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ASYMPTOTIC EQUIVALENCE OF NEUTRAL AND INFINITE RETARDED DIFFERENTIAL EQUATIONS

J. R. HADDOCK
Department of Mathematical Sciences, Memphis State University, Memphis, Tennessee 38152, U.S.A.

T. KRISZTIN*
Bolyai Institute, Aradi Vértanúk Tere 1, 6720 Szeged, Hungary

and

JIANHONG WU†
Institute of Applied Mathematics, Hunan University, Changsha, Hunan, P. R. of China

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1. INTRODUCTION

IN [7] STAFFANS demonstrates that a neutral functional differential equation (FDE) with stable D-operator often can be treated as a retarded FDE with infinite delay. The main purpose of this paper is to illustrate how Staffans' ideas can be employed effectively to study the asymptotic behavior of solutions of certain nonlinear, nonautonomous neutral FDEs. In particular, we consider the scalar neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - cx(t-r)] = F[t, x(t), x(t-r)](t \geq 0), \tag{1}$$

where

(i) $0 \leq c < 1$, $r > 0$,

(ii) $F : [0, \infty) \times R^2 \to R$ is continuous,

(iii) $x \geq y$ implies $F(t, x, y) \leq 0$,

(iv) $x \leq y$ implies $F(t, x, y) \geq 0$,

(v) for any compact interval I of R there exists $L = L(I) \geq 0$ such that

$$|F(t, x, y)| \leq L|x - y| \qquad (x, y \in I,\ t \geq 0).$$

The above assumptions imply that each constant function is a solution of (1), and, once the appropriate setting is established, we will prove that (i)–(v) are sufficient to guarantee that each

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solution of (1) tends to a constant as $t \to +\infty$. Among the consequences will be a partial answer to a conjecture in [1], but in a more general framework than considered in [5]. Likewise, the results given here are intended to supplement the convergence theorems in [4], where the setting was for autonomous retarded ($c = 0$) FDEs. So, for certain cases, our results generalize those given in [4].

It is standard to consider equation (1) in the state space C_r (the space of continuous functions on $[-r, 0]$ with norm $\|\phi\| = \max_{-r \le s \le 0}|\phi(s)|$). Existence, uniqueness and continuation of bounded solutions of (1) can be shown easily (see [2]). In Section 2 we prove the boundedness of solutions of (1) under conditions (i)–(v). Thus, we may assume that for any $\phi \in C_r$ the solution of (1) through $(0, \phi)$ exists on $[0, \infty)$. For our purpose the space

$$C_\gamma = \left\{\phi \in C((-\infty, 0], R): \lim_{s \to -\infty} e^{\gamma s}\phi(s) = 0\right\}$$

with $\gamma > 0$ such that $ce^{\gamma s} < 1$ and norm $\|\phi\|_\gamma = \sup_{s \le 0} e^{\gamma s}|\phi(s)|$ has more advantages. The initial function $\phi \in C_r$ can be extended to $(-\infty, 0]$ such that $\phi \in C_\gamma$. So, we assume that $\phi \in C_\gamma$ is given and consider a solution (1) through $(0, \phi)$ on the interval $[0, \infty)$. By introducing the natural transformation

$$y(t) = x(t) - cx(t - r) \qquad (t \in R), \tag{2}$$

the retarded equations with infinite delay

$$y'(t) = F\left(t, y(t) + \sum_{i=1}^{\infty} c^i y(t - ir), \sum_{i=1}^{\infty} c^{i-1} y(t - ir)\right) \qquad (t \ge 0) \tag{3}$$

can be obtained for y (see [6] and Section 3 of this paper for details). The initial function belonging to y at $t = 0$ given by (2) for $t \ge 0$ is an element of C_γ. Since C_γ satisfies the basic local state space axioms of [3], existence, uniqueness and continuation of solutions of (3) also are valid under conditions (i)–(v).

Therefore, a class of retarded equations with infinite delay arises from the investigation of solutions of neutral equations. However, this class of retarded equations is different from the types investigated in much of the literature, because we cannot find a separate ordinary part which dominates the functional part. Actually, both terms in the second and third variables of F on the right hand side of equation (3) are functionals depending on the values of y in the intervals $(-\infty, t]$. Thus, to investigate asymptotic behavior of solutions for this class of retarded equations with infinite delay, some modifications and improvements are required on methods and results developed for those with a dominating ordinary part.

