

## On the boundedness of solutions of nonautonomous differential equations

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*Dedicated to L. Pintér on his 60th birthday*

### 1. Introduction

In the study of existence of periodic solutions and almost periodic solutions as well as behavior of limiting sets of solutions of ordinary differential equations, the uniform boundedness and uniform ultimate boundedness of solutions are frequently needed [1—4, 9]. These properties of solutions can be regarded as either the instability of infinity or a special case of some kind of stability of a set. Therefore, there exists a close relation between Lyapunov's direct method and the boundedness of solutions. A typical result showing this relation is Theorem 10.4 in [3]. In this theorem the uniform ultimate boundedness is guaranteed by the existence of an appropriate Lyapunov function having a negative definite derivative along the solutions. However, in practice it is very difficult to construct such a Lyapunov function. For example, for mechanical systems the total mechanical energy, which is a typical Lyapunov function, never has a negative definite derivative along the motions with respect to the generalized coordinates.

The purpose of this paper is to study the boundedness and ultimate boundedness of solutions of nonautonomous differential equations by Lyapunov's direct method when the derivative of the Lyapunov function along the solutions is only semidefinite. The results generalize V. M. MATROSOV's theorem [5] on the asymptotic stability to the boundedness of solutions. An application is given to the boundedness of the motions of a holonomic scleronomic mechanical system of  $n$  degrees of freedom being under the action of potential, dissipative and gyroscopic forces.

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2. Notations and definitions

Consider the system

$$(2.1) \quad \dot{x} = X(t, x),$$

where  $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^n$ ,  $\mathbf{R}^+ = [0, \infty)$  and  $X: \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  is continuous. Throughout this paper, for simplicity, we assume that for any  $(t_0, x_0) \in \mathbf{R}^+ \times \mathbf{R}^n$ , there exists a unique solution  $x(t; t_0, x_0)$  of (2.1) through  $(t_0, x_0)$  defined for all  $t \geq t_0$ .

**Definition 2.1** [3]. A solution  $x(t; t_0, x_0)$  of (2.1) is *bounded*, if  $\sup_{t \geq t_0} |x(t; t_0, x_0)| < \infty$ .

The solutions of (2.1) are *uniformly bounded* (U.B.) if for every  $\alpha > 0$  there exists a  $\beta(\alpha) > 0$  such that  $[t_0 \geq 0, |x_0| < \alpha, t \geq t_0]$  imply  $|x(t; t_0, x_0)| < \beta(\alpha)$ .

The solutions of (2.1) are *equiultimately bounded* (E.U.B.) for some bound  $B$  if for every  $\alpha > 0$  and  $t_0 \geq 0$  there exists a  $T(t_0, \alpha) > 0$  such that  $[|x_0| < \alpha, t \geq t_0 + T(t_0, \alpha)]$  imply  $|x(t; t_0, x_0)| < B$ .

The solutions of (2.1) are *uniformly ultimately bounded* (U.U.B.) for some bound  $B$  if for every  $\alpha > 0$  there exists a  $T(\alpha) > 0$  such that  $[t_0 \geq 0, |x_0| < \alpha, t \geq t_0 + T(\alpha)]$  imply  $|x(t; t_0, x_0)| < B$ .

By a *pseudo wedge*  $W$  we mean a continuous and strictly increasing function  $W: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $W(r) > 0$  if  $r > 0$ . A pseudo wedge  $W$  is called *unbounded* if  $\lim_{r \rightarrow \infty} W(r) = +\infty$ .

Denote by  $[a]_+$  and  $[a]_-$  the positive and negative part of the real number  $a$ , respectively, that is,  $[a]_+ = \max\{a, 0\}$ ,  $[a]_- = \max\{-a, 0\}$ .

**Definition 2.2** [5]. A measurable function  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is said to be *integrally positive* if  $\int_J \lambda(t) dt = \infty$  holds on every set  $J = \bigcup_{m=1}^{\infty} [a_m, b_m]$  such that  $a_m < b_m \leq a_{m+1}$  and  $b_m - a_m \geq \delta > 0$  ( $m=1, 2, \dots$ ) for a constant  $\delta > 0$ .

**Definition 2.3** [7]. A measurable function  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is said to be *weakly integrally positive* if for every  $\delta > 0, \Delta > 0$  and for every set  $J = \bigcup_{m=1}^{\infty} [a_m, b_m]$  with  $a_m + \delta \leq b_m \leq a_{m+1} < b_m + \Delta$  ( $m=1, 2, \dots$ ) the relation  $\int_J \lambda(t) dt = \infty$  holds.

**Lemma 2.1.** *If a measurable function  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is integrally positive, then for every  $\alpha > 0$  and  $\delta > 0$  there exists a positive integer  $K(\alpha, \delta)$  such that for every set  $J = \bigcup_{m=1}^K [a_m, b_m]$  with  $a_m < a_m + \delta \leq b_m \leq a_{m+1}$  for  $1 \leq m \leq K-1$ , we have  $\int_J \lambda(t) dt \geq \alpha$ .*

**Proof.** It is easy to see that  $\lambda$  is integrally positive if and only if for every  $\delta > 0$  the inequality

$$(2.2) \quad \liminf_{t \rightarrow \infty} \int_t^{t+\delta} \lambda(s) ds > 0$$

holds. Consequently, for any given  $\delta > 0$  there are  $T = T(\delta) > 0$  and  $\mu(\delta) > 0$  such that  $t \geq T(\delta)$  implies

$$\int_t^{t+\delta} \lambda(s) ds \geq \mu(\delta).$$

Let  $\alpha > 0$  and  $\delta > 0$  be given, and define  $K(\alpha, \delta) = [T(\delta)/\delta] + 1 + [\alpha/\mu(\delta)] + 1$ , where  $[a]$  denotes the integer part of  $a \in \mathbf{R}$ , that is,  $[a] = \max \{z: z \text{ is an integer with } z \leq a\}$ . Then the number  $K(\alpha, \delta)$  has the property mentioned in the assertion.

The following assertion can be easily proved by making use of (2.2).

**Lemma 2.2.** *If a measurable function  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is integrally positive, then*

$$(2.3) \quad \lim_{T \rightarrow \infty} \int_{t_0}^{t_0+T} \lambda = \infty$$

*uniformly with respect to  $t_0 \in \mathbf{R}^+$ .*

**Remark 2.1.** The property of weak integral positivity and property (2.3) are independent of one another. E.g.  $\lambda(t) = 1/(1+t)$  is weakly integrally positive, but it does not satisfy (2.3) and so it is not integrally positive. On the other hand, weak integral positivity and (2.3) together do not imply integral positivity. E.g., the function

$$\lambda(t) = \begin{cases} 1/(1+t) & n \leq t \leq n+1/2 \\ 1 & n+1/2 < t < n+1 \end{cases}$$

is weakly integrally positive and satisfy (2.3) but it is not integrally positive.

With a continuous function  $V: \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$  we associate the function

$$\dot{V}_{(2.1)}(t, x) = \limsup_{h \rightarrow 0^+} (1/h) \{V(t+h, x+hX(t, x)) - V(t, x)\},$$

which called the derivative of  $V$  with respect to (2.1).

It can be proved (see [3], p. 3) that if  $V$  is locally Lipschitz, then for an arbitrary solution  $x(t)$  of (2.1) we have

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) = \int_{t_1}^{t_2} \dot{V}(t, x(t)) dt, \quad (t_1, t_2 \in \mathbf{R}^+).$$

**3. The theorems and their proofs**

**Theorem 3.1.** *Suppose that there exist nonnegative constants  $B$  and  $D$ , non-negative locally Lipschitz functions  $V(t, x)$ ,  $P(t, x)$  and continuous  $K(t, x)$  defined for  $t \geq 0$ ,  $|x| \leq B$  satisfying the following conditions:*

- (i)  $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ , where  $W_1$  and  $W_2$  are unbounded pseudo wedges;
- (ii) the derivative of  $V$  with respect to (2.1) satisfies the inequality

$$(3.1) \quad \dot{V}_{(2.1)}(t, x) \leq -K(t, x) \quad \text{for } t \geq 0, \quad |x| \leq B;$$

(iii) for each  $M > B$  there are  $k = k(M) > 0$  and  $H = H(M) \geq 0$  such that  $[t \geq 0, B \leq |x| \leq M, P(t, x) \geq H]$  imply  $K(t, x) \geq k$ ;

(iv) for each  $M > B$  there exists an  $L(M) > 0$  such that  $[t \geq 0, B \leq |x| \leq M, H(M) \leq P(t, x) \leq 2H(M)]$  imply  $\dot{P}_{(2.1)}(t, x) \leq L(M)$ ;

(v) for each  $M > B$  there is a  $T(M) > 0$  such that for any solution  $x(t)$  of (2.1) with  $B \leq |x(t)| \leq M$  and  $P(t, x(t)) \leq 2H(M)$  for  $t_0 \leq t \leq t_0 + T(M)$  there exists  $s \in [t_0, t_0 + T(M)]$  with  $|x(s)| < D$ .

