circ j: \chi \to Y$ is a continuous compact mapping. $\tilde{F}$ is called an *admissible perturbation* of L if $\tilde{F}$ is L-compact and $L_\lambda \neq \tilde{F}(\lambda, x)$ for all $x \in \mathrm{Dom}(L_\lambda) \cap \partial\Omega$ and $\lambda \in [0, 1]$, where L is defined by a continuous family $\{L_\lambda\}_{\lambda \in P}$ of closed Fredholm operators of index zero.

THEOREM 2.2. Let $P = [0, 1]$, $\Omega \subseteq X$ be an open bounded set, and let $\tilde{F}$: $[0, 1] \times \bar{\Omega} \to Y$ be an admissible perturbation of L, where L is defined by a continuous family $\{L_\lambda\}_{\lambda \in [0,1]}$ of closed Fredholm operators of index zero. Then the following homotopy invariance property for the coincidence degree holds

$$d[(L_0, \tilde{F}_0), \Omega] = \pm d[(L_1, \tilde{F}_1), \Omega].$$

Remark 2.2. Using the fact that the vector bundle $\pi: \mathcal{E} \to [0, 1]$ determined by the family $\{L_\lambda\}_{\lambda \in [0,1]}$ is trivial, one could expect that the above homotopy invariance is an immediate consequence of theorem 2.1. However, there are some problems. Firstly, since the imbeddings $j_\lambda: X_\lambda \to X$, in general, are not isomorphisms but sometimes are compact linear operators, the set $\chi = j^{-1}(P \times \bar{\Omega})$ can be unbounded and therefore, the usual definition of the degree can not be applied. Secondly, by using a trivialization Ψ: $[0, 1] \times X_1 \xrightarrow{\cong} \mathcal{E}$ of the bundle $\mathcal{E}$, we obtain a parametrized family of continuous Fredholm operators of index zero from X_1 to Y and a parametrized family $\{F_\lambda\}_{\lambda \in [0,1]}$ of compact perturbations. Because the trivialization Ψ does not necessarily transform the set χ into a product $[0, 1] \times \bar{\Omega}$, a problem arises. However, this problem can be overcome by using the excision property of the degree. Before presenting our proof, we should mention that the possibly unbounded regions (as in the case of χ) whose intersections with finite dimensional spaces are bounded were studied by Ma in [25]. This approach was applied in [5] for the construction of the coincidence degree theory on vector bundles for weakly upper semicontinuous, weakly completely continuous multivalued perturbations of a continuous family of Fredholm operators of index zero parametrized by P.

Proof of theorem 2.2. Put $\chi = j^{-1}([0, 1] \times \Omega)$, $\mathfrak{A} = j^{-1}([0, 1] \times \partial\Omega)$ and define the class $\mathcal{K}_L(\chi, \mathfrak{A})$ of all compact maps $T: \chi \to Y$ such that $L(\lambda, x) \neq T(\lambda, x)$ for all $(\lambda, x) \in \mathfrak{A}$ and there is a continuous map $\tilde{T}$: $[0, 1] \times \bar{\Omega} \to Y$ such that the following diagram

$$\begin{array}{ccc} \chi & \xrightarrow{T} & Y \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \tilde{T}} & \\ [0, 1] \times \bar{\Omega} & & \end{array}$$

commutes. By $\Phi_L(\chi, \mathfrak{A})$ we denote the subset of $\mathcal{K}_L(\chi, \mathfrak{A})$ consisting of all *finite dimensional* maps $T: \chi \to Y$, i.e. such that there is a continuous compact map $\tilde{T}$: $[0, 1] \times \bar{\Omega} \to Y$ such that $\tilde{T}([0, 1] \times \bar{\Omega})$ is contained in a finite dimensional subspace of Y and the diagram

$$\begin{array}{ccc} \chi & \xrightarrow{T} & Y \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \tilde{T}} & \\ [0, 1] \times \bar{\Omega} & & \end{array}$$

commutes. The classes $\mathcal{K}_L(\chi, \mathfrak{A})$ and $\Phi_L(\chi, \mathfrak{A})$ can be equipped with $\mathfrak{A}$-noncoincidental homotopy relation (see [5]). The corresponding sets of equivalence classes are denoted by $\mathcal{K}_L[\chi, \mathfrak{A}]$ and $\Phi_L[\chi, \mathfrak{A}]$, respectively. It can be verified in standard way (compare [5, lemma 3.10]) that the inclusion $i: \Phi_L(\chi, \mathfrak{A}) \hookrightarrow \mathcal{K}_L(\chi, \mathfrak{A})$ induces a bijection

$$i_*: \Phi_L[\chi, \mathfrak{A}] \stackrel{\cong}{\to} \mathcal{K}_L[\chi, \mathfrak{A}].$$

For $\Omega \subseteq X$ a bounded open subset, put $\chi_\lambda := j^{-1}(\{\lambda\} \times \bar{\Omega})$ and $\mathfrak{A}_\lambda := j^{-1}(\{\lambda\} \times \partial\Omega)$. An admissible perturbation $\tilde{F}$ determines an element $F = \tilde{F} \circ j: \chi \to Y$ of $\mathcal{K}_L(\chi, \mathfrak{A})$. Since i_* is a bijection, F is $\mathfrak{A}$-noncoincidentally homotopic to a finite dimensional map $F^0 \in \Phi_L(\chi, \mathfrak{A})$. This yields the commutative diagram:

$$\begin{array}{ccc} (\chi, \mathfrak{A}) & \xrightarrow{F^0} & Y \\ {\scriptstyle j} \searrow & & \nearrow {\scriptstyle \tilde{F}} \\ & (\bar{\Omega}, \partial\Omega) & \end{array}$$