In Section 2, first we introduce certain auxiliary functionals. By establishing monotonicity properties and limit behaviors of these functional along the solutions of (1) we conclude that the convergence of bounded solutions of (3) as $t \to \infty$ implies that of each solution of (1) as $t \to \infty$. A modification of the idea of [5] will lead to the asymptotic constancy of bounded solutions of (3), and consequently, to the asymptotic constancy of all solutions of (1), which is the main result of this paper. Therefore, as was mentioned, we can give a partial solution to a conjecture presented by the first author in [1].

In order not to hide the main idea behind general and technically complicated statements, we restrict our discussion to equation (1). In Section 3 possible extensions will be indicated.

2. THE MAIN RESULT

Define the functionals $V, W: C_\gamma \to R$ by

$$V(\phi) = \max\left\{\max_{-r\le s\le 0}\phi(s), (\phi(0) - c\phi(-r))/(1 - c)\right\},$$

$$W(\phi) = \min\left\{\min_{-r\le s\le 0}\phi(s), (\phi(0) - c\phi(-r))/(1 - c)\right\}.$$

For a function $x:(-\infty, a) \to R$ and $t < a$ let $x_t:(-\infty, 0] \to R$ be defined by $x_t(s) = (t + s)$, $s \le 0$.

Monotonicity properties of V and W along the solutions of (1) are given in the next lemma.

LEMMA 1. If x is a solution of (1) on $[0, \beta)$, $\beta > 0$, then $V(x_t)$ is nonincreasing and $W(x_t)$ is nondecreasing on $[0, \beta)$.

Proof. We show only the monotonicity of V, since the proof is similar for W.

Let $v(t) = V(x_t)$, $u_1(t) = \max_{-r\le s\le 0} x(t + s)$, $u_2(t) = (x(t) - cx(t - r))/(1 - c)$. We examine three cases:

case 1: $u_1(t) < u_2(t)$,

case 2: $u_1(t) > u_2(t)$,

case 3: $u_1(t) = u_2(t)$.

Case 1. Here $D^+v(t) = D^+u_2(t) \cdot u_1(t) < U_2(t)$ implies $x(t) < (x(t) - cx(t - r))/(1 - c)$ and $c \ne 0$. Hence $x(t) > x(t - r)$, from which one obtains $D^+u_2(t) \le 0$ by using (iii) and (1).

Case 2. Now $D^+v(t) = D^+u_1(t) \cdot u_1(t) > u_2(t)$ and $x(t) = u_1(t)$ are incompatible. $x(t) < u_1(t)$ implies $D^+u_1(t) \le 0$.

Case 3. In this case $D^+v(t) \le \max\{D^+u_1(t), D^+u_2(t)\} \cdot u_1(t) = u_2(t)$ gives $x(t - r) \le u_1(t) = (x(t) - cx(t - r))/(1 - c)$, from which $x(t) \ge x(t - r)$. Therefore, by (iv) and equation (1), $D^+u_2(t) \le 0$. If $c = 0$ then $u_2(t) = x(t)$ and by (iii), and (1) the equality $x'(t) \le 0$ holds. Since obviously $D^+u_1(t) \le \max\{0, D^+x(t)\}$, we also have $D^+u_1(t) \le 0$. Suppose $c \ne 0$. Clearly $x(t) < u_1(t)$ gives $D^+u_1(t) \le 0$. Assume $c \ne 0$, $x(t) = u_1(t) = u_2(t)$. Then $x(t) = \max_{-r\le s\le 0} x(t + s) = x(t + s) = x(t - r)$. In order to have $D^+u_1(t) \le 0$ it is enough to show that $D^+x(t) \le 0$. Suppose the contrary, i.e. $D^+x(t) > 0$. Then there is a sequence $\{h_n\}$ such that $h_n > 0$, $h_n \to 0 (n \to \infty)$, $\limsup_{n\to\infty}(1/h_n)(x(t + h_n) - x(t)) > 0$. Using that $D^+u_2(t) \le 0$ and $x(t - r) = \max_{-r\le s\le 0} x(t + s)$, we have

$$0 \ge (1 - c)D^+u_2(t) \ge \limsup_{n\to\infty} h_n^{-1}(x(t + h_n) - x(t) + c(x(t - r) - x(t + h_n - r)))$$

$$\ge \limsup_{n\to\infty} h_n^{-1}(x(t + h_n) - x(t)) > 0,$$

a contradiction. Thus $D^+x(t) \le 0$ and $D^+u_1(t) \le 0$.