Then the solutions of (2.1) are U.B. and U.U.B.

**Proof.** For any  $\alpha > 0$ , define  $\beta(\alpha) = W_1^{-1}(W_2(\max\{B, \alpha\}))$ . It is easy to prove that  $[t_0 \geq 0, |x_0| \leq \alpha]$  imply  $|x(t; t_0, x_0)| \leq \beta(\alpha)$  for  $t \geq t_0$ . Therefore, the solutions of (2.1) are U.B. Throughout the remainder of this proof we use the notations  $x(t) = x(t; t_0, x_0)$ ,  $V(t) = V(t, x(t))$  and  $\dot{V}(t) = \dot{V}_{(2.1)}(t, x(t))$ .

To prove the uniform ultimate boundedness, we consider the following two cases:

- (a) there exists a  $t_2 \geq t_0$  with  $|x(t_2)| \leq B$ ;
- (b)  $|x(t)| \geq B$  for all  $t \geq t_0$ .

In case (a)  $|x(t)| \leq \beta(B)$  for  $t \geq t_2$ .

In case (b) we have  $\dot{V}(t) \leq -K(t, x(t))$  for all  $t \geq t_0$ . By (iii) there exist  $k = k(\beta(\alpha)) > 0$  and  $H = H(\beta(\alpha)) > 0$  such that  $P(t, x(t)) \geq H$  implies  $K(t, x(t)) \geq k$ . Let  $\bar{t} \geq t_0$  be fixed, and choose a constant  $S = S(\alpha) > W_2(\beta(\alpha))/k$ . Then by (3.1) the nonnegativeness of  $V$  implies the existence of a  $t_3 \in [\bar{t}, \bar{t} + S(\alpha)]$  such that  $P(t_3, x(t_3)) < H$ . By (v), there exists  $T = T(\beta(\alpha)) > 0$  such that if  $P(t, x(t)) < 2H$  for  $t \in [t_3, t_3 + T]$ , then there is an  $s \in [t_3, t_3 + T]$  with  $|x(s)| < D$ , which implies  $|x(t)| < \beta(D)$  for  $t \geq t_3 + T$ , especially, for  $t \geq \bar{t} + S + T$ .

Therefore, only two cases may occur:

- (b<sub>1</sub>)  $P(t, x(t)) < 2H$  for all  $t \in [t_3, t_3 + T]$ .

In this case,  $|x(t)| < \beta(D)$  for  $t \geq \bar{t} + T + S$ .

- (b<sub>2</sub>) there exists  $t_4 \in [t_3, t_3 + T]$  with  $P(t_4, x(t_4)) \geq 2H$ .

In this case, there are  $t_5, t_6$  such that  $t_3 < t_5 < t_6 \leq t_4$ ,  $P(t_5, x(t_5)) = H$ ,  $P(t_6, x(t_6)) =$

$=2H$  and  $H < P(t, x(t)) < 2H$  for  $t_5 < t < t_6$ . By (iv), we get  $t_6 - t_5 \geq H/L(\beta(\alpha))$ . On the other hand, by  $\dot{V}(t) \leq -K(t, x(t)) \leq -k$  for  $t \in [t_5, t_6]$  we obtain

$$(3.2) \quad V(t_6) \leq V(t_5) - kH/L(\beta(\alpha)).$$

Since in case (b)  $\dot{V}(t) \leq -K(t, x(t)) \leq 0$  for all  $t \geq t_0$ , we get  $V(\bar{t} + S + T) \leq V(\bar{t}) - kH/L(\beta(\alpha))$ . Let  $\bar{t} = t_0 + m(S + T)$ , where  $m$  is a nonnegative integer. Then from the argument above we get either

$$(c_m) \quad |x(t)| \leq \max\{\beta(B), \beta(D)\} \quad \text{for } t \geq t_0 + (m+1)(S+T),$$

or

$$(d_m) \quad V(t_0 + (m+1)(S+T)) \leq V(t_0 + m(S+T)) - kH/L(\beta(\alpha)).$$

Choose a positive integer  $N = N(\alpha)$  such that

$$(3.3) \quad N(\alpha)kH/L(\beta(\alpha)) > W_2(\beta(\alpha)).$$

Then by the nonnegativeness of  $V$ , (d<sub>m</sub>) holds for at most  $m = 0, 1, \dots, N-1$ , and thus  $|x(t)| < \max\{\beta(B), \beta(D)\}$  for  $t \geq t_0 + N(S+T)$ . This completes the proof.

Remark 3.1. Using the same argument as one above, the comparison method and Lemma 2.1, we can prove the following assertion:

If conditions (i), (iii)–(v) of Theorem 3.1 are satisfied and if for each  $M > B$  there exists a weakly integrally positive function  $\lambda_M: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$\dot{V}_{(2.1)}(t, x) \leq -\lambda_M(t)K(t, x) + F(t, V(t, x)) \quad \text{for } t \geq 0$$

and  $B \leq |x| \leq M$ , where  $F: \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is continuous, the solutions of  $\dot{z} = F(t, z)$  are uniformly bounded, and  $\int_0^\infty \sup_{0 \leq z \leq r} F(t, z) dt < \infty$  for  $r \geq 0$ , then the solutions of (2.1) are U.B. and E.U.B. If, in addition,  $\lambda_M$  is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

Remark 3.2. If conditions (i), (iii) and (v) of Theorem 3.1 are satisfied and if

(a)  $\dot{V}_{(2.1)}(t, x) \leq -\lambda(t)K(t, x) + F(t, V(t, x))$  for  $t \geq 0$  and  $|x| \leq B$ , where  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is measurable and satisfies condition (2.3), and  $F$  is of the same kind as in Remark 3.1;

(b) for any  $M > 0$  there exists a  $\mu = \mu(M) > 0$  such that  $[B \leq |x| \leq M, H(M) \leq P(t, x) \leq 2H(M)]$  imply

$$\dot{V}_{(2.1)}(t, x) \leq -\mu \dot{P}_{(2.1)}(t, x) + F(t, V(t, x)),$$

then the solutions of (2.1) are U.B. and U.U.B.

To prove this remark it is sufficient to replace (3.2) and (3.3) in the proof of Theorem 3.1 by

$$V(t_0) \leq V(t_0) - \mu(\beta(\alpha))H(\beta(\alpha)) + \int_{t_0}^{t_0} \max \{F(t, z) : 0 \leq z \leq W_2(\beta(\alpha))\} dt$$

and

$$N\mu(\beta(\alpha))H(\beta(\alpha)) > W_2(\beta(\alpha)) + \int_0^{\infty} \max \{F(t, z) : 0 \leq z \leq W_2(\beta(\alpha))\} dt,$$

respectively.

Remark 3.3. Condition (iv) in Theorem 3.1 can be weakened as follows: for any  $M > B$  there exists a continuous function  $L_M: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that  $\int_0^t L_M$  is uniformly continuous on  $[0, \infty)$  and either

$$[\dot{P}_{(2.1)}(t, x)]_+ \leq L_M(t) \text{ for } t \geq 0, B \leq |x| \leq M \text{ and } H(M) \leq P(t, x) \leq 2H(M),$$

or

$$[\dot{P}_{(2.1)}(t, x)]_- \leq L_M(t) \text{ for } t \geq 0, B \leq |x| \leq M \text{ and } H(M) \leq P(t, x) \leq 2H(M).$$

Remark 3.4. Condition (i) in Theorem 3.1 can be replaced by  $0 \leq V(t, x) \leq W_2(|x|)$  if the solutions of (2.1) are U.B.