Let $\tilde{K}_\lambda \in CR(L_\lambda)$ be such that $\dim K_\lambda < \infty$. Then the equation

$$L_\lambda(x) = F_\lambda^0(x), \qquad x \in \chi_\lambda$$

is equivalent to the following fixed point problem

$$x = (L_\lambda + \tilde{K}_\lambda \circ j)^{-1}(F_\lambda^0 + \tilde{K}_\lambda)j_\lambda(x), \qquad x \in \chi_\lambda.$$

Let $Y_* \subset Y$ be a finite dimensional subspace such that $\tilde{F}_\lambda^0(\chi) \subset Y_*$ and $Im\, K_\lambda \subset Y_*$. Put $X_* = (L_\lambda + \tilde{K}_\lambda \circ j_\lambda)^{-1}(Y_*)$. The coincidence degree of L_λ with $\tilde{F}_\lambda$ on Ω can be defined by

$$d[(L_\lambda, \tilde{F}_\lambda), \Omega] = d[(L_\lambda, \tilde{F}_\lambda^0), \Omega] := \deg_{\mathrm{L.S.}}((L_\lambda + \tilde{K}_\lambda^0 \circ j_\lambda)^{-1}(\tilde{F}_\lambda^0 + K_\lambda)|_{X^*}, \chi \cap X_*, 0) \quad (2.3)$$

where $\chi \cap X_*$ is now a bounded set. In this definition the sign of $d[\cdot, \cdot]$ depends on the choice of the resolvent K_λ. By the construction and the excision properties of the coincidence degree of Mawhin, the above constructed coincidence degree coincides with the coincidence degree of Mawhin. Let us remark that the resolvent $\tilde{K}_\lambda$ is also a resolvent for $L_{\tilde{\lambda}}$, where $\tilde{\lambda}$ belongs to some open subinterval U containing λ. By using a trivilization over U and the finite dimensional reduction $\mathcal{E}_*|_U := \{(\tilde{\lambda}, x) \in \mathcal{E}|_U;\ x \in (L_{\tilde{\lambda}} + \tilde{K}_{\tilde{\lambda}})^{-1}(Y_*)\}$ we reduce the problem to a finite dimensional subbundle $\mathcal{E}_*|_U$ of the bundle $\mathcal{E}|_U$. Since $\Omega \cap (L_{\tilde{\lambda}} + \tilde{K}_{\tilde{\lambda}})^{-1}(Y_*)$ is an open bounded subset, it follows from theorem 2.1 that $|d[(L_{\tilde{\lambda}}, \tilde{F}_{\tilde{\lambda}},), \Omega]|$ is a constant function on U. Consequently, by the compactness of $[0, 1]$, the statement of theorem 2.2 follows. ■

Remark 2.3. We remark that, by using the language of bundles, we can extend the notion of an admissible perturbation of L to more general classes of mappings. For example, we can study L-condensing perturbations, or multivalued perturbations satisfying additional conditions (cf. [5]). The coincidence degree theory of Mawhin for those classes of perturbations were studied in [20, 21]. Using the standard approach, it can be proved that in all mentioned cases the homotopy invariance property, admitting continuous deformations of the linear part, is still valid.

3. A REDUCTION THEOREM FOR PERIODIC BOUNDARY VALUE PROBLEMS OF NEUTRAL EQUATIONS

Let $C([a, b], R^n)$ be the space of continuous functions from $[a, b]$ to R^n with the topology of uniform convergence. For a fixed $r \geq 0$, let $C = C([-r, 0], R^n)$ with norm $\|\varphi\| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$ for $\varphi \in C$. If $x \in C([\sigma - r, \sigma + \delta], R^n)$ for some $\delta > 0$, then $x_t \in C$, $t \in [\sigma, \sigma + \delta]$, is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Suppose $w > 0$ is fixed, $A: R \times C \to R^n$ is continuous, $A(t + w, \varphi) = A(t, \varphi)$, $A(t, \varphi)$ is linear in φ and there exists a continuous function $v: [0, \infty) \to R$, $v(0) = 0$, such that $|A(t, \varphi^s)| \leq v(s)\|\varphi^s\|$ for $0 \leq s \leq r$, $t \in R$ and for all functions $\varphi^s \in C$ such that $\varphi^s(\theta) = 0$ for $\theta \in [-r, -s]$. Let $D: R \times C \to R^n$ be defined by $D(t)\varphi = \varphi(0) - A(t, \varphi)$. The operator D is said to be *stable* if the zero solution of the functional equation $D(t)y_t = 0$ is *uniformly asymptotically stable*, that is, there are constants K, $\alpha > 0$ such that if $y(\varphi)$ is the solution of $D(t)y_t = 0$ with $y_0 = \varphi$, then

$$\|y_t(\varphi)\| \leq K e^{-\alpha t}\|\varphi\| \qquad \text{for } t \geq 0 \qquad \text{and} \qquad \varphi \in C.$$

Let $P_w = \{h \in C(R, R^n);\ h(t + w) = h(t) \text{ for } t \in R\}$ and $H_w = \{H \in C(R, R^n);\ H(0) = 0,\ H(t) = \alpha t + h(t) \text{ for some } \alpha \in R^n \text{ and } h \in P_w\}$. For any $h \in P_w$, let $|h| = \sup_{0 \leq t \leq w} |h(t)|$, and for any $H \in H_w$, $H(t) = \alpha t + h(t)$, $\alpha \in R^n$, $h \in P_w$, let $|H| = |\alpha| + |h|$. Then P_w and H_w are Banach spaces.

