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On J. R. Haddock's Conjecture

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Abstract **In this paper we prove the following conjecture of J.R. Haddock: each solution of the following neutral equation**

$$\frac{d}{dt}[x(t) - cx(t-r)] = -F(x(t)) + F(x(t-r))$$

tends to a constant, if $F : R \to R$ is continuous and increasing, $0 < c < 1$.

KEY WORDS: Convergence, neutral equations.
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1. *INTRODUCTION.*

In this paper, we give a proof of the following J. R. Haddock's conjecture ([1]):

Conjecture: If $o < c < 1$ and $\gamma > 0$ is a quotient of positive odd integers, then each solution of the following equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = -ax^{\gamma}(t) + ax^{\gamma}x(t-r); \qquad a \geq 0, r \geq 0 \tag{1}$$

tends to a constant as $t \to \infty$.

In fact, we will prove that the same conclusion holds for the following more general equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = -F(x(t)) + F(x(t-r)), \tag{2}$$

where $F : R \to R$ is increasing and continuous, that is, we prove the following

Theorem: Each solution of the equation (2) tends to a constant as $t \to \infty$.

2. *Lemmas*

In this section, we will establish three important lemmas to be used to prove our main theorem.

Lemma 1. Let $c > 0$ and

$$A_i = max_{ir \le t \le (i+1)r} max\{(1-c)x(t), x(t) - cx(t-r)\}.$$

Then

$$A_0 \ge A_1 \ge A_2 \ge ... \ge A_m \ge A_{m+1} \ge ...$$

Proof: By way of contradiction, if this is not true, then there exists an integer m so that $A_m < A_{m+1}$, that is,

$$max_{mr \le t \le (m+1)r} max\{(1-c)x(t), x(t) - cx(t-r)\}$$
$$\le max_{(m+1)r \le t \le (m+2)r} max\{(1-c)x(t), x(t) - cx(t-r)\}.$$

Let $\alpha_{m+1} \in [(m+1)r, (m+2)r]$ so that

$$max\{(1-c)x(\alpha_{m+1}), x(\alpha_{m+1}) - cx(\alpha_{m+1} - r)\} = A_{m+1}.$$

Then we have only two cases

Case 1: $A_{m+1} = (1-c)x(\alpha_{m+1}) > x(\alpha_{m+1}) - cx(\alpha_{m+1} - r)$.

In this case, we have

$$x(\alpha_{m+1}) \le x(\alpha_{m+1} - r),$$

and therefore

$$A_{m+1} = (1-c)x(\alpha_{m+1}) \le (1-c)x(\alpha_{m+1} - r) \le A_m$$

which is contrary to the assumption $A_m < A_{m+1}$.

Case 2: $A_{m+1} = x(\alpha_{m+1}) - cx(\alpha_{m+1} - r) \ge (1-c)c(\alpha_{m+1})$.

In this case, we have

$$x(\alpha_{m+1}) - cx(\alpha_{m+1} - r)$$
$$= max_{mr \le t \le (m+2)r}[x(t) - cx(t-r)],$$

and thus if $\alpha_{m+1} < (m+2)r$, then at $t = \alpha_{m+1}$ we have

$$\frac{d}{dt}[x(t) - cx(t-r)] = 0.$$

This means

$$-F(x(\alpha_{m+1})) + F(x(\alpha_{m+1} - r)) = 0,$$

and so

$$x(\alpha_{m+1}) = x(\alpha_{m+1} - r).$$

This implies

$$\begin{aligned} A_{m+1} &= x(\alpha_{m+1}) - cx(\alpha_{m+1} - r) \\ &= (1-c)x(\alpha_{m+1} - r) \\ &\leq A_m \end{aligned}$$

which is contrary to the assumption $A_{m+1} \leq A_m$. So $\alpha_{m+1} = (m+2)r$ and

$$\frac{d}{dt}[x((m+2)r) - cx((m+1)r)] > 0.$$

If there exists $\tau \in [(m+2)r, (m+3)r)$ such that

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0 \qquad \text{for } t \in [(m+2)r, \tau)$$

