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Remarks on "Periodic Solutions of Linear Volterra Equations"[1]

By

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Consider the Volterra equation

$$x'(t) = D(t)x(t) + \int_{-\infty}^{t} E(t, s)x(s)ds + f(t), \tag{1}$$

where x is an n-vector, the function $f(t)$ is continuous on $(-\infty, \infty)$ with the value in $\boldsymbol{R}^n$, D is an $n \times n$ matrix of functions continuous on $(-\infty, \infty)$ with $D(t+T) = D(t)$ for a fixed positive number T, $E(t, s)$ is an $n \times n$ matrix of functions continuous for $-\infty < s \leqslant t < \infty$ and $\int_{-\infty}^{t} |E(t, s)|ds$ is continuous and bounded on $(-\infty, \infty)$.

In [1], T. A. Burton has proved the following theorems (see [1], Theorems 1 and 6).

Theorem 1. *Suppose that*

(i) *if $x(t)$ is a solution of* (1) *on* $[a, \infty)$, *then $x(t+T)$ is also a solution of* (1) *on* $[a-T, \infty)$,

(ii) (1) *has one and only one solution $x^*(t)$ which is bounded on* $(-\infty, \infty)$.

Then $x^*(t)$ is the one and only one T-periodic solution of (1).

Theorem 2. *Suppose that for each $\delta > 0$ there is an $S > 0$ such that $t - t_1 \geqslant S$ implies $\int_{-\infty}^{t_1} |E(t, s)|ds \leqslant \delta$. Also, assume*

(i) *if $x(t)$ is a solution of* (1) *on* $[a, \infty)$, *then $x(t+T)$ is also a solution of* (1) *on* $[a-T, \infty)$,

(ii) *if* (1) *has a solution on* $(-\infty, \infty)$ *which is bounded, it is U.A.S.*,

(iii) *the solution $x(t, 0, 0)$ of* (1) *is bounded on* $[0, \infty)$ *and is equiasymptotically stable at $t_0 = 0$.*

Then (1) *has a T-periodic solution.*

In this paper we get

Theorem 3. *Suppose that*

(i) *if $x(t)$ is a solution of* (1) *on* $[a, \infty)$, *then $x(t+T)$ is also a solution of* (1) *on* $[a-T, \infty)$,

(ii) (1) *has at most one bounded solution on* $(-\infty, \infty)$, *and*

(iii) *there is a constant function* C *and the solution* $x(t, 0, C)$ *of* (1) *is bounded on* $[0, \infty)$.

Then (1) *has a* T*-periodic solution.*

Remark 1. Theorem 3 can be considered as a generalization of Theorem 2. Here we don't suppose the asymptotic stability of bounded solution of (1).

Remark 2. Consider the homogeneous equation

$$x'(t) = D(t)x(t) + \int_{-\infty}^{t} E(t, s)x(s)ds. \tag{2}$$

Obviously, if (2) has at most one solution which is bounded on $(-\infty, \infty)$, then so does (1).

Proof of Theorem 3. By (i), (ii) and Theorem 1, to prove (1) has a T-periodic solution, it is sufficient to prove that (1) has one bounded solution on $(-\infty, \infty)$.

Define a sequence of solutions of (1) on $[-nT, \infty)$ by

$$x_n(t) = \begin{cases} x(t+nT, 0, C), & t \geqslant -nT, \\ C, & t < -nT, \end{cases}$$

where n is a positive integer.

Now, $x(t)=x(t, 0, C)$ is bounded on $[0, \infty)$, so there is a constant M such that $|x(t)| \leqslant M$ for $t \in [0, \infty)$; thus, we also have

$$|x_n(t)| \leqslant M, \quad t \in (-\infty, \infty).$$

$$|x'(t)| \leqslant |D(t)|M + M\int_{-\infty}^{t} |E(t, s)|ds + |f(t)|$$

$$\leqslant B = \text{const}, \quad t \in [0, \infty).$$

Moreover, for $t > -nT$, $|x_n'(t)| \leqslant B$.

For $n \geqslant 2$, $\{x_n(t)\}$ is an equicontinuous and uniformly bounded function sequence on $[-T, T]$. By the Ascoli theorem, there exists a subsequence $\{x_{n_{k,1}}(t)\}$ of $\{x_n(t)\}$ which tends to some continuous function $z_1(t)$ uniformly on $[-T, T]$.

Choose a positive integer K such that for $k \geqslant K$, $n_{k,1} \geqslant 3$. Then $\{x_{n_{k,1}}(t), k \geqslant K\}$ is also an equicontinuous and uniformly bounded function sequence on $[-2T, 2T]$. So there is a subsequence $\{x_{n_{k,2}}(t)\}$ of $\{x_{n_{k,1}}(t)\}$ which converges to some continuous function $z_2(t)$ uniformly on $[-2T, 2T]$. Obviously, $z_1(t) = z_2(t)$ on $[-T, T]$.

Thus, by the same argument, we can choose a subsequence $\{x_{n_{k,m}}(t)\}$ which converges to some continuous function $z_m(t)$ uniformly on $[-mT, mT]$ with $\{x_{n_{k,m-1}}(t)\} \supseteq \{x_{n_{k,m}}(t)\}$ and $z_{m-1}(t) = z_m(t)$ on $[-(m-1)T, (m-1)T]$.

