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Globally Stable Periodic Solutions of Linear Neutral Volterra Integrodifferential Equations

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We shall associate the linear neutral Volterra integrodifferential equation

$$\frac{d}{dt}\left[x(t)-\int_{-\infty}^{t} C(t-s)\,x(s)\,ds-g(t)\right]=Ax(t)+\int_{-\infty}^{t} G(t-s)\,x(s)\,ds+f(t) \tag{1}$$

with

$$\frac{d}{dt}\left[x(t)-\int_{0}^{t} C(t-s)\,x(s)\,ds-g(t)\right]=Ax(t)+\int_{0}^{t} G(t-s)\,x(s)\,ds+f(t) \tag{2}$$

via the resolvent equation

$$\begin{aligned}&\frac{d}{dt}\left[Z(t)-\int_{0}^{t} C(t-s)\,Z(s)\,ds\right]=AZ(t)+\int_{0}^{t} G(t-s)\,Z(s)\,ds\\&Z(0)=I.\end{aligned} \tag{3}$$

Here and hereafter, $C(t)$ and $G(t)$ are $n\times n$ matrices continuous for $t\geqslant 0$ $g(t)$ and $f(t)$ n-vectors continuous for $t\in R$ with $f(t+T)=f(t)$ and $g(t+T)=g(t)$ for a constant $T>0$, A a constant $n\times n$ matrix, I the $n\times n$ identity matrix, and Z an $n\times n$ matrix.

In the case where $C(t)=0$ and $g(t)=0$, T. A. Burton [1, Theorem 5] proved that for any bounded solution $x(t)$ of (2) there exists an integer sequence $n_j\to\infty$ (as $j\to\infty$) such that $x(t+n_jT)$ converges to a solution $x^*(t)$ of (1), which, if $Z\in L^1[0,\infty)$, is T-periodic and has the following nice formula, $x^*(t)=\int_{-\infty}^{t} Z(t-s)\,f(s)\,ds$ (see [2, Theorem 1.1]). Similar results can be found in [3] for the case $g(t)=0$ under the assumptions $Z\in L^1[0,\infty)$ and $\lim_{t\to\infty} Z(t)=0$. However, [3] had not gotten sufficient

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conditions to ensure $Z \in L^1[0, \infty)$ and $\lim_{t \to \infty} Z(t) = 0$. In [4], the discussion of the T-periodic solution of (1) (in the case $g(t) = 0$) depended heavily on the behaviors of solutions of the integral equation $h(t) = \int_0^t C(t-s)\, h(s)\, ds + f(t)$.

The present paper is an extension of [1–4]. Using the variation of constants formula for (2), we prove that if $Z, \dot{Z} \in L^1[0, \infty)$, then there exists a unique globally stable T-periodic solution $g(t) + \int_{-\infty}^t \dot{Z}(t-s)\, g(s)\, ds + \int_{-\infty}^t Z(t-s)\, f(s)\, ds$. Some sufficient conditions ensuring $Z, \dot{Z} \in L^1[0, \infty)$ are also given.

The following variation of constants formula generalizes [2, Theorem 1.1] to neutral equations.

THEOREM 1. *There exists an $n \times n$ continuously differentiable matrix $Z(t)$ satisfying Eq.* (3) *with initial value $Z(0) = I$. Moreover, any solution $x(t)$ of* (2) *can be represented by*

$$x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)\, g(s)\, ds + \int_0^t Z(t-s)\, f(s). \tag{4}$$

Proof. The existence of the solution $Z(t)$ of (3) with initial value $Z(0) = I$ follows from the fundamental theory of neutral functional differential equations with infinite delay (cf. [5, 6]). Equation (3) with initial value $Z(0) = I$ is equivalent to

$$Z(t) = I + \int_0^t E(t-s)\, Z(s)\, ds,$$

where $E(t) = A + C(t) + \int_0^t G(v)\, dv$. It is easy to verify that $Z(t) = I + \int_0^t M(s)\, ds$ with $M(t)$ being the solution of $M(t) = E(t) + \int_0^t E(t-s)\, M(s)\, ds$, and so $\dot{Z}(t) = M(t)$ is continuous. On the other hand, Eq. (2) is equivalent to

$$x(t) = x(0) - g(0) + g(t) + \int_0^t f(s)\, ds + \int_0^t E(t-s)\, x(s)\, ds.$$