We consider the neutral functional differential equation

$$\frac{d}{dt} D(t)x_t = f(t, x_t) \tag{3.1}$$

where D is defined above and $f: R \times C \to R^n$ is completely continuous and w-periodic in the first argument. Let $L: P_w \to H_u$ be the continuous linear mapping defined by

$$Lx(t) = D(t)x_t - D(0)x_0, \qquad t \in R$$

and $G: P_w \to H_w$ be defined by

$$Gx(t) = \int_0^t f(s, x_s)\, ds, \qquad t \in R.$$

Then finding w-periodic solutions of equation (3.1) is equivalent to solving the operator equation

$$Lx = Gx \qquad \text{in } P_w.$$

The Fredholm alternative theory (cf. [13]) of the equation $D(t)x_t = H(t)$ for $H \in H_w$ implies that L is a continuous Fredholm operator of index zero, and an immediate application of the Arzela-Ascoli theorem shows that G is completely continuous (cf. [15]). Therefore, the coincidence degree $d[(L, G), \Omega]$ is well defined for any open bounded set $\Omega \subseteq P_w$ such that $0 \notin (L - G)(\partial\Omega)$. The following result, due to Hale and Mawhin [15], is an immediate consequence of the existence property of coincidence degree.

THEOREM 3.1. If there exists a bounded open set $\Omega \subseteq P_w$ whose boundary $\partial\Omega$ contains no w-periodic solution of the equation (3.1) and if the coincidence degree $d[(L, G), \Omega] \neq 0$, then equation (3.1) has at least one w-periodic solution.

Application of this general result requires solving two difficult problems—finding *a priori* bounds and estimating $d[(L, G), \Omega]$. The major purpose of this section is to reduce the estimation of $d[(L, G), \Omega]$ to the existence problem of periodic solutions to the retarded equation

$$\frac{d}{dt} x(t) = f(t, x_t) \tag{3.2}$$

by applying the homotopy invariance property in the previous section to the following family of equations

$$\frac{d}{dt}[x(t) - \lambda A(t, x_t)] = f(t, x_t), \qquad \lambda \in [0, 1]. \tag{3.3$_\lambda$}$$

More generally, we consider the following family of neutral equations

$$\frac{d}{dt}[x(t) - J(\lambda, t)x_t] = g(t, x_t, \lambda), \qquad \lambda \in [0, 1] \tag{3.4$_\lambda$}$$

where we assume that

(H1) $J: [0, 1] \times R \to \mathcal{L}(C, R^n)$ is continuous;

(H2) $g: R \times C \times [0, 1] \to R^n$ is completely continuous;

(H3) $J(\lambda, t + w)\varphi = J(\lambda, t)\varphi$, $g(t, \varphi, \lambda) = g(t + w, \varphi, \lambda)$ for $(t, \lambda, \varphi) \in R \times [0, 1] \times C$;

(H4) for any fixed $\lambda \in [0, 1]$, the operator $D_\lambda: R \times C \to R^n$ defined by $D_\lambda(t, \varphi) = \varphi(0) - J(\lambda, t)\varphi$ is stable, and there exists a continuous function $\nu_\lambda: [0, \infty) \to K$, $\nu_\lambda(0) = 0$ such that $|J(\lambda, t)\varphi^s| \leq \nu_\lambda(s)\|\varphi^s\|$ for $0 \leq s \leq r$, $t \in R$ and $\varphi^s \in C$ such that $\varphi^s(\theta) = 0$ on $[-r, -s]$;

(H5) $J(1, t)\varphi = A(t, \varphi)$ and $g(t, \varphi, 1) = f(t, \varphi)$ for $(t, \varphi) \in R \times C$.

As a consequence of theorem 2.2 and 3.1, we have the following.

THEOREM 3.2. Assume (H1)-(H5) hold and there exists an open bounded set $\Omega \subseteq P_w$ whose boundary $\partial\Omega$ contains no w-periodic solution of the equation (3.4)$_\lambda$ for $\lambda \in [0, 1]$ and that $d[(L_0, G_0), \Omega] \neq 0$, where L_0 and $G_0: P_w \to H_w$ are defined by $L_0x(t) = x(t) - A(0, t)x_t - [x(0) - A(0, 0)x_0]$ and $G_0x(t) = \int_0^t g(s, x_s, 0)\,ds$. Then there exists at least one ω-periodic solution to the equation (3.1).