Example 3.1. Consider a Liénard equation with forcing term

$$(3.4) \quad \ddot{x} + f(x)\dot{x} + g(t, x) = e(t),$$

where  $f(x), g(t, x), \partial g(t, x)/\partial t$  and  $e(t)$  are continuous for  $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$  and  $\int_0^{\infty} |e(s)| ds < \infty$ . Besides, we assume that there exist unbounded pseudo wedges  $W_1, W_2$ , a continuous  $W_3: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $W_3(r) > 0$  for  $r > 0$  and an integrally positive function  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$W_1(|x|) \leq \int_0^x g(t, x) dx \leq W_2(|x|),$$

$$g(t, x)F(x) - \int_0^x (\partial g(t, r)/\partial t) dr \leq \lambda(t)W_3(|x|),$$

where  $F(x) = \int_0^x f(s) ds$ . Obviously, (3.4) is equivalent to

$$(3.5) \quad \dot{x} = y - F(x), \quad \dot{y} = -g(t, x) + e(t).$$

Let  $V(t, x, y) = [y^2 + 2 \int_0^x g(t, r) dr]^{1/2} + \int_t^\infty |e(s)| ds$ , then

$$[y^2 + 2W_1(|x|)]^{1/2} \leq V(t, x, y) \leq [y^2 + 2W_2(|x|)]^{1/2} + \int_0^\infty |e(s)| ds$$

$$\dot{V}_{(3.5)}(t, x, y) \leq -\lambda(t)W_3(|x|)[y^2 + 2W_2(|x|)]^{-1/2}$$

Let  $K(t, x, y) = W_3(|x|)[y^2 + 2W_2(|x|)]^{-1/2}$ ,  $P(t, x, y) = |x|$ ,  $B = 1$  and  $H = 1$ . Then for each  $M > 1$  and for  $t \geq 0$ ,  $1 \leq |x| + |y| \leq M$  and  $|x| \geq 1$ , we have  $K(t, x, y) \geq \min \{W_3(r) : 1 \leq r \leq M\} (M^2 + 2W_2(M))^{-1/2}$ . Therefore, conditions (i)–(iv) of Theorem 3.1 hold (see also Remark 3.1). Now we check condition (v).

Let  $E = \max \{|F(x)| + 1 : |x| \leq 2\}$ ,  $D = E + 2$ , and for  $M > 1$  define  $T(M) = 2M + 1$ . Suppose that  $(x(t), y(t))$  is a solution of (3.5) with  $1 \leq |x(t)| + |y(t)| \leq M$  and  $|x(t)| \leq 2$  for  $t \in [t_0, t_0 + T(M)]$ . If  $|x(t)| + |y(t)| \geq E + 2$  for all  $t \in [t_0, t_0 + T(M)]$ , then  $|y(t)| \geq E$ , e.g.  $y(t) \geq E$ , and consequently  $\dot{x}(t) = y(t) - F(x(t)) \geq E - \max_{|x| \leq 2} F(x) \geq 1$ . Hence we obtain the inequality  $2M \geq |x(t_0 + T(M)) - x(t_0)| \geq T(M) = 2M + 1$ , which is a contradiction. Therefore, there is an  $s \in [t_0, t_0 + T(M)]$  with  $|x(s)| + |y(s)| < D = E + 2$ , i.e. condition (v) in Theorem 3.1 holds.

Consequently, under our conditions the solutions of (3.5) are U.B. and U.U.B.

Notice that if  $P(t, x) = |x|$ , then condition (iv) in Theorem 3.1 can be dropped. (Indeed, if condition (i)–(iii), (v) are satisfied for  $P(t, x) = |x|$ , then all the conditions of the theorem are satisfied for the new auxiliary function  $\tilde{P}(t, x) = V(t, x)$ . If, in addition,  $H$  in (iii) is constant, then (v) obviously holds. This special case initiates the following generalization of T. YOSHIZAWA's theorem ([3], Theorem 10.4):

**Theorem 3.2.** *Suppose that there exist a constant  $B \geq 0$ , a locally Lipschitz function  $V(t, x)$  and a continuous function  $K(t, x)$  defined for  $t \geq 0$  and  $|x| \leq B$  satisfying the following conditions:*

- (i)  $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ , where  $W_1$  and  $W_2$  are unbounded pseudo wedges;
- (ii)  $\dot{V}_{(2.1)}(t, x) \leq -\lambda(t)K(t, x)$  for  $t \geq 0$  and  $|x| \leq B$ , where  $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is measurable with  $\lim_{t \rightarrow \infty} \int_{t_0}^t \lambda(s) ds = \infty$  for any  $t_0 \geq 0$ ;

(iii) for each  $M > B$  there exists  $k(M) > 0$  such that  $B \leq |x| \leq M$  implies  $K(t, x) \geq k(M)$ .

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition,  $\lambda$  satisfies condition (2.3), then the solutions of (2.1) are U.B. and U.U.B.

**Proof.** For any  $\alpha > 0$ , define  $\beta(\alpha) = W_1^{-1}(W_2(\max \{B, \alpha\}))$ . Let  $x(t; t_0, x_0)$  be a solution of (2.1) with  $|x_0| < \alpha$ . Then  $|x(t; t_0, x_0)| < \beta(\alpha)$  for all  $t \geq t_0$ , i.e. the solutions are U.B.

For a given  $t_0 \geq 0$  choose  $T(t_0, \alpha) > 0$  such that

$$\int_{t_0}^{t_0+T(t_0, \alpha)} \lambda(s) ds > W_2(\beta(\alpha))/k(\beta(\alpha)).$$

It is easy to prove that  $|x(t; t_0, x_0)| < \beta(B)$  for all  $t \geq t_0 + T(t_0, \alpha)$ .

The second conclusion can be proved similarly.

The following theorem is a generalization of V. M. MATROSOV's stability theorem [5] to the boundedness of solutions.

**Theorem 3.3.** *Suppose that there exist a constant  $B \geq 0$  and nonnegative locally Lipschitz functions  $V(t, x), W(t, x), P(t, x)$ , a continuous function  $F(t, u)$  defined for  $t \geq 0, |x| \leq B, u \geq 0$  and such that*

- (i)  $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ , where  $W_1$  and  $W_2$  are unbounded pseudo wedges;
- (ii) for every  $M > B$  there is a measurable function  $\lambda_M: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$\dot{V}_{(2.1)}(t, x) \leq -\lambda_M(t)P(t, x) + F(t, V(t, x)) \quad \text{for } t \geq 0 \text{ and } B \leq |x| \leq M,$$

where

- (a)  $\lambda_M$  is weakly integrally positive;

(b) the solutions of the equation  $\dot{z} = F(t, z)$  are U.B.; and  $\int_0^\infty [\sup_{0 \leq z \leq r} F(t, z)] dt < \infty$  for every  $r > 0$ ;

(iii) for every  $M > B$  there exists a continuous function  $L_M: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that  $\int_0^t L_M$  is uniformly continuous on  $\mathbf{R}^+$  and either  $[\dot{P}_{(2.1)}(t, x)]_+ \leq L_M(t)$  or  $[\dot{P}_{(2.1)}(t, x)]_- \leq L_M(t)$  for  $t \geq 0, B \leq |x| \leq M$ ;

(iv) for every  $M > B$  there exists a constant  $A(M) > 0$  such that  $|W(t, x)| \leq A(M)$  for  $t \geq 0$  and  $B \leq |x| \leq M$ ;

(v) there exists a constant  $D \geq B$  and for any  $M > B$  there exists a continuous function  $W_3: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $W_3(r) > 0$  for  $r \geq D$  such that

$$\max \{P(t, x), |\dot{W}_{(2.1)}(t, x)|\} \leq W_3(|x|) \quad \text{for } t \geq 0 \text{ and } D \leq |x| \leq M.$$

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition,  $\lambda_M(t)$  is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

**Proof.** First we show that under the assumptions of the theorem condition (v) in Theorem 3.1 is satisfied.

For any  $M > D$ , choose  $H(M) > 0$  such that  $2H < \alpha(M) = \min_{D \leq r \leq M} W_3(r)$  and define  $T(M) = [2A(M) + 1]/\alpha$ . Let  $x(t)$  be a solution of (2.1) with  $B \leq |x(t)| \leq M$  and  $P(t, x(t)) \leq 2H(M)$  for  $t \in [t_0, t_0 + T(M)]$ . If  $|x(t)| \geq D$  for all  $t \in [t_0, t_0 + T(M)]$  then according to condition (v) we get  $|\dot{W}_{(2.1)}(t, x(t))| \leq \alpha$ , hence,  $2A(M) \leq$



$|W(t_0 + T(M), x(t_0 + T(M))) - W(t_0, x(t_0))| \cong \alpha T(M) = 2A(M) + 1$ , which is a contradiction. Therefore, condition (v) of Theorem 3.1 holds.