By a direct verification, we get

$$x(t) = x(0) - g(0) + g(t) + \int_0^t f(s)\, ds + \int_0^t M(t-s)$$
$$\times \left\{ [x(0) - g(0)] + g(s) + \int_0^s f(u)\, du \right\} ds$$

$$= x(0) - g(0) + g(t) + \int_0^t f(s)\,ds + \int_0^t \dot{Z}(t-s)$$

$$\times \left[x(0) - g(0) + g(s) + \int_0^s f(u)\,du \right] ds$$

$$= Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)\,g(s)\,ds$$

$$+ \int_0^t Z(t-s)\,f(s)\,ds$$

This completes the proof.

Following the similar argument to those of [2, Theorem 1.1; 4, Theorem 2; 7, pp. 171–178], we get

THEOREM 2. *Suppose $C, G \in L^1[0, \infty)$. Then for any bounded solution $x(t)$ of (2), there exists an integer sequence $n_j \to \infty$ as $j \to \infty$ such that $x(t + n_j T)$ converges to a solution of (1) on $(-\infty, +\infty)$ and the convergence is uniform on any compact subset of $(-\infty, +\infty)$.*

THEOREM 3. *If $C, G, Z, \dot{Z} \in L^1[0, \infty)$, then (1) has a T-periodic solution $g(t) + \int_{-\infty}^t \dot{Z}(t-s)\,g(s)\,ds + \int_{-\infty}^t Z(t-s)\,f(s)\,ds$, and all solutions of (1) defined for $t \geqslant 0$ with bounded continuous functions on $(-\infty, 0]$ as their initial values tend to this T-periodic solution as $t \to \infty$.*

Proof. $Z, \dot{Z} \in L^1[0, \infty)$ implies that $\lim_{t\to\infty} Z(t) = 0$. Therefore by (4) all solutions of (2) are bounded. On the other hand,

$$x(t + n_j T) = Z(t + n_j T)[x(0) - g(0)] + g(t) + \int_0^{t + n_j T} \dot{Z}(t + n_j T - s)\,g(s)\,ds$$

$$+ \int_0^{t + n_j T} Z(t + n_j T - s)\,f(s)\,ds$$

$$= Z(t + n_j T)[x(0) - g(0)] + g(t) + \int_{-n_j T}^t \dot{Z}(t-s)\,g(s)\,ds$$

$$+ \int_{-n_j T}^t Z(t-s)\,f(t)\,ds$$

$$\to g(t) + \int_{-\infty}^t \dot{Z}(t-s)\,g(s)\,ds + \int_{-\infty}^t Z(t-s)\,f(s)\,ds, \quad \text{as } j \to \infty.$$

By Theorem 2, $g(t) + \int_{-\infty}^t \dot{Z}(t-s)\,g(s)\,ds + \int_{-\infty}^t Z(t-s)\,f(s)\,ds$ is a solution of (1). Obviously, it is T-periodic.

Suppose $x(t)$ is a solution of (1) defined for $t \geqslant 0$ with initial value

$x_0=\varphi$, where φ is a continuous bounded R^n-valued function defined on $(-\infty, 0]$. Then from

$$\frac{d}{dt}\left[x(t)-\int_0^t C(t-s)\,x(s)\,ds-g(t)-\int_{-\infty}^0 C(t-s)\,\varphi(s)\,ds\right]$$
$$=Ax(t)+\int_0^t G(t-s)\,x(s)\,ds+\int_{-\infty}^0 G(t-s)\,\varphi(s)\,ds+f(t)$$

and by Theorem 1, we get

$$x(t)=Z(t)\left[x(0)-\int_{-\infty}^0 C(-s)\,\varphi(s)\,ds-g(0)\right]+g(t)$$
$$+\int_{-\infty}^0 C(t-s)\,\varphi(s)\,ds$$
$$+\int_0^t \dot{Z}(t-s)\left[g(s)+\int_{-\infty}^0 C(s-u)\,\varphi(u)\,du\right]ds$$
$$+\int_0^t Z(t-s)\left[f(s)+\int_{-\infty}^0 G(s-u)\,\varphi(u)\,du\right]ds.$$