So, $D^+v(t) \le 0$ for all cases, which implies our statement.

The following corollaries can be obtained from lemma 1.

COROLLARY 1. If x is a solution of (1) on $[0, \beta)$, $\beta > 0$, then x is bounded.

COROLLARY 2. Any constant solution of (1) is stable.

Our next lemma gives some limit relations between $V(x_t)$, $W(x_t)$, and the maximum and minimum of solutions on intervals $[t - 2r, t]$.

LEMMA 2. If x is a solution of (1) on $[0, \infty)$, and

$$v_0 = \lim_{t\to\infty} V(x_t), \qquad w_0 = \lim_{t\to\infty} W(x_t),$$

then

$$\lim_{t\to\infty} \max_{-2r\le s\le 0} x(t+s) = v_0, \qquad \lim_{t\to\infty} \max_{-2r\le s\le 0} x(t+s) = w_0.$$

Proof. The cases $v_0 = w_0$ and $c = 0$ are evident. Assume $v_0 > w_0$ and $c \neq 0$. Choose $\varepsilon > 0$ such that $\varepsilon c/(1 - c) < v_0 - w_0$.

If the first limit does not exist or is not equal to v_0, then there exists $T \ge r$ such that $W(x_t) \ge w_0 - \varepsilon$ for all $t \ge T - r$ and $\max_{-2r\le s\le 0} x(T + s) < v_0$. By lemma 1, $W(x_T) \le w_0 < v_0 \le V(x_T)$. Consequently, $\min_{-r\le s\le 0} x(T + s) \le w_0$. So, there is $t_0 \varepsilon [T - r, T]$ with $x(t_0) \le w_0$. Then

$$V_0 \le (x(t_0) - cx(t_0 - r))/(1 - c) \le (w_0 - c(w_0 - \varepsilon))/(1 - c) = w_0 + \varepsilon c/(1 - c) < v_0$$

is a contradiction. Therefore the first limit exists and equals v_0.

The proof of the second statement is similar.

LEMMA 3. If x is a solution of (1) on $[0, \infty)$, $\alpha \in R$ and

$$\lim_{t\to\infty} (x(t) - cx(t - r))/(1 - c) = \alpha,$$

then

$$\lim_{t\to\infty} x(t) = \alpha.$$

Proof. It is enough to prove that $v_0 = w_0$. Clearly, $v_0 \ge \alpha \ge w_0$. Assume $v_0 > \alpha$. Let $0 < \varepsilon < (v_0 - \alpha)(1 - c)/(1 + c)$. There is a sequence $\{t_n\}$ with $t_n \to \infty$ $(n \to \infty)$ and $x(t_n) \ge v_0 - \varepsilon$, $x(t_n - r) \le v_0 + \varepsilon$. Then $(x(t_n) - cx(t_n) - r))/(1 - c) \ge (v_0 - \varepsilon - c(v_0 + \varepsilon))/(1 - c) = v_0 - \varepsilon(1 + c)/(1 - c) > \alpha$, a contradiction. Likely, $\alpha > w_0$ is also impossible.

The proof is complete.

Therefore, in order to get asymptotic constancy of solutions of (1) it is sufficient to show that any bounded solution of (3) belonging to an initial function in C_γ tends to a finite constant as $t \to \infty$. This will be done in the following lemma by using the idea of [5].

LEMMA 4. If y is a bounded solution of (3) through $(0, \phi)$, $\phi \in C_\gamma$ on $[0, \infty)$, then y has a finite limit as $t \to \infty$.