An application of Theorem 3.1, Remark 3.1 and Remark 3.3 completes the proof.

Remark 3.5. Condition (v) of Theorem 3.3 can be weakened by asking there is a constant  $D \cong B$  such that for every  $M > D$  there are  $B_2(M) > 0$  and a continuous function  $\mu_M: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with property (2.3) and such that  $[t \cong 0, D \cong |x| \cong M, P(t, x) \cong B_2]$  imply  $|W_{(2.1)}(t, x)| \cong \mu_M(t)$ .

An application of this theorem to a holonomic scleronomous mechanical system will be given in Section 4.

As we have seen so far, the key step in the application of Theorem 3.1 is to check condition (v). Now we establish a sufficient condition for this property by Lyapunov's direct method.

Lemma 3.1. Suppose that there exist  $H_0 > 0, D > B$  and a locally Lipschitz function  $Q(t, x)$  defined on the set  $\{(t, x): t \cong 0, |x| \cong D, P(t, x) \cong 2H_0\}$  such that

(i) for each  $M > D$  there are continuous functions  $\gamma, g: \mathbf{R}^+ \rightarrow \mathbf{R}$  and a number  $H \in (0, H_0]$  such that  $\gamma$  has property (2.3), the function  $\int_0^t [g(s)]_+ ds$  is bounded on  $\mathbf{R}^+$ , and  $[t \cong 0, D \cong |x| \cong M, P(t, x) \cong 2H]$  imply  $\dot{Q}_{(2.1)}(t, x) \cong -\gamma(t) + g(t)$ ;

(ii) for each  $M > D$  there exists  $L(M) > 0$  with  $|Q(t, x)| \cong L(M)$  for  $t \cong 0$  and  $D \cong |x| \cong M$ .

Then condition (v) of Theorem 3.1 holds with these numbers  $H$  and  $D$ .

Proof. Let  $M > D$  be given and let a solution  $x(t)$  of (2.1) satisfy  $B \cong |x(t)| \cong M$  and  $P(t, x(t)) \cong 2H(M)$  for  $t \in [t_0, t_0 + T(M)]$ , where  $T(M) > 0$  is a constant such that

$$\int_{t_0}^{t_0 + T(M)} \gamma(s) ds > 2L(M) + \int_0^\infty [g(s)]_+ ds \quad \text{for all } t_0 \cong 0.$$

If  $|x(t)| \cong D$  for  $t \in [t_0, t_0 + T(M)]$ , then we get

$$-L(M) \cong Q(t_0 + T(M), x(t_0 + T(M))) \cong L(M) - \int_{t_0}^{t_0 + T(M)} \gamma(t) dt + \int_0^\infty [g(s)]_+ ds$$

which yields a contradiction to the choice of  $T(M)$ . Consequently, there is  $s \in [t_0, t_0 + T(M)]$  with  $|x(s)| < D$ , and the proof is complete.

Example 3.2. Consider the equation

$$(3.6) \quad \ddot{x} + a(t)\dot{x} + f(x) = e(t)$$

and suppose that the continuous functions  $a, e: \mathbf{R}^+ \rightarrow \mathbf{R}, f: \mathbf{R} \rightarrow \mathbf{R}$  satisfy the following conditions:

(i)  $a(t) \geq 0$  for  $t \in \mathbf{R}^+, a$  is weakly integrally positive, and there exist constant  $\tilde{a} > 0, T > 0$  such that  $[t_0 \geq 0, t \geq T]$  imply  $(1/t) \int_{t_0}^{t_0+t} a(s) ds \leq \tilde{a}$ ;

(ii)  $e \in L^1[0, \infty)$ ;

(iii) there is an  $r_0 > 0$  such that  $xf(x) > 0, |f(x)| > 0$  provided  $|x| > r_0$ , and  $F(x) = \int_0^x f(s) ds \rightarrow \infty$ , as  $|x| \rightarrow \infty$ .

Then the solutions of equation (3.6) and their derivatives are U.B. and E.U.B. If, in addition, the function  $a(t)$  is integrally positive, then the solutions and their derivatives are U.B. and U.U.B.

Equation (3.6) is equivalent to the system

$$(3.7) \quad \dot{x} = y, \quad \dot{y} = -f(x) - a(t)y + e(t).$$

Define  $V(t, x, y) = [y^2 + 2F(x)]^{1/2} + \int_t^\infty |e(s)| ds$ . Then

$$\dot{V}_{(3.7)}(t, x, y) \leq -a(t)y^2[y^2 + 2F(x)]^{-1/2}.$$

Choose  $K(t, x, y) = y^2[y^2 + 2F(x)]^{-1/2}, P(t, x, y) = y^2$ . Then

$$[\dot{P}_{(3.7)}(t, x, y)]_+ = [-f(x)y - a(t)y^2 + e(t)y]_+ \leq |f(x)||y| + |e(t)||y|.$$

Let  $B > 0$  be fixed arbitrarily. For  $M > B$  let  $K_M = \max \{|f(x)|: 0 \leq |x| \leq M\}$  and suppose  $B \leq |x| + |y| \leq M$ . Then  $[\dot{P}_{(3.7)}(t, x, y)]_+ \leq [K_M + |e(t)|]M$  and  $\int_0^t (K_M + |e(s)|)M ds$  is uniformly continuous in  $\mathbf{R}^+$ . Consequently, conditions (i)–(iv) of Theorem 3.1 (see also Remark 3.3) are met with arbitrary  $H > 0$ , and the solutions are U.B.

Now define  $D = r_0 + 1, H_0 = 1/2$ , and

$$Q(t, x, y) = \begin{cases} y & \text{if } x \geq r_0, \\ -y & \text{if } x \leq -r_0, \end{cases}$$

whose derivative is

$$\dot{Q}_{(3.7)}(t, x, y) = \begin{cases} -f(x) - a(t)y + e(t) & \text{if } x \geq r_0, \\ f(x) + a(t)y - e(t) & \text{if } x \leq -r_0. \end{cases}$$

For a given  $M > D$  introduce the notation  $m(M) = \min \{|f(x)|: r_0 \leq |x| \leq M\}$ . By the conditions,  $m(M) > 0$ , and  $[t \geq 0, D \leq |x| + |y| \leq M, y^2 \leq 2H]$  imply the inequality

$$\dot{Q}_{(3.7)}(t, x, y) \leq -m(M) + a(t)[2H]^{1/2} + e(t).$$

Let  $H = \min \left\{ \frac{1}{2} [m(M)/(\tilde{a}+1)]^2, \frac{1}{2} \right\}$ ,  $\gamma(t) = m(M) - (2H)^{1/2} a(t)$  and  $g(t) = |e(t)|$ .

Then  $\dot{Q}_{(3.7)}(t, x, y) \leq -\gamma(t) + g(t)$  and for sufficiently large  $T > 0$ ,

$$\int_{t_0}^{t_0+T} \gamma(t) dt = m(M)T - (2H)^{1/2} \int_{t_0}^{t_0+T} a(t) dt \geq m(M)\tilde{a}/(\tilde{a}+1)T \rightarrow \infty$$

as  $T \rightarrow \infty$  uniformly with respect to  $t_0 \geq 0$ , and so all the conditions of Lemma 3.1 are satisfied.

This completes the proof.