The boundedness of φ and $C\in L^1[0,\infty)$ imply $\int_{-\infty}^0 C(-s)\,\varphi(s)\,ds$ is a bounded real number, $\int_{-\infty}^0 C(t-s)\,\varphi(s)\,ds=\int_t^{+\infty} C(u)\,\varphi(t-u)\,du\to 0$, $\int_{-\infty}^t G(t-s)\,\varphi(s)\,ds=\int_t^{+\infty} G(u)\,\varphi(t-u)\,du\to 0$ as $t\to\infty$. These show

$$\int_0^t \dot{Z}(t-s)\int_{-\infty}^0 C(s-u)\,\varphi(u)\,du\,ds\to 0$$

and

$$\int_0^t Z(t-s)\int_{-\infty}^0 G(s-u)\,\varphi(u)\,du\,ds\to 0 \qquad \text{as} \quad t\to\infty.$$

On the other hand,

$$\int_{-\infty}^0 \dot{Z}(t-s)\,g(s)\,ds\leqslant\int_t^{+\infty}|\dot{Z}(v)|\,dv\cdot\max_{0\leqslant s\leqslant T}|g(s)|\to 0$$
$$\int_{-\infty}^0 Z(t-s)\,f(s)\,ds\leqslant\int_t^{+\infty}|Z(v)|\,dv\cdot\max_{0\leqslant s\leqslant T}|f(s)|\to 0$$

as $t\to\infty$. Therefore,

$$x(t)-g(t)-\int_{-\infty}^t \dot{Z}(t-s)\,g(s)\,ds-\int_{-\infty}^t Z(t-s)\,f(s)\,ds\to 0 \qquad \text{as} \quad t\to\infty.$$

This completes the proof.

So the key to prove the existence of a unique globally stable T-periodic solution for (1) is to verify Z and $\dot{Z} \in L^1[0, \infty)$. In the remainder of this paper, we shall give some sufficient conditions ensuring them.

LEMMA 1. *If* $\int_0^{+\infty} |C(t)|\, dt < 1$, *then* $\lim_{t\to\infty}[Z(t) - \int_0^t C(t-s) Z(s)\, ds] = 0$ *implies* $\lim_{t\to\infty} Z(t) = 0$.

Proof. The continuity of $Z(t)$ and $\lim_{t\to\infty}[Z(t) - \int_0^t C(t-s) Z(s)\, ds] = 0$ imply the existence of a constant $N > 1$ such that $|Z(t) - \int_0^t C(t-s), Z(s)\, ds| \leqslant N$. If there is a $u \geqslant 0$ with $|Z(u)| = \max_{0 \leqslant s \leqslant u} |Z(s)|$, then

$$|Z(u)| \leqslant \int_0^u |C(u-s)|\, |Z(s)|\, ds + N$$

$$\leqslant \int_0^{+\infty} |C(s)|\, ds\, |Z(u)| + N$$

and thus

$$|Z(u)| \leqslant N \Big/ \left[1 - \int_0^{+\infty} |C(s)|\, ds\right] = M.$$

This shows that $|Z(t)| \leqslant M$ for $t \geqslant 0$. For any $\varepsilon > 0$ choose h sufficiently large so that

$$|D(t)| + \int_h^{+\infty} |C(s)|\, ds\, M < \varepsilon \qquad \text{for} \quad t \geqslant h,$$

where $D(t) = Z(t) - \int_0^t C(t-s)\, Z(s)\, ds$. Therefore for $t \geqslant h$, we have

$$|Z(t)| \leqslant \int_{t-h}^{t} |C(t-s)|\, |Z(s)|\, ds + \int_0^{t-h} |C(t-s)|\, dsM + |D(t)|$$

$$\leqslant \varepsilon + \int_0^{+\infty} |C(t)|\, dt \cdot \max_{t-h \leqslant s \leqslant t} |Z(s)|.$$