Proof. From $y_0 \in C_\gamma$, $\gamma > 0$, and the boundedness of y on $[0, \infty)$, the existence of $K > 0$ follows such that $\|y_t\|_\gamma \le K$ $(t \ge 0)$. Then

$$\left| y(t) + \sum_{i=1}^{\infty} c^i y(t - ir) \right| \le K + \sum_{i=1}^{\infty} c^i K e^{\gamma i r} = K/(1 - ce^{\gamma r}) \qquad (t \ge 0)$$

and

$$\left|\sum_{i=1}^{\infty} c^{i-1}y(t-ir)\right| \le Ke^{\gamma r}/(1-ce^{\gamma r}) \qquad (t \ge 0),$$

where, and in the sequel, we mean 1 for 0^0.

Suppose $\lim_{t\to\infty} y(t)$ does not exist, that is

$$a \stackrel{\text{def}}{=} \liminf_{t\to\infty} y(t) < \limsup_{t\to\infty} y(t) \stackrel{\text{def}}{=} b.$$

Let $d \in (a, b)$ and $L = L([-M, M])$, where $M = Ke^{\gamma r}/(1-ce^{\gamma r})$. Define $\varepsilon > 0$ such that

$$b + \varepsilon + \tfrac{1}{2}(d - b - \varepsilon)\exp(-Lr/(1-c)) < b.$$

Let $t_1 \ge 0$ be given such that $t \ge t_1$ implies $y(t) \le b + \varepsilon$. Choose $t_2 \ge t_1$ such that $y(t_2) = d$ and

$$2LM(1-c)\int_{t_2}^{\infty} (ce^{\gamma r})^{[(t-t_1)/r]}\,\mathrm{d}t < \tfrac{1}{2}(b + \varepsilon - d)\exp(-Lr/(1-c)),$$

where $[\cdot]$ denotes the integer part.

Let

$$v(t) = \max_{t_2 \le s \le t} y(t) \qquad (t \ge t_2).$$

We are going to show that

$$D^+v(t) \le Lc^{[(t-t_2)/r]}(b + \varepsilon - v(t)) + 2LM(1-c)(ce^{\gamma r})^{[(t-t_1)/r]}. \tag{4}$$

In both the case where $y(t) < v(t)$ and the case where $y(t) = v(t)$ and $y'(t) \le 0$ we have $D^+v(t) \le 0$. Since the right hand side of (4) is nonnegative we obtain (4) for both cases. Now assume $y(t) = v(t)$ and $y'(t) > 0$. Then by assumption (iv)

$$y(t) + \sum_{i=1}^{\infty} c^i y(t-ir) < \sum_{i=1}^{\infty} c^{i-1}y(t-ir).$$

So, from condition (v) one has

$$y'(t) \le L\sum_{i=1}^{\infty}(c^{i-1} - c^i)(y(t-ir) - y(t)).$$

Now, using that $y(t) = v(t)$ implies $y(t) \ge y(t-ir)$ for $i = 1, \ldots, [(t-t_2)/r]$, $y(t) \le b + \varepsilon$ on $[t_1, \infty)$, $y(t-ir) \le Ke^{i\gamma r}$ for $t \ge 0$, we conclude that

$$\begin{aligned} y'(t) &\le L\sum_{[(t-t_2)/r]+1}^{\infty}(c^{i-1} - c^i)(y(t-ir) - y(t)) \\ &\le L\sum_{[(t-t_2)/r]+1}^{[(t-t_1)/r]}(c^{i-1} - c^i)(b + \varepsilon - y(t)) + L\sum_{[(t-t_1)/r]+1}^{\infty}(c^{i-1} - c^i)(Ke^{\gamma ir} + K) \\ &\le Lc^{[(t-t_2)/r]}(b + \varepsilon - y(t)) + 2LM(1-c)(ce^{\gamma r})^{[(t-t_1)/r]}. \end{aligned}$$

Since $y(t) = v(t)$ together with $y'(t) > 0$ implies $D^+v(t) = y'(t)$, we obtain (4) for all $t \ge t_2$. From well-known differential inequalities (see e.g. [6, theorem 1.10.2]), it follows that

$v(t) \le \omega(t)$ on $[t_2, \infty)$, where $\omega(t)$ is the unique solution of the initial value problem

$$\begin{cases} \omega'(t) = Lc^{[(t-t_2)/r]}(b + \varepsilon\,\omega(t)) + 2LM(1 - c)(ce^{\gamma r})^{[(t-t_1)/r]} & \text{a.e. on } [t_2, \infty) \\ \omega(t_2) = d. \end{cases}$$