and at $t = \tau$, we have

$$\frac{d}{dt}[x(t) - cx(t-r)] = 0,$$

then

$$-F(x(\tau)) + F(x(\tau - r)) = 0,$$

and thus

$$x(\tau) = x(\tau - r)$$

This implies

$$x(\tau) - cx(\tau - r) = (1-c)x(\tau - r) \leq A_{m+1}. \tag{3}$$

On the other hand

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0 \qquad \text{for } t \in [(m+2)r, \tau)$$

implies

$$\begin{aligned} A_{m+1} &= x((m+2)r) - cx((m+1)r) \\ &< x(\tau) - cx(\tau - r) \end{aligned}$$

which is contrary to (3). Therefore on the interval $t \in [(m+2)r, (m+3)r)$ we have

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0,$$

and thus

$$\begin{aligned} &x(t) - cx(t-r) \\ &> x((m+2)r) - cx((m+1)r) \\ &= A_{m+1} \qquad \text{for } t \in ((m+2)r, (m+3)r] \end{aligned}$$

This implies

$$A_{m+2} > Am + 1.$$

Using the same argument as above we can prove that

$$A_m < A_{m+1} < A_{m+2} \cdots$$

and

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0$$

for $t \in [(m+2)r, \infty)$. Then at $t = (m+3)r$ we have

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0.$$

This means

$$-F(x(m+3)r)) + F(x((m+2)r)) > 0,$$

and thus

$$x((m+3)r) < x((m+2)r).$$

This implies

$$\begin{aligned} &x((m+3)r) - cx((m+2)r) \\ &< (1-c)x((m+2)r) \\ &\leq x((m+2)r) - cx((m+1)r). \end{aligned}$$

This is contrary to

$$\frac{d}{dt}[x(t) - cx(t-r)] > 0 \qquad \text{on } [(m+2)r, (m+3)r].$$

So, there exists no integer m so that $A_m < A_{m+1}$, and thus

$$A_0 \geq A_1 \geq A_2 \geq \dots \geq A_m \geq A_{m+1} \geq \dots$$

This completes the proof.

Similarly, we can prove

Lemma 2. Let $c > 0$ and

$$B_i = min_{ir \leq t \leq (i+1)r} min\{(1-c)x(t), x(t) - cx(t-r)\}.$$

Then

$$B_0 \leq B_1 \leq B_2 \leq \dots \leq B_m \leq B_{m+1} \leq \dots$$

By Lemma 1 and Lemma 2, we can assert that all solutions of (2) are bounded.

Lemma 3. Let $x : (-r, \infty) \to R$ is a bounded continuous function with

$$lim_{t\to\infty}[x(t) - cx(t-r)] = d$$

exists, then $lim_{t\to\infty} x(t)$ exists and

$$lim_{t\to\infty} x(t) = \frac{d}{1-c}.$$

Proof: Let $|x(t)| \leq M$ for all $t \geq -r$ and for a constant $M > 0$. For any $\epsilon > 0$, there exists an integer $N = N(\epsilon) > 0$ so that for $n \geq N$ we have

$$\frac{|c|^n|d|}{1-c} + |c|^n M < \frac{\epsilon}{2}.$$

Let $y(t) = x(t) - cx(t-r)$ and $lim_{t\to\infty} y(t) = d$. By the assumption there exists $T_1(\epsilon) > 0$ so that

$$|y(t) - d| < \frac{(1-|c|)\epsilon}{2} \qquad \text{for } t \geq T_1(\epsilon).$$