Consider now the system

$$(3.8) \quad \dot{x} = X(t, x, y), \quad \dot{y} = Y(t, x, y)$$

where  $x \in \mathbb{R}^m, y \in \mathbb{R}^k; X: \mathbb{R}^+ \times \mathbb{R}^{m+k} \rightarrow \mathbb{R}^m$  and  $Y: \mathbb{R}^+ \times \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$  are continuous. The following theorem shows that the function  $Q$  in Lemma 3.1 can be constructed from the reduced subsystem

$$(3.9) \quad \dot{y} = Y(t, 0, y).$$

**Theorem 3.4.** *Suppose that*

(i) *There exist constants  $B, H \geq 0$  and a locally Lipschitz function  $V(t, x, y)$  defined for  $t \geq 0$  and  $|x| + |y| \leq B$  such that*

(a)  $W_1(|x| + |y|) \leq V(t, x, y) \leq W_2(|x| + |y|)$ , *where  $W_1$  and  $W_2$  are unbounded pseudo wedges;*

(b)  $\dot{V}_{(3.8)}(t, x, y) \leq -\lambda(t)K(x, y)$  *for  $t \geq 0$  and  $|x| + |y| \leq B$ , where  $\lambda(t)$  is weakly integrally positive,  $K(x, y) \geq 0$  for  $|x| + |y| \leq B$ , and for any  $M > B$  there exists  $k(M) > 0$  such that  $K(x, y) \geq k(M)$  for  $H \leq |x|, B \leq |x| + |y| \leq M$ ;*

(ii) *there exist a constant  $B_1 > 0$ , a continuous  $N: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $N(s) > 0$  for  $s \geq B_1$  and a locally Lipschitz function  $Q(t, y)$  defined for  $t \geq 0$  and  $|y| \geq B_1$  such that*

(c)  $0 \leq Q(t, y) \leq W_3(|y|)$ , *where  $W_3$  is a pseudo wedge;*

(d)  $\dot{Q}_{(3.9)}(t, y) \leq -W_4(|y|)$  *for  $|y| \geq B_1$ , where  $W_4$  is a pseudo wedge;*

(e)  $|Q(t, y) - Q(t, \tilde{y})| \leq N(\max\{|y|, |\tilde{y}|\})|y - \tilde{y}|$ ;

(iii) *for any  $M > 0$  there exists  $L(M) > 0$  such that  $|X(t, x, y)| \leq L(M)$  if  $|x| + |y| \leq M$ ;*

(iv) *there exist continuous  $P_1, P_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $P_1(s) > 0$  for  $s \geq B_1$  such that  $|Y(t, x, y) - Y(t, 0, y)| \leq P_1(|y|)P_2(|x|)$ ;*

(v)  $\lim_{r \rightarrow \infty} W_4(r)/(P_1(r)N(r)) = \infty$ .

*Then the solutions of (3.8) are U.B. and E.U.B. If, in addition,  $\lambda$  is integrally positive, then the solutions of (3.8) are U.B. and U.U.B.*

**Proof.** Obviously, (i)–(iv) of Theorem 3.1 hold with  $P(t, x, y) = |x|$ .

Choose  $D > 0$  such that  $D - 2H \geq B_1, W_4(r)/N(r)P_1(r) \geq \max\{P_2(s) : |s| \leq 2H\} + 1$

for  $s \cong D - 2H$ . Then if  $D \cong |x| + |y| \cong M$ ,  $|x| \cong 2H$ , then  $|y| \cong D - 2H \cong B_1$ , and thus

$$\begin{aligned} \dot{Q}_{(3.8)}(t, y) &\cong \dot{Q}_{(3.9)}(t, y) + N(|y|)|Y(t, x, y) - Y(t, 0, y)| \cong -W_4(|y|) + \\ &+ N(|y|)P_1(|y|)P_2(|x|) \cong -N(|y|)P_1(|y|) \left[ \frac{W_4(|y|)}{N(|y|)P_1(|y|)} - P_2(|x|) \right] - \\ &- N(|y|)P_1(|y|) \cong -\inf \{ N(r)P_1(r) : B_1 \cong r \cong M \}. \end{aligned}$$

Therefore, condition (v) of Theorem 3.1 holds by Lemma 3.1, and so the proof is complete.

Example 3.3. Consider now the system

$$(3.10) \quad \dot{x} = f_1(t, x) + by, \quad \dot{y} = f_2(t, x) + dy + e(t),$$

where  $f_1, f_2 \in C(\mathbf{R}^+ \times \mathbf{R}, \mathbf{R})$  with  $f_1(t, 0) = 0, f_2(t, 0) = 0, e(t)$  is a bounded continuous function on  $\mathbf{R}^+$  with  $e \in L^1[0, \infty)$ ,  $b, d$  are constants with  $db \neq 0$ . Besides, we assume

- (i)  $\sup \{ |f_1(t, x)| + |f_2(t, x)| : t \cong 0, |x| \cong M \} < \infty$  for any  $M > 0$ ;
- (ii)  $[df_1(t, x) - bf_2(t, x)]/x \cong \alpha(x) > 0$  for  $t \cong 0$  and  $x \neq 0$ , where  $\alpha$  is continuous

and  $\lim_{|x| \rightarrow \infty} \int_0^x \alpha(r)r \, dr = \infty$ ;

- (iii)  $[f_1(t, x) + dx][bf_2(t, x) - df_1(t, x)] - \int_0^x [(d\partial f_1(t, r)/\partial t) - (b\partial f_2(t, r)/\partial t)] \, dr \cong \lambda(t)\beta(x)$ , where  $\lambda(t)$  is integrally positive,  $\beta$  is continuous with  $\beta(x) > 0$  if  $x \neq 0$ .

Under these conditions the solutions of (3.10) are U.B. and U.U.B.

Indeed, let

$$V(t, x, y) = [(dx - by)^2 + 2 \int_0^x [df_1(t, r) - bf_2(t, r)] \, dr]^{1/2} + b \int_t^\infty |e(s)| \, ds.$$

Then

$$\begin{aligned} \dot{V}_{(3.10)}(t, x, y) &\cong \\ &- [bf_2(t, x) - df_1(t, x)][f_1(t, x) + dx] + \int_0^x \left[ d \frac{\partial}{\partial t} f_1(t, r) - b \frac{\partial}{\partial t} f_2(t, r) \right] \, dr \\ &\cong \frac{\int_0^x \left[ d \frac{\partial}{\partial t} f_1(t, r) - b \frac{\partial}{\partial t} f_2(t, r) \right] \, dr}{[(dx - by)^2 + 2 \int_0^x [df_1(t, r) - bf_2(t, r)] \, dr]^{1/2}} \cong \\ &\cong -\lambda(t)K(x, y), \end{aligned}$$

where

$$K(x, y) = \beta(x) \left[ (dx - by)^2 + 2 \sup_{t \cong 0} \int_0^x [df_1(t, r) - bf_2(t, r)] \, dr \right]^{-1/2}.$$

It is easy to prove that for any  $M > 0$  there exists  $k = k(M) > 0$  such that  $[|x| + |y| \leq M, |x| \geq H]$  imply  $K(x, y) > k(M)$ . Therefore, (i) of Theorem 3.4 holds.

On the other hand, for the subsystem

$$(3.11) \quad \dot{y} = dy + e(t)$$

and for  $Q(t, y) = y^2/2, N(r) = r$ , we have

$$\dot{Q}_{(3.11)}(t, y) \leq d|y| [ |y| + (1/d) \sup_{t \geq 0} |e(t)| ] \leq (1/2) dy^2 \quad \text{for } |y| \leq -(2/d) \sup_{t \geq 0} |e(t)|.$$

Therefore, after making the choice  $P_1(r) = 1, P_2(r) = \sup \{ |f_2(t, x)| : t \geq 0, |x| \leq r \}$  all the conditions of Theorem 3.4 are met, and our assertion is true.

**Theorem 3.5.** *For system (3.8), suppose that*

(i) *there exist continuous functions  $P_1, P_2: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $P_1(s) > 0$  for  $s > 0$  such that  $|Y(t, x, y) - Y(t, 0, y)| \leq P_1(|y|)P_2(|x|)$ ;*

(ii) *there exist a constant  $B_1 > 0$  and a locally Lipschitz function  $V_1(t, x, y)$  defined for  $t \geq 0, |x| \leq B_1$  and  $y \in \mathbf{R}^k$  such that*

$$W_1(|x|) \leq V_1(t, x, y) \leq W_2(|x|),$$

$$\dot{V}_{1(3.8)}(t, x, y) \leq -W_3(|x|) \quad \text{for } t \geq 0, |x| \leq B_1 \text{ and } y \in \mathbf{R}^k,$$

where  $W_1$  and  $W_2$  are unbounded pseudo wedges and  $W_3: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is continuous with  $W_3(r) > 0$  for  $r \geq B_1$ ;

(iii) *there exist a constant  $B_2 > 0$ , a locally Lipschitz function  $V_2(t, y)$  defined for  $t \geq 0$  and  $|y| \leq B_2$ , and a positive continuous function  $N: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $N(r) > 0$  for  $r \geq B_2$  and such that*

$$W_4(|y|) \leq V_2(t, y) \leq W_5(|y|),$$

$$\dot{V}_{2(3.9)}(t, y) \leq -W_6(|y|) \quad \text{for } |y| \leq B_2,$$

$$|V_2(t, y) - V_2(t, \tilde{y})| \leq N(\max \{ |y|, |\tilde{y}| \}) |y - \tilde{y}|,$$

where  $W_4, W_5$  are unbounded pseudo wedges,  $W_6$  is nonnegative and continuous with  $\lim_{r \rightarrow \infty} W_6(r)/(N(r)P_1(r)) = \infty$ .