Choose $t_n \in I_n = [nh, (n+1)h]$ so that $|Z(t_n)| = \max_{t \in I_n} |Z(t)|$, then

$$|Z(t_n)| \leqslant \begin{cases} \varepsilon + \int_0^{+\infty} |C(t)|\, dt\, |Z(t_{n-1})|, & \text{if there exists } u \in [t_n - h, nh] \text{ with } |Z(u)| = \max_{t \in [t_n - h, t_n]} |Z(t)| \\ \varepsilon + \int_0^{+\infty} |C(t)|\, dt\, |Z(t_n)|, & \text{if } \max_{s \in [t_n - h, t_n]} |Z(s)| = \max_{s \in [nh, t_n]} |Z(s)|. \end{cases}$$

Therefore if $|Z(t_{n-1})| \leqslant \varepsilon/[1-\int_0^{+\infty} |C(t)|\,dt]$, then $|Z(t_n)| \leqslant \varepsilon/[1-\int_0^{+\infty} |C(t)|\,dt]$, and thus $|Z(t_k)| \leqslant \varepsilon/[1-\int_0^{+\infty} |C(t)|\,dt]$ for all $k \geqslant n-1$, and if $|Z(t_n)| > \varepsilon/[1-\int_0^{+\infty} |C(t)|\,dt]$, then $|Z(t_n)| \leqslant \varepsilon + \int_0^{+\infty} |C(t)|\,dt\,|Z(t_{n-1})|$. This implies that either there exists a positive integer K such that $|Z(t)| \leqslant \varepsilon/[1-\int_0^{+\infty} |C(t)|\,dt]$ for $t \geqslant Kh$ or $|Z(t_n)| \leqslant \varepsilon + \int_0^{+\infty} |C(t)|\,dt\,|Z(t_{n-1})|$ for $n = 1, 2, \ldots$. If the latter case occurs, then

$$|Z(t_n)| \leqslant \varepsilon\left[1+\int_0^{+\infty} |C(t)\,dt + \cdots + \left(\int_0^{+\infty} |C(t)|\,dt\right)^n\right] + \left(\int_0^{+\infty} |C(t)|\,dt\right)^{n+1} |Z(t_0)|$$
$$\leqslant \varepsilon \Big/ \left[1-\int_0^{+\infty} |C(t)|\right] dt + \left(\int_0^{+\infty} |C(t)|\,dt\right)^{n+1} M.$$

Therefore these two cases imply $\lim_{t\to\infty} Z(t) = 0$, since ε is sufficiently small. This completes the proof.

LEMMA 2. *If* $\int_0^{+\infty} |C(t)|\,dt < 1$ *and* $G \in L^1[0, \infty)$ *then* $Z(t) - \int_0^t C(t-s)\,Z(s)\,ds \in L^1[0, \infty)$ *implies* $Z, \dot{Z} \in L^1[0, \infty)$.

Proof. Let $D(t) = Z(t) - \int_0^t C(t-s)\,Z(s)\,ds$. Then $Z(t)$ is a fixed point of the mapping T defined by $(TZ)(t) = \int_0^t C(t-s)\,Z(s)\,ds + D(t)$ for $Z \in L^1[0, \infty)$. It is easy to verify that T maps $L^1[0, \infty)$ into itself with

$$\int_0^{+\infty} |(TZ - T\tilde{Z})(t)|\,dt$$
$$= \int_0^{+\infty} \int_0^t |C(t-s)|\,|[Z(s) - \tilde{Z}(s)]|\,ds\,dt$$
$$\leqslant \int_0^{+\infty} |C(t)|\,dt \int_0^{+\infty} |Z(t) - \tilde{Z}(t)|\,dt \quad \text{for } Z, \tilde{Z} \in L^1[0, \infty),$$

that is, T is a contraction mapping from $L^1[0, \infty)$ into itself. Therefore T has a unique fixed point in $L^1[0, \infty)$, that is, $Z \in L^1[0, \infty)$. Z is continuously differentiable, so (3) is equivalent to

$$Z'(t) = \int_0^t C(s)\,Z'(t-s)\,ds + C(t) + AZ(t) + \textstyle\int_0^t G(t-s)\,Z(s)\,ds.$$

Using the same argument as above we get $Z' \in L^1[0, \infty)$. This completes the proof.