Therefore for any $t \geq T_1(\epsilon) + [N+1]r$, we have

$$\begin{aligned}
&|x(t) - \frac{d}{1-c}| \\
&= |y(t) + cx(t-r) - \frac{d}{1-c}| \\
&= |y(t) + cy(t-r) + c^2x(t-2r) - \frac{d}{1-c}| \\
&= \ldots\ldots \\
&= |y(t) + cy(t-r) + \ldots + c^N y(t-Nr) + C^{N+1}x(t-(N+1)r) - \frac{d}{1-c}| \\
&= |y(t) - d + c[y(t-r) - d] + \ldots + c^N[y(t-Nr) - d] \\
&+ c^{N+1}\frac{d}{1-c} + c^{N+1}x(t-(N+1)r)| \\
&\leq |y(t) - d| + |c||y(t-r) - d| + \ldots + |c|^N|y(t-Nr) - d| + \frac{|c|^{N+1}|d|}{1-c} + |c|^{N+1}M \\
&\leq \frac{1-|c|}{2}\epsilon + |c|\frac{1-|c|}{2}\epsilon + \ldots + |c|^N\frac{1-|c|}{2}\epsilon + \frac{\epsilon}{2} \\
&\leq \frac{1-|c|}{2}\epsilon + |c|\frac{1-|c|}{2}\epsilon + \ldots + |c|^N\frac{1-|c|}{2}\epsilon + \frac{\epsilon}{2} \\
&\leq \epsilon
\end{aligned}$$

This completes the proof.

The following two lemmas are taken from [2, Proportion 4, Propertion 5].

Lemma 4: Consider now the ordinary differential equation

$$\dot{u}(t) = -F(\frac{u(t)}{1-c}) + F(\frac{A+\epsilon}{1-c}), \tag{4}$$

where A is a constant and ϵ is a parameter with $0 \leq \epsilon \leq 1$, and the initial condition

$$u(t_0) = u_0 < A. \tag{5}$$

Let $u(t) = u(t; t_0, \epsilon)$ be the solution of the initial value problem (4)-(5), and $\alpha > 0$ be a given constant. Then there exists a positive constant μ independent of t_0 and ϵ such that

$$(A+\epsilon) - u(t; t_0, \epsilon) \geq \mu > 0 \qquad \text{for } t \in [t_0, t_0+\alpha].$$

Lemma 5. Consider the ordinary differential equation

$$\dot{u}(t) = -F(\frac{u(t)}{1-c}) + F(\frac{A-\epsilon}{1-c}), \tag{6}$$

where A is a constant and ϵ is a parameter with $0 \leq \epsilon \leq 1$, and the initial condition

$$u(t_0) = u_0 > A. \tag{7}$$

Let $u(t) = u(t; t_0, \epsilon)$ be the solution of the initial value problem (6)-(7), and $\alpha > 0$ be a given constant. Then there exists a positive constant ν independent of t_0 and ϵ such that

$$u(t; t_0, \epsilon) - (A-\epsilon) \geq \nu > 0 \qquad \text{for } t \in [t_0, t_0+\alpha].$$

3.*Proof of the Main Results*

Now we are in the position to prove our main theorem.

Proof of Theorem: By Lemma 1, we know

$$A = lim_{t\to\infty} sup max\{(1-c)x(t), x(t) - cx(t-r)\} = lim_{n\to\infty} A_n < +\infty.$$

By Lemma 2, we know that

$$B = lim_{t\to\infty} inf min\{(1-c)x(t), x(t) - cx(t-r)\} = lim_{n\to\infty} B_n > -\infty.$$

Therefore

$$-\infty < B \leq A < +\infty.$$

Let

$$E = lim_{t\to\infty} inf max\{(1-c)x(t), x(t) - cx(t-r)\}.$$

If $E < A$, then we can find a constant $H \in (E, A)$ and an increasing real number sequence $\{\tau_m\}$ so that $\tau_m \in [mr, (m+1)r] := I_m$ and

$$max\{(1-c)x(\tau_m), x(\tau_m) - cx(\tau_m - r)\} = H.$$

It is clear that

$$[\tau_m, \tau_m + 2r] \subseteq I_m \cup I_{m+1} \cup I_{m+2}$$

and so if $t \in [\tau_m, \tau_m + 2r]$, then

$$\begin{aligned}
&(1-c)x(t-r) \\
&\leq max_{\tau_m - r \leq t \leq \tau_m + 2r} max\{(1-c)x(t), x(t) - cx(t-r)\} \\
&\leq max_{(m-1)r \leq t \leq (m+3)r} max\{(1-c)x(t), x(t) - cx(t-r)\} \\
&\leq A_{m-1} =: A + \epsilon_m.
\end{aligned}$$