Then the solutions of (3.8) are U.B. and U.U.B.

**Proof.** First, we shall prove the uniform boundedness. For any  $\alpha > \max \{ B_1, B_2 \}$ , there exist  $\beta(\alpha), \beta_1(\alpha)$  and  $\beta_2(\alpha) > 0$  such that  $W_1(\beta(\alpha)) > W_2(\alpha), \beta_2(\alpha) > \beta_1(\alpha) > \alpha, W_6(s)/N(s)P_1(s) - \max_{r \leq \beta(\alpha)} P_2(r) \geq 1$  for  $s \geq \beta_1(\alpha)$ , and  $W_4(\beta_2(\alpha)) > W_5(\beta_1(\alpha))$ . Then for any solution  $(x(t), y(t))$  with  $|x(t_0)| < \alpha$ , and  $|y(t_0)| < \alpha$ , we have  $x(t) < \beta(\alpha)$  and  $|y(t)| < \beta_2(\alpha)$  for  $t \geq t_0$ .

If this is not true, then only two cases may occur:

*Case 1.* There exist  $t_2 > t_1 > t_0$  with  $|y(t_1)| = \beta_1(\alpha)$ ,  $|y(t_2)| = \beta_2(\alpha)$ ,  $\beta_1(\alpha) < |y(t)| < \beta_2(\alpha)$  for  $t \in (t_1, t_2)$  and  $|x(t)| < \beta(\alpha)$  for  $t \in [t_0, t_2)$ .

*Case 2.* There exist  $t_4 > t_3 > t_0$  such that  $|x(t_3)| = \alpha$ ,  $|x(t_4)| = \beta(\alpha)$ ,  $\alpha < |x(t)| < \beta(\alpha)$  for  $t \in (t_3, t_4)$  and  $|y(t)| \equiv \beta_2(\alpha)$  for  $t \in [t_3, t_4)$ .

In Case 1, for  $t \in [t_1, t_2]$ , we have

$$\begin{aligned} \dot{V}_{2(a.8)}(t, y(t)) &\equiv -W_6(|y(t)|) + N(|y(t)|)P_1(|y(t)|)P_2(|x(t)|) \equiv \\ &\equiv -N(|y(t)|)P_1(|y(t)|) \left[ \frac{W_6(|y(t)|)}{N(|y(t)|)P_1(|y(t)|) - P_2(|x(t)|)} \right] \equiv \\ &\equiv -N(|y(t)|)P_1(|y(t)|) \equiv 0. \end{aligned}$$

Therefore,  $W_4(\beta_2(\alpha)) \equiv V_2(t_2, y(t_2)) \equiv V_2(t_1, y(t_1)) \equiv W_5(\beta_1(\alpha))$ . This contradicts  $W_4(\beta_2(\alpha)) > W_5(\beta_1(\alpha))$ .

In Case 2, for  $t \in [t_3, t_4]$ , we have  $\dot{V}_1(t, x(t), y(t)) \equiv 0$ , thus

$$W_1(\beta(\alpha)) \equiv V_1(t_4, x(t_4), y(t_4)) \equiv V_1(t_3, x(t_3), y(t_3)) \equiv W_2(\alpha),$$

which contradicts  $W_1(\beta(\alpha)) > W_2(\alpha)$ .

Therefore,  $|x(t; t_0, x_0, y_0)| < \beta(\alpha)$  and  $|y(t; t_0, x_0, y_0)| < \beta_2(\alpha)$  for  $t \geq t_0$  if  $|x_0| < \alpha$  and  $|y_0| < \alpha$ . This completes the proof of uniform boundedness.

Let  $v_1(\alpha) = \min \{W_3(r) : B_1 + 1 \leq r \leq \beta(\alpha)\}$  and  $T_1(\alpha) = W_2(\alpha)/v_1(\alpha)$ . If  $|x(t)| \equiv \equiv B_1 + 1$  holds for  $t \in [t_0, \bar{t}]$  ( $\bar{t} > t_0 + T_1(\alpha)$ ) then

$$\begin{aligned} W_1(B_1 + 1) &\equiv V_1(\bar{t}, x(\bar{t}), y(\bar{t})) \equiv V_1(t_0, x(t_0), y(t_0)) - v_1(\alpha)(\bar{t} - t_0) < \\ &< W_2(\alpha) - v_1(\alpha)W_2(\alpha)/v_1(\alpha) = 0, \end{aligned}$$

which yields a contradiction. Therefore, there exists  $t_5 \in [t_0, t_0 + T_1(\alpha)]$  with  $|x(t_5)| \equiv \equiv B_1 + 1$ . Following the same argument as in the proof of uniform boundedness, we get  $|x(t)| < \beta(B_1 + 1)$  for  $t \geq t_5$ , especially for  $t \geq t_0 + T_1(\alpha)$ .

Choose  $B_3 > B_2$  with  $W_6(s)/N(s)P_1(s) - \max \{P_2(r) : |r| < \beta(B_1 + 1)\} \equiv \equiv 1$  for  $s \geq B_3$ . If  $|y(t)| \equiv \equiv B_3$  for  $t \geq t_0 + T_1(\alpha)$ , then there exists  $v_2(\alpha) > 0$  such that  $P_1(|y(t)|)N(|y(t)|) \equiv \equiv v_2(\alpha)$ , and so

$$\begin{aligned} \dot{V}_{2(a.8)}(t, y(t)) &\equiv -P_1(|y(t)|)N(|y(t)|) \left[ \frac{W_6(|y(t)|)}{N(|y(t)|)P_1(|y(t)|) - P_2(|x(t)|)} \right] \equiv \\ &\equiv -N(|y(t)|)P_1(|y(t)|) \equiv -v_2(\alpha). \end{aligned}$$

Therefore, if  $|y(t)| \equiv \equiv B_3$  for  $t \in [t_0 + T_1(\alpha), t_0 + T_1(\alpha) + \bar{t}]$ , then

$$\begin{aligned} V_2(t_0 + T_1(\alpha) + \bar{t}, y(t_0 + T_1(\alpha) + \bar{t})) &\equiv \\ &\equiv V_2(t_0 + T_1(\alpha), y(t_0 + T_1(\alpha))) - v_2(\alpha)\bar{t} \equiv W_5(\beta(\alpha)) - v_2(\alpha)\bar{t}. \end{aligned}$$

If  $\bar{t} \geq T_2(\alpha)$ , where  $T_2(\alpha) = (W_5(\beta_2(\alpha)) - W_4(B_3))/v_2(\alpha)$ , then

$$W_4(B_3) \equiv V_2(t_0 + T_1(\alpha) + \bar{t}, y(t_0 + T_1(\alpha) + \bar{t})) < W_5(\beta_2(\alpha)) - v_2(\alpha)T_2(\alpha) \equiv W_4(B_3),$$

which yields a contradiction, Therefore, there exists  $t_0 \in [t_0 + T_1(\alpha), t_0 + T_1(\alpha) + T_2(\alpha)]$  with  $|y(t_0)| < B_3$ , and thus  $|x(t_0)| < B_4$  and  $|y(t_0)| < B_4$ , where  $B_4 = \max\{B_3, \beta(B_1 + 1)\}$ . This implies  $|x(t)| < \beta(B_4)$  and  $|y(t)| < \beta_2(B_4)$  for  $t \in [t_0 + T_1(\alpha), t_0 + T_2(\alpha)]$ . This completes the proof.

Sometimes in practice it is very difficult to find a Lyapunov function satisfying the condition  $V_1(t, x, y) \equiv W_2(|x|)$  (see Example 3.4). Now we give a modification of Theorem 3.5 asking the much milder property  $V_1(t, x, y) \equiv W_2(|x| + |y|)$ .