For the case where (3) is a scalar equation, we have

THEOREM 4. *Suppose that* $\int_0^{+\infty} |C(t)|\, dt < 1$,

$$\frac{\int_0^{+\infty} |AC(t)+G(t)|\, dt}{1-\int_0^{+\infty} |C(t)|\, dt} + A = \alpha < 0.$$

Then $Z, \dot{Z}, D \in L^1[0, \infty)$, $\lim_{t\to\infty} Z(t) = 0$, $\lim_{t\to\infty} D(t) = 0$, *where* $D(t) = Z(t) - \int_0^t C(t-s)\, Z(s)\, ds$.

Proof. Let

$$F(t) = |AC(t)+G(t)| + \left(\int_0^{+\infty} |AC(t)+G(t)|\, dt \Big/ \left(1 - \int_0^{+\infty} |C(t)|\, dt\right)\right) |C(t)|$$

and

$$V(t, Z_t) = \left| Z(t) - \int_0^t C(t-s)\, Z(s)\, ds \right| + \int_0^t \int_t^{+\infty} F(u-s)\, du\, |x(s)|\, ds.$$

Rewriting (3) as

$$\frac{d}{dt}\left[Z(t) - \int_0^t C(t-s)\, Z(s)\, ds\right] = A\left[Z(t) - \int_0^t C(t-s)\, Z(s)\, ds\right] + \int_0^t [AC(t-s)+G(t-s)]\, Z(s)\, ds,$$

we get

$$\begin{aligned}
\dot{V}(t, Z_t) &\leqslant A\left|Z(t) - \int_0^t C(t-s)\, Z(s)\, ds\right| \\
&\quad + \int_0^t |AC(t-s)+G(t-s)|\, |Z(s)|\, ds \\
&\quad + \int_t^{+\infty} F(u-t)\, du\, |Z(t)| - \int_0^t F(t-s)\, |Z(s)|\, ds \\
&\leqslant A\left|Z(t) - \int_0^t C(t-s)\, Z(s)\, ds\right| \\
&\quad + \int_0^t |AC(t-s)+G(t-s)|\, |Z(s)|\, ds \\
&\quad + \int_0^{+\infty} F(u)\, du \left|Z(t) - \int_0^t C(t-s)\, Z(s)\, ds\right|
\end{aligned}$$

$$+\int_0^{+\infty} F(u)\,du \int_0^t |C(t-s)|\,|Z(s)|\,ds - \int_0^t F(t-s)\,|Z(s)|\,ds$$

$$\leqslant \left[\frac{\int_0^{+\infty} |AC(t)+G(t)|\,dt}{1-\int_0^{+\infty} |C(t)|\,dt} + A\right] \left|Z(t) - \int_0^t C(t-s)\,Z(s)\,ds\right|.$$

Therefore

$$0 \leqslant V(t, Z_t) \leqslant V(0, Z_0) - \alpha \int_0^t |D(s)|\,ds,$$

which implies $\int_0^{+\infty} |D(s)|\,ds \leqslant (1/\alpha)\,V(0, Z_0)$. That is, $D \in L^1[0, \infty)$, and thus $Z, Z' \in L^1[0, \infty)$ by Lemma 2, which implies $\lim_{t\to\infty} Z(t) = 0$ and $\lim_{t\to\infty} D(t) = 0$. This completes the proof.

For the general n-dimensional equation (3), we have

THEOREM 5. *Suppose that A is a stable matrix, B is a positive definite $n \times n$ matrix with $A^{\mathrm{T}}B + BA = -I$, and α, β are positive constants with $\alpha^2 x^{\mathrm{T}} x \leqslant x^{\mathrm{T}} B x \leqslant \beta^2 x^{\mathrm{T}} x$. If $\int_0^{+\infty} |C(t)|\,dt < 1$,*

$$\frac{\int_0^{+\infty} |BAC(t) + BG(t)|\,dt}{\alpha[1 - \int_0^{+\infty} |C(t)|\,dt]} < \frac{1}{2\beta},$$

then $Z, \dot{Z}, D \in L^1[0, \infty)$, $\lim_{t\to\infty} D(t) = 0$ and $\lim_{t\to\infty} Z(t) = 0$, where $D(t) = Z(t) - \int_0^t C(t-s)\,Z(s)\,ds$.