Proof. Consider the maps $\tilde{F}: [0, 1] \times P_w \to H_w$ and $L: [0, 1] \times P_w \to H_w$ defined by

$$\tilde{F}(\lambda, x)(t) = \int_0^t g(s, x_s, \lambda)\,ds, \qquad x \in P_w, \qquad t \in R, \qquad \lambda \in [0, 1]$$

and

$$L(\lambda, x)(t) = x(t) - J(\lambda, t)x_t - [x(0) - J(\lambda, 0)x_0], \qquad x \in P_w, \qquad t \in R, \qquad \lambda \in [0, 1].$$

By (H1) and (H4), $\{L_\lambda\}_{\lambda \in [0,1]}$ is a continuous Fredholm operator of index zero. Assumption (H2) guarantees that $\tilde{F}: [0, 1] \times P_w \to H_w$ is completely continuous, and thus $\tilde{F}|_{[0,1] \times \bar{\Omega}}: [0, 1] \times \bar{\Omega} \to H_w$ is a compact map. Since there is no w-periodic solution to (3.4)$_\lambda$ for $\lambda \in [0, 1]$ on $\partial\Omega$, $L(\lambda, x) \neq \tilde{F}(\lambda, x)$ for $(\lambda, x) \in [0, 1] \times \partial\Omega$. Therefore $\tilde{F}$ is an admissible perturbation of $\{L_\lambda\}_{\lambda \in [0,1]}$. By theorem 2.2 (or even corollary 2.1),

$$|d[(L_0, \tilde{F}_0), \Omega]| = |d[(L_1, \tilde{F}_1), \Omega]|,$$

that is, $|d[(L, G), \Omega]| = |d[(L_0, G_0), \Omega]|$. Therefore by the existence theorem of coincidence degree, if $\deg[(L_0, G_0), \Omega] \neq 0$, then there exists at least one w-periodic solution to the equation (3.1). This completes the proof. ∎

Let $J(\lambda, t)\varphi = \lambda A(t, \varphi)$ and $g(t, \varphi, \lambda) = f(t, \varphi)$ in equation (3.4)$_\lambda$ we get the following result by which we can reduce the problem of the solvability of periodic boundary value problems for neutral equations to a corresponding problem for retarded equations.

COROLLARY 3.1. Assume that

(H6) for any $\lambda \in [0, 1]$, the operator $D_\lambda: R \times C \to R^n$ defined by $D_\lambda(t, \varphi) = \varphi(0) - \lambda A(t, \varphi)$ is stable,

and that there exists an open boundary set $\Omega \subseteq P_w$ whose boundary $\partial\Omega$ contains no w-periodic solution of the equation $(3.3)_\lambda$ for $\lambda \in [0, 1]$ and $d[(L_0, G_0), \Omega] \neq 0$, where L_0 and $G_0: P_w \to H_w$ are defined by $L_0 x(t) = x(t) - x(0)$ and $G_0 x(t) = \int_0^t f(s, x_s)\, ds$. Then there exists at least one w-periodic solution to the neutral equation (3.1).

Remark 3.1. (H6) is a physically meaningful assumption. To illustrate this point, we consider the following D-operator defined by

$$D(t, \varphi) = \varphi(0) - \sum_{i=1}^{m} B_i(t)\varphi(-r_i) - \int_{-r}^{0} g(t, \theta)\varphi(\theta)\, d\theta$$

where $t \in R$, $\varphi \in C$, $r_i > 0$, $B_i: R \to R^{n\times n}$, $i = 1, \ldots, m$, and $g: R \times [-r, 0] \to R^{n\times n}$ are continuous and w-periodic in the first argument. It can be shown that the D-operator is stable if

$$\sum_{i=1}^{m} |B_i(t)| + \int_{-r}^{0} |g(t, \theta| \, d\theta < 1, \qquad t \in R \tag{3.5}$$

(cf. [2, 23, 24, 30]). Although it is possible to obtain some more general sufficient conditions guaranteeing the stability of the D-operator, Melvin [30] proved that (3.5) is the most physically meaningful condition for the stability of D-operator in the sense that D-operator preserves the stability under small perturbation of r_i. It is easy to verify that if (3.5) holds, then $D_\lambda: R \times C \to R^n$ defined by $D_\lambda(t, \varphi) = \varphi(0) - \lambda\sum_{i=1}^{m} B_i(t)\varphi(-r_i) - \lambda\int_{-r}^{0} g(t, \theta)\varphi(\theta)\, d\theta$ is stable for $\lambda \in [0, 1]$.

Remark 3.2. Reducing the solvability problem of the periodic boundary value problem for the neutral equation (2.1) to the problem of estimating the degree $d[(L_0, G_0), \Omega]$ is significant because there have been various results developed for the estimation of the coincidence degree $d[(L_0, G_0), \Omega]$ associated with the existence of periodic solutions to the retarded equation (3.2). To illustrate this significance, we present the following result.

COROLLARY 3.2. Assume (H6) holds, and there exist a constant $\rho > 0$ and a completely continuous map $g: R \times C \times [0, 1] \to R^n$ which is w-periodic in the first argument, $g(t, \varphi, 1) = f(t, \varphi)$ for $(t, \varphi) \in R \times C$ and such that

(i) $\partial B(\rho)$ contains no w-periodic solution of the equation $(3.3)_\lambda$, where $B(\rho) = \{x \in P_\omega;\ |x| < \rho\}$;

(ii) $\partial B(\rho)$ contains no w-periodic solution of the equation $\dot{x} = \lambda g(t, x_t, \lambda)$ for $\lambda \in [0, 1]$;

(iii) $\partial B_{R^n}(\rho)$ contains no zero of the mapping $g_0: R^n \to R^n$ defined by $g_0(a) := 1/w\int_0^w g(t, \hat{a}, 0)\, dt$, where $B_{R^n}(\rho)$ denotes the open ball centred at 0 of R^n and of radius ρ, $\hat{a}$ denotes a constant mapping in C with the value $a \in R^n$;

(iv) the Brouwer degree $d(g_0(a), \overline{B_{R^n}(\rho)}, 0) \neq 0$.