By a diagonal process, we can choose a subsequence $\{x_{n_k}(t)\} \subseteq \{x_n(t)\}$ which converges uniformly on compact subsets of $(-\infty, \infty)$ to some continuous function $z(t)$. Obviously, $|z(t)| \leqslant M$ on $(-\infty, \infty)$.

For any $m > 0$, find a positive integer K such that for any $k \geqslant K$, $n_k > m$. By (i), we have

$$x'_{n_k}(t) = D(t)x_{n_k}(t) + \int_{-\infty}^{t} E(t, s)x_{n_k}(s)ds + f(t), \quad t > -n_k T.$$

So, for $t \in [-mT, mT]$, $k \geqslant K$,

$$(*) \qquad x_{n_k}(t) = x_{n_k}(0) + \int_0^t D(v)x_{n_k}(v)dv + \int_0^t f(v)dv + \int_0^t \left(\int_{-\infty}^{v} E(v, s)x_{n_k}(s)ds \right) dv,$$

$$|E(v, s)x_{n_k}(s)| \leqslant M|E(v, s)|.$$

Since $\int_{-\infty}^{v} |E(v, s)| M ds$ is continuous on $(-\infty, \infty)$, by the Lebesgue-dominated convergence theorem we may take the limit as $k \to \infty$ in $(*)$ and obtain

$$z(t) = z(0) + \int_0^t D(v)z(v)dv + \int_0^t f(v)dv + \int_0^t \left(\int_{-\infty}^{v} E(v, s)z(s)ds \right) dv,$$

for $t \in [-mT, mT]$, and then

$$z'(t) = D(t)z(t) + \int_{-\infty}^{t} E(t, s)z(s)ds + f(t).$$

The relation above holds on every $[-mT, mT]$ and hence on $(-\infty, \infty)$. Thus, $z(t)$ is a bounded solution of (1) on $(-\infty, \infty)$. This completes the proof.

Example 1. Consider the scalar equations

$$(3) \qquad Z'(t) = -Z(t) + 4\int_0^t e^{-(t-s)}Z(s)ds, \quad Z(0) = 1,$$

$$(4) \qquad x'(t) = -x(t) + 4\int_{-\infty}^{t} e^{-(t-s)}x(s)ds + 10\cos t.$$

It is easy to see that the unique solution $Z(t)$ of (3) is

$$Z(t) = (e^t + e^{-3t})/2.$$

By the variation of parameters formula, for any constant k the solution $x(t, 0, k)$ of (4) is given by

$$x(t, 0, k) = Z(t)k + \int_0^t Z(t-s)f(s)ds,$$

where

$$f(t) = 10\cos t + 4k\int_{-\infty}^0 e^{-(t-s)}ds = 10\cos t + 4ke^{-t}.$$

Thus

$$\begin{aligned} x(t, 0, k) &= k(e^t + e^{-3t})/2 \\ &\quad + \int_0^t ((e^{(t-s)} + e^{-3(t-s)})(10\cos s + 4ke^{-s})/2)ds \\ &= 3\sin t - \cos t + (3k+5)e^t/2 - (3+k)e^{-3t}/2. \end{aligned}$$

We have that $x(t, 0, -5/3)$ is bounded on $[0, \infty)$ and that for any $k \neq -5/3$ $x(t, 0, k) \to \infty$ as $t \to \infty$. Obviously, $x(t, 0, -5/3)$ is not equiasymptotically stable at $t_0 = 0$.

Now we want to show that (4) has at most one bounded solution on $(-\infty, \infty)$. Suppose that there are two bounded solutions $x_1(t)$ and $x_2(t)$ of (4) on $(-\infty, \infty)$ with $x_1(t_0) \neq x_2(t_0)$ for some $t_0 \in (-\infty, \infty)$. Let $y(t) = x_1(t) - x_2(t) \not\equiv 0$. Then $y(t)$ is bounded on $(-\infty, \infty)$ and satisfies the homogeneous equation

$$y'(t) = -y(t) + 4\int_{-\infty}^t e^{-(t-s)}y(s)ds,$$

and then differentiation yields a second-order linear ordinary differential equation

$$y''(t) + 2y'(t) - 3y(t) = 0.$$

Thus, we have

$$y(t) = ae^t + be^{-3t}$$

with $a \neq 0$ or $b \neq 0$. This implies $y(t) \to \infty$ as $t \to \infty$ or $t \to -\infty$, a contradiction.

Now, all the conditions of Theorem 3 hold and (4) has one and only one periodic solution

$$x^*(t) = 3\sin t - \cos t.$$

Finally, we want to point out that $\int_{-\infty}^t |E(t, s)|ds$ bounded and continuous on $(-\infty, \infty)$ implies that there is an $M > 0$ such that

$$\int_{-\infty}^t |E(t, s)|ds = \int_0^\infty |E(t, t-u)|du \leqslant M.$$

Then for each $\delta > 0$, there exists an $S > 0$ such that $t - t_1 \geqslant S$ implies

$$\int_{-\infty}^{t_1} |E(t, s)| ds = \int_{t-t_1}^{\infty} |E(t, t-u)| du < \delta.$$

References

[1] Burton, T. A., Periodic solutions of linear Volterra equations, Funckial. Ekvac., **27** (1984), 229–253.

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