Proof. Let

$$K(t) = \frac{\int_0^{+\infty} |BAC(t) + BG(t)|\,dt}{\alpha[1 - \int_0^{+\infty} |C(t)|\,dt]}\,|C(t)| + \frac{1}{\alpha}\,|BAC(t) + BG(t)|$$

and

$$V(t, x_t) = \left\{\left[x(t) - \int_0^t C(t-s)\,x(s)\,ds\right]^{\mathrm{T}} B \left[x(t) - \int_0^t C(t-s)\,x(s)\,ds\right]\right\}^{1/2}$$
$$+ \int_0^t \int_t^{\infty} K(u-s)\,du\,|x(s)|\,ds,$$

where $x(t)$ is a solution of

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)\,x(s)\,ds\right] = Ax(t) + \int_0^t G(t-s)\,x(s)\,ds.$$

Then

$$
\begin{aligned}
&V(t, x_t)\\
&= \frac{[Ax(t)+\int_0^t G(t-s)\,x(s)\,ds]^{\mathrm{T}} B[x(t)-\int_0^t C(t-s)\,x(s)\,ds]}{2\{(x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B[x(t)-\int_0^t C(t-s)\,x(s)\,ds]\}^{1/2}}\\
&\quad+\frac{[x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B[Ax(t)+\int_0^t G(t-s)\,x(s)\,ds]}{2[\{x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B[x(t)-\int_0^t C(t-s)\,x(s)\,ds]\}^{1/2}}\\
&\quad+\int_t^{+\infty} K(u-t)\,du\,\left|x(t)\right|-\left|\int_0^t K(t-s)\,|x(s)|\,ds\right|\\
&= \frac{-[x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}}[x(t)-\int_0^t C(t-s)\,x(s)\,ds]}{2\{[x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B[x(t)-\int_0^t C(t-s)\,x(s)\,ds]\}^{1/2}}\\
&\quad+\frac{[x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B\int_0^t [AC(t-s)+G(t-s)]\,x(s)\,ds}{\{[x(t)-\int_0^t C(t-s)\,x(s)\,ds]^{\mathrm{T}} B[x(t)-\int_0^t C(t-s)\,x(s)\,ds]\}^{1/2}}\\
&\quad+\int_0^{+\infty} K(u)\,du\left|x(t)-\int_0^t C(t-s)\,x(s)\,ds\right|\\
&\quad+\int_0^{\infty} K(u)\,du\int_0^t |C(t-s)|\,|x(s)|\,ds-\int_0^{+\infty} K(t-s)\,|x(s)|\,ds\\
&\leqslant -\left[\frac{1}{2\beta}-\int_0^{+\infty} K(u)\,du\right]\left|x(t)-\int_0^t C(t-s)\,x(s)\,ds\right|\\
&\quad+\frac{1}{\alpha}\int_0^t |BAC(t-s)+BG(t-s)|\,|x(s)|\,ds\\
&\quad+\int_0^{+\infty} K(u)\,du\int_0^t |C(t-s)|\,|x(s)|\,ds-\int_0^t K(t-s)\,|x(s)|\,ds\\
&= -\left[\frac{1}{2\beta}-\frac{\int_0^{+\infty} |BAC(t)+BG(t)|\,dt}{\alpha[1-\int_0^{\infty} |C(t)|\,dt]}\right]\left|x(t)-\int_0^t C(t-s)\,x(s)\,ds\right|.
\end{aligned}
$$

The remainder of the proof is the same as that of Theorem 4 and so we leave it to the readers.

Remark. Making a change of variable $x(t)=y(t)+k(t)$, where $k(t)$ is the unique T-periodic solution of the following integral equation

$$k(t)=\int_{-\infty}^{t} C(t-s)\,k(s)\,ds+g(t). \tag{5}$$

Equation (1) is equivalent to

$$\frac{d}{dt}\left[y(t)-\int_{-\infty}^{t} C(t-s)\,y(s)\,ds\right]=Ay(t)+\int_{-\infty}^{t} G(t-s)\,y(s)\,ds+f^*(t) \tag{6}$$

with

$$f^*(t) = f(t) + Ak(t) + \int_{-\infty}^{t} G(t-s)\, k(s)\, ds. \tag{7}$$

Therefore, by Theorem 3, (1) has a T-periodic solution

$$k(t) + \int_{-\infty}^{t} Z(t-s)\, f^*(s)\, ds$$

provided that $C, G, Z, \dot{Z} \in L^1[0, \infty)$.

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