Then there exists at least one w-periodic solution to the neutral equation (3.1).

Proof. According to the proof of [26, theorem 4], $d([L_0, G_0], B(\rho)) = d(g_0(a), \overline{B_{R^n}(\rho)}, 0)$ if the assumption (ii) and (iii) hold. Therefore by corollary 3.1, if (i) and (iv) hold then there exists at least one w-periodic solution to the equation (3.1).

Remark 3.3. In [15], Hale and Mawhin proved a similar result replacing $g_0(a)$ by $1/w\int_0^w g(t, (Ma)_t, 0)\,dt$, where Ma is the unique solution of the functional equation $D(t)x_t = a$. The advantage of our result is obvious since it is usually difficult, if not impossible, to find an explicit expression for Ma.

Remark 3.4. The proof of corollary 3.2 indicates that once an existence result is proved for a retarded equation by using the coincidence degree theoretical method, and *a priori* bounds are established for neutral equations, this existence result holds automatically for neutral equations. More results can be obtained in this way. We list in the following some of these results for the convenience of application.

COROLLARY 3.3. Assume that (H6) holds and there exists an open bounded set $\Omega \subseteq P_w$ symmetric with respect to the origin, containing it and such that $\partial\Omega$ contains no w-periodic solution of each equation

$$\frac{d}{dt}D_\lambda(t)x_t = g(t, x_t, \lambda), \qquad \lambda \in [0, 1],$$

where $g: R \times C \times [0, 1] \to R^n$ is completely continuous, w-periodic in the first argument, $g(t, \varphi, 0) = -g(t, -\varphi, 0)$ and $g(t, \varphi, 1) = f(t, \varphi)$ for $(t, \varphi) \in R \times C$. Then equation (3.1) has at least one w-periodic solution in Ω.

This is an immediate consequence of corollary 3.2 and [26, theorem 3]. We should mention that this result was established in [15] by using an extension of the Borsuk theorem given in [27, theorem 7.2].

COROLLARY 3.4. Assume (H6) holds, and there exists an open bounded set $G \subseteq R^n$ such that

(i) $\partial\tilde{G}$ contains no w-periodic solution of the equation $(3.1)_\lambda$ for $\lambda \in [0, 1]$, where $\tilde{G} = \{x \in P_w;\ x(t) \in G \text{ for } t \in R\}$;

(ii) for each $u \in \partial G$ there exists a $V_u \in C^1(R^n, R)$ such that $V_u(u) = 0$, $G \subseteq \{v \in R^n;\ V_u(v) < 0\}$ and for any $x \in P_w$ with $x([0, w]) \subseteq \bar{G}$, at any $t \in R$ with $x(t) = u$, one has $(\text{grad } V_u, f(t, x_t)) \neq 0$.

Then there exists at least one w-periodic solution to the neutral equation (3.1) if the Brouwer degree $d(\bar{f}, G, 0) \neq 0$, where $\bar{f}(u) = 1/w\int_0^w f(s, \hat{u})\,ds$ and $\hat{u}$ is a constant function on R^n with value $u \in R^n$.

This is a consequence of corollary 3.1 and a corresponding existence result for retarded equation (cf. [29, theorem V II.9]). We note that the existence of such a set G can be guaranteed by a guiding function. For details, we refer to [29] and the next section.

Finally, we point out that the solvability of periodic boundary value problems of neutral equations can also be reduced to the corresponding problem of an ordinary differential equation in some cases by using corollary 3.1 or theorem 3.2. For example, let $J(\lambda, t)\varphi = \lambda A(t, \varphi)$ and $g(t, \varphi, \lambda) = (1 - \lambda)h(t, \varphi(0)) + \lambda f(t, \varphi)$ in theorem 3.2, and we get the following.

COROLLARY 3.5. Assume (H6) holds, and there exists an open bounded set $\Omega \subseteq P_w$ whose boundary $\partial\Omega$ contains no w-periodic solution to the equation

$$\frac{d}{dt}[x(t) - \lambda A(t, x_t)] = (1 - \lambda)h(t, x(t)) + \lambda f(t, x_t), \qquad \lambda \in [0, 1],$$

where $h: R \times R^n \to R^n$ is continuous, and w-periodic in the first argument. If $d[(L_0, G_0], \Omega] \neq 0$, where $L_0x(t) = x(t) - x(0)$ and $G_0x(t) = \int_0^t h(s, x(s))\,ds$ Then there exists at least one w-periodic solution to the neutral equation (3.1).

Note that $L_0x = G_0x$, in the above result, is equivalent to the ordinary differential equation $\dot{x} = h(t, x)$. Therefore there are lots of results about the estimation of $d[(L_0, G_0), \Omega]$. For details, we refer to [10, 29, 41].