**Theorem 3.6.** *Suppose that*

- (i) *conditions (i), (iii) of Theorem 3.5 hold;*
- (ii) *there exist a constant  $B_1 > 0$  and a continuous function  $V_1(t, x, y)$  defined for  $t \geq 0, (x, y) \in \mathbf{R}^{m+k}$  and such that*

$$W_1(|x|) \equiv V_1(t, x, y) \equiv W_2(|x| + |y|),$$

$$\dot{V}_{1(3,8)}(t, x, y) \equiv -W_3(x, y),$$

where  $W_1$  and  $W_2$  are unbounded pseudo wedges, and  $W_3: \mathbf{R}^{m+k} \rightarrow \mathbf{R}^+$  is continuous and  $|x| \geq B_1$  implies  $W_3(x, y) > 0$ ;

- (iii) *for any  $M > 0$  there exists  $L(M) > 0$  such that  $[t \geq 0, |x| + |y| \leq M]$  imply  $|X(t, x, y)| \leq L(M)$ ;*

*Then the solutions of (3.8) are U.B. and U.U.B.*

**Proof.** Obviously, by (ii) for any  $\alpha > 0$ , if  $|x_0| + |y_0| < \alpha$ , then  $|x(t; t_0, x_0, y_0)| < W_1^{-1}(W_2(\alpha)) = \beta(\alpha)$  provided that  $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$  exists. Following the same argument as in the proof of Theorem 3.5, there exists  $\beta_2(\alpha) > 0$  such that  $|y(t; t_0, x_0, y_0)| < \beta_2(\alpha)$  provided that  $|x_0| + |y_0| < \alpha$  and  $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$  exists. Then the solutions of (3.8) are U.B. Throughout the remainder of the proof denote  $x(t) = x(t; t_0, x_0, y_0), y(t) = y(t; t_0, x_0, y_0)$ .

Let  $T_1(\alpha) = W_2(\beta(\alpha) + \beta_2(\alpha)) / \min\{W_3(x, y) : B_1 + 1 \leq |x| \leq \beta(\alpha), |y| \leq \beta_2(\alpha)\}$ . Then by (ii), for any  $\bar{t} \geq t_0$  there is a  $t_1 \in [\bar{t}, \bar{t} + T_1(\alpha)]$  with  $|x(t_1)| < B_1 + 1$ .

Suppose that for all  $t \in [t_1, \bar{t} + T_1(\alpha) + t^*]$  we have  $|x(t)| < B_1 + 2$  and  $|y(t)| \geq B_3$ , where  $B_3 = B_2$  is a fixed constant such that

$$W_6(r) / N(r) P_1(r) - \max\{P_2(s) : 0 \leq s \leq B_1 + 2\} \geq 1 \quad \text{for } r \geq B_3.$$

Then from

$$\begin{aligned} \dot{V}_{2(3,8)}(t, y(t)) &\equiv -N(|y(t)|) P_1(|y(t)|) \left[ \frac{W_6(|y(t)|)}{N(|y(t)|) P_1(|y(t)|)} - P_2(|x(t)|) \right] \equiv \\ &\equiv -\min\{N(r) P_1(r) : B_3 \leq r \leq \beta_2(\alpha)\} = -m \end{aligned}$$

we get

$$\begin{aligned} 0 &\equiv V_2(\bar{t} + T_1(\alpha) + t^*, y(\bar{t} + T_1(\alpha) + t^*)) \equiv \\ &\equiv V_2(t_1, y(t_1)) - m[t^* + T_1(\alpha) + \bar{t} - t_1] \equiv W_5(\beta_2(\alpha)) - m[t^* + T_1(\alpha) + \bar{t} - t_1]. \end{aligned}$$

Therefore,  $t^* < T_2(\alpha) = [W_6(\beta_2(\alpha)) + 1]/m$ . This shows only two cases may occur:

*Case 1.*  $|x(t)| < B_1 + 2$  for all  $t \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$  and there exists  $t_2 \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$  with  $|y(t_2)| < B_3$ . In this case,  $|x(t)| < \beta(B_1 + B_3 + 2)$  and  $|y(t)| < \beta_2(B_1 + B_3 + 2)$  for  $t \geq \bar{t} + T_1(\alpha) + T_2(\alpha)$ .

*Case 2.* There exists  $t_3 \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$  such that  $|x(t_3)| \geq B_1 + 2$ . In this case, there exist  $t_4, t_5 \in [t_1, t_3]$  with  $|x(t_4)| = B_1 + 1$  and  $|x(t_5)| = B_1 + 2$  and  $B_1 + 1 < |x(t)| < B_1 + 2$  for  $t \in (t_4, t_5)$ . By condition (iii)  $t_5 - t_4 \geq 1/L(\beta(\alpha) + \beta_2(\alpha))$ , and (ii) implies  $V_1(\bar{t} + T_1(\alpha) + T_2(\alpha)) \leq V_1(t_5) \leq V_1(t_4) - (t_5 - t_4)m(\alpha) \leq V_1(\bar{t}) - v(\alpha)$ , where  $V_1(t) = V_1(t, x(t), y(t))$ ,  $v(\alpha) = [L(\beta(\alpha) + \beta_2(\alpha))]^{-1}m(\alpha)$ , and  $m(\alpha) = \min\{W_3(x, y) : B_1 + 1 \leq |x| \leq \beta(\alpha), |y| \leq \beta_2(\alpha)\}$ . Making the choice  $\bar{t} = t_m = t_0 + m[T_1(\alpha) + T_2(\alpha)]$  ( $m = 0, 1, 2, \dots$ ) we get that either  $|x(t)| < \beta(B_1 + B_3 + 2)$  and  $|y(t)| < \beta_2(B_1 + B_3 + 2)$  for  $t \geq t_{m+1}$ , or

$$(3.12) \quad V_1(t_{m+1}) \leq V_1(t_m) - v(\alpha).$$

On the other hand,  $0 \leq V_1(t) \leq W_2(\beta(\alpha) + \beta_2(\alpha))$  for  $t \geq t_0$ , and so (3.12) can not be true for  $m = 0, 1, \dots, N$ , where  $N = N(\alpha)$  is a positive integer such that  $N(\alpha)v(\alpha) > W_2(\beta(\alpha) + \beta_2(\alpha))$ . Therefore,  $|x(t)| < \beta(B_1 + B_3 + 2)$  and  $|y(t)| < \beta_2(B_1 + B_3 + 2)$  for  $t \geq t_0 + [N(\alpha) + 1][T_1(\alpha) + T_2(\alpha)]$ . This completes the proof.

**Example 3.4.** Consider the Liénard equation with forcing term

$$(3.13) \quad \ddot{x} + f(x)\dot{x} + g(x) = p(t),$$

where  $f(x)$  and  $g(x)$  are continuous for  $x \in \mathbb{R}$  and  $p(t)$  is continuous for  $t \geq 0$ . Besides, we assume that

- (i)  $f(x) > 1$ ;
- (ii)  $x\{g(x) - x[f(x) - 1]\} \geq 0$ ;
- (iii)  $\int_0^\infty |p(s)| ds < \infty$ .

Then the solutions of (3.13) are U.B. and U.U.B.

**Proof.** System (3.13) is equivalent to

$$(3.14) \quad \dot{x} = -x + y, \quad \dot{y} = -\{g(x) - x[f(x) - 1]\} - [f(x) - 1]y + p(t).$$

Let  $V(t, x, y) = [y^2 + 2 \int_0^x \{g(r) - r[f(r) - 1]\} dr]^{1/2} + \int_t^\infty |p(s)| ds$ .

Then

$$\dot{V}_{(3.14)}(t, x, y) \leq \frac{-[f(x) - 1]y^2 - x\{g(x) - x[f(x) - 1]\}}{[y^2 + 2 \int_0^x \{g(r) - r[f(r) - 1]\} dr]^{1/2}} = -W(x, y).$$



Then  $|y| > 0$  implies  $W(x, y) > 0$ . On the other hand, for the subsystem  $\dot{x} = -x$  the auxiliary function  $V_2(t, x) = x^2$ ,  $N(r) = 2r$  and  $W_6(r) = 2r^2$  satisfy condition (iii) of Theorem 3.5 and so the solutions of (3.13) are U.B. and U.U.B. by Theorem 3.6.