That is,

$$\begin{aligned} v(t) &\le b + \varepsilon + (d - b - \varepsilon)\exp\left(-\int_{t_2}^{t} Lc^{[(s-t_2)/r]}\right) \mathrm{d}s \\ &\quad + \int_{t_2}^{t} 2LM(1 - c)(ce^{\gamma r})^{[(s-t_1)/r]} \exp\left(-\int_{s}^{t} Lc^{[(\tau-t)/r]}\,\mathrm{d}\tau\right) \mathrm{d}s \\ &\le b + \varepsilon + (d - b - \varepsilon)\exp\left(-\int_0^\infty Lc^{[s/r]}\,\mathrm{d}s\right) + 2LM(1 - c)\int_{t_2}^{\infty}(ce^{\gamma r})^{[(s-t_1)/r]}\,\mathrm{d}s \\ &< b + \varepsilon + (d - b - \varepsilon)\exp(-Lr/(1 - c)) + \tfrac{1}{2}(b + \varepsilon - d)\exp(-Lr/(1 - c)) \\ &< b \qquad (t \ge t_2). \end{aligned}$$

Hence $b = \limsup_{t\to\infty} y(t) \le \limsup_{t\to\infty} v(t) < b$ follows, a contradiction. This proof is complete.

Out main result below now follows readily from the previous lemmas and the fact that we have set $y(t) = x(t) - cx(t - r)$.

THEOREM 1. Under conditions (i)–(v) each solution of (1) tends to a finite limit as $t \to \infty$.

Example. Consider the equation.

$$(\mathrm{d}/\mathrm{d}t)(x(t) - cx(t - r)) = (1 + \sin t)(-h(x(t)) + h(x(t - r))), \tag{5}$$

where $0 \le c < 1$, $r > 0$, $h : R \to R$ is nondecreasing and locally Lipschitzean. By theorem 1, each solution of (5) tends to a finite constant as $t \to \infty$.

3. REMARKS

1. The ideas of lemmas 1, 2 and 3 also work for more general equations than (1). For example, consider the scalar neutral equation

$$(\mathrm{d}/\mathrm{d}t)\left(x(t) - c\int_{-r}^{0} x(t + s)\,\mathrm{d}\nu(s)\right) = F\left(t, x(t), \int_{-r}^{0} x(t + s)\,\mathrm{d}\nu(s)\right) \qquad (t \ge 0), \tag{6}$$

where conditions (i)–(v) also are assumed, $\nu : [-r, 0] \to R$ is nondecreasing and $\int_{-r}^{0} \mathrm{d}\nu = 1$.

If we define the functionals by

$$V(\phi) = \max\left\{\max_{-r\le s\le 0} \phi(s), \left(\phi(0) - c\int_{-r}^{0}\phi(s)\,\mathrm{d}\nu(s)\right)\Big/(1 - c)\right\},$$

$$W(\phi) = \min\left\{\min_{-r\le s\le 0} \phi(s), \left(\phi(0) - c\int_{-r}^{0}\phi(s)\,\mathrm{d}\nu(s)\right)\Big/(1 - c)\right\},$$

then lemmas 1, 2 and 3 are valid for equation (6).

Let $\gamma > 0$ be defined such that

$$c\int_{-r}^{0} e^{-\gamma s}\,\mathrm{d}\nu(s) < 1.$$

Extend ν to $(-\infty, 0]$ with $\mathrm{d}\nu(s) = 0$ for $s < -r$. Let $\mu = \delta - c\nu$, where $\delta(s) = 0$ for $s < 0$, $\delta(0) = 1$. From Staffans' result [7] it follows that the convolution operator $\mu*$, defined by

$$\mu * \phi(u) = \int_{-\infty}^{0} \phi(u + s)\,\mathrm{d}\mu(s) \qquad (u \le 0),$$

maps C_γ continuously into C_γ, its inverse $\mu^{-1}*$ also exists and maps C_γ continuously into itself, and is given by

$$\mu^{-1}* = \delta* + c\nu* + c^2\nu*\nu* + c^3\nu*\nu*\nu* + \cdots.$$

In the same way as in Section 1, the solution x of (6) can be extended to $(-\infty, 0]$ with $x_0 \in C_\gamma$. By using the transformation $y = \mu * x$, that is $y(t) = x(t) - c\int_{-r}^{0} x(t + s)\,\mathrm{d}\nu(s)$, we obtain $x = \mu^{-1} * y$ and

$$y'(t) = F\left(t, (\mu^{-1} * y_t)(0), \int_{-r}^{0} (\mu^{-1} * y_t)(s)\,\mathrm{d}\nu(s)\right) \qquad (t \ge 0). \tag{7}$$