4. METHOD OF GUIDING FUNCTIONS AND LIAPUNOV-RAZUMIKHIN TECHNIQUE FOR *A PRIORI* BOUNDS

In this section, we develop a generalization of the basic theorem of the method of guiding functions and the Liapunov–Razumikhin technique to provide *a priori* bounds for the periodic boundary value problem of neutral equations.

The following definition is a generalization to neutral equations of the guiding function introduced and developed by Krasnosel'skii and others (see [9, 18, 19] and references therein) for ordinary differential equations, by Mawhin, Hetzer, Gustafson, Schmitt and others (see [9, 10, 11, 16, 17, 29] and references therein) for retarded equations.

Definition 4.1. A continuously differentiable function $V: R^n \to [0, \infty)$ is said to be a *guiding function* for the periodic boundary value problem of equation (3.1), if there exists a constant $\rho > 0$ such that

$$\langle \operatorname{grad} V(D_\lambda(t, x_t)), f(t, x_t)\rangle < 0$$

where $x \in P_w$, $\lambda \in [0, 1]$, and $t \in R$ is such that $|D_\lambda(t, x_t)| \geq \rho$ and $V(D_\lambda(s, x_s)) \leq V(D_\lambda(t, x_t))$ for all $s \in R$, where $D_\lambda(t, \varphi) = \varphi(0) - \lambda A(t, \varphi)$.

According to the definition, we have the following simple criterion for a function to be a guiding function.

LEMMA 4.1. A continuous differentiable function $V: R^n \to [0, \infty)$ is a guiding function for the periodic boundary value problem of neutral equation (3.1) if there exists a constant $\rho > 0$ and a continuous function $P: [0, \infty) \to [0, \infty)$ such that:

(i) for any given $h > 0$, $\lambda \in [0, 1]$ and any $x \in P_w$, if $V(D_\lambda(t, x_t)) \leq h$ for $t \in R$, then $|x(t)| \leq P(h)$ for $t \in R$;

(ii) $\langle \operatorname{grad} V(D_\lambda(t, x_t)), f(t, x_t)\rangle < 0$ for every $x \in P_w$ and $t \in R$ with $|D_\lambda(t, x_t)| \geq \rho$ and $\|x_t\| \leq P(V(D_\lambda(t, x_t)))$.

Example 4.1. Consider the following neutral equation

$$\frac{d}{dt}\left[x(t) - \int_0^r d\eta(\theta)x(t-\theta)\right] = f(t, x_t) \tag{4.1}$$

where η is an $n \times n$ matrix function whose entries are of bounded variation on $[0, r]$ and $\text{Var}_{[0,r]} < 1$. We claim that the function $V: R^n \to [0, \infty)$ defined by $V(x) = \langle x, x\rangle$ is a guiding function for the periodic boundary value problem of neutral equation (3.1) if there exists a constant $\rho > 0$ such that $\langle D_\lambda(x_t), f(t, x_t)\rangle < 0$ for every $x \in P_w$ and $t \in R$ with $|D_\lambda(x_t)| \geq \rho$ and $\|x_t\| \leq |D_\lambda(x_t)|/(1 - \text{Var}_{[0,r]}\eta)$, where $D_\lambda(\varphi) = \varphi(0) - \lambda\int_0^r d\eta(\theta)\varphi(-\theta)$.

Proof. Denote by BC the space of bounded continuous functions on R with the norm $\|\varphi\|_{BC} = \sup_{s \in R} |\varphi(s)|$, and by E the set of the $n \times n$ matrix measure ξ satisfying

$$\|\xi\|_0 = \int_{-\infty}^{+\infty} \mathrm{d}|\xi|(t) < \infty.$$

For $\xi \in E$, we define the operator $\xi * : BC \to BC$ by

$$\xi * \varphi(t) = \int_{-\infty}^{+\infty} [\mathrm{d}\xi(s)]\varphi(t - s) \qquad \text{for } t \in R.$$

Let δ denote the measure

$$\delta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ Id & \text{if } t > 0 \end{cases}$$

and extend η to an $n \times n$ matrix measure on $(-\infty, +\infty)$ by defining $\eta(t) = 0$ for $t \leq 0$ and $\eta(t) = \eta(r)$ for $t \geq r$, then

$$D_\lambda \varphi = \mu_\lambda * \varphi(0)$$

with

$$\mu_\lambda = \delta - \lambda\eta.$$

In [37], Staffans proved that $\mu_\lambda *$ continuously maps BC into itself, it is invertible and its inverse operator $\mu_\lambda^{-1} * \delta$ is also continuous. Moreover, $\mu_\lambda^{-1} * \mu_\lambda = \mu_\lambda * \mu_\lambda^{-1} = \delta$ and $\mu_\lambda, \mu_\lambda^{-1} \in E$. Using these notations, the equation $D_\lambda(x_t) = g(t)$ for $g \in BC$ can be solved and $x(t) = (\mu_\lambda^{-1} * g)(t)$. Noting that $(\delta - \lambda\eta)^{-1}$ can be expressed by the series $\delta + \lambda\eta + \lambda^2\eta * \eta + \lambda^3\eta * \eta * \eta + \cdots$ which is uniformly convergent for $\lambda \in [0, 1]$, we see that if $g \in P_w$, then

$$\begin{aligned} |x(t)| &\leq (1 + \lambda\|\eta\|_0 + \lambda^2\|\eta_0\|^2 + \cdots) \max_{t \in [0,w]} |g(t)| \\ &\leq \frac{\max_{t \in [0,w]}|g(t)|}{1 - \|\eta\|_0} \\ &\leq \frac{\max_{t \in [0,w]}|g(t)|}{1 - \mathrm{Var}_{[0,r]}\eta} \end{aligned}$$

which implies that if $\langle D_\lambda(x_t), D_\lambda(x_t)\rangle \leq h$ for $t \in R$, then $|x(t)| \leq \sqrt{h}/(1 - \mathrm{Var}_{[0,r]}\eta)$. Therefore our conclusion follows trivially from lemma 4.1.