This implies

$$x(t-r) \leq \frac{A_{m-1}}{1-c} = \frac{A+\epsilon_m}{1-c}.$$

Now on the interval $[\tau_m, \tau_m + 2r]$, define

$$y(t) = x(t) - cx(t-r).$$

If $x(t) < x(t-r)$, then

$$(1-c)x(t) > x(t) - cx(t-r) = y(t)$$

and thus

$$x(t) > \frac{y(t)}{1-c}$$

This implies

$$\begin{aligned}
\dot{y}(t) &= -F(x(t)) + F(x(t-r)) \\
&\leq -F(\frac{y(t)}{1-c}) + F(\frac{A+\epsilon_m}{1-c})
\end{aligned}$$

If $x(t) \geq x(t-r)$, then

$$\begin{aligned}
&\dot{y}(t) = -F(x(t)) + F(x(t-r)) \\
&\leq 0 \\
&\leq -F(\frac{y(t)}{1-c}) + F(\frac{A+\epsilon_m}{1-c})
\end{aligned}$$

and so on the interval $[\tau_m, \tau_m + 2r]$, we always have

$$\dot{y}(t) = -F(\frac{y(t)}{1-c}) + F(\frac{A+\epsilon_m}{1-c})$$

and

$$y(\tau_m) \leq H < A$$

By Lemma 4, there exists a constant $\mu > 0$ so that

$$(A + \epsilon_m) - u(t; \tau_m, H) \geq \mu > 0,$$

where $u(t; \tau_m, H)$ is the solution of the following initial value problem

$$\dot{u}(t) =_F (\frac{u(t)}{1-c}) + F(\frac{A+\epsilon_m}{1-c})$$
$$u(\tau_m) = H$$

Using the usual comparison principle, we obtain

$$u(t; \tau_m, H) \geq y(t) \qquad \text{on } [\tau_m, \tau_m + 2r],$$

and thus

$$A + \epsilon_m - y(t) \geq \mu > 0 \qquad \text{for } t \in [\tau_m, \tau_m + 2r]. \tag{8}$$

On the other hand, on the interval $[\tau_m, \tau_m + 2r] \subseteq I_m \cup I_{m+1} \cup I_{m+2}$, we have

$$max\{(1-c)x(t), y(t)\} \geq A_{m+2},$$

and thus either

(A) there exists $\tau \in [\tau_m, \tau_m + 2r]$ so that $y(\tau) \geq A_{m+2}$, or

(B) $(1-c)x(t) \geq A_{m+2}$ for all $t \in [\tau_m, \tau_m + 2r]$.

In case (A), from (8) we get

$$A_{m-1} - A_{m+2} \geq \mu > 0. \tag{9}$$

In case (B), we have

$$x(t) \geq \frac{A_{m+2}}{1-c},$$

and so using the following equality

$$x(t-r) \leq \frac{A_{m-1}}{1-c},$$

we obtain

$$y(t) = x(t) - cx(t-r) \geq \frac{A_{m+2}}{1-c} - c\frac{A_{m-1}}{1-c}.$$

Substituting the above inequality into (8), we get

$$A_{m-1} - [\frac{A_{m+2}}{1-c} - c\frac{A_{m-1}}{1-c}] \geq \mu > 0,$$

that is

$$\frac{A_{m-1}}{1-c} - \frac{A_{m+2}}{1-c} \geq \mu > 0 \quad (10)$$

Combining (9) and (10), we get

$$A_{m-1} - A_{m+2} \geq (1-c)\mu > 0.$$

The inequality above holds for all $m \geq 2$. This is contrary to $lim_{m\to\infty} A_m = A$.

So $E = A$. That is

$$lim_{t\to\infty} max\{(1-c)x(t), x(t) - cx(t-r)\} = A$$

exists. Similarly,

$$lim_{t\to\infty} min\{(1-c)x(t), x(t) - cx(t-r)\} = B$$

exists. Therefore

$$lim_{t\to\infty}\{(1-c)x(t) + x(t) - cx(t-r)\} = A + B$$

exists. That is

$$lim_{t\to\infty}[x(t) - \frac{cx(t-r)}{2-c}] = \frac{A+B}{2-c}$$

exists. Obviously $|\frac{c}{2-c}| < 1$ and so $lim_{t\to\infty}x(t)$ exists by Lemma 3. This completes the proof.

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