If $y_0 \in C_\gamma$ and y is a bounded solution of (7) on $[0, \infty)$ then there is $K > 0$ with $\|y_t\|_\gamma \le K$, $t \ge 0$. In addition,

$$\begin{aligned}|(\mu^{-1} * y_t)(0)| &= |y(t) + c(\nu * y_t)(0) + c^2(\nu^2 * y_t)(0) + \cdots|\\ &\le K(1 + q + q^2 + \cdots) = K/(1 - q)\end{aligned}$$

and

$$\begin{aligned}\left|\int_{-r}^{0} (\mu^{-1} * y_t)(s)\right| &= |(\nu * y_t)(0) + c(\nu^2 * y_t)(0) + c^2(\nu^3 * y_t)(0) + \cdots|\\ &\le K(q + q^2 + q^3 + \cdots) = Kq/(1 - q),\end{aligned}$$

where $q = c\int_{-r}^{0} e^{-\gamma r}\,\mathrm{d}\nu(s) < 1$, $\nu^1 * y_t = \nu * y_t$, $\nu^{k=1} * y_t = \nu * (\nu^k * y_t)$. It also follows that

$$\int_{-r}^{0} (\mu^{-1} * y_t)(s)\,\mathrm{d}\nu(s) - (\mu^{-1} * y_t)(0) = \sum_{i=1}^{\infty} (c^{i-1} - c^i)((\nu^i * y_t)(0) - y(t)).$$

Therefore the symptotic constancy of bounded solutions of (7) can be obtained analogously to that of (3).

2. We employ the phase space C_γ only for the purpose of extending an initial function of $\phi \in C_r$ to $(-\infty, 0]$ such that the natural transformation (2) is invertible and the equation (1) can be related to the retarded equation (3) with infinite delay. Such a technical procedure can be avoided if we restrict our discussion of asymptotic behavior of solutions to a compact invariant subset Ω of C_γ, invariance means that any solution in Ω can be uniquely extended to $(-\infty, +\infty)$. For example, for any $\phi \in C_r$, the orbit of (1) through $(0, \phi)$ is relatively compact by corollary 1, and thus the ω-limit set $\omega(\phi)$, is compact and invariant. If $\psi \in \omega(\phi)$, we can find a bounded continuous function $x : (-\infty, +\infty) \to R$ with $x_0 = \psi$ and such that (1) holds for all $t \in (-\infty, +\infty)$. Evidently, the transformation $y(t) = x(t) - cx(t - r)$ is invertible with $x(t) = y(t) + \sum_{i=1}^{\infty} c^i y(t - ir)$, and $y(t)$ satisfies (3). Therefore $y(t)$ has a finite limit by lemma 4,

which in turn indicates $x(t)$ is convergent to a finite number according to lemma 3. Therefore convergence of solution through $(0, \phi)$ follows from the convergence of solutions in $\omega(\phi)$ and the stability of any constant solution.

3. The statement of lemma 4 is interesting in itself for retarded differential equations with infinite delay. It can be extended partly as follows.

LEMMA 4′. Assume that conditions (ii)–(v) hold; the functions $\eta, \vartheta : (-\infty, 0] \to R$ are non-decreasing, $\int_{-\infty}^0 d\vartheta < \infty$, $\int_{-\infty}^0 d\eta = 1$ and $\int_{-\infty}^0 |s|\, d\eta(s) < \infty$. If the function $y : R \to R$ is continuous, and bounded on $(-\infty, 0]$, continuously differentiable on $[0, \infty)$, and satisfies

$$y'(t) = F\left(t, y(t) + \int_{-\infty}^0 y(t+s)\, d\vartheta(s), \int_{-\infty}^0 y(t+s)\, d\eta(s) + \int_{-\infty}^0 y(t+s)\, d\vartheta(s)\right) \qquad (t \ge 0),$$

then $y(t)$ tends to a finite limit as $t \to \infty$.