Example 4.2. Consider the following neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - B(t)x(t - r)] = f(t, x_t) \tag{4.2}$$

where $B: R \to R^{n \times n}$ is a w-periodic and continuous map and $|B(t)| \leq k < 1$ for $t \in R$ and for a constant k. We claim that the function $V: R^n \to [0, \infty)$ defined by $V(x) = \langle x, x\rangle$ is a guiding function for the periodic boundary value problem of equation (4.2) if there exists a constant $\rho > 0$ such that

$$\langle x(t) - \lambda B(t)x(t - r), f(t, x_t)\rangle < 0$$

for every $x \in P_w$, $\lambda \in [0, 1]$, $t \in R$ with $|x(t) - \lambda B(t)x(t - r)| \geq \rho$ and

$$\|x_t\| \leq \frac{|x(t) - \lambda B(t)x(t - r)|}{1 - k}.$$

Proof. It is easy to verify that for any $g \in BC$ the functional equation $x(t) - \lambda B(t)x(t-r) = g(t)$ has a solution

$$x(t) = \sum_{i=1}^{\infty} \lambda^i \prod_{j=0}^{i-1} B(t-jr)g(t-ir) + g(t).$$

Therefore if $|g(t)| \le h$ for $t \in R$, then $|x(t)| \le [\sum_{i=1}^{\infty} \lambda^i k^i + 1] \le h/(1-k)$. Our conclusion then follows from lemma 4.1.

The method of guiding functions and Liapunov-Razumikhin technique have been used for the study of stability of both retarded equations and neutral equations (cf. [12, 14, 23, 24, 35, 36, 40]) and for the study of *a priori* bounds of retarded equations (cf. [10, 11, 16, 17, 22, 29]). The following result is a generalization of these results to *a priori* bound estimate of periodic solutions of neutral equations.

LEMMA 4.2. If there exists a guiding function $V: R^n \to [0, \infty)$ for the periodic boundary value problem of equation (3.1) such that

$$\limsup_{|x| \to \infty} \sup_{t \in \mathbb{R}} \inf_{\lambda \in [0,1]} V(D_\lambda(t, x_t)) = \infty \qquad \text{for } x \in P_w, \tag{4.3}$$

then there exists a constant $\rho^* > 0$ such that any w-periodic solution to the equation $(3.3)_\lambda$, $\lambda \in [0, 1]$, satisfies $|x(t)| < \rho^*$ for $t \in R$.

Proof. Let $x(t)$ be a w-periodic solution to equation $(3.3)_\lambda$ for some $\lambda \in [0, 1]$. Then $V(D_\lambda(t, x_t))$ is also a w-periodic function, and thus there exists $\tau > 0$ such that $V(D_\lambda(\tau, x_\tau)) = \max_{t \in [0,w]} V(D_\lambda(t, x_t))$ and $\langle \text{grad } V(D_\lambda(\tau, x_\tau)), f(\tau, x_\tau)\rangle = 0$. Therefore by the definition of a guiding function, $|D_\lambda(\tau, x_\tau)| < \rho$. This implies that $V(D_\lambda(t, x_t)) \le \max_{|z| \le \rho} V(z)$ from which we obtain $|x(t)| < \rho^*$, where $\rho^* > 0$ is a given constant such that for any $x \in P_\omega$ and $\lambda \in [0, 1]$, if $|x| \ge \rho^*$, then $V(D_\lambda(t, x_t)) > \max_{|z| \le \rho} V(z)$ for some $t \in [0, \omega]$.

We are now in the position to state our major result in this section.

THEOREM 4.1. Assume (H6) holds and suppose that there exists a guiding function $V: R^n \to [0, \infty)$ for the periodic boundary value problem of equation (3.1) such that (4.3) holds. If the Brouwer degree $d(\text{grad } V, G, 0) \ne 0$, where $G = \{u \in R^n; |V(u)| < \nu\}$ and ν is a constant such that $\nu > \max_{|u| \le \rho^*} |V(u)|$, then there exists at least one w-periodic solution of the equation (3.1).