#### 4. An application to a holonomic scleronomic mechanical system

Consider a holonomic scleronomic mechanical system of  $n$  degrees of freedom being under the action of potential, dissipative and gyroscopic forces. The motions such a system can be described by the Lagrangian equation

$$(4.1) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = - \frac{\partial \pi}{\partial q} - B\dot{q} + G\dot{q},$$

where  $q, \dot{q} \in \mathbf{R}^n$  are the vectors of the generalized coordinates and velocities, respectively,  $\pi = \pi(t, q)$  is the potential energy,  $T = T(q, \dot{q}) = (1/2)\dot{q}^T A(q)\dot{q}$  is the kinetic energy where  $A(q)$  is a symmetric  $n \times n$  matrix function ( $v^T$  denotes the transposed of  $v \in \mathbf{R}^n$ );  $B = B(t, q)$  is the symmetric positive semi-definite  $n \times n$  matrix function of dissipation, and  $G = G(t, q)$  is the antisymmetric  $n \times n$  matrix of the gyroscopic coefficients.

By the Hamiltonian variables  $q, p = A(q)\dot{q}$  system (4.1) can be rewritten into the form

$$(4.2) \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = - \frac{\partial H}{\partial q} + (G - B) \frac{\partial H}{\partial p},$$

where  $H = H(t, p, q)$  is the total mechanical energy:

$$H = H(t, q, p) = T + \pi = (1/2)p^T A^{-1}(q)p + \pi(t, q).$$

Choose the auxiliary functions  $V = H(t, p, q)$ ,  $W = p^T q$ . Their derivatives with respect to (4.2) read as follows:

$$\begin{aligned} \dot{H} &= \left( \frac{\partial H}{\partial p} \right)^T (G - B) \frac{\partial H}{\partial p} + \frac{\partial \pi}{\partial t} = -p^T A^{-1}(q)B(t, q)A^{-1}(q)p + \frac{\partial \pi(t, q)}{\partial t} \cong \\ &\cong -\beta(t, q)A^{-1}(q)p^T A^{-1}(q)p + \left[ \frac{\partial \pi(t, q)}{\partial t} \right]_+ \end{aligned}$$

where  $\beta(t, q)$  denotes the smallest eigenvalue of the matrix  $B(t, q)$ ;  $\Lambda(q)$  denotes the largest eigenvalue of  $A(q)$ . It is known from the mechanics that the kinetic energy is a positive definite quadratic form of the velocities, consequently  $\Lambda(q) > 0$  for all  $q \in \mathbf{R}^n$ .

Let

$$A^{-1}(q) = (a_{ij}^{-1}(q))_{n \times n},$$

$$d_{ij} = \left( \frac{\partial a_{ij}^{-1}(q)}{\partial q_1}, \dots, \frac{\partial a_{ij}^{-1}(q)}{\partial q_n} \right) A^{-1}(q)p, \quad D = (d_{ij})_{n \times n},$$

$$e_k = \sum_{i,j=1}^n \frac{\partial a_{ij}^{-1}(q)}{\partial q_k} p_i p_j, \quad e = (e_1, \dots, e_n)^T.$$

Then for  $P := p^T A^{-1}(q)p$ , its derivative with respect to (4.2) is

$$\begin{aligned} \dot{P} &= \left[ -\frac{\partial \pi}{\partial q} - \frac{1}{2} \frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p \right]^T A^{-1}(q)p + \\ &+ p^T A^{-1}(q) \left[ -\frac{\partial \pi}{\partial q} - \frac{1}{2} \frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p \right] + p^T Dp = \\ &= -2 \left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T A^{-1}(q)p + p^T A^{-1}(q)[(G-B)^T + (G-B)]A^{-1}(q)p - \\ &- p^T A^{-1}(q) \frac{\partial}{\partial q} [p^T A^{-1}(q)p] + p^T Dp = -2 \left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T A^{-1}(q)p - \\ &- 2p^T A^{-1}(q)BA^{-1}(q)p - p^T A^{-1}(q)e + p^T Dp; \\ [\dot{P}]_+ &\equiv \left| \frac{\partial}{\partial q} \pi(t, q) \right| F_2(q, p) + F_3(q, p), \end{aligned}$$

where

$$F_2(q, p) = 2|A^{-1}(q)p|, \quad F_3(q, p) = |p||A^{-1}(q)||e| + |D|p^2.$$

Similarly,

$$\begin{aligned} \dot{W} &= \dot{p}^T q + p^T \dot{q} = - \left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T q + \frac{1}{2} e^T q + p^T A^{-1}(q)(G-B)^T q + p^T A^{-1}(q)p, \\ |\dot{W}| &\equiv \left| q^T \frac{\partial \pi(t, q)}{\partial q} \right| - |G(t, q) - B(t, q)|F_5(q, p) - F_4(q, p), \end{aligned}$$

where

$$F_4(q, p) = \frac{1}{2} |e||q| + |A^{-1}(q)|p^2, \quad F_5(q, p) = |A^{-1}(q)||q||p|.$$

It is easy to prove that  $F_i(q, p)$  are continuous for  $p, q \in \mathbb{R}^n$ , and for every  $M > 0$ ,  $\lim_{p \rightarrow 0} \sup_{|q| \leq M} F_i(q, p) = 0$  for  $i = 2, \dots, 5$ . Therefore, from Theorem 3.3 and Remark 3.5, we get the following

**Corollary 4.1.** *Suppose that there are  $B \geq 0$  and unbounded pseudo wedges  $W_1, W_2$  such that*

- (i)  $W_1(|q|) \leq \pi(t, q) \leq W_2(|q|)$  for  $t > 0$  and  $q \in \mathbf{R}^n$ ;  
 (ii) for every  $M > 0$  the function  $\beta_M(t) = \min \{\beta(t, q) : 0 \leq |q| \leq M\}$  is weakly integrally positive;  
 (iii) there is a continuous function  $r : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that  $r(t, u)$  is increasing with respect to  $u$  for every  $t \in \mathbf{R}^+$  and  $[\partial\pi(t, q)/\partial t]_+ \leq r(t, \pi(t, q))$  for  $t \in \mathbf{R}^+$  and  $q \in \mathbf{R}^n$ ;

(iv) for every  $u_0 > 0$  there is a  $u_1 > u_0$  with  $\int_0^\infty r(s, u_1) ds < u_1 - u_0$ ;

(v) for every  $M > 0$  the function  $|\partial\pi(t, q)/\partial q|$  is bounded for  $t \geq 0$  and  $|q| \leq M$ ;

(vi) for every  $M > B$  there are  $\mu_M > 0$  and  $K_M > 0$  such that  $|q^T \partial\pi(t, q)/\partial q| \leq \mu_M |G(t, q) - B(t, q)| \leq K_M$  for  $t \geq 0$  and  $B \leq |q| \leq M$ .

Then the motions are U.B. and E.U.B.

If, in addition,  $\beta_M(t)$  is integrally positive, then the motions are U.B. and U.U.B.

## References

- [1] T. YOSHIKAWA, Liapunov's function and boundedness of solutions, *Funkcial. Ekvac.*, **2** (1959), 95—112.  
 [2] T. YOSHIKAWA, Stability and boundedness of systems, *Arch. Rational Mech. Anal.*, **6** (1960), 409—421.  
 [3] T. YOSHIKAWA, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan (Tokyo, 1966).  
 [4] T. A. BURTON, *Volterra Integral and Differential Equations*, Academic Press (New York, 1983).  
 [5] V. M. MATROSOV, On the stability of motion, *Prikl. Mat. Mekh.*, **26** (1962), 885—895.  
 [6] L. HATVANI, Attractivity theorems for nonautonomous systems of differential equations, *Acta Sci. Math.*, **40** (1978), 271—283.  
 [7] L. HATVANI, A generalization of the Barbashin—Krasovskii theorems to the partial stability in nonautonomous systems, in: *Qualitative Theory of Differential Equations*, Coll. Math. Soc. J. Bolyai, **30** (Szeged, 1979), pp. 381—409.  
 [8] LI SENLIN and WU JIANHONG, On the stability of differential equations, *Ann. Differential Equations*, **1** (1985), 171—187.  
 [9] J. KATO, Boundedness theorems on Liénard type differential equations with damping, *J. Northeast Normal Univ.* (China), to appear.

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