The proof can be carried out similarly to that of lemma 4. So, it is omitted.

Let us remark that $y_0 \in C_\gamma$ in lemma 4, while y_0 is bounded and continuous in lemma 4′. In order to get asymptotic constancy of solutions either of (1) or (6), it is enough to consider (3) and (7) with bounded y_0, since we can extend x to $(-\infty, 0]$ such that y_0 is bounded. It is possible to give another extension of lemma 4, where y_0 is in a phase space used in the theory of differential equations with infinite delay (e.g. C_γ, see [5]).

Notice that lemma 4′ is not true without the monotonicity property of η. To show this, consider

$$y'(t) = -\left(y(t) + \frac{\pi}{4} y(t-1)\right) + \left(-\frac{\pi}{4} y(t-1) + \frac{\pi}{4} y(t-3) + y(t-4) + \frac{\pi}{4} y(t-1)\right) \qquad (t \ge 0)$$

and observe that $y(t) = \sin(\pi t/2)$ is a solution.

4. The Lipschitz property (v) of F is used only in the proof of lemma 4. It can be replaced by the following condition:

(v') $c < \frac{1}{2}$ and there is $u_0 \in R$ such that for any compact interval I, which is either in $(-\infty, u_0)$ or in (u_0, ∞) there exists $L = L(I) \ge 0$ such that $|F(t, x, y)| \le L|x - y|$ $(x, y \in I,\ t \ge 0)$.

Under condition (v') for all $a < b$ (see the proof of lemma 4 for the definition of a and b) one can choose suitable $d \in (a, b)$ and $\varepsilon > 0$ such that the intervals

$$I_1 = [a + ac/(1-c) - \varepsilon, d + bc/(1-c) + \varepsilon], \qquad I_2 = [d + ac/(1-c) - \varepsilon, b + bc/(1-c) + \varepsilon]$$

are disjoint, and therefore F is Lipschitzean at least on one of the intervals. Assume that F is Lipschitzean on I_2. It is easy to see that if t is large enough and $y(t) \ge d$, then

$$y(t) + \sum_{i=1}^{\infty} c^i y(t - ir), \qquad \sum_{i=1}^{\infty} c^{i-1} y(t - ir) \in I_2.$$

Consequently, we can do the same as in the proof of lemma 4 for sufficiently large t. The case, when F is Lipschitzean of I_1, is analogous.

A similar remark is valid for lemma 4′.

5. In [1] the first author considered the scalar neutral equation

$$(\mathrm{d}/\mathrm{d}t)(x(t) - cx(t - r)) = -ax^{\gamma}(t) + ax^{\gamma}(t - r) \qquad (t \geq 0), \tag{8}$$

where $|c| < 1$, $a \geq 0$, $r \geq 0$, $\gamma > 0$ is a quotient of positive odd integers, and conjectures that each solution of (8) belonging to the space of continuously differentiable initial functions tends to a finite limit as $t \to \infty$. On the basis of theorem 1 and the above remarks we can show the convergence of the solutions of (8), wherever $0 \leq c < \frac{1}{2}$.

6. Out technique does not work for neutral equations, where there are different measures of delay on the left and right hand sides of the equation

$$(\mathrm{d}/\mathrm{d}t)(x(t) - cx(t - r)) = -h(x(t)) + h(x(t - q)) \qquad (t \geq 0) \tag{9}$$

where $|c| < 1$, $r > 0$, $q > 0$, $h : R \to R$ is continuous and nondecreasing. Staffan's transformation [7] $y(t) = x(t) - cx(t - r)$ can be used to obtain

$$y'(t) = -h\left(\sum_{i=1}^{\infty} c^{i}y(t - ir)\right) + h\left(\sum_{i=1}^{\infty} c^{i-1}y(t - q - ir)\right) \qquad (t \geq 0). \tag{10}$$

As far as we know analogous results to lemmas 1, 2, 3 and 4 are not available at the moment for equations (9) and (10). Stability, boundedness and convergence of solutions of both equations (9) and (10) still are interesting and open problems.

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