Proof. Let $\lambda = 0$ in the definition of a guiding function, we get $\langle \text{grad } V(x(t)), f(t, x_t)\rangle < 0$ for every $x \in P_\omega$ $t \in R$ with $|x(t)| \ge \rho$ and $V(x(s)) \le V(x(t))$ for $s \in R$. It is easy, from the definition of a guiding function, to verify that $G = \{v \in R^n; |V(v)| < \nu\}$ satisfies (ii) in corollary 3.4, and $\langle \text{grad } V(u), \tilde{f}(u)\rangle < 0$ for every $u \in R^n$ with $|u| \ge \rho$. Therefore by the generalized Poincaré-Bohl theorem (see [29, proposition II.9]), $d\langle \text{grad } V, G, 0) = d(\tilde{f}, G, 0) \ne 0$. Moreover by lemma 4.2, for any possible w-periodic solution $x(t)$ to the equation $(3.3)_\lambda$, $\lambda \in [0, 1]$, $|x(t)| < \rho^*$ and thus $|V(x(t))| < \nu$ which implies that $x \notin \partial\tilde{G}$, where $\tilde{G} := \{x \in P_w; x(t) \in G$ for $t \in R\}$. Therefore by corollary 3.4, there exists at least one w-periodic solution to the equation (3.1). ■

For illustrative purposes, we consider the following equation

$$\frac{d}{dt}\left[x(t) - \int_0^r d\eta(\theta)x(t-\theta)\right] = Ax(t) + f(t, x_t) \tag{4.4}$$

where

(i) η is an $n \times n$ matrix function whose entries are of bounded variation on $[0, r]$ and $\mathrm{Var}_{[0,r]}\eta < \frac{1}{2}$;

(ii) there exists a constant $\rho_1 > 0$ such that

$$1 - 2\left[|A|\mathrm{Var}_{[0,r]}\eta + \sup_{|\varphi(0)| \geq \rho_1} \frac{|f(t,\varphi)|}{\|\varphi\|}\right][1 - \mathrm{Var}_{[0,r]}\eta]^{-1} > 0;$$

(iii) $A^T + A = -Id$.

COROLLARY 4.1. Equation (4.4) has at least one w-periodic solution.

Proof. By remark 3.1, (H6) holds with $D_\lambda(\varphi) = \varphi(0) - \lambda\int_0^r d\eta(\theta)\varphi(-\theta)$. Let $V(x) = \langle x, x\rangle$. It is easy to prove that for any $x \in P_\omega$ and $\lambda \in [0, 1]$, if $|x(t^*)| = \max_{t\in R}|x(t)|$ for some $t^* \in R$, then

$$|D_\lambda(x_{t^*})| = \left|x(t^*) - \lambda\int_0^r d\eta(\theta)x(t^* - \theta)\right| \geq [1 - \mathrm{Var}_{[0,r]}\eta]|x(t^*)|.$$

This implies that

$$\lim_{|x|\to\infty}\ \inf_{\lambda\in[0,1]}\ \sup_{t\in[0,w]} |D_\lambda(x_t)| = \infty,$$

and therefore, V satisfies (4.3).

Let $\rho = [1 - \mathrm{Var}_{[0,r]}\eta \cdot (1 - \mathrm{Var}_{[0,1]}\eta)^{-1}]^{-1}\rho_1$. Suppose $x \in P_w$ and $t \in R$ are such that $|D_\lambda(x_t)| \geq \rho$ and $|D_\lambda(x_s)| \leq |D_\lambda(x_t)|$ for $s \in R$. Then putting $H(t) = D_\lambda(x_t)$, we get

$$x(t) = [\delta - \lambda\eta]^{-1} * H(t) = H(t) + \lambda\eta * H(t) + \lambda^2\eta * \eta * H(t) + \cdots$$

from which we have

$$|x(t)| \geq |H(t)| - \max_{s\in[0,\omega]}|H(s)|\mathrm{Var}_{[0,r]}\eta(1 - \mathrm{Var}_{[0,r]}\eta)^{-1}$$
$$= [1 - \mathrm{Var}_{[0,r]}\eta(1 - \mathrm{Var}_{[0,r]}\eta)^{-1}]|H(t)| \geq \rho_1$$

and

$$|x(t)| \leq |H(t)| + \max_{s\in[0,\omega]}|H(s)|\mathrm{Var}_{[0,r]}\eta(1 - \mathrm{Var}_{[0,r]}\eta)^{-1}$$
$$\leq (1 - \mathrm{Var}_{[0,r]}\eta)^{-1}|H(t)| \leq (1 - \mathrm{Var}_{[0,r]}\eta)^{-1}|D_\lambda(x_t)|.$$

Therefore

$$\langle \mathrm{grad}\, V(D_\lambda(x_t)), f(t, x_t) + Ax(t)\rangle$$
$$= -\langle D_\lambda(x_t), D_\lambda(x_t)\rangle + 2\langle D_\lambda(x_t), f(t, x_t)\rangle + 2\langle D_\lambda(x_t), \lambda A\int_0^r d\eta(\theta)x(t-\theta)\rangle$$
$$\leq -|D_\lambda(x_t)|^2 + 2\left[|A|\mathrm{Var}_{[0,r]}\eta + \frac{|f(t,x_t)|}{\|x_t\|}\right]\|x_t\|\,|D_\lambda(x_t)|$$
$$\leq -|D_\lambda(x_t)|^2 + 2\left[|A|\mathrm{Var}_{[0,r]}\eta + \sup_{|\varphi(0)|\geq\rho_1}\frac{|f(t,\varphi)|}{\|\varphi\|}\right][1 - \mathrm{Var}_{[0,r]}\eta]^{-1}|D_\lambda(x_t)|^2 < 0.$$

Hence, V is a guiding function for the periodic boundary value problem of equation (4.4). Since the Brouwer degree

$$d(\text{grad } V, B_{R^n}(v), 0) = (-1)^n \neq 0$$

for any $v > 0$, by theorem 4.1, there exists at least one w-periodic solution.

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